

# A Novel Formulation of Cabibbo-Kobayashi-Maskawa Matrix Renormalization

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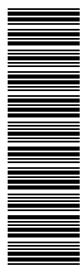
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## Abstract

We present a gauge-independent quark mass counterterm for the on-shell renormalization of the Cabibbo-Kobayashi-Maskawa (CKM) matrix in the Standard Model that is directly expressed in terms of the Lorentz-invariant self-energy functions, and automatically satisfies the hermiticity constraints of the mass matrix. It is very convenient for practical applications and leads to a gauge-independent CKM counterterm matrix that preserves unitarity and satisfies other highly desirable theoretical properties, such as flavor democracy.

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Recently, a new approach to renormalize the Cabibbo-Kobayashi-Maskawa (CKM) matrix [1] at the one-loop level in the Standard Model (SM) framework was proposed [2, 3]. It is based on a simple procedure to separate the external-leg mixing corrections generated by the Feynman diagrams of Fig. 1 into gauge-independent *self-mass* (sm) and gauge-dependent *wave-function renormalization* (wfr) contributions, and to adjust non-diagonal counterterm matrices to cancel the sm contributions, subject to constraints imposed by the hermiticity of the mass matrices. Diagonalization of the complete mass matrices for up-type and down-type quarks leads then to an explicitly gauge-independent CKM counterterm matrix that preserves unitarity.

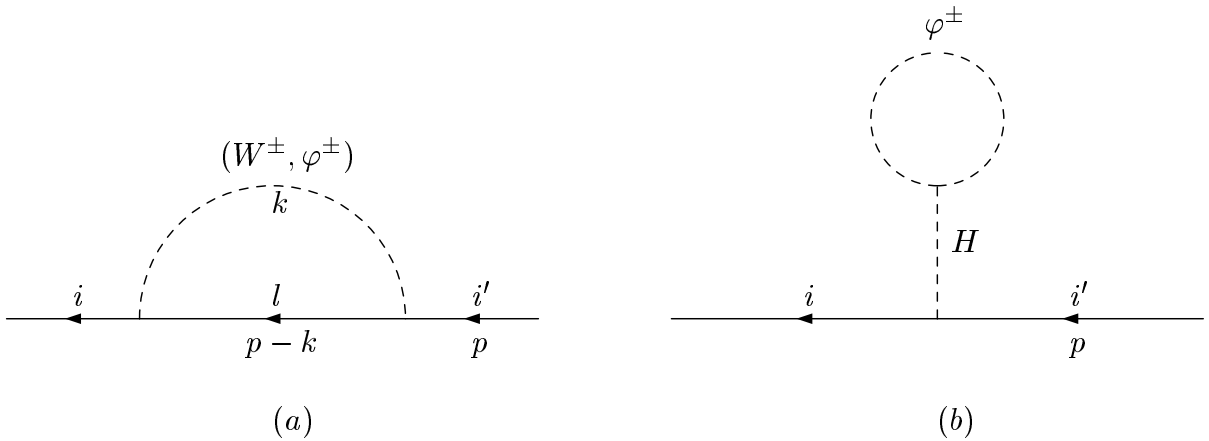


Figure 1: Fermion self-energy diagrams.

In this paper we discuss an alternative on-shell approach that presents especially attractive features. It is based on a gauge-independent mass counterterm matrix that is directly expressed in terms of the Lorentz-invariant self-energy functions and automatically satisfies the hermiticity constraints of the mass matrix.

On covariance grounds, the self-energy  $\Sigma_{ii'}(\not{p})$  associated with Fig. 1 is of the form

$$\Sigma_{ii'}(\not{p}) = \not{p} a_- \Sigma_{ii'}^L(p^2) + \not{p} a_+ \Sigma_{ii'}^R(p^2) + a_- A_{ii'}^L(p^2) + a_+ A_{ii'}^R(p^2), \quad (1)$$

where  $a_\pm = (1 \pm \gamma_5)/2$  are the chiral projectors and  $\Sigma_{ii'}^{L,R}(p^2)$  and  $A_{ii'}^{L,R}(p^2)$  are the invariant self-energy functions. At one loop in the SM, we have

$$m_{i'} A_{ii'}^L(p^2) = m_i A_{ii'}^R(p^2). \quad (2)$$

Explicit one-loop expressions for the SM in the  $R_\xi$  gauges are given in the Appendix of Ref. [4] in combination with the tadpole contributions in Eq. (B.3) of Ref. [5] and Eq. (A5) of Ref. [6].

The corresponding self-energy corrections to an external leg involving an outgoing quark is

$$\Delta \mathcal{M}_{ii'}^{\text{leg}} = \bar{u}_i(p) \left[ \Sigma_{ii'}(\not{p}) - \delta m_{ii'} \right] \frac{1}{\not{p} - m_{i'}}, \quad (3)$$

where  $i$  denotes the flavor of the external quark of mass  $m_i$  and four-momentum  $p$ ,  $i'$  that of the virtual quark of mass  $m_{i'}$ , and  $\delta m_{ii'}$  is the mass counterterm matrix. For definiteness, we first consider the case in which  $i$  and  $i'$  in Fig. 1 are up-type quarks and  $l$  in the loop is a down-type quark. In this case, the proposed mass counterterm is

$$\begin{aligned} \delta m_{ii'} = & V_{il} V_{li'}^\dagger \operatorname{Re} \left\{ a_+ \left[ \frac{m_{i'}}{2} \tilde{\Sigma}_{ii'}^L(m_i^2) + \frac{m_i}{2} \tilde{\Sigma}_{ii'}^R(m_i^2) + \tilde{A}_{ii'}^R(m_i^2) \right] \right. \\ & \left. + a_- \left[ \frac{m_i}{2} \tilde{\Sigma}_{ii'}^L(m_{i'}^2) + \frac{m_{i'}}{2} \tilde{\Sigma}_{ii'}^R(m_{i'}^2) + \tilde{A}_{ii'}^L(m_{i'}^2) \right] \right\}, \end{aligned} \quad (4)$$

where  $\tilde{\Sigma}_{ii'}^{L,R}(p^2)$  and  $\tilde{A}_{ii'}^{L,R}(p^2)$  are the invariant self-energies with  $V_{il} V_{li'}^\dagger$  factored out and, following standard conventions,  $V_{il}$  is the CKM matrix element involving the up-type quark  $i$  and the down-type quark  $l$ .

An explicit expression for  $\delta m_{ii'}$  in the SM can be obtained by using Eq. (21) of Ref. [3], which provides the Feynman amplitude  $M_{ii'}^{(1)}$  corresponding to Fig. 1 in the  $R_\xi$  gauges. Recalling that  $\Sigma_{ii'}(\not{p}) = i M_{ii'}^{(1)}$  and taking into account Eq. (1), one can readily determine the contributions of each term of Eq. (21) to the invariant functions and, via Eq. (4), to  $\delta m_{ii'}$ . One finds that only the first three terms of Eq. (21) give non-vanishing contributions to  $\delta m_{ii'}$ . Separating out the chiral components,

$$\delta m_{ii'} = a_+ \delta m_{ii'}^{(+)} + a_- \delta m_{ii'}^{(-)}, \quad (5)$$

we obtain the SM expressions

$$\begin{aligned} \delta m_{ii'}^{(+)} = & \frac{g^2}{32\pi^2} V_{il} V_{li'}^\dagger \operatorname{Re} \left\{ m_i \left( 1 + \frac{m_i^2}{2m_W^2} \Delta \right) - \frac{m_{i'} m_l^2}{2m_W^2} [3\Delta + I(m_i^2, m_l) + J(m_i^2, m_l)] \right. \\ & \left. + m_{i'} \left( 1 + \frac{m_i^2}{2m_W^2} \right) [I(m_i^2, m_l) - J(m_i^2, m_l)] \right\}, \end{aligned} \quad (6)$$

$$\begin{aligned} \delta m_{ii'}^{(-)} = & \frac{g^2}{32\pi^2} V_{il} V_{li'}^\dagger \operatorname{Re} \left\{ m_{i'} \left( 1 + \frac{m_{i'}^2}{2m_W^2} \Delta \right) - \frac{m_i m_l^2}{2m_W^2} [3\Delta + I(m_{i'}^2, m_l) + J(m_{i'}^2, m_l)] \right. \\ & \left. + m_i \left( 1 + \frac{m_{i'}^2}{2m_W^2} \right) [I(m_{i'}^2, m_l) - J(m_{i'}^2, m_l)] \right\}, \end{aligned} \quad (7)$$

where  $g$  is the SU(2) gauge coupling,  $\Delta = 1/(n-4) + [\gamma_E - \ln(4\pi)]/2 + \ln(m_W/\mu)$  the ultraviolet divergence,  $n$  the space-time dimensionality,  $\gamma_E$  the Euler-Mascheroni constant,  $\mu$  the 't Hooft mass scale, and

$$\{I(p^2, m_l); J(p^2, m_l)\} = \int_0^1 dx \{1; x\} \ln \frac{m_l^2 x + m_W^2(1-x) - p^2 x(1-x) - i\varepsilon}{m_W^2}. \quad (8)$$

The mass counterterms  $\delta m_{ii'}^{(\pm)}$  and endowed with very important properties:

1. They are gauge independent. Although  $\Sigma_{ii'}(\not{p})$  contains several gauge-dependent terms, they do not contribute to Eq. (4). As explained in Ref. [3], such gauge-dependent terms cancel the  $(\not{p} - m_{i'})^{-1}$  propagator in Eq. (3) and contribute to the wfr.

2. Equations (6) and (7) automatically satisfy the hermiticity constraint of the mass matrix, namely

$$\delta m_{i'i}^{(-)} = \delta m_{ii'}^{(+)*}, \quad \delta m_{i'i}^{(+)} = \delta m_{ii'}^{(-)*}. \quad (9)$$

The gauge independence of  $\delta m_{ii'}$  is also easily verified by inserting in Eq. (4) the expressions for  $\Sigma_{ii'}^{L,R}(p^2)$  and  $A_{ii'}^{L,R}(p^2)$  given in Refs. [4, 5, 6]. Alternatively, this can be established by means of Nielsen identities [7]. In fact, these identities were employed in Ref. [8] to show that the  $p^2$ -dependent combination

$$m_i m_{i'} \Sigma_{ii'}^L(p^2) + p^2 \Sigma_{ii'}^R(p^2) + m_{i'} A_{ii'}^L(p^2) + m_i A_{ii'}^R(p^2) \quad (10)$$

is gauge independent. Inserting Eq. (2) in Eq. (10) and evaluating the resulting expression at  $p^2 = m_i^2$  and  $p^2 = m_{i'}^2$ , one immediately observes that  $\delta m_{ii'}$  is gauge independent.

In the SM the functions  $I(m_i^2, m_l)$ ,  $J(m_i^2, m_l)$ ,  $I(m_{i'}^2, m_l)$ , and  $J(m_{i'}^2, m_l)$  are real when  $i, i' \neq t$ . Thus, in such cases the Re instruction is not necessary. On the other hand, when  $i = t$  ( $i' = t$ ), the first two (last two) develop imaginary parts, and the Re instruction tells us that only the real parts of  $I$  and  $J$  are included in the definition of  $\delta m_{ii'}^{(\pm)}$ .

Inserting Eqs. (6) and (7) in Eq. (3), we find

$$\Delta \mathcal{M}_{ii'}^{\text{leg}} = \Delta \mathcal{M}_{ii'}^{\text{wfr}} + \Delta \mathcal{M}_{ii'}^{\text{res}}, \quad (11)$$

where  $\Delta \mathcal{M}_{ii'}^{\text{wfr}}$  is the wfr given in Eq. (30) of Ref. [3], and

$$\begin{aligned} \Delta \mathcal{M}_{ii'}^{\text{res}} = & \frac{g^2}{32\pi^2} V_{il} V_{li'}^\dagger \bar{u}_i(p) \left\{ a_+ m_{i'} i \text{Im} \left[ \left( 1 + \frac{m_i^2}{2m_W^2} \right) (I - J)(m_i^2, m_l) \right. \right. \\ & - \left. \frac{m_l^2}{2m_W^2} (I + J)(m_i^2, m_l) \right] + a_- m_i \left[ \left( 1 + \frac{m_{i'}^2}{2m_W^2} \right) ((I - J)(m_i^2, m_l) \right. \\ & - \left. \left. \text{Re}(I - J)(m_{i'}^2, m_l) \right) - \frac{m_l^2}{2m_W^2} ((I + J)(m_i^2, m_l) - \text{Re}(I + J)(m_{i'}^2, m_l)) \right] \left. \right\} \\ & \times \frac{1}{\not{p} - m_{i'}} \end{aligned} \quad (12)$$

is a residual contribution that arises because the  $I$  and  $J$  functions are evaluated at  $p^2 = m_i^2$  in  $\Sigma_{ii'}(\not{p})$  and  $\delta m_{ii'}^{(+)}$  [cf. Eqs. (3) and (6)], at  $p^2 = m_{i'}^2$  in  $\delta m_{ii'}^{(-)}$  [cf. Eq. (7)], and only their real parts are included in the counterterms. When  $i, i' \neq t$ , the  $I$  and  $J$  functions are real in the SM and Eq. (12) greatly simplifies: the  $a_+$  component vanishes and the  $a_-$  component involves differences of real functions evaluated at  $p^2 = m_i^2$  and  $p^2 = m_{i'}^2$ .

It is important to note that  $\Delta \mathcal{M}_{ii'}^{\text{res}}$  is finite and gauge independent. Furthermore, it is non-singular in the limit  $m_{i'} \rightarrow m_i$ , provided that  $m_i < m_W$ .<sup>1</sup> In contrast,  $\Delta \mathcal{M}_{ii'}^{\text{wfr}}$  is

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<sup>1</sup>This does not preclude the possibility of a mass-degeneracy singularity involving two quarks with the same charges and masses  $m_i, m_{i'} > m_W$ . However, this hypothetical scenario is not realized in the SM with three generations.

gauge dependent and divergent, a standard property of wfrs. However, as explained in Refs. [2, 3], its contribution to the physical  $W^+ \rightarrow u_i + \bar{d}_j$  amplitude does not involve CKM matrix elements except for an overall factor  $V_{ij}$ , and only depends on the masses  $m_i$  and  $m_j$  of the external particles, in complete analogy with the proper vertex corrections. As a consequence, the proof of finiteness and gauge independence of the  $W^+ \rightarrow u_i + \bar{d}_j$  amplitude is reduced to that in the unmixed, single-generation case.

For an incoming up-type quark of flavor  $i'$ , mass  $m_{i'}$ , and four-momentum  $p$ , the external-leg correction is obtained by multiplying  $\Sigma_{ii'}(\not{p}) - \delta m_{ii'}$  by  $u_{i'}(p)$  on the right and by  $(\not{p} - m_i)^{-1}$  on the left, where  $i$  denotes now the virtual up-type quark of flavor  $i$  and mass  $m_i$ , and  $\Sigma_{ii'}(\not{p}) - \delta m_{ii'}$  is the same amplitude discussed before. It is then easy to see that the residual contributions in the incoming case are obtained by interchanging  $a_+ \leftrightarrow a_-$  and  $m_i \leftrightarrow m_{i'}$  between the curly brackets of Eq. (12), and multiplying the resulting expression by  $u_{i'}(p)$  on the right hand and by  $(\not{p} - m_i)^{-1}$  on the left. Similarly, the wfr for an incoming up-type quark of flavor  $i'$  is obtained by interchanging  $a_+ \leftrightarrow a_-$  and  $m_i \leftrightarrow m_{i'}$  between the curly brackets of Eq. (30) in Ref. [3] and multiplying the resulting expression by  $u_{i'}(p)$  on the right. Finally, the expressions for an outgoing down-type quark of flavor  $j$  are obtained from those of an outgoing up-type quark by substituting  $i \rightarrow j$ ,  $i' \rightarrow j'$ , and  $V_{il}V_{li'}^\dagger \rightarrow V_{jl}^\dagger V_{lj'}$ , where  $j'$  is the flavor of the virtual down-type quark and  $l$  that of the up-type quark in the loop. In the case in which the external particle is a down-type quark, the  $I$  and  $J$  functions are real, and the Re instruction in Eqs. (4), (6), and (7) is not necessary.

As discussed in Refs. [2, 3], diagonalization of the complete mass matrices for both up-type and down-type quarks generates a CKM counterterm matrix that is gauge independent, preserves unitarity in the sense that both the bare and renormalized CKM matrices are unitary, and leads to renormalized amplitudes that are non-singular in the limit  $m_{i'} \rightarrow m_i$  for  $m_i < m_W$ . A comparative analysis of the calculations of the  $W$ -boson hadronic widths in various CKM renormalization schemes, including the ones proposed here and in Refs. [2, 3], is presented in Ref. [9].

In summary, we have presented a novel mass counterterm for CKM renormalization that is endowed with very attractive features:

1. It is expressed in terms of the invariant self-energy functions, a property that is very useful for practical applications, since such functions are routinely evaluated in computer codes.
2. It is gauge independent, which is a crucial property to ensure the gauge independence of the associated CKM counterterm matrix.
3. It leads to renormalized amplitudes that are non-singular in the limit  $m_{i'} \rightarrow m_i$  for  $m_i < m_W$ .
4. It automatically satisfies the hermiticity constraints of the mass matrix, a property that eliminates the need for special and somewhat arbitrary adjustments of the counterterms in specific transition channels. In fact, the counterterm presented

in Eqs. (5)–(7) can be applied as it stands to all diagonal and off-diagonal CKM amplitudes and, in this sense, it is flavor-democratic since it does not single out particular flavor channels.

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