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# The GL(1|1)-symplectic fermion correspondence

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**Abstract**

In this note we prove a correspondence between the Wess-Zumino-Novikov-Witten model of the Lie supergroup GL(1|1) and a free model consisting of two scalars and a pair of symplectic fermions. This model was discussed earlier by LeClair. Vertex operators for the symplectic fermions include twist fields, and correlation functions of GL(1|1) agree with the known results for the scalars and symplectic fermions. We perform a detailed study of boundary states for symplectic fermions and apply them to branes in GL(1|1). This allows us to compute new amplitudes of strings stretching between branes of different types and confirming Cardy's condition.

# 1 Introduction

Conformal field theories with supersymmetric target space has become an important area of current research. They are essential in a variety of significant problems both in string theory and in disordered systems.

Understanding sigma models on supersymmetric spaces deep in the strongly coupled regime is of primary importance. In many models one believes that there exists a dual description which is better accessible in such a regime. The most prominent example is certainly the celebrated AdS/CFT correspondence [1, 2], but there are, of course, other interesting dualities involving sigma models on supersymmetric spaces. For example, recently a strong-weak duality between the OSp(2N+2|2N) Gross-Neveu model and the principal chiral model on the supersphere $S^{2N|2N+1}$ was conjectured [3, 4].

There are various ways to find and to test such correspondences. Many supersymmetric spaces possess a family of conformally invariant field theories, and points in the moduli space that are exactly solvable e.g. the Wess-Zumino-Novikov-Witten point on supergroup manifolds or the infinite radius limit of the principal chiral model on the supersphere. In these cases, one way to test a duality is to compute certain quantities, e.g. some boundary spectra, at this solvable point and perform the perturbation to other points in the moduli space exactly [5]. This method has successfully been applied to the supersphere/Gross-Neveu correspondence [6]. The question remains how to actually prove such a correspondence. The case $N = 0$ in the Gross-Neveu-supersphere duality is the well-known correspondence between the O(2) Gross-Neveu model, that is the massless Thirring model, and a free boson on the circle, i.e. bosonization [7]. Unfortunately, the proof does not generalize straightforwardly, but still we believe that bosonization techniques will turn out to be crucial in understanding the correspondence.

If there is a simpler model at hand, it is a good idea to study it in detail to gain insight and to establish techniques for the more complicated models. This leads us to the GL(1|1)-symplectic fermion correspondence. The GL(1|1) WZNW model is probably the best understood CFT with supersymmetric target space that is not free. There exists another CFT with GL(1|1) current symmetry [8], which was used to study the spin quantum Hall transition. This CFT is constructed from the OSp(2|2) Gross-Neveu model at the free point via bosonization and it automatically has GL(1|1) symmetry since the OSp(2|2) Gross-Neveu model is constructed from a spin one half vector transforming in the adjoint representation of GL(1|1). This model consists of two free scalars and a set of symplectic fermions. The symplectic fermions were first analyzed in detail in [9, 10]. The first part of this note is devoted to showing the correspondence between these models. The technique we use is based on bosonization, but in addition we use the affine currents as a guideline which we hope is also useful for other Gross-Neveu-like models.

The correspondence, we find, is remarkable in its own right since the GL(1|1) WZNW model is an interacting theory, while the corresponding model is free, and the bosons are completely decoupled from the fermions. The non-triviality is hidden in the vertex operators, i.e. the GL(1|1) vertex operators in the free description contain twist fields of the fermions and it turns out that the computation of bulk correlation functions in both descriptions is of a similar complexity. Still our method provides a new approach to WZNW models on Lie supergroups. So far, the models have been investigated either algebraically [11] or in terms of fermionic ghost systems [12, 13, 14, 15]. Hopefully, there exist generalizations of our approach to other Lie supergroups leading to a better understanding of them.

In the second part of this note, we apply the correspondence to branes in GL(1|1). For the understanding of Cardy boundary states, the free description is better adapted than the original one. GL(1|1) possesses two classes of branes. One of them, the so-called untwisted branes which geometrically describe superconjugacy classes in the supergroup manifold, have been studied in detail in [16]. It was found that amplitudes of boundary states satisfy Cardy conditions [17] and that they agree with fusion, as expected from experience with rational CFT [18] and also logarithmic CFT [19]. The second class of branes contains just one volume-filling brane. This brane has been investigated in [15], i.e. its correlation functions have been computed, but also the boundary state has been constructed and tested. For a complete description of boundary states one still needs to understand amplitudes of strings stretching between a twisted and an untwisted brane. In our new description this can be done straightforwardly. The final result is that Cardy conditions are still satisfied, and essentially all known results of branes on Lie groups carry over to the Lie supergroup GL(1|1).

The structure of this note is as follows. In section 2 we verify the correspondence in detail. We explain how the currents are used as a guideline to prove the correspondence, and we check the correspondence by comparing correlation functions. Section 3 gives a detailed discussion of boundary states in the symplectic fermions, including twist fields. In section 4 we apply the results of the two previous sections to complete the discussion of GL(1|1) boundary states.

# 2 The GL(1|1)-symplectic fermion correspondence

In this section we will set up the notation and show the relation between the GL(1|1) WZNW model and the free scalars and symplectic fermions. Finally, we will comment on the bulk correlation functions.

## 2.1 The GL(1|1) WZNW model

Our starting point for the relation between the GL(1|1) WZNW model and the free theory will be the first order action for GL(1|1) found in [12]. To set up the notation used in this paper we recall a few facts about the gl(1|1) superalgebra. It is generated by two bosonic elements $E, N$ and two fermionic $\psi^\pm$ which have the following non-zero (anti)commutator relations

$$[N, \psi^\pm] = \pm\psi^\pm, \quad \{\psi^-, \psi^+\} = E. \tag{2.1}$$

Further, we have a family of supersymmetric bilinear forms, but below we will always work with

$$\mathrm{str}(NE) = \mathrm{str}(\psi^+\psi^-) = -1. \tag{2.2}$$

For the GL(1|1) supergroup we choose a Gauss-like decomposition of the form

$$g \;=\; e^{c_-\psi^-}\, e^{XE+YN}\, e^{-c_+\psi^+}.$$

The WZNW model thus has two bosonic fields $X(z,\bar{z}), Y(z,\bar{z})$ and two fermionic fields $c_\pm(z,\bar{z})$, and the action takes the form

$$\begin{aligned} S_{\mathrm{WZNW}}[g(X,Y,c_\pm)] \;&=\; \frac{k}{4\pi}\int_\Sigma d^2z\langle g^{-1}\partial g, g^{-1}\bar{\partial} g\rangle + \frac{k}{24\pi}\int_B \langle g^{-1}dg, [g^{-1}dg, g^{-1}dg]\rangle \\ &=\; \frac{k}{4\pi}\int_\Sigma d^2z\ \left(-\partial X\bar{\partial}Y - \partial Y\bar{\partial}X + 2e^Y\partial c_+\bar{\partial}c_-\right)\ , \end{aligned} \tag{2.3}$$

where $k$ is the level. Variation of the action leads to the usual bulk equations of motion [16].

The holomorphic current of the GL(1|1) WZNW model is in our notation given by $k\partial g g^{-1}$. The components corresponding to the generators are

$$\begin{aligned} J^E &= -k\partial Y, & J^N &= -k\partial X + kc_-\partial c_+\, e^Y\ , \\ J^- &= ke^Y\partial c_+, & J^+ &= -k\partial c_- - kc_-\partial Y\ , \end{aligned} \tag{2.4}$$

Similarly, for the anti-holomorphic current $-kg^{-1}\bar{\partial}g$ the components are

$$\begin{aligned} \bar{J}^E &= k\bar{\partial}Y, & \bar{J}^N &= k\bar{\partial}X - k\bar{\partial}c_-\, c_+\, e^Y\ , \\ \bar{J}^+ &= ke^Y\bar{\partial}c_-, & \bar{J}^- &= -k\bar{\partial}c_+ - kc_+\bar{\partial}Y\ . \end{aligned} \tag{2.5}$$

Let us also mention that the modes of this affine algebra satisfy

$$[J^E_n, J^N_m] = -km\delta_{n+m}, \quad [J^N_n, J^\pm_m] = \pm J^\pm_{n+m}, \quad \{J^-_n, J^+_m\} = J^E_{n+m} + km\delta_{n+m}, \tag{2.6}$$

where we note that the modes can be rescaled such that the algebra is independent of the level $k$. Equation (2.6) corresponds to the OPE

$$J^A(z)J^B(w) \sim -k\frac{\mathrm{str}(AB)}{(z-w)^2} + \frac{[A,B\}}{z-w}. \tag{2.7}$$

## 2.2 First order formulation

Following [12] we will now pass to a first order formalism by introducing two additional fermionic auxiliary fields $b_\pm$ of weight $\Delta(b_\pm) = 1$. Naively, the action would be

$$\frac{1}{4\pi}\int_\Sigma d^2z\ \left(-k\partial X\bar{\partial}Y - k\partial Y\bar{\partial}X + 2b_+\partial c_+ + 2b_-\bar{\partial}c_- + \frac{2}{k}e^{-Y}b_-b_+\right). \tag{2.8}$$

This reduces to (2.3) if we integrate out $b_\pm$ using their equations of motion

$$b_- = k\partial c_+ \exp Y, \qquad b_+ = -k\bar{\partial} c_- \exp Y. \tag{2.9}$$

However, we get a quantum correction in going from the GL(1|1) invariant measure used for the action in (2.3) to the free-field measure $\mathcal{D}X\mathcal{D}Y\mathcal{D}c_-\mathcal{D}c_+\mathcal{D}b_-\mathcal{D}b_+$ that we want to use for our first order formalism. In analogy with [20] the correction is

$$\ln\det\left(|\rho|^{-2}e^{-Y}\partial e^{Y}\bar{\partial}\right) = \frac{1}{4\pi}\int d^2z\left(\partial Y\bar{\partial}Y + \frac{1}{4}\sqrt{G}\mathcal{R}Y\right). \tag{2.10}$$

Here $G$ is the determinant of the world-sheet metric and $\mathcal{R}$ its Gaussian curvature. $|\rho|^2$ is the metric and we have the relation $\sqrt{G}\mathcal{R} = 4\partial\bar{\partial}\log|\rho|^2$. We thus get a correction to the kinetic term and a background charge for $Y$. The first order action including the correction is

$$\begin{aligned} S(X,Y,b_\pm,c_\pm) = \frac{1}{4\pi}\int_\Sigma d^2z\Big( & -k\partial X\bar{\partial}Y - k\partial Y\bar{\partial}X + \partial Y\bar{\partial}Y + \frac{1}{4}\sqrt{G}\mathcal{R}Y \\ & + 2c_+\partial b_+ + 2c_-\bar{\partial}b_- + \frac{2}{k}e^{-Y}b_-b_+\Big). \end{aligned} \tag{2.11}$$

We also get a quantum correction to the current. This will happen where we have to choose a normal ordering of the terms in the current (2.4). We fix this by demanding that the currents obey the OPEs (2.7). Indeed, we have to add $\partial Y$ to $J^N$ to ensure that it has a regular OPE with itself. Thus the holomorphic currents in the free field formalism are

$$\begin{aligned} J^E &= -k\partial Y, & J^N &= -k\partial X + c_-b_- + \partial Y\ , \\ J^- &= b_-, & J^+ &= -k\partial c_- - kc_-\partial Y\ , \end{aligned}$$

where we here and in the following suppress the normal ordering. We get similar expressions for the anti-holomorphic currents.

## 2.3 The correspondence

If we integrate out $b_\pm$ in (2.11) we would simply obtain the GL(1|1) WZNW model. We will now show that if we instead bosonize the $bc$ system to obtain a system of three scalars, it is possible to perform a field redefinition such that one of the scalars decouples. We can then return to a new $b'c'$ formalism and integrate out $b'_\pm$ to arrive at a decoupled theory of two scalars and a set of symplectic fermions.

In this process the current becomes more symmetric and simple. It can be seen as a guideline for which transformations to perform and we will therefore explicitly follow the transformation of the current in each step.

We will start by only discussing the transformation of the action and the current. The map of the vertex operators will be determined in the next subsection.

To begin we bosonize the $bc$ system in (2.11) in the standard way [21]

$$\begin{aligned} c_\pm = e^{\rho^{R,L}}, \qquad b_\pm &= e^{-\rho^{R,L}}, \\ c_+\partial b_+ + c_-\bar{\partial} b_- &= -\frac{1}{2}\partial\rho\bar{\partial}\rho + \frac{1}{8}\sqrt{G}\mathcal{R}\rho, \\ b_-c_- &= -\partial\rho^L, \end{aligned} \tag{2.12}$$

where we denote left and right components of scalars by superscripts $L, R$. In the currents we likewise have to introduce left and right indices and the holomorphic currents then become

$$\begin{aligned} J^E &= -k\partial Y^L, & J^N &= -k\partial X^L + \partial\rho^L + \partial Y^L \,, \\ J^- &= e^{-\rho^L}, & J^+ &= -k\partial(\rho^L + Y^L)e^{\rho^L} \,, \end{aligned} \tag{2.13}$$

and the action is

$$\begin{aligned} S(X, Y, b_\pm, c_\pm) \;=\; & \frac{1}{4\pi}\int_\Sigma d^2z\Big(-k\partial X\bar{\partial}Y - k\partial Y\bar{\partial}X + \partial Y\bar{\partial}Y + \\ & - \partial\rho\bar{\partial}\rho + \frac{1}{4}\sqrt{G}\mathcal{R}(Y+\rho) + \frac{2}{k}e^{-Y-\rho}\Big)\,. \end{aligned} \tag{2.14}$$

We observe, both from the current and the action, that it is very natural to go to variables $Y, Z, \rho'$ where

$$\rho' = Y + \rho, \qquad Z = kX - \rho - Y = kX - \rho'. \tag{2.15}$$

The currents and the action in these variables are

$$\begin{aligned} J^E &= -k\partial Y^L, & J^N &= -\partial Z^L, \\ J^- &= e^{Y^L - \rho'^L}, & J^+ &= -k\partial\rho'^L e^{\rho'^L - Y^L}, \end{aligned} \tag{2.16}$$

$$S(X, Y, b_\pm, c_\pm) = \frac{1}{4\pi}\int_\Sigma d^2z\left(-\partial Z\bar{\partial}Y - \partial Y\bar{\partial}Z - \partial\rho'\bar{\partial}\rho' + \frac{1}{4}\sqrt{G}\mathcal{R}\rho' + \frac{2}{k}e^{-\rho'}\right)\,. \tag{2.17}$$

Hence we got a theory of two scalars decoupled from a Coulomb gas with screening charge. For calculation of correlation functions this is a very efficient formulation of the theory. We will, however, go one step further and rewrite the screened Coulomb gas in terms of symplectic fermions.

We thus return to a $b'c'$ system using again (2.12), but now for the field $\rho'$. This gives us the following simple expressions

$$\begin{aligned} J^E &= -k\partial Y^L, & J^N &= -\partial Z^L, \\ J^- &= e^{Y^L}b'_-, & J^+ &= -ke^{-Y^L}\partial c'_- \,, \end{aligned} \tag{2.18}$$

$$S(X,Y,b'_\pm,c'_\pm)=\frac{1}{4\pi}\int_\Sigma d^2z\left(-\partial Z\bar\partial Y-\partial Y\bar\partial Z+2c'_+\partial b'_+ +2c'_-\bar\partial b'_- +\frac{2}{k}b'_-b'_+\right)\,. \quad (2.19)$$

We can now integrate out the fields $b'_\pm$ getting the equations of motion

$$b'_+=-k\bar\partial c'_-,\quad b'_-=k\partial c'_+\,, \quad (2.20)$$

and arrive at

$$S(X,Y,c_\pm)=\frac{1}{4\pi}\int_\Sigma d^2z\left(-\partial Z\bar\partial Y-\partial Y\bar\partial Z+2k\partial c'_+\bar\partial c'_-\right)\,. \quad (2.21)$$

Of course, we have to be careful when the vertex operators depend on $b'$. As we will see below, the vertex operators for typical representations will be twist operators which we interpret as not containing $b$.

To remove the dependence on the level $k$ in the action we introduce $\chi^a$ by

$$\sqrt{k}c'_+=\chi^1,\quad \sqrt{k}c'_-=\chi^2, \quad (2.22)$$

and the currents and action are then

$$\begin{aligned} J^E&=-k\partial Y^L, & J^N&=-\partial Z^L,\\ J^-&=\sqrt{k}e^{Y^L}\partial\chi^1, & J^+&=-\sqrt{k}e^{-Y^L}\partial\chi^2, \end{aligned} \quad (2.23)$$

$$S(X,Y,\chi^a)=\frac{1}{4\pi}\int_\Sigma d^2z\left(-\partial Z\bar\partial Y-\partial Y\bar\partial Z+\epsilon_{ab}\partial\chi^a\bar\partial\chi^b\right)\,. \quad (2.24)$$

where the anti-symmetric symbol is defined by $\epsilon_{12}=-\epsilon_{21}=1$. This gives the OPEs

$$\begin{aligned} \chi^a(z,\bar z)\chi^b(w,\bar w)&\sim-\epsilon^{ab}\ln|z-w|^2\,,\\ Z(z,\bar z)Y(w,\bar w)&\sim\ln|z-w|^2\,. \end{aligned} \quad (2.25)$$

where $\epsilon^{12}=-1$. This is the action and current that was constructed in [8]. In that reference it was also found that the action has an enlarged OSp(2|2) symmetry.

For future reference, let us sum up the correspondence between the symplectic fermions and the underlying $b',c'$ system. We have

$$\begin{aligned} \bar\partial\chi^1&=\sqrt{k}\bar\partial c'_+, & \bar\partial\chi^2&=\sqrt{k}\bar\partial c'_-=-\frac{1}{\sqrt{k}}b'_+,\\ \partial\chi^1&=\sqrt{k}\partial c'_+=\frac{1}{\sqrt{k}}b'_-, & \partial\chi^2&=\sqrt{k}\partial c'_-, \end{aligned} \quad (2.26)$$

which will be useful in the next section where we study what happens to the vertex operators.

## 2.4 Mapping of the vertex operators

We now consider the mapping of the GL(1|1) vertex operators under the transformation that we found in the last subsection. The basis of vertex operators to be used with the first order action (2.11) were found in [12] by a minisuperspace analysis. We will here use the notation of [15] and write the operators as

$$V_{\langle -e,-n+1\rangle} = \; : e^{eX+nY} : \begin{pmatrix} 1 & c_- \\ c_+ & c_-c_+ \end{pmatrix}, \tag{2.27}$$

and the conformal dimension is

$$\Delta_{(e,n)} = \frac{e}{2k}(2n-1+\frac{e}{k}). \tag{2.28}$$

For $e \neq mk$, where $m$ is an integer, the columns of this matrix will correspond to the two-dimensional representation $\langle -e,-n+1\rangle$ for the left-moving currents while the rows correspond to the representation $\langle e,n\rangle$ under the right-moving currents.

Let us first consider the transformation giving us (2.17):

$$\begin{gathered} X = \frac{1}{k}(\rho'+Z), \\ c_- = e^{\rho_1'^L - Y^L}, \quad b_- = e^{-\rho_1'^L + Y^L}. \end{gathered} \tag{2.29}$$

This maps the vertex operators to

$$V_{\langle -e,-n+1\rangle} = \; : e^{\frac{e}{k}\rho' + \frac{e}{k}Z + nY} \begin{pmatrix} 1 & e^{\rho'^L - Y^L} \\ e^{\rho'^R - Y^R} & e^{\rho' - Y} \end{pmatrix} : . \tag{2.30}$$

Here we generally split scalar fields into the left and right handed part as $\rho' = \rho'^L + \rho'^R$. Some comments are in order here: Firstly, rather than thinking of e.g. $c_-$ in (2.27) as a function to be evaluated under the path integral, we have here used bosonization and will think about the vertex operators in the operator formalism. This means that $c_-$ is a holomorphic operator. Secondly, for the $YZ$ system the vertex operators are

$$V^{\mathrm{B}}_{\langle -e,-n+1\rangle} = \begin{pmatrix} : e^{\frac{e}{k}Z+nY} : & : e^{\frac{e}{k}Z+(n-1)Y^L+nY^R} : \\ : e^{\frac{e}{k}Z+nY^L+(n-1)Y^R} : & : e^{\frac{e}{k}Z+(n-1)Y} : \end{pmatrix}, \tag{2.31}$$

whereas for the $\rho'$ system they are

$$V^{\mathrm{F}}_{\langle -e,-n+1\rangle} = \begin{pmatrix} : e^{\frac{e}{k}\rho'} : & : e^{(\frac{e}{k}+1)\rho'^L + \frac{e}{k}\rho'^R} : \\ : e^{\frac{e}{k}\rho'^L + (\frac{e}{k}+1)\rho'^R} : & : e^{(\frac{e}{k}+1)\rho'} : \end{pmatrix}. \tag{2.32}$$

Thus in the off-diagonal terms, the splitting into holomorphic and anti-holomorphic parts means that the correlation functions calculated in respectively the $YZ$ system and the $\rho'$ system are not separately real, but only the combined correlation function can be expressed in the absolute values of the insertions $z_i$. Also, we see that around the off-diagonal terms in the operator (2.31) the field $Z$ gets an additive twist. The overall twist vanishes due to charge conservation for $Y$.

Since $\rho'$ now appears with non-integer momenta, we see that in going to the $b', c'$ system with action (2.19) we get twist operators. Precisely, the vertex operator (2.32) maps into

$$V^{\mathrm{F}}_{\langle -e,-n+1\rangle} = \begin{pmatrix} \tilde{\mu}^L_{e/k}\tilde{\mu}^R_{e/k} & \tilde{\mu}^L_{e/k+1}\tilde{\mu}^R_{e/k} \\ \tilde{\mu}^L_{e/k}\tilde{\mu}^R_{e/k+1} & \tilde{\mu}^L_{e/k+1}\tilde{\mu}^R_{e/k+1} \end{pmatrix}, \tag{2.33}$$

where the twist states are defined by

$$c'_-(e^{2\pi i}z)\tilde{\mu}^L_\lambda(0) = e^{2\pi i\lambda}\tilde{\mu}^L_\lambda(0). \tag{2.34}$$

This is solved by

$$\tilde{\mu}^L_\lambda \equiv \ : e^{\lambda\rho'^L} : , \tag{2.35}$$

but only uniquely in $\lambda$ modulo integers and, naturally, up to a normalisation. The conformal dimension is $-\frac{1}{2}\lambda(1-\lambda)$ so the ground states have $0<\lambda<1$. We can step $\lambda$ up and down with respectively $c'_-$ and $b'_-$ e.g.

$$c'_-(z)\tilde{\mu}^L_\lambda(0) \sim \frac{1}{z^{-\lambda}}\tilde{\mu}^L_{\lambda+1}(0). \tag{2.36}$$

Also note that

$$\tilde{\mu}^R_\lambda \equiv \ : e^{\lambda\rho'^R} : , \tag{2.37}$$

fulfills

$$c'_+(e^{-2\pi i}\bar{z})\tilde{\mu}^R_\lambda(0) = e^{-2\pi i\lambda}\tilde{\mu}^R_\lambda(0). \tag{2.38}$$

Since $\tilde{\mu}^R_\lambda$ gives opposite transformations compared to the holomorphic operator $\tilde{\mu}^L_\lambda$, but has the same dimension $-\frac{1}{2}\lambda(1-\lambda)$, it in many ways compares to $\tilde{\mu}^L_{1-\lambda}$.

To obtain the symplectic fermions requires integrating out $b'$. This means that the anti-holomorphic part of $c'_-$ is non-trivial in the OPEs. As an example, $c'_+$ and $c'_-$ with action (2.21) have a singular OPE that is $\sim \frac{1}{k}\ln|z-w|^2$. However, using equations (2.26) we get the mapping of $\partial c'_-$ and $b'_-$ to the holomorphic operators $\partial\chi^2$ and $\partial\chi^1$. Likewise, $\bar{\partial}c'_+$ and $b'_+$ will correspond to the anti-holomorphic operators $\bar{\partial}\chi^1$ and $\bar{\partial}\chi^2$.

One has to be careful since we in principle can not integrate out $b'$ when the vertex operators depend on $b'_-b'_+$. However, for the twist operators it seems plausible since, at least for $\lambda>0$, we can naively think of $\mu_\lambda$ as $c'^\lambda$. To check this we will in the next section compare the correlation functions to the already known calculation for the symplectic fermions. The twist fields in the $b', c'$ system then directly translates into twist fields of the symplectic fermions. The symplectic fermion twist fields are defined by [10]

$$\begin{aligned} \chi^1(e^{2\pi i}z)\mu_\lambda(0) &= e^{-2\pi i\lambda}\chi^1(z)\mu_\lambda(0), & \chi^2(e^{2\pi i}z)\mu_\lambda(0) &= e^{2\pi i\lambda}\chi^2(z)\mu_\lambda(0), \\ \bar{\chi}^1(e^{-2\pi i}\bar{z})\mu_\lambda(0) &= e^{-2\pi i\lambda}\bar{\chi}^1(\bar{z})\mu_\lambda(0), & \bar{\chi}^2(e^{-2\pi i}\bar{z})\mu_\lambda(0) &= e^{2\pi i\lambda}\bar{\chi}^2(\bar{z})\mu_\lambda(0), \end{aligned} \tag{2.39}$$

where $\chi^1$ and $\chi^2$ has to transform oppositely to give a symmetry of the Lagrangian. Here we have split the symplectic fermions into their chiral and anti-chiral parts $\chi^a(z,\bar{z}) = \chi^a(z) + \bar{\chi}^a(\bar{z})$. The anti-holomorphic part must transform in the same way under $\bar{z} \mapsto e^{-2\pi i}\bar{z}$, but importantly $\lambda$ can differ by an integer between the holomorphic and anti-holomorphic sector. The condition (2.39) is fulfilled by $\tilde{\mu}^L_\lambda \tilde{\mu}^R_\lambda$ and the other operators in (2.33). However, we have done the rescaling (2.22) so if we think of the twist operator as $(c'_-)^\lambda$ we should choose the following normalisation:

$$\mu^L_\lambda = \sqrt{k}^\lambda \tilde{\mu}^L_\lambda = \sqrt{k}^\lambda : e^{\lambda \rho'^L} : , \tag{2.40}$$

and similarly for the anti-holomorphic part. Thus the vertex operator (2.33) maps into

$$V^{\mathrm{F}}_{\langle -e,-n+1\rangle} \mapsto k^{-\frac{e}{k}} \begin{pmatrix} \mu^L_{e/k}\mu^R_{e/k} & \frac{1}{\sqrt{k}}\mu^L_{e/k+1}\mu^R_{e/k} \\ \frac{1}{\sqrt{k}}\mu^L_{e/k}\mu^R_{e/k+1} & \frac{1}{k}\mu^L_{e/k+1}\mu^R_{e/k+1} \end{pmatrix} . \tag{2.41}$$

A notation with splitting into left and right part, like in the $b'c'$ system, turns out to be useful. The twist values can be stepped up and down using the following OPEs:

$$\partial\chi^1(z)\mu^L_\lambda(0) \sim \frac{1}{z^\lambda}\mu^L_{\lambda-1}(0), \quad \partial\chi^2(z)\mu^L_\lambda(0) \sim \frac{\lambda}{z^{1-\lambda}}\mu^L_{\lambda+1}(0), \tag{2.42}$$

and correspondingly

$$\bar{\partial}\bar{\chi}^1(\bar{z})\mu^R_\lambda(0) \sim \frac{\lambda}{\bar{z}^{1-\lambda}}\mu^R_{\lambda+1}(0), \quad \bar{\partial}\bar{\chi}^2(\bar{z})\bar{\mu}^R_\lambda(0) \sim -\frac{1}{\bar{z}^\lambda}\mu^R_{\lambda-1}(0). \tag{2.43}$$

We note here again that up to a sign the anti-holomorphic side is understood by seeing $\mu^R_\lambda$ as $\mu^L_{1-\lambda}$.

To conclude the total vertex operator $V_{\langle -e,-n+1\rangle}$ in the $YZ$ and symplectic fermion system with action (2.24) takes the form

$$V_{\langle -e,-n+1\rangle} \mapsto k^{-\frac{e}{k}} \begin{pmatrix} :e^{\frac{e}{k}Z+nY}: \mu^L_{e/k}\mu^R_{e/k} & \frac{1}{\sqrt{k}} :e^{\frac{e}{k}Z+(n-1)Y^L+nY^R}: \mu^L_{e/k+1}\mu^R_{e/k} \\ \frac{1}{\sqrt{k}} :e^{\frac{e}{k}Z+nY^L+(n-1)Y^R}: \mu^L_{e/k}\mu^R_{e/k+1} & \frac{1}{k} :e^{\frac{e}{k}Z+(n-1)Y}: \mu^L_{e/k+1}\mu^R_{e/k+1} \end{pmatrix} \tag{2.44}$$

We note that equations (2.42) can be used to check that the columns of this operator transform in the $\langle -e, -n+1\rangle$ representation of GL(1|1) under the left-moving currents (2.23). These operators are indeed close to the operators found in [8]. Let us now check the operators in correlation functions.

## 2.5 Bulk correlation functions

We will now compare the correlation functions of the primary fields (2.27) obtained in the GL(1|1) model to the calculations done for the symplectic fermions in [10]. The similarity was already noted in [12].

Let us first note that from equations (2.31) and (2.32) the vertex operators (2.27) in the $Y, Z, \rho'$ picture (2.17) takes the form

$$V_{\langle -e,-n+1\rangle}{}^{\bar{\sigma}}_{\sigma} = \,:e^{\frac{e}{k}Z+(n-\sigma)Y^L+(n-\bar{\sigma})Y^R}e^{(\frac{e}{k}+\sigma)\rho'^L+(\frac{e}{k}+\bar{\sigma})\rho'^R}:, \tag{2.45}$$

where $\sigma, \bar{\sigma} \in \{0, 1\}$ labels respectively the columns and the rows.

We consider the three-point function

$$A = \langle V_{\langle -e_1,-n_1+1\rangle}{}^{\bar{\sigma}_1}_{\sigma_1}(z_1)V_{\langle -e_2,-n_2+1\rangle}{}^{\bar{\sigma}_2}_{\sigma_2}(z_2)V_{\langle -e_3,-n_3+1\rangle}{}^{\bar{\sigma}_3}_{\sigma_3}(z_3)\rangle. \tag{2.46}$$

The correlation function splits into a $YZ$ and a $\rho'$ part, $A = A^{\mathrm{B}}A^{\mathrm{F}}$. The $YZ$ part is easily evaluated to be

$$\begin{aligned} A^{\mathrm{B}} = \delta\big(\sum_i \frac{e_i}{k}\big)\delta\big(\sum_i(n_i-\sigma_i)\big)\delta\big(\sum_i(n_i-\bar{\sigma}_i)\big) \\ \times\prod_{i<j}(z_i-z_j)^{\frac{e_i}{k}(n_j-\sigma_j)+\frac{e_j}{k}(n_i-\sigma_i)}(\bar{z}_i-\bar{z}_j)^{\frac{e_i}{k}(n_j-\bar{\sigma}_j)+\frac{e_j}{k}(n_i-\bar{\sigma}_i)}, \end{aligned} \tag{2.47}$$

where the indices run from 1 to 3. The $\delta$-functions follow directly from the $J^E$ and $J^N$ currents. The $\rho'$ part is also easily evaluated. Here one has to remember that the overall $\rho'$ charge has to sum to one due to the background charge of $\rho'$. This means that we can maximally have two insertions of the interaction term of the action (2.17). However, as was commented in [12], the part with two interaction terms vanish. The part with one interaction term is calculated using the Dotsenko-Fateev like integral used in [12]. We get

$$\begin{aligned} A^{\mathrm{F}} =\; & \delta\big(\sum_i\sigma_i-1\big)\delta\big(\sum_i\bar{\sigma}_i-1\big)\prod_{i<j}(z_i-z_j)^{(\frac{e_i}{k}+\sigma_i)(\frac{e_j}{k}+\sigma_j)}(\bar{z}_i-\bar{z}_j)^{(\frac{e_i}{k}+\bar{\sigma}_i)(\frac{e_j}{k}+\bar{\sigma}_j)} \\ & -\frac{1}{k}\delta\big(\sum_i\sigma_i-2\big)\delta\big(\sum_i\bar{\sigma}_i-2\big)(-1)^{\sigma_3+\bar{\sigma}_3}\frac{\Gamma(1-\frac{e_1}{k}-\sigma_1)\Gamma(1-\frac{e_2}{k}-\sigma_2)\Gamma(1-\frac{e_3}{k}-\bar{\sigma}_3)}{\Gamma(\frac{e_3}{k}+\sigma_3)\Gamma(\frac{e_1}{k}+\bar{\sigma}_1)\Gamma(\frac{e_2}{k}+\bar{\sigma}_2)} \\ & \times\prod_{i<j}(z_i-z_j)^{(\frac{e_i}{k}+\sigma_i-1)(\frac{e_j}{k}+\sigma_j-1)}(\bar{z}_i-\bar{z}_j)^{(\frac{e_i}{k}+\bar{\sigma}_i-1)(\frac{e_j}{k}+\bar{\sigma}_j-1)}, \end{aligned} \tag{2.48}$$

where the first part is for no interaction term and the second part for one interaction term. We have here used that $\sum_i e_i = 0$ due to the delta-function from the $YZ$ part of the correlation function in (2.47).

If we combine the two parts in (2.47) and (2.48) the symmetry between the holomorphic and anti-holomorphic sector is restored and we arrive at

$$A = \delta\big(\sum_i \frac{e_i}{k}\big)\delta\big(\sum_i (n_i - \sigma_i)\big)\delta\big(\sum_i (n_i - \bar\sigma_i)\big)$$
$$\Bigg(\delta\big(\sum_i \sigma_i - 1\big)\delta\big(\sum_i \bar\sigma_i - 1\big)\prod_{i<j}|z_i - z_j|^{2\frac{e_i}{k}n_j + 2\frac{e_j}{k}n_i + 2\frac{e_ie_j}{k^2}}$$
$$-\frac{1}{k}\delta\big(\sum_i \sigma_i - 2\big)\delta\big(\sum_i \bar\sigma_i - 2\big)(-1)^{\sigma_3+\bar\sigma_3}\frac{\Gamma(1-\frac{e_1}{k}-\sigma_1)\Gamma(1-\frac{e_2}{k}-\sigma_2)\Gamma(1-\frac{e_3}{k}-\bar\sigma_3)}{\Gamma(\frac{e_3}{k}+\sigma_3)\Gamma(\frac{e_1}{k}+\bar\sigma_1)\Gamma(\frac{e_2}{k}+\bar\sigma_2)}$$
$$\times\prod_{i<j}|z_i - z_j|^{2\frac{e_i}{k}(n_j-1) + 2\frac{e_j}{k}(n_i-1) + 2\frac{e_ie_j}{k^2}}\Bigg), \quad (2.49)$$

as was derived in [12]. This indeed supports the validity of our decoupling of the GL(1|1) WZNW model into a set of free scalars and the $\rho'$ system with action (2.17). The result may not look local, e.g. does not seem to be symmetric in interchanging operator 2 and 3, due to the asymmetric-looking $\Gamma$ functions. However, these can be rewritten in the following symmetric form

$$(-1)^{\sigma_3+\bar\sigma_3}\frac{\Gamma(1-\frac{e_1}{k}-\sigma_1)\Gamma(1-\frac{e_2}{k}-\sigma_2)\Gamma(1-\frac{e_3}{k}-\bar\sigma_3)}{\Gamma(\frac{e_3}{k}+\sigma_3)\Gamma(\frac{e_1}{k}+\bar\sigma_1)\Gamma(\frac{e_2}{k}+\bar\sigma_2)} = \prod_i \frac{\Gamma(1-\frac{e_i}{k})}{\Gamma(\frac{e_i}{k})}\left(\frac{-e_i}{k}\right)^{-\sigma_i-\bar\sigma_i}. \quad (2.50)$$

As we see from the result (2.49) one has to be careful in the limit when $e_i$ is an integer multiple of $k$. As was shown in [12] this gives logarithmic correlation functions. For now let us not consider these limits. Thus we get genuine twist operators when going to the symplectic fermions and the twists are $\lambda_i = e_i/k + \sigma_i$ in the holomorphic sector and $\bar\lambda_i = e_i + \bar\sigma_i$ in the anti-holomorphic sector when we compare equation (2.45) with (2.41). As we see from the vertex operators in (2.41), the results that we expect from the symplectic fermions to comply with correlation function (2.48) are

$$\langle\mu^L_{\lambda_1}(z_1)\mu^R_{\bar\lambda_1}(\bar z_1)\mu^L_{\lambda_2}(z_2)\mu^R_{\bar\lambda_2}(\bar z_2)\mu^L_{\lambda_3}(z_3)\mu^R_{\bar\lambda_3}(\bar z_3)\rangle_{\mathrm{SF}} = \prod_{i<j}(z_i - z_j)^{\lambda_i\lambda_j}(\bar z_i - \bar z_j)^{\bar\lambda_i\bar\lambda_j}$$
$$\text{for } \sum_i \lambda_i = \sum_i \bar\lambda_i = 1, \quad (2.51)$$

and

$$\langle\mu^L_{\lambda_1}(z_1)\mu^R_{\bar\lambda_1}(\bar z_1)\mu^L_{\lambda_2}(z_2)\mu^R_{\bar\lambda_2}(\bar z_2)\mu^L_{\lambda_3}(z_3)\mu^R_{\bar\lambda_3}(\bar z_3)\rangle_{\mathrm{SF}}$$
$$= -(-1)^{\lambda_3-\bar\lambda_3}\frac{\Gamma(\lambda_1^*)\Gamma(\lambda_2^*)\Gamma(\bar\lambda_3^*)}{\Gamma(\bar\lambda_1)\Gamma(\bar\lambda_2)\Gamma(\lambda_3)}\prod_{i<j}(z_i - z_j)^{\lambda_i^*\lambda_j^*}(\bar z_i - \bar z_j)^{\bar\lambda_i^*\bar\lambda_j^*} \quad \text{for } \sum_i \lambda_i = \sum_i \bar\lambda_i = 2, \quad (2.52)$$

where $\lambda^* = 1 - \lambda$ and the subscript SF means that the expectation value is calculated using the symplectic fermion part of the action (2.24). Here $\mu_\lambda$ are the twist operators defined in eq. (2.39). We have also used that in going to this expectation value under the rescaling (2.22) we have to multiply the correlation functions with an overall factor of $k$. This is because the correlation function normalisation is relative to the correlator of $\bar{\chi}^1\chi^2$ or $c'_+c'_-$ in the $b'c'$ system in eq. (2.19). This simply means that the dependence on $k$ disappears due to the normalisation in eq. (2.40) as is expected.

We want to compare this to the calculation of bulk twist correlators done by Kausch in [10]. In that paper, of course, only twist fields with identical twist in the holomorphic and anti-holomorphic sector are treated so we take $\lambda_i = \bar{\lambda}_i$. Further, we have to remember that the twist fields are only defined up to normalisation. To compare with Kausch we use one of the equations (2.51), (2.52) to fix the normalisation and can then compare to the second one. The normalisation is fixed by defining

$$\mu^L_\lambda \mu^R_\lambda = -\sqrt{\frac{\Gamma(\lambda^*)}{\Gamma(\lambda)}}\mu_\lambda. \tag{2.53}$$

Then we get

$$\begin{aligned}\langle \mu_{\lambda_1}(z_1,\bar{z}_1)\mu_{\lambda_2}(z_2,\bar{z}_2)\mu_{\lambda_3}(z_3,\bar{z}_3)\rangle_{\mathrm{SF}} &= \prod_i \sqrt{\frac{\Gamma(\lambda_i)}{\Gamma(\lambda_i^*)}} \prod_{i<j} |z_i - z_j|^{2\lambda_i\lambda_j} \quad \text{for } \sum_i \lambda_i = 1, \\ &= \prod_i \sqrt{\frac{\Gamma(\lambda_i^*)}{\Gamma(\lambda_i)}} \prod_{i<j} |z_i - z_j|^{2\lambda_i^*\lambda_j^*} \quad \text{for } \sum_i \lambda_i = 2,\end{aligned} \tag{2.54}$$

which is exactly as in [10]. We can also compare with the two-point function which is easily calculated and also get a match here. Note, however, that in [10] only ground state twist fields with $0 < \lambda < 1$ are considered. Our results thus compare precisely in this range, and are the analytic continuation of the twists $\lambda$ for the results in [10].

In the case where we allow the $e_i$ to be zero or an integer multiple of $k$, we have to take into account the zero modes of the symplectic fermions. This gives four different ground states in the symplectic model - two fermionic and two bosonic, where the last two span a Jordan block for $L_0$. The result is that we get logarithmic branch cuts in the correlation functions. This can be seen from the GL(1|1) side where the $\Gamma$ functions diverge when $\lambda$ becomes integer [12]. Thus we also get agreement from the two sides of the correspondence here.

# 3 Branes in the symplectic fermions

Now, having established the correspondence, we want to apply it. There are two apparent applications. For point-like branes in the GL(1|1) WZNW model, so far it could be argued that correlators containing only boundary fields behave like untwisted symplectic fermions [15], but it was not possible to handle insertions of bulk fields. Now, we are in a position to approach the problem of computing correlation functions involving bulk and boundary fields. We will refrain from this problem for now, but keep it in mind for future research. Instead, we reconsider the study of boundary states. Recall that the group of outer automorphisms of GL(1|1) is $\mathbb{Z}_2$. The branes corresponding to the trivial gluing automorphism we call untwisted and their boundary states have been studied in [16]. The non-trivial automorphism only admits one volume-filling brane, which we call twisted. Its boundary state has been studied, with quite some effort, in [15]. With the GL(1|1)-symplectic fermion correspondence, we can easily reproduce these results, but also compute spectra of strings stretching between an untwisted and a twisted brane. This gives, finally, a complete discussion of Cardy boundary states. It will turn out that the boundary states indeed satisfy Cardy's condition, i.e. the amplitude is a true character.

As we have seen, the GL(1|1) WZNW model can equally well be understood in a theory of scalars and symplectic fermions. Since boundary states with symplectic fermions have not been discussed in completeness before, we start by a quite general analysis of these. For earlier works on boundary models of symplectic fermions see [22, 23, 24, 25].

## 3.1 Boundary conditions

We start our considerations by investigating possible boundary conditions. The energy momentum tensors are

$$T(z) \;=\; -\tfrac{1}{2}\epsilon_{ab} :\partial\chi^a\partial\chi^b: , \qquad \bar{T}(\bar{z}) \;=\; -\tfrac{1}{2}\epsilon_{ab} :\bar{\partial}\chi^a\bar{\partial}\chi^b: . \tag{3.1}$$

They preserve the symplectic fermion symmetry and coincide along the boundary if

$$\partial\chi \;=\; A\,\bar{\partial}\chi \qquad\qquad \text{for } z \;=\bar{z}\,, \tag{3.2}$$

where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a matrix in SL(2) and for convenience we combined the two fermions in the vector $\chi = \left(\begin{smallmatrix} \chi^1 \\ \chi^2 \end{smallmatrix}\right)$. In terms of Dirichlet and Neumann derivatives ($\partial = \frac{1}{2}\partial_u - i\frac{1}{2}\partial_n$ and $\bar{\partial} = \frac{1}{2}\partial_u + i\frac{1}{2}\partial_n$) the boundary conditions are

$$-i\partial_n\chi \;=\; \frac{A-1}{A+1}\,\partial_u\chi \tag{3.3}$$

provided $1+A$ is invertible. Then the action on the upper half-plane is

$$S \;=\; -\frac{1}{4\pi}\int d^2z\;\partial\chi^t\,J\,\bar{\partial}\chi \;+\; \frac{i}{8\pi}\int_{z=\bar{z}} du\;\chi^t\,J\,\frac{A-1}{A+1}\,\partial_u\chi\,, \tag{3.4}$$

where the matrix $J$ is $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The variation of this action vanishes provided the above boundary conditions hold as well as the bulk equations of motion $\partial\bar{\partial}\chi^{\pm} = 0$. If $1+A$ is not invertible it has characteristic polynomial $\lambda^2$, i.e. if $1+A=0$ there are Dirichlet conditions in both directions while otherwise there is one Dirichlet and one Neumann condition. Note that these cases resemble the atypical branes in GL(1|1) [26].

## 3.2 The Ramond sector

We first consider the Ramond sector, by which we mean the symplectic fermions without any twist fields, or in the language of modes meaning only integer modes appear. The explicit mode expansion is

$$\chi^a(z,\bar z) \;=\; \xi^a + \chi_0^a \ln|z|^2 - \sum_{n\neq 0} \frac{1}{n}\chi_n^a z^{-n} + \frac{1}{n}\bar\chi_n^a \bar z^{-n}, \tag{3.5}$$

where the modes satisfy

$$\{\chi_m^a, \chi_n^b\} \;=\; -m\epsilon^{ab}\,\delta_{m,-n} \quad , \quad \{\bar\chi_m^a, \bar\chi_n^b\} \;=\; -m\epsilon^{ab}\,\delta_{m,-n} \;\text{ and }\; \{\xi^a, \chi_0^b\} \;=\; \epsilon^{ab} \quad . \tag{3.6}$$

All other anti-commutators vanish. Note that for locality we have required $\chi_0^a = \bar\chi_0^a$.

In this section we construct the boundary states in the Ramond sector, compute the amplitudes and construct the corresponding open string model. We start the discussion of boundary states by investigating Dirichlet conditions in the two fermionic directions.

### 3.2.1 Dirichlet conditions

Let us first remind ourselves that if we have an extended chiral algebra given by $W(z)$ and $\bar W(\bar z)$ we need an gluing automorphism, $\Omega$, for the boundary [27]:

$$W(z) = \Omega(\bar W)(\bar z) \qquad \text{for } z = \bar z\,. \tag{3.7}$$

This is as in equation (3.2) for the gluing of the currents. We now pass to closed strings via the world-sheet duality. The gluing conditions then become the following Ishibashi conditions for the boundary states $|\alpha\rangle\!\rangle_\Omega$ in the CFT on the full plane:

$$\big(W_n - (-1)^{h_W}\Omega(\bar W_{-n})\big)\,|\alpha\rangle\!\rangle_\Omega\,, \tag{3.8}$$

where $h_W$ is the conformal dimension of $W$.

Using (3.8) we see that for the Dirichlet boundary conditions ($A=-1$ in (3.2)) the corresponding Ishibashi states have to satisfy

$$\big(\chi_n^a - \bar\chi_{-n}^a\big)\,|D\rangle\!\rangle \;=\; 0 \qquad\qquad \text{for } a = 1,2\;, \tag{3.9}$$

note that there is no condition on $\chi_0^a$ because of the locality constraint $\chi_0^a - \bar\chi_0^a = 0$. The Ishibashi states are explicitly constructed as

$$|D_0\rangle\!\rangle \;=\; \sqrt{2\pi}\exp\Big(\textstyle\sum_{m>0}\frac{1}{m}\big(\chi_{-m}^2\bar\chi_{-m}^1 - \chi_{-m}^1\bar\chi_{-m}^2\big)\Big)|0\rangle\,, \tag{3.10}$$

$$|D_\pm\rangle\!\rangle \;=\; \xi^\pm\,\exp\Big(\textstyle\sum_{m>0}\frac{1}{m}\big(\chi_{-m}^2\bar\chi_{-m}^1 - \chi_{-m}^1\bar\chi_{-m}^2\big)\Big)|0\rangle\,, \tag{3.11}$$

$$|D_2\rangle\!\rangle \;=\; \tfrac{\xi^-\xi^+}{\sqrt{2\pi}}\,\exp\Big(\textstyle\sum_{m>0}\frac{1}{m}\big(\chi_{-m}^2\bar\chi_{-m}^1 - \chi_{-m}^1\bar\chi_{-m}^2\big)\Big)|0\rangle\,, \tag{3.12}$$

where the ground state $|0\rangle$ is defined by $\chi^a_n|0\rangle = 0$ for $n \geq 0$. The dual Ishibashi state is obtained by dualizing the modes using (here $m > 0$)

$$\chi^1_{-m}{}^\dagger \;=\; \chi^1_m \qquad \text{and} \qquad \chi^2_{-m}{}^\dagger \;=\; -\chi^2_m \;. \tag{3.13}$$

For the computation of amplitudes we need the Virasoro generators, they are

$$L_n \;=\; -\tfrac{1}{2}\epsilon_{ab}\sum_m :\chi^a_{n-m}\chi^b_m: \tag{3.14}$$

and the central charge is $c = -2$. Define $q = \exp 2\pi i\tau$ and $\tilde{q} = \exp(-2\pi i/\tau)$ as usual, where $\tau$ takes values in the upper half plane. Then the non-vanishing overlaps are

$$\begin{aligned}
\langle\!\langle D_0|q^{L^c_0+\frac{1}{12}}(-1)^{F^c}|D_2\rangle\!\rangle \;&=\; \langle\!\langle D_2|q^{L^c_0+\frac{1}{12}}(-1)^{F^c}|D_0\rangle\!\rangle \;=\; \eta(\tau)^2,\\
\langle\!\langle D_-|q^{L^c_0+\frac{1}{12}}(-1)^{F^c}|D_+\rangle\!\rangle \;&=\; -\langle\!\langle D_+|q^{L^c_0+\frac{1}{12}}(-1)^{F^c}|D_-\rangle\!\rangle \;=\; \eta(\tau)^2,\\
\langle\!\langle D_2|q^{L^c_0+\frac{1}{12}}(-1)^{F^c}|D_2\rangle\!\rangle \;&=\; -i\tau\eta(\tau)^2 \;=\; \eta(\tilde{\tau})^2\;,
\end{aligned} \tag{3.15}$$

where $L^c_0 = L_0 + \bar{L}_0$. Thus only $|D_2\rangle$ makes sense as a boundary state.

### 3.2.2 Neumann conditions

Next we would like to display the boundary state $|A\rangle$ for our general boundary conditions (3.2). It has to satisfy the Ishibashi condition (3.8)

$$\begin{aligned}
\chi^1_n + a\,\bar{\chi}^1_{-n} + b\,\bar{\chi}^2_{-n}|A\rangle\!\rangle \;&=\; 0\,,\\
\chi^2_n + c\,\bar{\chi}^1_{-n} + d\,\bar{\chi}^2_{-n}|A\rangle\!\rangle \;&=\; 0\,,
\end{aligned} \tag{3.16}$$

which are satisfied by

$$|A\rangle\!\rangle \;=\; \mathcal{N}\;\exp\Big(-\sum_{m>0}\frac{1}{m}\big(a\chi^2_{-m}\bar{\chi}^1_{-m} + b\chi^2_{-m}\bar{\chi}^2_{-m} - c\chi^1_{-m}\bar{\chi}^1_{-m} - d\chi^1_{-m}\bar{\chi}^2_{-m}\big)\Big)|0\rangle\,. \tag{3.17}$$

The dual state is

$$\langle\!\langle A| \;=\; \mathcal{N}\;\langle\!\langle 0|\exp\Big(-\sum_{m>0}\frac{1}{m}\big(-a\chi^2_m\bar{\chi}^1_m + b\chi^2_m\bar{\chi}^2_m - c\chi^1_m\bar{\chi}^1_m + d\chi^1_m\bar{\chi}^2_m\big)\Big)\,. \tag{3.18}$$

It will turn out that the normalization should be fixed to be

$$\mathcal{N} \;=\; \sqrt{2\pi}\,2\,\sin\pi\mu\,, \tag{3.19}$$

where we introduce $\mu$ via $\alpha = \exp 2\pi i\mu$ by $-\mathrm{tr}(A) = \alpha + \alpha^{-1}$.

Now it is straightforward to compute amplitudes between two boundary states. Any non-zero amplitude requires the zero modes of $\chi^1$ and $\chi^2$ hence only the Dirichlet boundary state has non-vanishing overlap with any Neumann state:

$$\langle\!\langle A|\,q^{\frac{1}{2}L^c_0+\frac{1}{12}}\,(-1)^{F^c}\,|D_2\rangle\!\rangle \;=\; \frac{\mathcal{N}}{\sqrt{2\pi}}q^{\frac{1}{12}}\prod_{m>0}(1-\alpha_{12}q^m)(1-\alpha_{12}^{-1}q^m)\,. \tag{3.20}$$

Upon modular transformation this amplitude is the spectrum of an open string stretching between two branes with respectively Neumann boundary conditions given by $A$ and Dirichlet conditions. Using the formulas provided in the appendix equation (3.20) becomes

$$\frac{\mathcal{N}}{\sqrt{2\pi}} q^{\frac{1}{12}} \prod_{m>0} (1-\alpha q^m)(1-\alpha^{-1}q^m) \;=\; \tilde{q}^{\frac{1}{2}(\mu-\frac{1}{2})^2-\frac{1}{24}} \prod_{n=0}^{\infty} \big(1-\tilde{q}^{n+1-\mu}\big)\big(1-\tilde{q}^{n+\mu}\big) . \quad (3.21)$$

Now, we construct the boundary theory of a string stretching between these two branes and check that its spectrum is indeed given by the amplitude we just computed, we follow [28]. Therefore consider the upper half plane, and demand boundary condition $A$ for the negative real line, i.e.

$$\partial\chi \;=\; A\,\bar{\partial}\chi \qquad \text{for } z \;=\bar{z} \qquad \text{and } z+\bar{z} \;<\; 0\,; \quad (3.22)$$

and Dirichlet conditions for the positive real axis

$$\partial_u\chi \;=\; 0 \qquad \text{for } z \;=\bar{z} \qquad \text{and } z+\bar{z} \;>\; 0\,. \quad (3.23)$$

Then the fields have the following SL(2) monodromy (counterclockwise)

$$\partial\chi(ze^{2\pi i}) \;=\; -A\partial\chi(z)\,, \quad (3.24)$$

and similar for the bared quantities. Denote by $S$ the matrix that diagonalizes the monodromy, i.e. $S(-A)S^{-1}$ is diagonal. We denote the eigenvalues by $\alpha^{\pm 1}$. Further, call the eigenvectors $\partial\chi^{\pm}$, they then have the usual mode expansion [10]

$$\chi^{\pm}(z) \;=\; \sum_{n\in\mathbb{Z}} \frac{1}{n\pm\mu} \chi^{\pm}_{n\pm\mu} z^{-(n\pm\mu)}\,. \quad (3.25)$$

The original fields are then explicitly

$$\begin{pmatrix}\chi^1\\ \chi^2\end{pmatrix} \;=\; S^{-1}\begin{pmatrix}\chi^+\\ \chi^-\end{pmatrix}\,. \quad (3.26)$$

Their partition function is

$$\mathrm{tr}(\,q^{L_0-\frac{c}{24}}(-1)^F\,) \;=\; q^{\frac{1}{2}(\mu-\frac{1}{2})^2-\frac{1}{24}} \prod_{n=0}^{\infty} \big(1-q^{n+1-\mu}\big)\big(1-q^{n+\mu}\big)\,. \quad (3.27)$$

The computation has been done similarly by Kausch [10]. We see that the result fits with (3.20) and the Cardy condition is fulfilled. Thus, we nicely established our boundary state and the open string theory it describes.

If we want to investigate amplitudes involving Neumann boundary states on both ends, we learnt [28] that it is necessary to insert additional zero modes in order to obtain a non-vanishing amplitude. Also introduce $\alpha_{12}$ via $\mathrm{tr}(A_1A_2^{-1}) = \alpha_{12} + \alpha_{12}^{-1}$ then we get

$$\begin{aligned}
\langle\!\langle A_1|\,\chi^2\chi^1\,q^{\frac{1}{2}L_0^c+\frac{1}{12}}\,(-1)^{F^c}\,|A_2\rangle\!\rangle \;&=\; \mathcal{N}_1\mathcal{N}_2\,q^{\frac{1}{12}}\prod_{m>0}(1-\alpha_{12}q^m)(1-\alpha_{12}^{-1}q^m) \\
&=\; \mathcal{N}_{12}\,\tilde{q}^{\frac{1}{2}(\mu_{12}-\frac{1}{2})^2-\frac{1}{24}}\prod_{n=0}^{\infty}\big(1-\tilde{q}^{n+1-\mu_{12}}\big)\big(1-\tilde{q}^{n+\mu_{12}}\big)\,,
\end{aligned} \tag{3.28}$$

where

$$\mathcal{N}_{12} \;=\; 4\pi\,\frac{\sin\pi\mu_1\,\sin\pi\mu_2}{\sin\pi\mu_{12}}\,. \tag{3.29}$$

The open string theory is constructed almost exactly as above and again resembles [28]. We demand boundary condition $A_1$ for the negative real line and $A_2$ for the positive one,

$$\partial\chi \;=\; \begin{cases} A_1\,\bar\partial\chi & \text{if } z\;=\bar z \quad \text{and } z+\bar z\;<\;0 \\ A_2\,\bar\partial\chi & \text{if } z\;=\bar z \quad \text{and } z+\bar z\;>\;0\,. \end{cases} \tag{3.30}$$

The fields have the following SL(2) monodromy

$$\partial\chi(ze^{2\pi i}) \;=\; A_1A_2^{-1}\,\partial\chi(z)\,. \tag{3.31}$$

Let $S$ diagonalize the monodromy, then its eigenvalues are $\alpha_{12}^{\pm1}$ and we call the eigenvectors again $\partial\chi^\pm$. They have the mode expansion

$$\chi^\pm(z) \;=\; \sqrt{\mathcal{N}_{12}}\,\xi^\pm + \sum_{n\in\mathbb{Z}}\frac{1}{n\pm\mu_{12}}\chi^\pm_{n\pm\mu_{12}}z^{-(n\pm\mu_{12})}\,, \tag{3.32}$$

note the extra zero mode, since the monodromy does only concern derivatives. Its partition function with appropriate insertion is

$$\mathrm{tr}(\chi^2\chi^1q^{L_0-\frac{c}{24}}(-1)^F) \;=\; \mathcal{N}_{12}\,q^{\frac{1}{2}(\mu_{12}-\frac{1}{2})^2-\frac{1}{24}}\prod_{n=0}^{\infty}\big(1-q^{n+1-\mu_{12}}\big)\big(1-q^{n+\mu_{12}}\big)\,, \tag{3.33}$$

and coincides with (3.28) as desired.

## 3.3 The Neveu-Schwarz sector

In this section we study the boundary states in the Neveu-Schwarz sector. The states have to satisfy the usual Ishibashi condition

$$\begin{aligned}
\chi^1_n + a\bar\chi^1_{-n} + b\bar\chi^2_{-n}|A\rangle\!\rangle_{NS} \;&=\; 0\,, \\
\chi^2_n + c\bar\chi^1_{-n} + d\bar\chi^2_{-n}|A\rangle\!\rangle_{NS} \;&=\; 0\,,
\end{aligned} \tag{3.34}$$

where the modes are half-integer, i.e. $n$ in $\mathbb{Z}+1/2$. The conditions are satisfied by

$$|A\rangle\!\rangle \;=\; \exp\Big(-\sum_{\substack{m>0\\ m\,\in\,\mathbb{Z}+1/2}}\frac{1}{m}\big(a\chi^2_{-m}\bar\chi^1_{-m}+b\chi^2_{-m}\bar\chi^2_{-m}-c\chi^1_{-m}\bar\chi^1_{-m}-d\chi^1_{-m}\bar\chi^2_{-m}\big)\Big)|0\rangle\,. \tag{3.35}$$

We introduce $\alpha_{12}$ as before, that is $\mathrm{tr}(A_1 A_2^{-1}) = \alpha_{12} + \alpha_{12}^{-1}$, and get

$$\begin{aligned} {}_{NS}\langle\langle A_1 | q^{L_0^c + \frac{1}{12}} (-1)^{F_c} | A_2 \rangle\rangle_{NS} &= q^{-\frac{1}{24}} \prod_{\substack{m>0 \\ m\in\mathbb{Z}+1/2}} (1 - \alpha_{12} q^m)(1 - \alpha_{12}^{-1} q^m) \\ &= \tilde{q}^{\frac{1}{2}(\mu - \frac{1}{2})^2 - \frac{1}{24}} \prod_{n>0} (1 + \tilde{q}^{n-\mu})(1 + \tilde{q}^{n-\mu^*}) \,, \end{aligned} \tag{3.36}$$

where $\alpha_{12} = e^{2\pi i \mu}$. This is the spectrum of an open string constructed similarly as before, and in addition demanding antisymmetric boundary conditions in the time-direction.

## 3.4 The twisted sectors

Given any twisted sector we can diagonalize it and thus we can restrict to twists that are diagonal. Call the ground state of the sector for $\mu_\lambda$ on which $\chi^a$ has twists

$$\chi^1 \longrightarrow e^{-2\pi i \lambda} \chi^1 \qquad \text{and} \qquad \chi^2 \longrightarrow e^{2\pi i \lambda} \chi^2 \,. \tag{3.37}$$

Then the mode expansions of the fields in these sectors are

$$\begin{aligned} \partial \chi^1(z) &= -\sum_{n\in\mathbb{Z}} \chi^1_{n+\lambda} z^{-(n+\lambda)-1} \qquad \text{and} \qquad \bar{\partial}\bar{\chi}^1(\bar{z}) = -\sum_{n\in\mathbb{Z}} \bar{\chi}^1_{n-\lambda} \bar{z}^{-(n-\lambda)-1} \\ \partial \chi^2(z) &= -\sum_{n\in\mathbb{Z}} \chi^2_{n-\lambda} z^{-(n-\lambda)-1} \qquad \text{and} \qquad \bar{\partial}\bar{\chi}^2(\bar{z}) = -\sum_{n\in\mathbb{Z}} \bar{\chi}^2_{n+\lambda} \bar{z}^{-(n+\lambda)-1} \,. \end{aligned} \tag{3.38}$$

Whenever $\lambda \neq 1/2$ the boundary conditions are parameterized by just one parameter $\alpha$ according to the boundary conditions

$$\partial \chi^1 = \alpha \bar{\partial} \chi^1 \qquad \text{and} \qquad \partial \chi^2 = \alpha^{-1} \bar{\partial} \chi^2 \,. \tag{3.39}$$

Only to these conditions there exist twisted Ishibashi states. The boundary state has to satisfy the usual Ishibashi condition

$$\begin{aligned} \chi^1_{n+\lambda} + \alpha \bar{\chi}^1_{-n-\lambda} |\alpha\rangle\rangle_\lambda &= 0 \,, \\ \chi^2_{n-\lambda} + \alpha^{-1} \bar{\chi}^2_{-n+\lambda} |\alpha\rangle\rangle_\lambda &= 0 \,, \end{aligned} \tag{3.40}$$

and these are solved by ($\lambda^* = 1 - \lambda$)

$$|\alpha\rangle\rangle_\lambda = \mathcal{N} \exp\Big( -\sum_{m>0} \frac{\alpha}{m - \lambda^*} \chi^2_{-m+\lambda^*} \bar{\chi}^1_{-m+\lambda^*} - \frac{\alpha^{-1}}{m-\lambda} \chi^1_{-m+\lambda} \bar{\chi}^2_{-m+\lambda} \Big) \mu_\lambda \,. \tag{3.41}$$

where we fix the normalization to be $\mathcal{N} = e^{-2\pi i(\lambda - 1/2)(\mu - 1/4)}$ and $\alpha = e^{2\pi i \mu}$. The dual boundary state is

$${}_\lambda\langle\langle \alpha | = \bar{\mathcal{N}} \mu_\lambda^\dagger \exp\Big( \sum_{m>0} \frac{\alpha}{m - \lambda} \chi^2_{m-\lambda} \bar{\chi}^1_{m-\lambda} - \frac{\alpha^{-1}}{m - \lambda^*} \chi^1_{m-\lambda^*} \bar{\chi}^2_{m-\lambda^*} \Big) \,. \tag{3.42}$$

Now we are prepared to compute the amplitudes (note that the conformal dimension of the twist state is $h_\lambda = -\lambda\lambda^*/2$ and we use the shorthand $\alpha_1\alpha_2^{-1} = e^{2\pi i\mu}$)

$$\begin{aligned}
{}_\lambda\langle\alpha_1|q^{L_0^c+\frac{1}{12}}(-1)^{F_c}|\alpha_2\rangle_\lambda &= e^{-2\pi i(\lambda-\frac{1}{2})(\mu-\frac{1}{2})}q^{\frac{1}{2}(\lambda-\frac{1}{2})^2-\frac{1}{24}}\prod_{n>0}(1-\alpha_1\alpha_2^{-1}q^{n-\lambda^*})(1-\alpha_2\alpha_1^{-1}q^{n-\lambda}) \\
&= \tilde{q}^{\frac{1}{2}(\mu-\frac{1}{2})^2}\theta\Big(\tilde{\tau}(\frac{1}{2}-\mu)-(\lambda-\frac{1}{2}),\tilde{\tau}\Big)/\eta(\tilde{\tau}) \\
&= \tilde{q}^{\frac{1}{2}(\mu-\frac{1}{2})^2-\frac{1}{24}}\prod_{n>0}(1-u^{-1}\tilde{q}^{n-\mu})(1-u\tilde{q}^{n-\mu^*})\,,
\end{aligned} \tag{3.43}$$

where $u = e^{2\pi i\lambda}$. This is the character of a boundary theory twisted by $\mu_{12}$ in an orbifold by an abelian subgroup $\mathcal{G}$ of SL(2), which is generated by $u$, see [10] for a detailed discussion.

# 4 Branes in the GL(1|1) WZNW model

We are finally in a position to apply the symplectic fermion GL(1|1) correspondence to boundary states in GL(1|1). GL(1|1) possesses two non-trivial gluing conditions. One condition, which we call untwisted since it is given by a trivial gluing automorphism, has a two-parameter family of branes corresponding to super conjugacy classes. They have been discussed in detail in [16]. The other gluing condition, which we call twisted, consists of one volume filling brane. Its boundary state has been discussed in [15] as a rather complicated perturbative expansion from a free scalar times a symplectic fermion boundary state. In [15] it was shown that for a particular amplitude this expansion did not contribute and computations reduced to computations in the decoupled free scalar free fermion model. This was already a first hint of the correspondence now found.

We can now compute amplitudes between boundary states of different gluing conditions and construct the corresponding open string theory explicitly. There is another puzzle we can unreveal and that is the role of atypical Ishibashi states and their $\log q$ dependent overlaps. While Ishibashi states corresponding to typical representations have true characters as overlap, atypicals might have a $\log q$ prefactor, as seen in (3.15) and e.g. [29]. We will see that in the GL(1|1) story, this $\log q$ dependence arises by a limiting procedure from characters of typical representations. The understanding of the atypical Ishibashi states in our context is important for amplitudes involving two branes with different gluing conditions.

## 4.1 Untwisted Branes

Let us recall the analysis performed in [16]. Untwisted branes correspond to the gluing conditions

$$J(z) = \bar{J}(\bar{z}) \qquad \text{for } z = \bar{z}\,. \tag{4.1}$$

Insertion of the explicit formulas (2.23) for the currents into above gluing conditions gives Dirichlet conditions for the bosonic currents

$$\partial_u Y \;=\; 0 \qquad , \qquad \partial_u Z \;=\; 0 \qquad , \qquad \text{for } z \;=\; \bar{z}\,, \tag{4.2}$$

While the fermionic ones generically satisfy Neumann conditions

$$e^{Y_0^L}\partial\chi^1 \;=\; -e^{-Y_0^R}\bar{\partial}\chi^1 \qquad \text{and} \qquad e^{-Y_0^L}\partial\chi^2 \;=\; -e^{Y_0^R}\bar{\partial}\chi^2\,. \tag{4.3}$$

Thus one parameterizes the branes by their positions labeled by $(y_0, z_0)$. But whenever[1] $Y_0^L + Y_0^R = iy_0 = 2\pi i s, s \in \mathbb{Z}$ we obtain Dirichlet boundary conditions in all directions, bosonic and fermionic ones,

$$\partial_u Y \;=\; \partial_u Z \;=\; \chi^a \;=\; 0 \;\; \text{for } z \;=\; \bar{z}\,. \tag{4.4}$$

These branes will be called non-generic (untwisted) branes in the following.

Let us shortly describe the main results of the minisuperspace analysis performed in [16]. The minisuperspace or particle limit describes the behaviour of full field theory quantities in the large level limit, $k \to \infty$. In this limit the zero modes of the fields dominate and thus fields are interpreted as functions on the supergroup, and the action of the currents is mimicked by the right and left invariant vector fields $R_X$ and $L_X$. We are interested in semiclassical analogua of Ishibashi states. Minisuperspace Ishibashi states are those states invariant under the adjoint action $\text{ad}_X = R_X + L_X$ since the gluing automorphism is the identity. What is interesting for our further consideration is that there exist two kinds of atypical Ishibashi states. One of them has vanishing overlap with itself and is thus associated with (3.10) in our correspondence to symplectic fermions, while the other one is obtained from the first kind by the action of the fermionic functions associated to the fermionic fields $c_\pm$, and thus it should be identified with (3.12). Further, in the minisuperspace limit boundary states become distributions concentrated on the super conjugacy class they correspond to, and this distribution can be expressed in terms of minisuperspace Ishibashi states. It turns out that the first kind of Ishibashi state contributes to the generic boundary states while the second kind to the non-generic one. For more details, we refer the reader to [16], but we will also illustrate the lift of this minisuperspace story to the full field theory in the following subsections.

### 4.1.1 Ishibashi states

We now construct the Ishibashi states using our symplectic fermion correspondence. Recall that the currents take the form (2.23)

$$J^E = -k\partial Y, \qquad J^N = -\partial Z, \qquad J^- = \sqrt{k}e^{Y^L}\partial\chi^1, \qquad J^+ = -\sqrt{k}e^{-Y^L}\partial\chi^2, \tag{4.5}$$

$$\bar{J}^E = k\bar{\partial}Y, \qquad \bar{J}^N = \bar{\partial}Z, \qquad \bar{J}^- = -\sqrt{k}e^{-Y^R}\bar{\partial}\chi^1, \qquad \bar{J}^+ = \sqrt{k}e^{Y^R}\bar{\partial}\chi^2. \tag{4.6}$$

[1] In this section we allow for imaginary brane positions exactly as done in [16].

Further, the fermions have mode expansion as in equation (3.5) and relations (3.6) (or the twisted versions thereof) while the two scalars have expansion

$$
\begin{aligned}
Y^L(z) &= Y_0^L + p_Y^L \ln z - \sum_{n\neq 0} \frac{1}{n} Y_n^L z^{-n}, \\
Y^R(z) &= Y_0^R + p_Y^R \ln \bar{z} - \sum_{n\neq 0} \frac{1}{n} Y_n^R \bar{z}^{-n}, \\
Z^L(z) &= Z_0^L + p_Z^L \ln z - \sum_{n\neq 0} \frac{1}{n} Z_n^L z^{-n}, \\
Z^R(z) &= Z_0^R + p_Z^R \ln \bar{z} - \sum_{n\neq 0} \frac{1}{n} Z_n^R \bar{z}^{-n},
\end{aligned}
\tag{4.7}
$$

and relations

$$
[Y_n^{L,R}, Z_m^{L,R}] \;=\; -m\delta_{n,-m} \qquad \text{and} \qquad [Z_0^{L,R}, p_Y^{L,R}] \;=\; [Y_0^{L,R}, p_Z^{L,R}] \;=\; -1\,. \tag{4.8}
$$

To ensure locality we have $p_Y^L = p_Y^R$ and also $Z_0^L = Z_0^R$ for the conjugate modes. However, we will not demand $p_Z^L = p_Z^R$ and correspondingly not $Y_0^L = Y_0^R$ since $Z$ has an additive twist around our winding states (2.44).

The energy momentum tensor is

$$
T(z) \;=\; \partial Y \partial Z - \tfrac{1}{2}\epsilon_{ab}\partial\chi^a\partial\chi^b \qquad \text{and} \qquad \bar{T}(\bar{z}) \;=\; \bar{\partial} Y \bar{\partial} Z - \tfrac{1}{2}\epsilon_{ab}\bar{\partial}\chi^a\bar{\partial}\chi^b\,, \tag{4.9}
$$

and thus the Virasoro modes are

$$
\begin{aligned}
L_n &= -\sum_{m\,\in\,\mathbb{Z}} :\chi^1_{n-m}\chi^2_m: \;+ \sum_{m\,\neq\,0,n} :Y^L_{n-m}Z^L_m: + \\
&\quad + \sum_{m\,\neq\,0} (\,:p_Y^L Z_m^L:\; + \;:p_Z^L Y_m^L:\,) + \delta_{n,0}\, p_Y^L p_Z^L\,, \\
\bar{L}_n &= -\sum_{m\,\in\,\mathbb{Z}} :\bar{\chi}^1_{n-m}\bar{\chi}^2_m: \;+ \sum_{m\,\neq\,0,n} :Y^R_{n-m}Z^R_m: + \\
&\quad + \sum_{m\,\neq\,0} (\,:p_Y^R Z_m^R:\; + \;:p_Z^R Y_m^R:\,) + \delta_{n,0}\, p_Y^R p_Z^R\,.
\end{aligned}
\tag{4.10}
$$

We also need the zero modes of the currents corresponding to the Cartan generators $J^E$ and $J^N$:

$$
E_0 \;=\; -kp_Y^L, \qquad \bar{E}_0 \;=\; kp_Y^R, \qquad N_0 \;=\; -p_Z^L, \qquad \bar{N}_0 \;=\; p_Z^R\,. \tag{4.11}
$$

Let us now consider the Ishibashi states. We start by spelling out the Ishibashi conditions for the untwisted case. As noted above, the gluing condition $J = \bar{J}$ means that the bosonic fields simply satisfy Dirichlet conditions

$$
\partial_u Y \;=\; \partial_u Z \;=\; 0\,. \tag{4.12}
$$

Using these Dirichlet conditions for the field $Y = Y^L + Y^R$ the fermionic ones can be written as follows

$$e^{Y_0^L}\partial\chi^1 \;=\; -e^{-Y_0^R}\bar{\partial}\chi^1 \qquad \text{and} \qquad e^{-Y_0^L}\partial\chi^2 \;=\; -e^{Y_0^R}\bar{\partial}\chi^2 \,. \tag{4.13}$$

Then correspondingly the Ishibashi conditions for the bosonic fields are

$$\begin{aligned} \left(Y_n^L - Y_{-n}^R\right) |\, I\,\rangle\!\rangle \;&=\; \left(Z_n^L - Z_{-n}^R\right) |\, I\,\rangle\!\rangle \;=\; 0 \qquad n \neq 0 \\ \left(p_Z^L - p_Z^R\right) |\, I\,\rangle\!\rangle \;&=\; \left(p_Y^L - p_Y^R\right) |\, I\,\rangle\!\rangle \;=\; 0\,, \end{aligned} \tag{4.14}$$

note that there is no conditions on the zero modes $Y_0^L$ and $Y_0^R$. Further, the conditions for the fermionic ones are

$$\left(e^{Y_0^L}\chi_n^1 - e^{-Y_0^R}\bar{\chi}_{-n}^1\right) |\, I\,\rangle\!\rangle \;=\; \left(e^{-Y_0^L}\chi_n^2 - e^{Y_0^R}\bar{\chi}_{-n}^2\right) |\, I\,\rangle\!\rangle \;=\; 0\,. \tag{4.15}$$

The Ishibashi states clearly factorize into a bosonic and a fermionic part and are easily constructed as follows. The typical primary of GL(1|1), $\langle e, n\rangle_R$, is the representation with ground state $|n, \mu_\lambda\rangle$ where $\lambda = e/k$ satisfying

$$\begin{aligned} p_Z^L|n,\mu_\lambda\rangle \;&=\; p_Z^R|n,\mu_\lambda\rangle \;=\; n|n,\mu_\lambda\rangle\,, \\ p_Y^L|n,\mu_\lambda\rangle \;&=\; p_Y^R|n,\mu_\lambda\rangle \;=\; \lambda|n,\mu_\lambda\rangle\,. \end{aligned} \tag{4.16}$$

Further, recall that the fermions have the mode expansion in the presence of the ground state $\mu_\lambda$ (3.38)

$$\begin{aligned} \chi^1(z,\bar{z}) \;&=\; \sum_{n\,\in\,\mathbb{Z}+\lambda} \frac{1}{n}\,\chi_n^1\, z^{-n} \;+\; \sum_{n\,\in\,\mathbb{Z}+\lambda^*} \frac{1}{n}\,\bar{\chi}_n^1\,\bar{z}^{-n}\,, \\ \chi^2(z,\bar{z}) \;&=\; \sum_{n\,\in\,\mathbb{Z}+\lambda^*} \frac{1}{n}\,\chi_n^2\, z^{-n} \;+\; \sum_{n\,\in\,\mathbb{Z}+\lambda} \frac{1}{n}\,\bar{\chi}_n^2\,\bar{z}^{-n}\,, \end{aligned} \tag{4.17}$$

where $\lambda^* = 1 - \lambda$. Then the bosonic Ishibashi state is

$$|n,e\rangle\!\rangle_B \;=\; \exp\Big(\sum_{m>0}\frac{1}{m}\big(Y_{-m}^L Z_{-m}^R + Z_{-m}^L Y_{-m}^R\big)\Big)|n,\mu_\lambda\rangle_B\,, \tag{4.18}$$

and the fermionic one is computed as (3.41)

$$|n,e\rangle\!\rangle_F \;=\; \exp\Big(-\sum_{m\,>\,0}\frac{e^{Y_0^L+Y_0^R}}{m-\lambda}\chi_{-m+\lambda}^1\bar{\chi}_{-m+\lambda}^2 - \frac{e^{-Y_0^L-Y_0^R}}{m-\lambda^*}\chi_{-m+\lambda^*}^2\bar{\chi}_{-m+\lambda^*}^1\Big)|n,\mu_\lambda\rangle_F\,. \tag{4.19}$$

and the Ishibashi state is then the product of the two. The following simple computations are crucial

$$\begin{aligned} q^{L_0}e^{\pm Y_0^L} \;&=\; e^{\pm Y_0^L}q^{L_0\mp\frac{E_0}{k}} \qquad , \qquad Z^{N_0}e^{\pm Y_0^L} \;=\; e^{\pm Y_0^L}Z^{N_0\mp 1}\,, \\ q^{\bar{L}_0}e^{\pm Y_0^R} \;&=\; e^{\pm Y_0^R}q^{\bar{L}_0\pm\frac{\bar{E}_0}{k}} \qquad , \qquad Z^{\bar{N}_0}e^{\pm Y_0^R} \;=\; e^{\pm Y_0^R}Z^{\bar{N}_0\pm 1}\,, \end{aligned} \tag{4.20}$$

Introduce $L_0^c = \frac{1}{2}(L_0 + \bar{L}_0)$ and $N_0^c = \frac{1}{2}(N_0 - \bar{N}_0)$ as usual. Then we get the fermionic contribution of the overlap, that is

$$ {}_F\langle\langle n,e|q^{L_0^c+\frac{1}{12}}z^{N_0^c}(-1)^{F^c}|n,e\rangle\rangle_F \;=\; z^n(1-z^{-1})q^{\frac{1}{2}(\lambda-\frac{1}{2})^2-\frac{1}{24}}\prod_{n>0}(1-z^{-1}q^n)(1-zq^n)\,, \quad (4.21) $$

and the bosonic

$$ {}_B\langle\langle n,e|q^{L_0^c-\frac{1}{12}}z^{N_0^c}(-1)^{F^c}|n,e\rangle\rangle_B \;=\; -\frac{q^{n\lambda}}{\eta(\tau)^2}\,, \quad (4.22) $$

where we normalized the dual state such that we get the minus sign. Then in total, we arrive at

$$ \begin{aligned} \langle\langle n,e|q^{L_0^c}z^{N_0^c}(-1)^{F^c}|n,e\rangle\rangle \;&=\; z^{n-1}(1-z)\frac{q^{n\lambda+\frac{1}{2}(\lambda-\frac{1}{2})^2-\frac{1}{24}}}{\eta(\tau)^2}\prod_{n>0}(1-z^{-1}q^n)(1-zq^n) \\ &=\; \hat{\chi}_{<e,n>}(z,\tau)\,. \end{aligned} \quad (4.23) $$

So far we assumed $0<\lambda<1$, whenever $\lambda$ becomes zero our Dirichlet symplectic fermion boundary states come into the game. There are four of them. Denote by $|n,0\rangle$ the ground state with $N_0$ eigenvalue $n$, i.e.

$$ \begin{aligned} &N_0|n,0\rangle = n|n,0\rangle, \qquad E_0|n,0\rangle = 0\,, \\ &Y_m|n,0\rangle = Z_m|n,0\rangle = \chi_m^a|n,0\rangle = \chi_0^a|n,0\rangle = 0, \quad \text{for } m>0. \end{aligned} \quad (4.24) $$

Then the Ishibashi states are

$$ \begin{aligned} |n_0\rangle\rangle \;&=\; \exp\Big(\sum_{m>0}\frac{1}{m}\big(Y_{-m}^L Z_{-m}^R + Z_{-m}^L Y_{-m}^R - e^{Y_0^L+Y_0^R}\chi_{-m}^1\bar{\chi}_{-m}^2 + e^{-Y_0^L-Y_0^R}\chi_{-m}^2\bar{\chi}_{-m}^1\big)\Big)|n,0\rangle\,, \\ |n_\pm\rangle\rangle \;&=\; \xi^\pm|n_0\rangle\rangle\,, \\ |n\rangle\rangle \;&=\; \xi^-\xi^+|n_0\rangle\rangle\,, \end{aligned} \quad (4.25) $$

and we arrive at the following amplitudes

$$ \begin{aligned} \langle\langle n_0|q^{L_0^c}z^{N_0^c}(-1)^{F^c}|n\rangle\rangle \;&=\; \chi_0(\mu,\tau), \\ \langle\langle n|q^{L_0^c}z^{N_0^c}(-1)^{F^c}|n_0\rangle\rangle \;&=\; -\chi_0(\mu,\tau), \\ \langle\langle n_\pm|q^{L_0^c}z^{N_0^c}(-1)^{F^c}|n_\mp\rangle\rangle \;&=\; -\chi_0(\mu,\tau), \\ \langle\langle n|q^{L_0^c}z^{N_0^c}(-1)^{F^c}|n\rangle\rangle \;&=\; -2\pi i\tau\chi_0(\mu,\tau), \end{aligned} \quad (4.26) $$

where

$$ \chi_0(\mu,\tau) = z^{n-1}q^{\frac{1}{12}}\prod_{n>0}(1-z^{-1}q^n)(1-zq^n)/\eta(\tau)^2. \quad (4.27) $$

All other amplitudes vanish unless zero modes are inserted.

Let us now consider twist states $\mu_{\tilde{\lambda}}$ where $\tilde{\lambda} \notin \,]0,1[$. We already saw in the second section that such states are simply descendants of $\mu_\lambda$ where $\tilde{\lambda} = \lambda + m$ for some integer $m$ and $\lambda \in ]0,1[$. The state $|n, \mu_{\tilde{\lambda}}\rangle$ satisfies the following conditions

$$N_0|n,\mu_{\tilde{\lambda}}\rangle \;=\; n|n,\mu_{\tilde{\lambda}}\rangle \qquad \text{and} \qquad E_0|n,\mu_{\tilde{\lambda}}\rangle \;=\; k(\lambda+m)|n,\mu_{\tilde{\lambda}}\rangle\,. \tag{4.28}$$

The Ishibashi state $|e,n\rangle\rangle$ (with $e/k = \tilde{\lambda} = \lambda + m$) in this representation is obtained from the previously constructed ones as

$$|e,n\rangle\rangle \;=\; e^{m(Z_0^L - Z_0^R)} e^{m(Y_0^L + Y_0^R)} |e - mk, n\rangle\rangle\,. \tag{4.29}$$

The amplitude is computed using

$$q^{L_0^c} e^{m(Z_0^L - Z_0^R)} \;=\; e^{m(Z_0^L - Z_0^R)} q^{L_0^c - mN_0^c}\,, \tag{4.30}$$

and the spectral flow formulas provided in appendix B

$$\langle\langle n, e| q^{L_0^c} z^{N_0^c} (-1)^{F^c} |n, e\rangle\rangle \;=\; \hat{\chi}_{<e-mk,n+m>}(z - m\tau, \tau) \;=\; (-1)^m \hat{\chi}_{<e,n>}(z,\tau)\,. \tag{4.31}$$

A similar construction holds also for the atypical part.

## 4.2 The untwisted boundary states

Untwisted boundary states were studied in detail in [16], here we recall the states and their properties. Atypical Ishibashi states contribute to amplitudes only by a set of measure zero, therefore their role has not been fully investigated previously. Here we will fill this gap. As we will see in a moment boundary states are represented by an integral of Ishibashi states, hence any amplitude is given by an integral of GL(1|1) characters. We fix the role of the Ishibashi states by requiring that any integrand of any amplitude is a smooth function. Let us be more precise.

The minisuperspace analysis [16] already suggests that the Ishibashi states $|n_0\rangle\rangle$ contribute to generic branes, while the states $|n\rangle\rangle$ contribute to non-generic branes. We will see that this is correct. The boundary state corresponding to a generic brane localized at $(z_0, y_0)$ with $y_0 \neq 2\pi s$ is

$$\begin{aligned} |z_0, y_0\rangle \;=\; & \sqrt{\frac{2i}{k}} \int_{\substack{e\neq mk \\ m\in\mathbb{Z}}} de dn \;\exp\bigl(i(n-1/2)y_0 + iez_0\bigr)\; \sin^{1/2}(\pi e/k)\; |e,n\rangle\rangle \;- \\ & \frac{\sqrt{2\pi i}}{k} \sum_{m\in\mathbb{Z}} \int dn\;\exp\bigl(i(n-1/2)y_0 + imkz_0\bigr)\; |n_0\rangle\rangle^{(m)}, \end{aligned} \tag{4.32}$$

where the superscript $m$ denotes the $\gamma_m$ spectral flowed state. In order to check the consistency of our proposal for the boundary states with world-sheet duality, we compute the spectrum between a pair of generic branes,

$$\begin{aligned} \langle z_0, y_0|(-1)^{F^c} \tilde{q}^{L_0^c} \tilde{z}^{N_0^c} |z_0', y_0'\rangle = & \frac{2i}{k} \int de' dn' e^{i(n'-\frac{1}{2})(y_0'-y_0) + ie'(z_0'-z_0)} \sin(\pi e'/k) \hat{\chi}_{\langle e', n'\rangle}(\tilde{\mu}, \tilde{\tau}) \\ = & \;\hat{\chi}_{\langle e,n\rangle}(\mu,\tau) \;-\; \hat{\chi}_{\langle e,n+1\rangle}(\mu,\tau)\,, \end{aligned} \tag{4.33}$$

where the momenta $e, n$ are related to the coordinates of the branes according to

$$e = \frac{k(y_0' - y_0)}{2\pi} \quad , \quad n = \frac{k(z_0' - z_0)}{2\pi} - \frac{y_0' - y_0}{2\pi} \, .$$

Let us now turn to the boundary states of non-generic untwisted branes in the GL(1|1) WZNW model. The boundary states of elementary branes associated with non-generic position parameters $z_0$ and $y_0 = 2\pi s, s \in \mathbb{Z}$, are given by

$$\begin{aligned} |z_0; s\rangle = & \frac{1}{\sqrt{2ki}} \int_{e \neq mk} dedn \ \exp\bigl(2\pi i(n - 1/2)s + iez_0\bigr) \ \sin^{-1/2}(\pi e/k) \ |e, n\rangle\rangle \\ & - \frac{1}{\sqrt{2\pi i}} \sum_{m \in \mathbb{Z}} \int dn \ \exp\bigl(2\pi i(n - 1/2)s + imkz_0\bigr) \ |n\rangle\rangle^{(m)} \quad . \end{aligned} \tag{4.34}$$

We also here verify that the proposed boundary states produce a consistent open string spectrum by calculating the overlap between two non-generic boundary states $|z_0; s\rangle$ and $|z_0'; s'\rangle$,

$$\begin{aligned} \langle z_0; s|(-1)^{F^c} \tilde{q}^{L_0^c} \tilde{z}^{N_0^c} |z_0'; s'\rangle & = \int \frac{de'dn'}{2ki} \frac{e^{2\pi i(n'-1/2)(s'-s)+ie'(z_0'-z_0)}}{\sin(\pi e'/k)} \ \hat{\chi}_{\langle e', n'\rangle}(\tilde{\mu}, \tilde{\tau}) \\ & = \hat{\chi}^{(m)}_{\langle n\rangle}(\mu, \tau) \, , \end{aligned} \tag{4.35}$$

where the labels $n$ and $m$ in the character are related to the branes' parameters through

$$n = \frac{k(z_0' - z_0)}{2\pi} + s - s' \quad , \quad m = s' - s \ . \tag{4.36}$$

The superscript on the character $\hat{\chi}^{(m)}_{\langle n\rangle}(\mu, \tau)$ again means we have used the spectral flow $\gamma_m$ see ref. [16] and appendix B. The following limit for $t$ any integer shows that in equation (4.35) is indeed a hidden $\tau$-dependence

$$\lim_{e \to mk} \frac{1}{2ki} \int dn \ \frac{e^{2\pi itn}}{\sin(\pi e/k)} \ \hat{\chi}_{\langle e, n\rangle}(\tilde{\mu}, \tilde{\tau}) = \int dn \ \tau \, e^{2\pi itn} \hat{\chi}^{(m)}_{\langle n\rangle}(\tilde{\mu}, \tilde{\tau}) \, . \tag{4.37}$$

Thus we observe that the Ishibashi state $|n\rangle\rangle$ (4.26) with its $\tau$-dependence is the natural atypical Ishibashi state contributing to the atypical boundary state.

Further, the overlap between a generic and a non-generic state is

$$\begin{aligned} \langle z_0, y_0|(-1)^{F^c} \tilde{q}^{L_0^c} \tilde{z}^{N_0^c} |z_0'; s\rangle & = \int \frac{de'dn'}{k} \ e^{i(n'-1/2)(2\pi s - y_0)+ie'(z_0'-z_0)} \ \hat{\chi}_{\langle e', n'\rangle}(\tilde{\mu}, \tilde{\tau}) \\ & = \hat{\chi}_{\langle e, n\rangle}(\mu, \tau) \, , \end{aligned} \tag{4.38}$$

where

$$n = \frac{k(z_0' - z_0)}{2\pi} + \frac{y_0}{2\pi} - s + \frac{1}{2} \quad , \quad \frac{e}{k} = s - \frac{y_0}{2\pi} \ . \tag{4.39}$$

## 4.3 Twisted boundary state

The group of outer automorphisms of the Lie superalgebra gl(1|1) is of order 2. We already discussed the boundary states belonging to the trivial one. The non-trivial one defines the following gluing conditions on the currents

$$J^E = -\bar{J}^E \quad , \quad J^N = -\bar{J}^N \quad , \quad J^+ = -\bar{J}^- \quad , \quad J^- = \bar{J}^+ \qquad \text{for } z = \bar{z} \,. \tag{4.40}$$

This translates into Neumann conditions for the bosonic and the fermionic fields, that is

$$\partial_n Y = \partial_n Z = 0 \qquad \text{for } z = \bar{z} \tag{4.41}$$

implying especially that the left movers of $Y$ coincide with its right movers up to the zero modes

$$Y^L - Y^R = Y_0^L - Y_0^R \qquad \text{for } z = \bar{z} \,. \tag{4.42}$$

Thus the gluing conditions for the fermions are

$$e^{Y_0^L}\partial\chi^1 = e^{Y_0^R}\bar{\partial}\chi^2 \qquad , \qquad e^{-Y_0^L}\partial\chi^2 = -e^{-Y_0^R}\bar{\partial}\chi^1 \qquad \text{for } z = \bar{z} \,. \tag{4.43}$$

The boundary state $|\,\Omega\rangle\rangle$ is easily constructed as before. It has to satisfy

$$\begin{aligned} (Y_n^L + Y_{-n}^R)\,|\,\Omega\rangle\rangle \ = (p_Y^L + p_Y^R)\,|\,\Omega\rangle\rangle \ &= \ 0\,, \\ (Z_n^L + Z_{-n}^R)\,|\,\Omega\rangle\rangle \ = (p_Z^L + p_Z^R)\,|\,\Omega\rangle\rangle \ &= \ 0\,, \\ (e^{Y_0^L}\chi_n^1 + e^{Y_0^R}\bar{\chi}_{-n}^2)\,|\,\Omega\rangle\rangle \ &= \ 0\,, \\ (e^{-Y_0^L}\chi_n^2 - e^{-Y_0^R}\bar{\chi}_{-n}^1)\,|\,\Omega\rangle\rangle \ &= \ 0\,, \end{aligned} \tag{4.44}$$

which can be computed to be

$$|\,\Omega\,\rangle\rangle \ = \ \sqrt{\pi/i}\,\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}\big(Y_{-n}^L Z_{-n}^R + Z_{-n}^L Y_{-n}^R - e^{Y_0^R - Y_0^L}\chi_{-n}^2\bar{\chi}_{-n}^2 + e^{Y_0^L - Y_0^R}\chi_{-n}^1\bar{\chi}_{-n}^1\big)\Big)|0,0\,\rangle\,. \tag{4.45}$$

Here, $|0,0\,\rangle$ denotes the vacuum defined by $\chi_n^a|0,0\,\rangle = 0$ for $n \geq 0$ and $Z_n^{L,R}|0,0\,\rangle = Y_n^{L,R}|0,0\,\rangle = p_Y^{L,R}|0,0\,\rangle = p_Z^{L,R}|0,0\,\rangle = 0$ for $n > 0$. The dual boundary state is constructed analogously.

Our main aim now is to compute some non-vanishing overlap of the twisted boundary state $|\Omega\rangle\rangle$. This requires the insertion of the invariant bulk field $\chi^1\chi^2$, i.e.

$$\langle\langle\Omega\,|\,\tilde{q}^{L_0^c}(-1)^{F^c}\,\tilde{z}^{N_0^c}\,\chi^1\chi^2\,|\,\Omega\rangle\rangle \ = \ \frac{\pi}{2k}\int dedn\,\frac{\hat{\chi}_{\langle e,n\rangle}(\tau,\mu)}{\sin(\pi e/k)}\,. \tag{4.46}$$

where $L_0^c = (L_0 + \bar{L}_0)/2$ and $N_0^c = (N_0 + \bar{N}_0)/2$ are obtained from the zero modes of the Virasoro field and the current $N$. Here the normalization in (4.45) by $\sqrt{\pi/i}$ was important.[2] This amplitude has been tested in detail in [15].

[2] Note the difference of a $\sqrt{2}$ compared to [15] which is due to a misprint there.

## 4.4 Mixed amplitudes and their open strings

Using the GL(1|1)-symplectic fermion correspondence we were able to construct boundary states explicitly, and the amplitudes fits with the previously known results calculated in GL(1|1). The new explicit formulation also allows us to compute new quantities such as overlaps for atypicals

$$\begin{aligned}\langle\langle \Omega \,|\, \tilde{q}^{L_0^c}(-1)^{F^c}\, \tilde{z}^{N_0^c} \,|\, z_0; s\rangle\rangle &= \sqrt{\frac{1}{2}}(-1)^s \prod_{n=0}^{\infty} \frac{(1-\tilde{q}^n)}{(1+\tilde{q}^n)} \\ &= (-1)^s\, q^{\frac{1}{32}} \prod_{n=0}^{\infty} \frac{(1-q^{n+\frac{1}{4}})(1-q^{n+\frac{3}{4}})}{(1-q^{n+\frac{1}{2}})^2}\,.\end{aligned} \tag{4.47}$$

Note the independence on $z$, no matter whether we take $N_0^c$ as in the previous section or as in the untwisted case, which is natural since there does not exist a distinguished choice for $N_0^c$ for mixed amplitudes.

The corresponding open string theory is easily constructed using our previous experience. That is, we demand untwisted gluing conditions on the negative real line

$$\begin{aligned}\partial_u Y &= \partial_u Z = 0\ , \\ e^{Y_0^L}\partial\chi^1 &= -e^{-Y_0^R}\bar{\partial}\chi^1\ , \\ e^{-Y_0^L}\partial\chi^2 &= -e^{Y_0^R}\bar{\partial}\chi^2 \qquad \text{for } z = \bar{z} \qquad \text{and } z+\bar{z} < 0\,;\end{aligned} \tag{4.48}$$

and twisted on the positive one

$$\begin{aligned}\partial_n Y &= \partial_n Z = 0\ , \\ e^{Y_0^L}\partial\chi^1 &= e^{Y_0^R}\bar{\partial}\chi^2\ , \\ e^{-Y_0^L}\partial\chi^2 &= -e^{-Y_0^R}\bar{\partial}\chi^1 \qquad \text{for } z = \bar{z} \qquad \text{and } z+\bar{z} > 0\,,\end{aligned} \tag{4.49}$$

Then the fermions have a monodromy of order four around the origin

$$\partial\chi^1(ze^{2\pi i}) = i\,\partial\chi^1(z) \qquad , \qquad \partial\chi^2(ze^{2\pi i}) = -i\,\partial\chi^2(z)\,, \tag{4.50}$$

and the bosons a monodromy of order two

$$\partial Y(ze^{2\pi i}) = -\partial Y \qquad , \qquad \partial Z(ze^{2\pi i}) = -\partial Z(z)\,. \tag{4.51}$$

Thus the fermions have mode expansion

$$\begin{aligned}\chi^1(z) &= \sum_{n\in\mathbb{Z}+\frac{3}{4}} \frac{1}{n}\chi^1_n z^{-n}\,, \\ \chi^2(z) &= \sum_{n\in\mathbb{Z}+\frac{1}{4}} \frac{1}{n}\chi^2_n z^{-n}\,,\end{aligned} \tag{4.52}$$

and the bosons

$$
\begin{aligned}
Y(z) &= \sum_{n \in \mathbb{Z}+\frac{1}{2}} \frac{1}{n} Y_n z^{-n}, \\
Z(z) &= \sum_{n \in \mathbb{Z}+\frac{1}{2}} \frac{1}{n} Z_n z^{-n},
\end{aligned}
\tag{4.53}
$$

We define the ground state to be bosonic if $s$ (the position parameter of the non-generic brane) is even and fermionic if it is odd. The partition function is then

$$
\mathrm{tr}(q^{L_0}(-1)^F) = (-1)^s q^{\frac{1}{32}} \prod_{n=0}^{\infty} \frac{(1-q^{n+\frac{1}{4}})(1-q^{n+\frac{3}{4}})}{(1-q^{n+\frac{1}{2}})^2}. \tag{4.54}
$$

The amplitude involving typical fields requires as usual zero mode insertions, i.e.

$$
\begin{aligned}
\langle\!\langle \Omega \,|\, \tilde{q}^{L_0^c}(-1)^{F^c} \tilde{z}^{N_0^c} \chi^1\chi^2 \,|\, z_0, y_0 \rangle\!\rangle &= \frac{\sqrt{2}\pi}{k} e^{-iy_0/2} \frac{\prod_{n=0}^{\infty}(1-\tilde{q}^n)}{\prod_{n=0}^{\infty}(1+\tilde{q}^n)} \\
&= \frac{2\pi}{k} e^{-iy_0/2} q^{\frac{1}{32}} \prod_{n=0}^{\infty} \frac{(1-q^{n+\frac{1}{4}})(1-q^{n+\frac{3}{4}})}{(1-q^{n+\frac{1}{2}})^2},
\end{aligned}
\tag{4.55}
$$

and its open string spectrum can be constructed as in the symplectic fermion case.

In summary, we have been able to give a complete discussion of Cardy boundary states in the GL(1|1) WZNW model. This was only possible due to the new formulation in terms of symplectic fermions. As a result, we saw that indeed also for the Lie supergroup GL(1|1) Cardy's condition holds, i.e. any amplitude of two boundary states indeed describes an open string spectrum.

# 5 Outlook

In this note, we have established a correspondence between the Wess-Zumino-Novikov-Witten model on the Lie supergroup GL(1|1) and free scalars plus symplectic fermions. This correspondence introduces a new efficient way to study the WZNW model. A natural question is whether there exist generalizations of the procedure. GL(1|1) is special in the sense that it is level-independent, i.e. rescaling the current $J^E$ by $\lambda^2$, the fermionic currents $J^{\pm}$ by $\lambda$ and leaving $J^N$ invariant simply changes the level $k$ by a factor of $\lambda^2$. Because of this peculiarity we do not expect our procedure to extend in full generality, but still we believe that for other supergroups at special levels such a prescription also applies. A first attempt would be to look for free field descriptions. This one can do immediately by taking the free Gross-Neveu model of a dimension one half vector transforming in the adjoint representation of the desired supergroup similar to what was done by LeClair for GL(1|1) [8].

For standard groups the procedure could also be useful. In the case of the $H_3^+$ model one can use the procedure to arrive at a model of two free scalars and the Liouville action. However, the vertex operators will take a very complicated form.

A main motivation to study the correspondence was to serve as a toy model for more sophisticated dualities. The guideline in our approach was to rewrite the GL(1|1) currents in such a form that they are very symmetric as the currents of a Gross-Neveu model are. We hope that this guiding principle can serve as an important step in understanding the dualities between the OSp(2N+2|2N) Gross-Neveu model and the principal chiral model of the supersphere $S^{2N+1|2N}$ [3].

Finally, we used the correspondence to construct the boundary states of GL(1|1) and verify Cardy's condition, completing the series of investigations [16, 15]. Especially, we got a picture of atypical Ishibashi states and their contributions. As in other logarithmic conformal field theories there exists more than just one Ishibashi state corresponding to each atypical representation. Their overlaps might give $\tau$-dependent contributions, but these Ishibashi states only contributed to atypical boundary states. Based on the insights of this note and former work, we should be able to investigate boundary states of other Lie supergroups such as SU(1|2).

We remark, however, that there is still one open problem for branes in GL(1|1). There exist branes whose geometry is not a superconjugacy class and which are rather special since their spectra are representations that are indecomposable but reducible. Further they are peculiar since their dual states are projected out [26]. These objects are not understood and it would be interesting to study these in the light of the GL(1|1)-symplectic fermion correspondence.

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# A Some formulas concerning theta functions

Let us recall some facts about the theta function in one variable, a good reference is Mumford's first book [30]. $\theta(\mu,\tau)$ is the unique holomorphic function on $\mathbb{C}\times\mathbb{H}$, such that

$$\begin{aligned} \theta(\mu+1,\tau) &= \theta(\mu,\tau), \\ \theta(\mu+\tau,\tau) &= e^{-\pi i\tau}e^{-2\pi i\mu}\theta(\mu,\tau), \\ \theta(\mu+\frac{1}{2},\tau+1) &= \theta(\mu,\tau), \\ \theta(\mu/\tau,-1/\tau) &= \sqrt{-i\tau}e^{\pi i\mu^2/\tau}\theta(\mu,\tau)\,, \\ \lim_{\mathrm{Im}(\tau)\to\infty}\theta(\mu,\tau) &= 1\quad. \end{aligned} \tag{A.1}$$

The theta functions has a simple expansion as an infinite product,

$$\theta(\mu,\tau) \;=\; \prod_{m=0}^{\infty}\left(1-q^m\right)\prod_{n=0}^{\infty}\left(1+u^{-1}q^{n+1/2}\right)\left(1+uq^{n+1/2}\right)\quad, \tag{A.2}$$

where $q=e^{2\pi i\tau}$ and $u=e^{2\pi i\mu}$. The following variant is of concern to us

$$\theta\Big(\mu-\frac{1}{2}(\tau+1),\tau\Big)/\eta(\tau) \;=\; (1-u)q^{\frac{-1}{24}}\prod_{n=1}^{\infty}\left(1-uq^n\right)\left(1-u^{-1}q^n\right)\quad. \tag{A.3}$$

Its behavior under modular $S$ transformations which send the arguments of the theta function to $\tilde{\tau}=-1/\tau$ and $\tilde{\mu}=\mu/\tau$ can be deduced from the properties above. One simply finds

$$\begin{aligned} \theta\Big(\mu-\frac{1}{2}(\tau+1),\tau\Big)/\eta(\tau) &= -ie^{\pi i\mu}q^{-\frac{1}{8}}\theta\Big(\tilde{\tau}(\frac{1}{2}-\mu)-\frac{1}{2},\tilde{\tau}\Big)/\eta(\tilde{\tau})\tilde{q}^{\frac{1}{2}(\mu-\frac{1}{2})^2} \\ &= ie^{\pi i\mu}q^{-\frac{1}{8}}\tilde{q}^{\frac{1}{2}(\mu-\frac{1}{2})^2-\frac{1}{24}}\prod_{m=0}^{\infty}\left(1-\tilde{q}^{n+1-\mu}\right)\left(1-\tilde{q}^{n+\mu}\right). \end{aligned} \tag{A.4}$$

# B Representation theory of $\mathrm{GL}(1|1)$

We recall some facts of the representation theory of $\widehat{\mathrm{gl}}(1|1)$. A more detailed discussion is given in the Appendix of [16].

A useful tool for the investigation of the affine Lie superalgebra $\widehat{\mathrm{gl}}(1|1)$ and its representations are automorphisms that do not leave the horizontal subalgebra invariant, the spectral flow automorphisms. The relevant one for our purposes [16], $\gamma_m$, leaves the modes $N_n$ invariant and acts on the remaining ones as

$$\gamma_m(E_n) \;=\; E_n+km\delta_{n0}\;,\quad \gamma_m(\Psi^\pm_n) \;=\; \Psi^\pm_{n\pm m}\,. \tag{B.1}$$

These transformations induce a modification of the energy momentum tensor

$$\gamma_m(L_n) \;=\; L_n + mN_n \quad . \tag{B.2}$$

The characters of two representations $\rho$ and $\gamma_m(\rho)$ that are related by spectral flow satisfy

$$\chi_{\gamma_m(\rho)}(\mu,\tau) \;=\; \chi_\rho(\mu + m\tau,\tau)\,. \tag{B.3}$$

Finally, we state the relevant characters, the typical one is

$$\hat{\chi}_{\langle e,n\rangle}(\mu,\tau) \;=\; \hat{\chi}_{\overline{\langle e,n\rangle}}(\mu,\tau) \;=\; u^{n-1}q^{\frac{e}{2k}(2n-1+e/k)+1/8}\theta\Big(\mu-\frac{1}{2}(\tau+1),\tau\Big)/\eta(\tau)^3 \tag{B.4}$$

and the atypical one is following [31]

$$\begin{aligned}\hat{\chi}^{(m)}_{\langle n\rangle}(\mu,\tau) \;&=\; \sum_{l=0}^{\infty}\hat{\chi}_{\langle mk,n+l+1\rangle}(\mu,\tau)\\ &=\; \frac{u^n}{1-uq^m}\,\frac{q^{\frac{m}{2}(2n+m+1)+1/8}\theta\Big(\mu-\frac{1}{2}(\tau+1),\tau\Big)}{\eta(\tau)^3}\,.\end{aligned} \tag{B.5}$$

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