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$AdS_3 \times_w (S^3 \times S^3 \times S^1)$ Solutions of Type IIB String Theory

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Abstract

We analyse a recently constructed class of local solutions of type IIB supergravity that consist of a warped product of AdS_3 with a seven-dimensional internal space. In one duality frame the only other non-vanishing fields are the NS three-form and the dilaton. We analyse in detail how these local solutions can be extended to globally well-defined solutions of type IIB string theory, with the internal space having topology $S^3 \times S^3 \times S^1$ and with properly quantised three-form flux. We show that many of the dual $(0, 2)$ SCFTs are exactly marginal deformations of the $(0, 2)$ SCFTs whose holographic duals are warped products of AdS_3 with seven-dimensional manifolds of topology $S^3 \times S^2 \times T^2$.

Contents

1 Introduction

Supersymmetric solutions of string or M-theory that contain AdS_{d+1} factors are dual to supersymmetric conformal field theories in d spacetime dimensions. Starting with the work of [1], general characterisations of the geometries underlying such solutions, using G-structure techniques [2, 3], have been achieved for various d and for various amounts of supersymmetry [4]–[24]. With a few exceptions, mostly with sixteen supersymmetries, many of these geometries are still poorly understood, and it has proved difficult to find explicit solutions.

One notable exception is the class of AdS_3 solutions of type IIB string theory with non-vanishing five-form flux, dual to $d = 2$ conformal field theories with $(0, 2)$ supersymmetry, that were classified in [7]. It was shown that the seven-dimensional internal space has a Killing vector which is dual to the R-symmetry of the dual SCFT. The Killing vector defines a foliation and the solution is completely determined, locally, by a Kähler metric on the six-dimensional leaf space whose Ricci tensor satisfies an additional differential condition. Moreover, a rich set of explicit solutions have been constructed in [25, 14, 26] and the corresponding central charges of the dual SCFTs have also been calculated.

More recently, it was understood how to generalise this class of type IIB AdS_3 solutions to also include three-form flux [27]. The solutions are again locally determined by a six-dimensional Kähler metric and a choice of a closed, primitive $(1,2)$-form on the Kähler space. Once again additional explicit solutions were constructed with the six-dimensional Kähler space having a two-torus factor and the three-form flux being parametrised by a real parameter Q. After two T-dualities on the two-torus it was also shown that these explicit solutions give type IIB AdS_3 solutions with non-vanishing dilaton and RR three-form flux only. After an additional S-duality the solutions only involve NS fields.

In [27] these explicit solutions were examined in more detail for the special case of $Q = 0$. It was shown that the parameters and ranges of the coordinates could be chosen to give globally defined supergravity solutions consisting of a warped product of AdS_3 with a seven-dimensional internal manifold that is diffeomorphic to $S^2 \times S^3 \times T^2$. It was shown that the solutions, with properly quantised three-form flux, are specified by a pair of positive coprime integers p, q.

The purpose of this paper is to carry out a similar analysis when we switch on the parameter Q. We will find that we are led to infinite classes of solutions, with the seven-dimensional internal space being diffeomorphic to $S^3 \times S^3 \times S^1$. Furthermore, we will see that the central charge is independent of Q and hence the Q deformation in many cases is dual to an exactly marginal perturbation in the dual SCFT.

While the final topology of the solutions is simple, it is not easy to see this in the local coordinates in which the solutions are presented. When $Q = 0$ the $S^2 \times S^3$ factor is realised in a manner very similar to the $Y^{p,q}$ Sasaki-Einstein spaces [28]. When $Q \neq 0$ one of the circles in the T^2 factor is fibred over the $S^2 \times S^3$ and we need to carefully check that the circle fibration is globally well-defined, leading to $S^3 \times S^3$. Furthermore we need to check that the three-form flux is properly quantised. This is not straightforward since it is not clear "where" the two S^3 factors are in the local coordinates. After some false starts we developed a workable prescription for ensuring that the three-form is properly quantised, as we shall explain.

The plan of the paper is as follows. In section 2, we begin by recalling the local solutions of [27] and then discuss how, after suitable choices of parameters and periods for the coordinates, the seven-dimensional internal manifold has topology $S^3 \times S^3 \times$

S^1. We discuss some aspects of the topology in detail, leading to a prescription for carrying out flux quantisation which is dealt with in section 3. Our method uses a quotient construction, which is explained in section 2, as well as explicit coordinate patches. In sections 2 and 3, the solutions depend on a pair of coprime positive integers p, q, the electric three-form flux, n_1, the magnetic three-form flux through each of the two S^3 factors, M_1 and M_2, and the parameter Q. For these solutions, it turns out that M_1 and M_2 are not independent and are given by $M_1 = M(p+q)^2$ and $M_2 = Mq^2$, where M is an integer. We calculate the central charge and show that it is given by the simple formula

$$c = 6n_1 \frac{(M_1 - M_2)M_2}{M_1} \ . \tag{1.1}$$

In particular it is independent of Q, and since the solutions are specified by the same number of parameters as for the $Q = 0$ solutions that were analysed in [27] we conclude that these solutions correspond to exactly marginal deformations of those with $Q = 0$.

In section 4, we generalise our construction by making more general identifications on the coordinates, obtaining solutions that involve more parameters. We show that the central charge has exactly the same form as in (1.1), but now, however, the integers M_1 and M_2 labelling the three-form flux through the two S^3's are no longer constrained. Thus not all of these more general solutions correspond to exactly marginal deformations of those that we consider in sections 2 and 3. We conclude in section 5.

We noted above that the $S^2 \times S^3$ factor in the AdS_3 solutions constructed in [27], with $Q = 0$, is realised in a similar way to the $Y^{p,q}$ Sasaki-Einstein spaces found in [28]. In particular, in both cases the metrics on $S^2 \times S^3$ are cohomogeneity one. Given that the $Y^{p,q}$ metrics can be generalised to cohomogeneity two Sasaki-Einstein metrics $L^{a,b,c}$ on $S^2 \times S^3$ [29] (see also [30]), it is natural to suspect that there are analogous AdS_3 solutions, with five-form flux only, with internal space having topology $S^2 \times S^3 \times T^2$ and with the metric on the $S^2 \times S^3$ factor having cohomogeneity two. This is indeed possible, and moreover it is also possible to find generalisations with non-zero three-form flux and with the internal manifold having topology $S^3 \times S^3 \times S^1$. We will present such solutions in appendix C, but we will leave a detailed analysis of the regularity and flux quantisation conditions for future work.

2 The AdS_3 solutions

2.1 The local solutions

We start with the explicit class of AdS_3 solutions of section 4.3 of [27]. The string frame metric is given by

$$\frac{1}{L^2}ds^2 = \frac{\beta}{y^{1/2}}[ds^2(AdS_3) + ds^2(X_7)] \tag{2.1}$$

where

$$\begin{aligned} ds^2(X_7) &= \frac{\beta^2 - 1 + 2y - Q^2y^2}{4\beta^2}Dz^2 + \frac{U(y)}{4(\beta^2 - 1 + 2y - Q^2y^2)}D\psi^2 + \frac{dy^2}{4\beta^2y^2U(y)} \\ &\quad + \frac{1}{4\beta^2}ds^2(S^2) + (du^1 - \frac{Qy}{2\beta}[(1-g)D\psi - Dz])^2 + (du^2)^2 \ , \end{aligned} \tag{2.2}$$

where β, Q are positive constants, L is an arbitrary length scale and

$$U(y) = 1 - \frac{1}{\beta^2}(1-y)^2 - Q^2y^2 \ . \tag{2.3}$$

In addition, $ds^2(S^2)$ is the standard[1] metric on a two-sphere, $ds^2(S^2) = d\theta^2 + \sin^2\theta d\phi^2$, and we have defined

$$D\psi = d\psi + P \tag{2.4}$$

with

$$dP = \mathrm{Vol}(S^2) = \sin\theta d\theta \wedge d\phi \equiv J \ . \tag{2.5}$$

Note that P is only a locally defined one-form on S^2. In fact, more precisely, P is a connection one-form on the $U(1)$ principal bundle associated to the tangent bundle of S^2. The two-form J introduced in (2.5) may be regarded as a Kähler form on S^2. We also have

$$Dz = dz - g(y)D\psi \tag{2.6}$$

with

$$g(y) = \frac{y(1-Q^2y)}{\beta^2 - 1 + 2y - Q^2y^2} \ . \tag{2.7}$$

The only other non-trivial type IIB supergravity fields are the dilaton and the RR three-form. The dilaton is given by

$$e^{2\phi} = \frac{\beta^2}{y} \tag{2.8}$$

[1] Note that we have rescaled the metric on S^2 appearing in [27] by a factor of 4.

while the RR three-form field strength is given by

$$\begin{aligned}\frac{1}{L^2}F^{(3)} &= -\frac{1}{4\beta^2}dy\wedge D\psi\wedge Dz - \frac{y}{4\beta^2}J\wedge Dz + \left[\frac{1-yg}{4\beta^2}\right]J\wedge D\psi \\ &\quad +\frac{Q}{2\beta}du^1\wedge[dy\wedge Dz - yJ - (1-g)dy\wedge D\psi] + 2\mathrm{Vol}(AdS_3)\ . \end{aligned}\tag{2.9}$$

This is closed. After a further S-duality transformation we obtain AdS_3 solutions with only NS fields non-vanishing, but we will continue to work with the above solution.

In order to simplify some of the formulae it will be helpful to introduce

$$Z \equiv 1 - \sqrt{1+Q^2(\beta^2-1)}\ . \tag{2.10}$$

We next change coordinates via

$$\begin{aligned} dz &= dw + \frac{2Q\beta}{Z-2}dv \\ du^1 &= dv + \frac{Q(1-\beta^2)}{2\beta(Z-2)}dw \end{aligned}\tag{2.11}$$

to bring the metric to the form

$$\begin{aligned} ds^2(X_7) = \frac{2(1-Z)(1-\beta^2-yZ)}{(2-Z)(1-\beta^2)}Dv^2 + \frac{(1-Z)(2y-yZ-1+\beta^2)}{2\beta^2(2-Z)}Dw^2 \\ +\frac{(1-\beta^2)U(y)}{4(1-\beta^2-Zy)(\beta^2-1+2y-Zy)}D\psi^2 + \frac{dy^2}{4\beta^2y^2U(y)} + \frac{1}{4\beta^2}ds^2(S^2) + (du^2)^2 \end{aligned}\tag{2.12}$$

where

$$\begin{aligned} Dv &= dv - A_v\,D\psi \\ Dw &= dw - A_w\,D\psi \end{aligned}\tag{2.13}$$

and

$$\begin{aligned} A_v &= \frac{Q(1-\beta^2)y}{4\beta(1-\beta^2-yZ)} \\ A_w &= \frac{(2-Z)y}{2(2y-yZ-1+\beta^2)}\ . \end{aligned}\tag{2.14}$$

The three-form in the new coordinates is given by

$$\begin{aligned} \frac{1}{L^2}F^{(3)} &= 2\,\mathrm{Vol}\,(AdS_3) + \frac{(1-\beta^2)U(y)}{4(1-\beta^2-yZ)(\beta^2-1+2y-yZ)}\,J\wedge D\psi \\ &- Dw\wedge\left\{\frac{(Z-1)(1-\beta^2)}{4\beta^2(Zy-1+\beta^2)}dy\wedge D\psi + \frac{y(1-Z)}{4\beta^2}J\right\} \\ &- QDv\wedge\left\{\frac{(1-Z)(1-\beta^2)}{2\beta(Z-2)(-Zy-1+\beta^2+2y)}dy\wedge D\psi + \frac{y(1-Z)}{2\beta(2-Z)}J\right\} \\ &- \frac{Q(1-Z)}{\beta(2-Z)}dy\wedge Dv\wedge Dw\ . \end{aligned}\tag{2.15}$$

Finally, we note that the canonical Killing vector related to supersymmetry is given by

$$\partial_\psi + \partial_z \ . \tag{2.16}$$

In the new coordinates this reads

$$\partial_\psi + \frac{(Z-2)}{2(Z-1)}\partial_w + \frac{Q(\beta^2-1)}{4\beta(Z-1)}\partial_v \ . \tag{2.17}$$

We now would like to find the restrictions on the parameters β, Q so that these local solutions extend to global solutions on a globally well-defined manifold X_7. Having achieved that goal, we will analyse the additional constraints imposed by ensuring that the three-form is properly quantised. Note that when $Q = 0$ the corresponding analysis was carried out in [27] and in particular it was shown that there were infinite classes of solutions, labelled by a pair of positive coprime integers, p, q, with X_7 having the topology of $S^3 \times S^2 \times T^2$.

Our strategy is to build X_7 in stages. The u^2 coordinate is taken to paramaterise an S^1: for now the period of u^2 is arbitrary but it will later be fixed by flux quantisation. We therefore write $X_7 = M_6 \times S^1$ with

$$ds^2(M_6) \equiv \frac{2(1-Z)(1-\beta^2-yZ)}{(2-Z)(1-\beta^2)}Dv^2 + ds^2(M_5) \ , \tag{2.18}$$

where

$$ds^2(M_5) \equiv \frac{(1-Z)(2y-yZ-1+\beta^2)}{2\beta^2(2-Z)}Dw^2 + ds^2(B_4) \tag{2.19}$$

and

$$ds^2(B_4) \equiv \frac{(1-\beta^2)U(y)}{4(1-\beta^2-Zy)(\beta^2-1+2y-Zy)}D\psi^2 + \frac{dy^2}{4\beta^2y^2U(y)} + \frac{1}{4\beta^2}ds^2(S^2) \ . \tag{2.20}$$

We will first analyse $ds^2(B_4)$, showing that, by taking ψ to be a periodic coordinate with period 2π, B_4 is a smooth manifold diffeomorphic to $S^2\times S^2$. We then show that, by taking w to be a periodic coordinate with a suitable period, with the parameter β fixed by two relatively prime positive integers p, q, M_5 is the total space of a circle fibration over B_4, and has topology $S^3 \times S^2$. Here p and q have a topological interpretation as Chern numbers of the circle bundle over B_4. These steps are familiar from the construction of the Sasaki-Einstein manifolds $Y^{p,q}$ [28]. The final step is to show that, by taking v to be periodic with a suitable period, M_6 is the total space of a circle fibration over M_5, and has topology $S^3 \times S^3$.

It will be useful in the following to observe that the function $U(y)$ is a quadratic function of y with roots y_1 and y_2 given by

$$
\begin{aligned}
y_1 &= \frac{1-\beta^2}{1+\beta(1-Z)} \\
y_2 &= \frac{1-\beta^2}{1-\beta(1-Z)} \ .
\end{aligned}
\tag{2.21}
$$

It will also be useful to know the values of the functions A_w and A_v appearing in $ds^2(M_6)$ at y_1 and y_2. We find

$$
\begin{aligned}
A_w(y_1) &= \frac{2-Z}{2(1-Z)(1-\beta)} \\
A_w(y_2) &= \frac{2-Z}{2(1-Z)(1+\beta)} \\
A_v(y_1) &= \frac{Q(1-\beta)}{4\beta(1-Z)} \\
A_v(y_2) &= \frac{Q(1+\beta)}{4\beta(1-Z)} \ .
\end{aligned}
\tag{2.22}
$$

2.2 $B_4 = S^2 \times S^2$

B_4 is parametrised by θ, ϕ, y and ψ. We take the coordinate y to lie in the interval $y \in [y_1, y_2]$ where y_i are the two distinct positive[2] roots of $U(y)$, given by (2.21). This requires that we demand

$$
0 < \beta < 1, \qquad 0 \leq Z < 1 \ . \tag{2.23}
$$

We next observe that if we choose the period of ψ to be 2π, then y, ψ parametrise a smooth two-sphere, with y a polar coordinate and ψ an azimuthal coordinate on the metrically squashed S^2 fibre. In particular, fixing a point on the round two-sphere, one can check that $ds^2(B_4)$ is free from conical singularities at the poles $y = y_1$ and $y = y_2$. B_4 is then a smooth S^2 bundle over the round S^2. The transition functions are in $U(1)$, acting in the obvious way on the fibre. The first Chern number of the $U(1)$ fibration is -2 and thus, as explained in [28], B_4 is diffeomorphic to $S^2 \times S^2$.

We have $H_2(B_4, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Three obvious two-spheres in B_4 are the sections $\Sigma_1 = \{y = y_1\}$ and $\Sigma_2 = \{y = y_2\}$, each a copy of the two-sphere base, and a copy of the fibre Σ_f at some point on the two-sphere base (for concreteness, say, the north pole $\{\theta = 0\}$). Call the corresponding homology classes $[\Sigma_1]$, $[\Sigma_2]$ and $[\Sigma_f]$,

[2] We need y to be positive to ensure that the warp factor is real.

respectively. We can take $[\Sigma_2]$ and $[\Sigma_f]$ to generate $H_2(B_4,\mathbb{Z})$, but we note that this is *not* the natural basis of $S^2\times S^2$. In particular, the intersections of the 2-cycles are

$$[\Sigma_f]\cap[\Sigma_f]=0,\quad [\Sigma_f]\cap[\Sigma_2]=1,\quad [\Sigma_2]\cap[\Sigma_2]=2\ . \tag{2.24}$$

The only non-obvious equality above is the last. This follows since the self-intersection of a 2-cycle in a 4-manifold is equal to the Chern number of the normal bundle. Similar calculations show that

$$[\Sigma_1]=[\Sigma_2]-2[\Sigma_f]. \tag{2.25}$$

Later it will be useful to use a more natural basis given by $[C_1]=[\Sigma_2]-[\Sigma_f]$ and $[C_2]=[\Sigma_f]$: indeed one can then check that $[C_1]\cap[C_1]=[C_2]\cap[C_2]=0$ and $[C_1]\cap[C_2]=1$.

By Poincaré duality we have $H_2(B_4,\mathbb{Z})\cong H^2(B_4,\mathbb{Z})$. Recall that, by definition, the Poincare dual η_Σ of a submanifold $\Sigma\subset M$ satisfies

$$\int_\Sigma \omega=\int_M \omega\wedge\eta_\Sigma \tag{2.26}$$

for any closed form ω. We introduce the closed two-forms on B_4

$$\begin{aligned}\sigma_2 &= \frac{1}{4\pi}J\\ \sigma_f &= \frac{1}{2\pi[A_w(y_1)-A_w(y_2)]}[(A_w(y_2)-A_w)J-\partial_y(A_w)dy\wedge D\psi]\ .\end{aligned} \tag{2.27}$$

These forms satisfy

$$\int_{\Sigma_2}\sigma_2=\int_{\Sigma_f}\sigma_f=1,\qquad \int_{\Sigma_2}\sigma_f=\int_{\Sigma_f}\sigma_2=0\ , \tag{2.28}$$

and one finds that Poincaré duality maps $\Sigma_f\mapsto\sigma_2$ and $\Sigma_2\mapsto\sigma_f+2\sigma_2$.

2.3 $M_5=S^3\times S^2$

We next construct M_5 as the total space of a circle bundle over B_4, by letting w be periodic with period $2\pi l_w$, for a suitably chosen l_w. We begin by observing from (2.12) that the norm of the Killing vector ∂_w is nowhere-vanishing, and so the size of the S^1 fibre doesn't degenerate anywhere. Recalling that $Dw=dw-A_w\,D\psi$, we require that $l_w^{-1}A_wD\psi$ is a connection on a *bona fide* $U(1)$ fibration with first Chern class represented by $(2\pi l_w)^{-1}d(A_w\,D\psi)$.

It is straightforward to first check that $(2\pi l_w)^{-1}d(A_w\, D\psi)$ is indeed a globally defined two-form on B_4. We next impose that it has integer valued periods:

$$\begin{aligned}
\frac{1}{2\pi l_w}\int_{\Sigma_2} d\left(A_w D\psi\right) &= \frac{2}{l_w}A_w\left(y_2\right) = p \\
\frac{1}{2\pi l_w}\int_{\Sigma_f} d\left(A_w D\psi\right) &= \frac{1}{l_w}\left[A_w\left(y_2\right) - A_w\left(y_1\right)\right] = -q,
\end{aligned} \tag{2.29}$$

where p, q are positive integers. One can then calculate

$$\frac{1}{2\pi l_w}\int_{\Sigma_1} d\left(A_w D\psi\right) = \frac{2}{l_w}A_w\left(y_1\right) = p + 2q \tag{2.30}$$

as expected from (2.25). We then deduce that

$$\beta = \frac{q}{p+q} \tag{2.31}$$

which, remarkably, is independent of Q, and

$$l_w = \frac{2-Z}{p(1-Z)(1+\beta)}\ . \tag{2.32}$$

With these choices we have that M_5 is the total space of a circle bundle with first Chern class given by

$$c_1 = p[\sigma_2] - q[\sigma_f] \in H^2(B_4, \mathbb{Z})\ . \tag{2.33}$$

As in [28], taking p and q to be relatively prime, as we shall henceforth do, one can show that M_5 is simply-connected with $H_2(M_5, \mathbb{Z}) \cong \mathbb{Z}$. Using Smale's theorem for five-manifolds [31], it follows that M_5 is diffeomorphic to $S^3 \times S^2$.

Having constructed M_5, it will be useful later to know various topological properties of this manifold in terms of the coordinate system above. In the remainder of this subsection we write down explicit generators for $H^2(M_5, \mathbb{Z}) \cong \mathbb{Z}$, which will be useful for constructing circle bundles over M_5, and for $H^3(M_5, \mathbb{Z}) \cong \mathbb{Z}$, which will be useful both for integration using Poincaré duality and also for quantising the three-form flux. We also find representatives of the generating 2-cycle and 3-cycle in $H_2(M_5, \mathbb{Z}) \cong \mathbb{Z}$ and $H_3(M_5, \mathbb{Z}) \cong \mathbb{Z}$, respectively.

The generator of $H^2(M_5, \mathbb{Z}) \cong \mathbb{Z}$ may be taken to be the pull-back of the class

$$\tau = b[\sigma_2] + a[\sigma_f] \in H^2(B_4, \mathbb{Z}) \tag{2.34}$$

under the projection

$$\pi : M_5 \to B_4\ , \tag{2.35}$$

where a and b are (any) integers satisfying

$$pa + qb = 1 \ . \tag{2.36}$$

These exist and are unique up to $b \to b + mp$, $a \to a - mq$, for any integer m, by Bezout's lemma. The non-uniqueness simply corresponds to the fact that the Chern class $c_1 = p\sigma_2 - q\sigma_f$ of the circle bundle over B_4 is trivial when pulled back to M_5, as is the Chern class of any tensor power of this circle bundle (the power corresponds to the integer m above).

To see that $\pi^*\tau$ is the generator of $H^2(M_5, \mathbb{Z})$ as claimed, note that, *a priori*, $\pi^*\tau$ is necessarily β times the generator, for some integer $\beta \in \mathbb{Z}$. Thus we write $\pi^*\tau = \beta \in H^2(M_5, \mathbb{Z}) \cong \mathbb{Z}$. Next note that the circle bundle π trivialises over any[3] smooth submanifold $S \subset B_4$ that represents the cycle

$$[S] = q[\Sigma_2] + p[\Sigma_f] \ . \tag{2.37}$$

This is simply because the first Chern class c_1 evaluated on $[S]$ is zero, as one sees using (2.28). Hence we may take a section s of π over S:

$$s : S \to M_5 \ . \tag{2.38}$$

This defines a 2-cycle $[s(S)]$ in $H_2(M_5, \mathbb{Z}) \cong \mathbb{Z}$, which we may take to be α times the generator, for some integer α. But then by construction

$$\int_{s(S)} \pi^*\tau = \int_S \tau = 1 \ , \tag{2.39}$$

implying that $\alpha\beta = 1$, and thus α and β are both ± 1. Hence $\pi^*\tau$ generates $H^2(M_5, \mathbb{Z})$, and $s(S)$ generates $H_2(M_5, \mathbb{Z})$.

The only other non-trivial homology group is $H_3(M_5, \mathbb{Z}) \cong \mathbb{Z}$. There are three natural three-submanifolds of M_5, which we call E_1, E_2 and E_f. These are the restriction of the circle bundle π to the submanifolds Σ_1, Σ_2 and Σ_f of B_4, respectively. These three-manifolds are all Lens spaces[4]. Indeed, Σ_1, Σ_2, Σ_f are all two-spheres. The Chern numbers are easily read off from c_1 above to be $p + 2q$, p and $-q$. Thus

$$E_1 \cong S^3/\mathbb{Z}_{p+2q}, \quad E_2 \cong S^3/\mathbb{Z}_p, \quad E_f \cong S^3/\mathbb{Z}_q \ . \tag{2.40}$$

We may take the generator of $H_3(M_5, \mathbb{Z})$ to be

$$E = k[E_1] + l[E_f] \tag{2.41}$$

[3] Although S certainly exists, in practice it is not easy to define such a smooth submanifold in the above coordinate system.

[4] See appendix A for some discussion.

where k and l are (any) integers satisfying

$$pk + ql = 1 \ . \tag{2.42}$$

Notice this is the same as (2.36), so one could choose $k = a$ and $l = b$. A simple way to check this is to note that the generator has intersection number 1 with $[s(S)]$. One computes

$$[s(S)] \cap E = pk + ql = 1 \tag{2.43}$$

which uniquely identifies E as the generator. We then have

$$[E_1] = pE, \qquad [E_2] = (p + 2q)E, \qquad [E_f] = qE \ , \tag{2.44}$$

which again can be shown by taking intersection numbers with $[s(S)]$.

Finally, we may also write down a representative Φ of the generator of $H^3(M_5, \mathbb{Z})$. By definition this is a closed three-form on M_5 that integrates to 1 over E. We choose

$$\begin{aligned}\Phi = \frac{1}{(2\pi)^2 \, l_w^2} & \{ Dw \wedge [(A_w(y_1) + A_w(y_2) - A_w(y)) \, J - \partial_y A_w dy \wedge D\psi] \\ & - \left[A_w^2(y) - A_w(y)(A_w(y_1) + A_w(y_2)) + A_w(y_1) A_w(y_2) \right] \, J \wedge D\psi \} \ .\end{aligned} \tag{2.45}$$

The three-form Φ is Poincaré dual to the non-trivial two-cycle in M_5.

2.4 $M_6 = S^3 \times S^3$

We now construct M_6 as a circle bundle over M_5. Since $H^2(M_5, \mathbb{Z}) \cong \mathbb{Z}$, such circle bundles are determined, up to isomorphism, by an integer. Since $M_5 \cong S^3 \times S^2$, taking this integer to be 1 (or -1) gives a total space $M_6 \cong S^3 \times S^3$. Taking the Chern number to be n would instead lead to an M_6 with $\pi_1(M_6) \cong \mathbb{Z}_n$, which we may always lift to the simply-connected cover with $n = \pm 1$. So, we will do this. However, as we shall see later, in fixing the three-form flux quantisation it will be helpful to consider such quotients of M_6.

Observe from (2.12) that the norm of the Killing vector ∂_v is nowhere-vanishing, and so the size of the S^1 fibre doesn't degenerate anywhere. The period of v is taken to be $2\pi l_v$, where l_v will be fixed shortly. Recalling that $Dv = dv - A_v \, D\psi$, we require that $l_v^{-1} A_v D\psi$ is a connection on a $U(1)$ fibration with first Chern class represented by $(2\pi l_v)^{-1} d(A_v \, D\psi)$. It is straightforward to check that $(2\pi l_v)^{-1} d(A_v \, D\psi)$ is a globally defined two-form on M_5. We next impose that it has unit period. To do this we would like to integrate $(2\pi l_v)^{-1} d(A_v\, D\psi)$ over a smooth submanifold in the same homology class as $s(S)$, the generator of $H_2(M_5,\mathbb{Z})$. However, as we have already noted, finding such a smooth submanifold is not so easy. Luckily, we can use Poincaré duality to calculate the period instead. Recalling that $[\Phi]$ is Poincaré dual to $[s(S)]$, we demand that

$$\begin{aligned}
\frac{1}{2\pi l_v}\int_{s(S)} d(A_v D\psi) &= \frac{1}{2\pi l_v}\int_{M_5} d(A_v D\psi)\wedge\Phi \\
&= \frac{2}{l_v l_w}[A_v(y_2)A_w(y_1) - A_v(y_1)A_w(y_2)] \\
&= \frac{1}{l_v}[2qA_v(y_2) - p(A_v(y_1) - A_v(y_2))] = 1, \qquad (2.46)
\end{aligned}$$

so that the circle bundle has Chern number 1, which can be achieved by setting

$$l_v = \frac{Q(p+q)}{1-Z}\ . \qquad (2.47)$$

Let us denote this circle bundle over M_5 by L, with corresponding projection

$$\Pi : M_6 \to M_5\ . \qquad (2.48)$$

Recalling that the generator of $H^2(M_5,\mathbb{Z})$ may be taken to be the pull-back of τ in (2.34) under the projection $\pi : M_5 \to B_4$, we see that L may be regarded as the pull-back of the circle bundle L_τ over B_4 with first Chern class given by $\tau \in H^2(B_4,\mathbb{Z})$. We write this as $L = \pi^* L_\tau$.

Since $M_6 \cong S^3\times S^3$, it follows that the only non-trivial homology group is $H_3(M_6,\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$. The two generators are clearly the two copies of S^3, at a fixed point on the other copy. However, because of the way we have constructed M_6 above, it is not easy to see the diffeomorphism of M_6 with $S^3\times S^3$ explicitly. Nevertheless, we observe that one three-cycle is represented by the total space of the circle bundle L over the S^2 in $M_5 \cong S^3\times S^2$. Since $s(S)$ is homologous to the S^2 in M_5, it follows[5] that taking the total space of L over both submanifolds gives homologous three-submanifolds of M_6, which is the total space of L. Thus the total space of the L circle bundle over $s(S)$ is one of the generators of the homology of M_6. It should be pointed out, though, that finding a smooth representative of this generator is not straightforward. For the other generator, the obvious thing to try is to take a representative for E, which afterall is represented by $S^3 \subset M_5$, and then try to

[5]Being homologous in M_5 means there is a three-dimensional chain in M_5 with boundary $s(S) - S^2$. By taking the total space of L over this chain, one obtains a chain in M_6 with boundary given by the total space of L over $s(S) - S^2$.

take a section of Π over this representative. However, unfortunately just because two submanifolds are homologous in M_5, with L trivial over one of them, this does not necessarily guarantee that the circle bundle L is trivial over the other submanifold[6]. So, we cannot necessarily do this. An additional observation is that, while a section of Π exists over E, it does not exist, in general, over the submanifolds E_1, E_2 and E_f, as we explain in the appendix.

In order to carry out the flux quantisation of the three-form in the supergravity solutions, we need a prescription to integrate three-forms over a basis of $H_3(M_6, \mathbb{Z})$. The comments in the last paragraph indicate that this is not as straightforward as it might seem. Our approach, employing a quotient construction[7], will be explained in the next subsection.

2.5 A quotient of M_6 and integral three-forms

In this section we want to explain how considering the periods of the three-form on the quotient $\hat{M}_6 = M_6/\mathbb{Z}_{(p+q)q}$ leads to a practical procedure for ensuring that a three-form, such as the suitably normalised RR three-form, has integral periods.

In order to obtain more insight into the topology of M_6, it will be helpful to think of it as a group manifold,

$$M_6 = S^3 \times S^3 \cong SU(2) \times SU(2) \ , \tag{2.49}$$

and observe that taking the quotient by the maximal torus $T^2 \subset SU(2) \times SU(2)$ leads to B_4:

$$M_6/T^2 = S^2 \times S^2 = B_4. \tag{2.50}$$

Now, recall that we constructed M_5 as the total space of a circle bundle over B_4 with winding numbers p and $-q$ over Σ_2 and Σ_f, respectively. With respect to the

[6] As a simple example, consider the five-manifold $T^{1,1} \cong S^2 \times S^3$, which recall is naturally a circle bundle over $S^2 \times S^2$. For our two three-submanifolds we take a contractible S^3, say the equatorial S^3 on a contractible S^4 that links a point, and the "diagonally embedded" Lens space $S^3/\mathbb{Z}_2$. Since $T^{1,1}$ is a circle bundle over $S^2 \times S^2$, we may describe the latter three-submanifold more precisely as the restriction of this circle bundle to the diagonal S^2 in $S^2 \times S^2$, which is the easiest way to see that the topology is indeed $S^3/\mathbb{Z}_2$. Both three-cycles are trivial – to see this for the latter construct the generator of $H^3(T^{1,1}, \mathbb{Z})$ and integrate over the three-cycle. However, if we pull back the complex line bundle $\mathcal{O}(1,0)_{S^2\times S^2}$ with winding numbers 1 and 0 on $S^2 \times S^2$ to $T^{1,1}$, this is trivial over the S^3 but non-trivial over the contractible $S^3/\mathbb{Z}_2$ (the latter follows using arguments similar to those in appendix B).

[7] We thank Dominic Joyce for suggesting this approach.

natural basis $[C_1] = [\Sigma_2] - [\Sigma_f]$ and $[C_2] = [\Sigma_f]$ of $B_4 \cong S^2 \times S^2$ introduced in section 2.2, we thus have Chern numbers

$$\int_{[C_1]} c_1 = p + q, \qquad \int_{[C_2]} c_1 = -q \ . \tag{2.51}$$

In this section we make the $U(1)$ fibration structure of M_5 explicit in the notation by denoting the latter as $M_5(p, q)$.

The key observation is that we may realise $M_5(p, q)$ as a quotient by the $U(1)$ subgroup of T^2 with charges $(q, p + q)$, as illustrated in the following diagram:

$$\begin{array}{ccccc} & U(1)_{q,p+q} & & & \\ \swarrow & & \searrow & & \\ T^2 & \hookrightarrow & M_6 & \rightarrow & B_4 \\ \downarrow & & \downarrow & & \\ T^2/U(1)_{q,p+q} & \hookrightarrow & M_5(p,q) & \rightarrow & B_4 \end{array} \ . \tag{2.52}$$

To see this more explicitly, we introduce Euler angles, ψ_1, θ_1, ϕ_1 and ψ_2, θ_2, ϕ_2 for each of the two $SU(2)$ factors. We also introduce the corresponding left-invariant one-forms σ_i^α for each factor, respectively, where $\alpha = 1, 2$; $i = 1, 2, 3$. Thus

$$\begin{aligned} \sigma_1^\alpha &= \cos\psi_\alpha d\theta_\alpha + \sin\theta_\alpha \sin\psi_\alpha d\phi_\alpha \\ \sigma_2^\alpha &= -\sin\psi_\alpha d\theta_\alpha + \sin\theta_\alpha \cos\psi_\alpha d\phi_\alpha \\ \sigma_3^\alpha &= d\psi_\alpha + \cos\theta_\alpha d\phi_\alpha \ . \end{aligned} \tag{2.53}$$

Now $\psi_1, \psi_2 \in [0, 4\pi)$ parametrise the T^2. The $U(1)_{q,p+q}$ circle action is then given explicitly by

$$(\psi_1, \psi_2) \mapsto (\psi_1 + q\psi, \psi_2 + (p + q)\psi) \tag{2.54}$$

where $\psi \in [0, 4\pi)$ parametrises the circle subgroup. If we introduce coordinates $\tilde{v}, \tilde{w}$ defined by

$$\tilde{v} = -\frac{1}{q}\psi_1, \qquad \tilde{w} = (p + q)\psi_1 - q\psi_2 \tag{2.55}$$

then the T^2 is parametrised by taking $\tilde{v}, \tilde{w} \in [0, 4\pi)$. In these coordinates the $U(1)_{q,p+q}$ circle action reads

$$(\tilde{v}, \tilde{w}) \mapsto (\tilde{v} - \psi, \tilde{w}) \tag{2.56}$$

and hence $\tilde{w}$ parametrises the circle $T^2/U(1)_{q,p+q}$. The globally defined connection one-form on the total space of the circle bundle on the bottom line of (2.52) is given by

$$\begin{aligned}\eta &= \frac{1}{2}((p+q)\sigma_3^1 - q\sigma_3^2) \\ &= \frac{1}{2}(d\tilde{w} + (p+q)\cos\theta_1 d\phi_1 - q\cos\theta_2 d\phi_2) \ . \end{aligned} \tag{2.57}$$

We can define two natural copies of S^2 in B_4 to be C_1 and C_2, which are round S^2s at the north pole of the other. So, $C_1 = \{\theta_2 = 0\}$, $C_2 = \{\theta_1 = 0\}$. We observe that (2.57) gives rise to Chern numbers $p+q$ and $-q$ for C_1 and C_2, respectively, as required for $M_5(p,q)$.

Let us denote the total space over each sphere C_1 and C_2 in $M_5(p,q)$ to be F_1 and F_2, respectively. Then by following similar arguments as in (2.40)–(2.44) we deduce that

$$F_1 \cong S^3/\mathbb{Z}_{p+q}, \qquad F_2 \cong S^3/\mathbb{Z}_q \tag{2.58}$$

and also the homology relations

$$[F_1] = (p+q)[S^3], \qquad [F_2] = q[S^3] \ . \tag{2.59}$$

In fact one can see (2.58) rather explicitly from the above quotient construction. We define $W_1 \cong S^3$ and $W_2 \cong S^3$ to be the two natural copies of S^3 in M_6 given by $W_1 = \{\theta_2 = 0, \psi_2 = 0\}$, $W_2 = \{\theta_1 = 0, \psi_1 = 0\}$. Consider now $\{\theta_2 = 0\} \subset M_6$. This is

$$W_1 \times S^1 \cong S^3 \times S^1 \ , \tag{2.60}$$

where the S^1 is parametrised by ψ_2. When we take the quotient by the $U(1)_{q,p+q}$ circle action (2.54) we may set $\psi_2 = 0$. However, there is then a remaining gauge freedom given by setting

$$\psi = \frac{4\pi k}{p+q} \ , \tag{2.61}$$

with $k = 1, \ldots, p+q$, since this also fixes $\psi_2 = 0$. This then acts on ψ_1, which is the Hopf fibre of W_1 realised as an S^1 bundle over S^2, and we see explicitly that $F_1 \cong S^3/\mathbb{Z}_{p+q}$. A similar argument applies to F_2.

We next observe that

$$\Phi = \frac{1}{8\pi^2}[(p+q)\eta\wedge\sigma_1^1\wedge\sigma_2^1 + q\eta\wedge\sigma_1^2\wedge\sigma_2^2] \tag{2.62}$$

is a closed globally defined three-form on $M_5(p,q)$. We see explicitly that

$$\int_{F_1} \Phi = \frac{p+q}{8\pi^2} \int_{F_1} \eta \wedge \sigma_1^1 \wedge \sigma_2^1 = p+q \ . \tag{2.63}$$

which shows that Φ generates $H^3(M_5(p,q),\mathbb{Z})$.

Next it is convenient to define $\hat{M}_6$ to be

$$\hat{M}_6 = M_6/\mathbb{Z}_{(p+q)q} \tag{2.64}$$

where we embed $\mathbb{Z}_{(p+q)q}$ along $U(1)_{q,p+q}$. This defines a quotient

$$f : M_6 \to \hat{M}_6 \ . \tag{2.65}$$

The action on the Euler angles is

$$\begin{aligned} (\psi_1,\psi_2) &\mapsto \left(\psi_1 + \frac{4\pi k q}{(p+q)q}, \psi_2 + \frac{4\pi k(p+q)}{(p+q)q}\right) \\ &= \left(\psi_1 + \frac{4\pi k}{p+q}, \psi_2 + \frac{4\pi k}{q}\right) \ . \end{aligned} \tag{2.66}$$

Here $k = 1,\ldots,(p+q)q$. This realises the $\mathbb{Z}_{(p+q)q}$ action as a $\mathbb{Z}_{p+q}\times\mathbb{Z}_q$ action (the groups are isomorphic as $p+q$ and q are coprime) and we have

$$\hat{M}_6 \cong (S^3/\mathbb{Z}_{p+q}) \times (S^3/\mathbb{Z}_q) \ . \tag{2.67}$$

In terms of $\tilde{v}$,$\tilde{w}$ we have

$$(\tilde{v},\tilde{w}) \mapsto (\tilde{v} - \frac{4\pi k}{(p+q)q}, \tilde{w}) \ . \tag{2.68}$$

Thus on $\hat{M}_6$ we can introduce a new coordinate $\hat{v} = (p+q)q\tilde{v}$ with period 4π and we also have

$$\hat{M}_6 \cong (S^3/\mathbb{Z}_{(p+q)q}) \times S^3 \ , \tag{2.69}$$

A key point is that the $\hat{v}$ circle bundle trivialises over both F_1 and F_2. One way to see this is to observe that the $\hat{v}$ circle bundle has first Chern class being $q(p+q)$ times the generator of $H^2(M_5(p,q),\mathbb{Z})$ and then following the arguments in the appendices. We can also see this directly. Consider again

$$W_1 \times S^1 \tag{2.70}$$

where the S^1 is coordinatised by ψ_2. The action of $\mathbb{Z}_{p(p+q)}$ is given by (2.66). We first set $k = nq$, with $n = 1,\ldots,p+q$. This defines a $\mathbb{Z}_{p+q}$ subgroup that acts trivially on

ψ_2, but acts non-trivially on W_1, with quotient $W_1/\mathbb{Z}_{p+q} \cong S^3/\mathbb{Z}_{p+q} = F_1$. We may then set $k = 1, \ldots, q$ in the identification. This now acts trivially on $W_1/\mathbb{Z}_{p+q}$, but acts non-trivially on S^1 to give $S^1/\mathbb{Z}_q \cong S^1$. This shows explicitly that

$$(W_1 \times S^1)/\mathbb{Z}_{(p+q)q} \cong F_1 \times S^1 \tag{2.71}$$

which in turn shows that the $\hat{v}$ bundle restricted to F_1 is trivial, as it is manifestly a product. Obviously, similar reasoning applies[8] to F_2.

Let us now define V_1 and V_2 to be the obvious 2 factors of $\hat{M}_6$ in (2.67). Because of the discrete identification (2.66), W_1 is a $(p+q)$-fold cover of V_1, and W_2 is a q-fold cover of V_2. Thus for any three-form Ψ on $\hat{M}_6$ we have

$$\begin{aligned} \int_{W_1} f^*\Psi &= (p+q)\int_{V_1} \Psi \\ \int_{W_2} f^*\Psi &= q\int_{V_2} \Psi \ . \end{aligned} \tag{2.72}$$

Here $f^*\Psi$ is obtained by simply replacing $\hat{v}$ in Ψ with $(p+q)q\tilde{v}$.

For example, if we let $\Pi : M_6 \to M_5(p,q)$ be the projection for the fibration in the second column in (2.52), then $\Pi^*\Phi$ is a three-form on M_6 that is invariant under f (it has no dependence on the coordinate $\tilde{v}$). It is therefore obviously the pull-back of a three-form on the quotient $\hat{M}_6$, and hence we may use (2.72) to calculate

$$\begin{aligned} \int_{W_1} \Pi^*\Phi &= (p+q)^2 \\ \int_{W_2} \Pi^*\Phi &= q^2. \end{aligned} \tag{2.73}$$

Finally, we are in a position to provide our prescription for quantising the RR flux. We first observe that while we may take $C_2 = \Sigma_f$, we cannot quite take C_1 to be $\Sigma_2 \cup (-\Sigma_f)$, because the two submanifolds intersect at a point and we don't have a smooth submanifold. We may remedy this by cutting out a small neighbourhood of the intesection point and gluing in a cylinder. This results in a two-sphere, which we can take to be C_1. We may then identify

$$\begin{aligned} F_1 &= E_2 \cup (-E_f) \\ F_2 &= E_f \cong S^3/\mathbb{Z}_q \ , \end{aligned} \tag{2.74}$$

[8]A point we shall return to later, in passing, is that the above arguments show that for the quotient $M_6/\mathbb{Z}_{p+q}$ the corresponding circle bundle trivialises over F_1, while for $M_6/\mathbb{Z}_q$ it trivialises over F_2. We consider $M_6/\mathbb{Z}_{(p+q)q}$ as it trivialises over both.

with the understanding that F_1 is to be smoothed out into $S^3/\mathbb{Z}_{p+q}$, rather than the union of $S^3/\mathbb{Z}_q$ with $S^3/\mathbb{Z}_p$ over the circle where they intersect. As we have shown, on $\hat{M}_6$ the $\hat{v}$ circle fibration trivialises over F_1 and F_2, and hence we may take sections giving submanifolds V_1 and V_2. The correct quantisation condition for an integral three-form on M_6 (such as our appropriately normalised RR three-form), in a workable form, is then given by (2.72), where the integrals over W_1 and W_2 are integers M_1, M_2.

3 Flux Quantisation

In order to obtain a good solution to string theory, we need to impose that both the electric and magnetic RR three-form charges are properly quantised.

3.1 Electric and magnetic charges

For the electric charge we require

$$n_1 = \frac{1}{(2\pi l_s)^6 g_s} \int_{X_7} *F^{(3)} \in \mathbb{Z} \ . \tag{3.1}$$

Since

$$\frac{1}{L^6} * F^{(3)} = \frac{(Z-1)}{8(Z-2)\beta^2 y^2} J \wedge dy \wedge D\psi \wedge Dw \wedge dv \wedge du^2 + \mathrm{Vol}(AdS_3) \wedge (\dots) \tag{3.2}$$

we have

$$n_1 = \left(\frac{L}{l_s}\right)^6 \frac{1}{g_s 8\pi^2} \frac{Q(p+q)^5}{p^2 q(p+2q)^2} \Delta u^2 \ , \tag{3.3}$$

which we interpret as fixing the period of the u^2 circle Δu^2.

We next turn to the magnetic three-form charge. We require that

$$\frac{1}{(2\pi l_s)^2 g_s} \int_W F^{(3)} \in \mathbb{Z} \tag{3.4}$$

when integrated over any three-cycle $W \subset X_7 = M_6 \times S^1$. The relevant three-cycles are in M_6, and so the quantisation condition amounts to quantising the restriction of $F^{(3)}$ to M_6 at a point on the S^1 coordinatised by u^2. In the previous subsection we gave a prescription for performing such integrals by instead calculating integrals on submanifolds of the quotient space $\hat{M}_6$. In the next subsection we will calculate these integrals by introducing explicit coordinate patches. This will illuminate and confirm many of our observations about the topology in the previous section. Furthermore, the techniques will be essential for the generalisation that we consider in section 4.

In the present case, however, there is a much simpler way to impose flux quantisation. The key observation is that, remarkably, the relevant part of $F^{(3)}$ is in the same cohomology class as[9] Φ. Indeed we have

$$\frac{1}{L^2}F^{(3)} - 2\mathrm{Vol}(AdS_3) = \frac{(2\pi)^2 l_w (1-Z)}{(Z-2)q\beta}\Phi + d\left\{K_1 Dv\wedge Dw + K_2 Dw\wedge D\psi\right\} \quad (3.5)$$

where

$$\begin{aligned} K_1 &= \frac{Q(-2y+3Zy-Z^2y+1-\beta^2-Z+Z\beta^2)}{\beta(Z-2)^2} \\ K_2 &= \frac{(1-\beta^2)(1-Z)U(y)}{(-1+\beta^2+2y-Zy)(-1+\beta^2+Zy)(2-Z)} \ . \end{aligned} \quad (3.6)$$

Note in particular that the function K_2 vanishes at y_1 and y_2, ensuring that the two-form $K_1 Dv\wedge Dw + K_2 Dw\wedge D\psi$ is globally defined. We thus conclude that

$$\frac{1}{(2\pi l_s)^2 g_s}\int_W F^{(3)} = -\frac{L^2}{l_s^2 g_s}\frac{(p+q)^2}{pq^2(p+2q)}\int_W \Phi \ . \quad (3.7)$$

Furthermore, we have already calculated the periods of Φ (more precisely, $\Pi^*\Phi$) over a basis of three-cycles on M_6 in (2.73). We find that if the length scale is taken to be

$$\frac{L^2}{l_s^2 g_s} = \frac{pq^2(p+2q)M}{(p+q)^2} \quad (3.8)$$

for some positive integer M, then

$$\begin{aligned} M_1 &\equiv \frac{-1}{(2\pi l_s)^2 g_s}\int_{W_1} F^{(3)} = M(p+q)^2 \\ M_2 &\equiv \frac{-1}{(2\pi l_s)^2 g_s}\int_{W_2} F^{(3)} = Mq^2 \ . \end{aligned} \quad (3.9)$$

We may now calculate the central charge of the dual SCFT. It is given by [32]

$$c = \frac{3R_{AdS_3}}{2G_{(3)}} \quad (3.10)$$

where $G_{(3)}$ is the three-dimensional Newton's constant and R_{AdS_3} is radius of the AdS_3 space. In our conventions the type IIB supergravity Lagrangian has the form

$$\frac{1}{(2\pi)^7 g_s^2 l_s^8}\sqrt{-\det g}e^{-2\phi}R + \ldots \quad (3.11)$$

[9]Here we are not distiguishing between Φ and $\Pi^*\Phi$.

and after a short calculation we find

$$\begin{aligned} c &= 6n_1 \left(\frac{L}{l_s}\right)^2 \frac{1}{g_s} \\ &= 6n_1 \frac{pq^2(p+2q)M}{(p+q)^2} = 6n_1 \frac{(M_1 - M_2)M_2}{M_1} \ . \end{aligned} \tag{3.12}$$

This result is exactly the same as for the $Q = 0$ case [27]. We thus conclude that switching on Q is an exactly marginal deformation. Note that when $Q = 0$ the topology of X_7 changes to $S^3 \times S^2 \times T^2$. Thus the marginal deformation away from $Q = 0$ changes the topology of the solution[10].

3.2 Computing periods using coordinate patches

In this subsection we directly compute the flux of $F^{(3)}$ through the two three-cycles of M_6 using coordinate patches. This provides a nice cross-check on various calculations carried out so far. Furthermore, we will use this method in the next section when we construct more general type IIB string theory solutions – there we will not be able to use the approach in the last subsection since the three-form flux will no longer be in the same cohomology class as $\Pi^*\Phi$.

Recall from section 2.5 that instead of considering the circle bundle L over M_5 with total space M_6 we should consider the circle bundle $\hat{L} = L^{(p+q)q}$ with total space $\hat{M}_6 = M_6/\mathbb{Z}_{(p+q)q}$. This is useful since $\hat{L}$ trivialises over both the submanifolds F_1, a smoothed out version of $E_2 \cup -E_f$, and $F_2 \equiv E_f$ of M_5. We may thus take sections of $\hat{L}$ over these submanifolds to obtain submanifolds V_1 and V_2 of $\hat{M}_6$. Then the quantisation of the three-form flux on M_6, through the two three-cycles W_1, W_2, is related to that on $\hat{M}_6$ via the general formulae (2.72).

In particular, this procedure involves trivialising the circle bundle $\hat{L}$ over F_1 and F_2. Concretely, this means that the corresponding connection one-form is a globally-defined one-form over F_1 and F_2. However, to see this requires carefully covering the manifold with coordinate patches, so that the connection form is represented by a globally defined one-form on each patch, and then gluing these forms together on overlaps using $U(1)$ transition functions. Only when one has picked a gauge where the connection one-form is globally defined on F_1, F_2 can one then represent a section by taking the (appropriately gauge transformed) v coordinate to be constant in the three-form flux $F^{(3)}$. This might sound overly-technical, but if one does not follow this carefully one obtains incorrect periods for the flux.

[10] There is an analogous change of topology in the exactly marginal family of AdS_5 solutions found in [33].

We begin by covering M_5 with 4 coordinate patches: U_{1N}, U_{2N}, U_{1S}, U_{2S}. Here, for example, U_{1N} is defined by removing $\{y = y_2\}$ and $\{\theta = \pi\}$, while U_{1S} is defined by removing $\{y = y_2\}$ and $\{\theta = 0\}$. On B_4 the points we remove in each case are two S^2s that intersect over a point. It follows that, regarded as defining subsets of B_4, the above conditions give 4 patches diffeomorphic to $\mathbb{R}^4$. On M_5 we thus obtain patches diffeomorphic to $S^1 \times \mathbb{R}^4$, with the S^1 in each patch parametrised by a coordinate $w_{1N}, w_{2N}, w_{1S}, w_{2S}$, respectively.

Recall that B_4 is constructed as an S^2 bundle over S^2, where the fibre S^2 has poles $\{y = y_1\}$, $\{y = y_2\}$. Removing these, one can define a global one-form:

$$\begin{aligned} D\psi &= D\psi_N = d\psi_N + (1 - \cos\theta)d\phi \\ &= D\psi_S = d\psi_S - (1 + \cos\theta)d\phi \ . \end{aligned} \tag{3.13}$$

The corresponding space is an $I \times S^1$ bundle over S^2, where $I = (y_1, y_2)$ is an open interval, and the circle S^1 is parametrised by ψ_N and ψ_S, each with period 2π. Here the first expression is valid on the complement of the south pole $\{\theta = \pi\}$, while the second is valid on the complement of the north pole $\{\theta = 0\}$. This is because the azimuthal coordinate ϕ degenerates at the poles of the base S^2. On the overlap one has

$$\psi_S - \psi_N = 2\phi \tag{3.14}$$

which shows that the S^1 bundle has Chern number -2. This is because the connection form is locally $\cos\theta d\phi$, and so has curvature form $-\sin\theta d\theta \wedge d\phi$, which integrates to $-2 \cdot 2\pi$ over the S^2. It is important that $D\psi$ is *not* defined at $\{y = y_i\}$, since these are coordinate singularities.

Recalling (2.13), we next define the global one-form on M_5:

$$\begin{aligned} Dw &= Dw_{1N} = dw_{1N} + A_w(y_1)d\psi_N - A_w D\psi_N \\ &= Dw_{2N} = dw_{2N} + A_w(y_2)d\psi_N - A_w D\psi_N \\ &= Dw_{1S} = dw_{1S} + A_w(y_1)d\psi_S - A_w D\psi_S \\ &= Dw_{2S} = dw_{2S} + A_w(y_2)d\psi_S - A_w D\psi_S \ . \end{aligned} \tag{3.15}$$

These are defined on the 4 patches U_{1N}, U_{2N}, U_{1S}, U_{2S}, respectively. Take, for example, Dw_{1N}. ψ_N is a coordinate on the complement of the south pole of the base S^2, although it degenerates at $y = y_1$. However, at $y = y_1$ we have

$$Dw_{1N} \mid_{\{y=y_1\}} = dw_{1N} - A_w(y_1)(1 - \cos\theta)d\phi \ . \tag{3.16}$$

and we see that w_{1N} is indeed a good coordinate on the S^1 of $U_{1N} \cong S^1 \times \mathbb{R}^4$. The period of all the w coordinates above is $2\pi l_w$.

One can immediately see the fibration structure of the w circle bundle, with total space M_5, from the above formulae. For example, on the overlap region where both are defined, we have

$$\frac{1}{l_w}(w_{2N} - w_{1N}) = q\psi_N \ . \tag{3.17}$$

In particular, restricting to $\{\theta = 0\}$, which is E_f, we see that the circle bundle has Chern number $-q$ and thus $E_f \cong S^3/\mathbb{Z}_q$. Similarly,

$$\frac{1}{l_w}(w_{2S} - w_{2N}) = -p\phi \tag{3.18}$$

showing that the Chern number over $E_2 = \{y = y_2\}$ is p, thus proving that $E_2 \cong S^3/\mathbb{Z}_p$.

In each of the patches we define the connection one-form that appears in the v circle fibration over M_5 to give M_6. Recalling (2.13) we write $Dv \equiv dv - A'$ and define

$$\begin{aligned}
A'_{1N} &= -A_v(y_1)d\psi_N + A_v D\psi_N + l_v\lambda_{1N}\frac{dw_{1N}}{l_w} \\
A'_{2N} &= -A_v(y_2)d\psi_N + A_v D\psi_N + l_v\lambda_{2N}\frac{dw_{2N}}{l_w} \\
A'_{1S} &= -A_v(y_1)d\psi_S + A_v D\psi_S + l_v\lambda_{1S}\frac{dw_{1S}}{l_w} \\
A'_{2S} &= -A_v(y_2)d\psi_S + A_v D\psi_S + l_v\lambda_{2S}\frac{dw_{2S}}{l_w} \ .
\end{aligned} \tag{3.19}$$

Here $\lambda_{1N}, \lambda_{2N}, \lambda_{1S}, \lambda_{2S}$ are constants to be fixed by the requirement that the $(1/l_v)A'$ patch together to give a connection one-form. We choose $\lambda_{1N} = \lambda_{2N} = \lambda_{1S} = \lambda_{2S} \equiv \lambda$ with

$$\begin{aligned}
\frac{A_v(y_1) - A_v(y_2)}{l_v} + \lambda q &= -a \\
\frac{2A_v(y_2)}{l_v} + \lambda p &= b.
\end{aligned} \tag{3.20}$$

where a, b are integers satisfying $ap + bq = 1$, which is possible because of (2.46). Consider first the overlap of U_{1N} with U_{2N}. On this overlap we have

$$\frac{1}{l_v}\left[A'_{2N} - A'_{1N}\right] = -ad\psi_N \ . \tag{3.21}$$

Since ψ_N has period 2π and a is an integer, we see that the two connections do indeed differ by a $U(1)$ gauge transformation. Next consider the overlap of U_{2S} with U_{2N}. Here we have

$$\frac{1}{l_v}\left[A'_{2S} - A'_{2N}\right] = -b d\phi \ . \tag{3.22}$$

It is illuminating to compare with equations (2.36) and (B.4), (B.7) in appendix B. In particular, we see that (3.21) and (3.22) give[11] the torsion Chern classes over E_f and E_2, respectively. As a check on this, we compute

$$\frac{1}{l_v}\left[A'_{1S} - A'_{1N}\right] \ = \ -(b-2a) d\phi \ , \tag{3.23}$$

which is equivalent to the Chern number of the w-fibration over Σ_1 being $p+2q$ and agrees with (B.6). Note that, conversely, if one allows general λ in (3.19) and instead imposes that the connections differ by $U(1)$ gauge transformations (3.21), (3.22) on the overlaps, then one finds the solution (3.20).

Now consider $\hat{M}_6$, where we divide the period of v by $q(p+q)$. Note immediately that the connection form on $U_{2N} \cap U_{1N}$ is

$$\frac{q(p+q)}{l_v}\left[A'_{2N} - A'_{1N}\right] = -a(p+q)\left[\frac{dw_{2N}}{l_w} - \frac{dw_{1N}}{l_w}\right] \ . \tag{3.24}$$

Thus we may define

$$\begin{aligned} \frac{q(p+q)}{l_v}\hat{A}'_{1N} \ &= \ \frac{q(p+q)}{l_v}A'_{1N} + a(p+q)\frac{dw_{1N}}{l_w} \\ \frac{q(p+q)}{l_v}\hat{A}'_{2N} \ &= \ \frac{q(p+q)}{l_v}A'_{2N} + a(p+q)\frac{dw_{2N}}{l_w} \ . \end{aligned} \tag{3.25}$$

These are good gauge transformations on each patch. We see that $\hat{A}'_{1N}$ and $\hat{A}'_{2N}$ agree on the overlap, and thus define a *globally* defined one-form on the complement of $\{\theta = \pi\}$. In particular, this shows explicitly that the v bundle over E_f (with the period above) is trivial[12]. A globally defined connection one-form is provided by $\frac{q(p+q)}{l_v}\hat{A}'$ above, restricted to $\{\theta = 0\}$.

Remarkably, the factors of a and b in $\hat{A}'$ now cancel, and the connection form reduces to

$$\hat{A}'_{1N} = -A_v(y_1)d\psi_N + A_v D\psi_N + \frac{l_v}{2q(p+q)}\frac{dw_{1N}}{l_w} \ . \tag{3.26}$$

[11]For a more detailed explanation of the relation between the transition functions (3.21), (3.22) and torsion Chern classes, we refer to appendix A.

[12]Note that we only need to quotient the period of v by q to be able to do this, not $q(p+q)$, as expected from the comment in footnote 8.

We are now in a position to calculate the integral of the three-form flux over V_2. In $F^{(3)}$ we set $\theta = 0$ and replace $Dv = dv - A'$ with $-\hat{A}'_{1N}$. After some calculation we obtain

$$\frac{1}{(2\pi l_s)^2 g_s} \int_{V_2} F^{(3)} = -\frac{L^2}{l_s^2 g_s} \frac{(p+q)^2}{pq(p+2q)} \equiv -\frac{M_2}{q} \ . \tag{3.27}$$

where M_2 is a positive integer.

It remains to calculate the integral over V_1. We cover V_1 by 3 patches: U_{1N}, U_{2N} and U_{2S}. These will cover the northern hemisphere, equatorial strip, and southern hemisphere patches, respectively, of the S^2 we get by gluing Σ_2 to $-\Sigma_f$. This is illustrated in Figure 1. To be more precise we will cover most of Σ_f in U_{1N} by setting $\theta = 0$, letting $y \in [y_1, y_2 - \epsilon]$ with ψ_N the azimuthal angle. We will cover most of Σ_2 in U_{2S} by setting $y = y_2$, letting $\theta \in [\delta, \pi]$ with ϕ the azimuthal angle. Here $\epsilon, \delta > 0$ are small. On the overlap in U_{2N} the equatorial strip, Eq, is the line in the δ, y plane stretching from $(\theta, y) = (\delta, y_2)$ to $(\theta, y) = (0, y_2 - \epsilon)$, over which there is an azimuthal angle – at the first end of this line it is ϕ and at the other end it is ψ_N. In fact, on this strip the azimuthal angles get identified via

$$\phi = -\psi_N \ , \tag{3.28}$$

with the sign corresponding to an orientation flip.

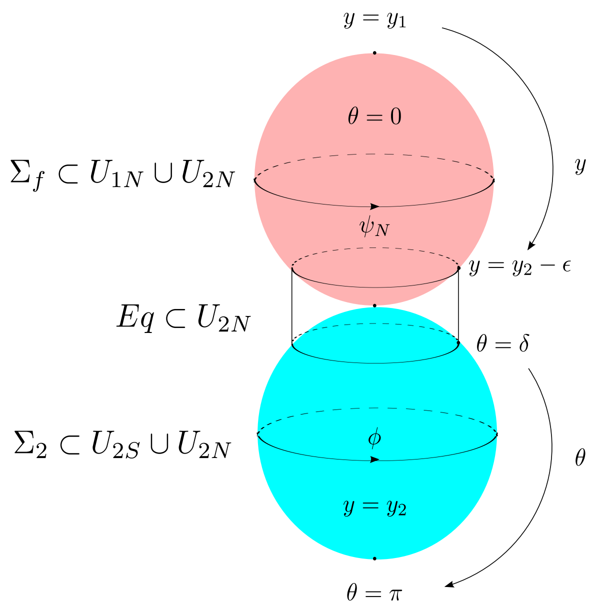

Figure 1: Desingularisation of $\Sigma_2 \cup -\Sigma_f$.

We first examine the overlaps

$$\begin{aligned}\frac{q(p+q)}{l_v}\left[A'_{2N}-A'_{1N}\right] &= -aq(p+q)d\psi_N = q((b-a)q-1)d\psi_N \\ \frac{q(p+q)}{l_v}\left[A'_{2S}-A'_{2N}\right] &= -bq(p+q)d\phi = -q((b-a)p+1)d\phi\ . \end{aligned} \tag{3.29}$$

This leads us to define

$$\frac{q(p+q)}{l_v}\tilde{A}'_{2N} = \frac{q(p+q)}{l_v}A'_{2N} + qd\psi_N - q(b-a)\frac{dw_{2N}}{l_w}\ , \tag{3.30}$$

which is obtained via a good gauge transformation on this patch. We then find

$$\begin{aligned}\frac{q(p+q)}{l_v}\left[\tilde{A}'_{2N}-A'_{1N}\right] &= q^2(b-a)d\psi_N - q(b-a)\frac{dw_{2N}}{l_w} \\ &= -q(b-a)\frac{dw_{1N}}{l_w} \end{aligned} \tag{3.31}$$

$$\begin{aligned}\frac{q(p+q)}{l_v}\left[A'_{2S}-\tilde{A}'_{2N}\right] &= -pq(b-a)d\phi + q(b-a)\frac{dw_{2N}}{l_w} \\ &= q(b-a)\frac{dw_{2S}}{l_w}\ . \end{aligned} \tag{3.32}$$

This prompts us to define

$$\begin{aligned}\frac{q(p+q)}{l_v}\tilde{A}'_{1N} &= \frac{q(p+q)}{l_v}A'_{1N} - q(b-a)\frac{dw_{1N}}{l_w} \\ \frac{q(p+q)}{l_v}\tilde{A}'_{2S} &= \frac{q(p+q)}{l_v}A'_{2S} - q(b-a)\frac{dw_{2S}}{l_w}\ , \end{aligned} \tag{3.33}$$

which are again obtained via good gauge transformations on the patches. After all this, $\tilde{A}'$ is a globally defined one-form on F_1, and thus we see explicitly that the v bundle trivialises over it since[13] we have divided the period by $(p+q)q$. Moreover, one finds that a and b end up completely cancelling, and that the correct connection form to use on U_{1N} and U_{2S} is

$$\begin{aligned}\tilde{A}'_{1N} &= -A_{\bar{u}}(y_1)d\psi_N + A_{\bar{u}}D\psi_N - \frac{l_v}{2q(p+q)}\frac{dw_{1N}}{l_w} \\ \tilde{A}'_{2S} &= -A_{\bar{u}}(y_2)d\psi_S + A_{\bar{u}}D\psi_S - \frac{l_v}{2q(p+q)}\frac{dw_{1S}}{l_w}\ . \end{aligned} \tag{3.34}$$

By taking $\epsilon, \delta \to 0$ we effectively use the gauge $\tilde{A}'_{1N}$ over E_f and $\tilde{A}'_{2S}$ over E_2 and then consider the result for E_2 minus the result for E_f. After some calculation this gives the period

$$\frac{1}{(2\pi l_s)^2 g_s}\int_{V_1} F^{(3)} = -\frac{L^2}{l_s^2 g_s}\frac{(p+q)^3}{pq^2(p+2q)} \equiv -\frac{M_1}{p+q}\ , \tag{3.35}$$

[13] Note that to obtain this result we only needed to quotient the period of v by $p+q$ here, not $(p+q)q$. In particular, all of the above gauge transformations are well-defined.

where M_1 is a positive integer.

Consistency of (3.27) and (3.35) implies that we choose

$$M_1 = M(p+q)^2, \qquad M_2 = Mq^2 \tag{3.36}$$

for some positive integer M and

$$\frac{L^2}{l_s^2 g_s} = \frac{pq^2(p+2q)M}{(p+q)^2} \ . \tag{3.37}$$

We have thus recovered the results (3.8) and (3.9), which is very satisfying.

4 More general identifications

In this section we will generalise the class of solutions that we have already constructed. We return to the local solution (2.12), (2.8), (2.15) and then employ the general linear coordinate transformation

$$\begin{aligned} w =& \quad h\, w' + r\tfrac{Q}{Z}\, v' \\ v =& \quad s\, w' + \tfrac{t}{2\beta}\, v' \end{aligned} \tag{4.1}$$

for constant r, t, s, h with

$$\Delta = h\frac{t}{2\beta} - r\frac{Q}{Z}s \neq 0 \ . \tag{4.2}$$

The idea is to now make appropriate periodic identifications of the new coordinates v', w'. As we shall see this will embed our solutions of type IIB string string theory of the last two sections into larger families.

We first observe that

$$\begin{aligned} Dw &= h\, Dw' + r\frac{Q}{Z}\, Dv' \\ Dv &= s\, Dw' + \frac{t}{2\beta}\, Dv' \end{aligned} \tag{4.3}$$

where we have defined

$$\begin{aligned} Dw' &= dw' - A_{w'}\, D\psi \\ Dv' &= dv' - A_{v'}\, D\psi \end{aligned} \tag{4.4}$$

with

$$\begin{aligned} A_{w'} &= \frac{t}{2\beta\Delta}\, A_w - r\frac{Q}{Z\Delta}\, A_v \\ A_{v'} &= \frac{h}{\Delta}\, A_v - \frac{s}{\Delta}\, A_w \ . \end{aligned} \tag{4.5}$$

We now construct M_5 as a circle fibration, with circle parametrised by w', over B_4 and then construct M_6 as a circle fibration, with circle parametrised by v', over M_5. It is straightforward to write the metric in the primed coordinates and then appropriately "complete the square" to make this fibration structure manifest in the metric. However, we will not need the explicit details. Observe that what will become the globally defined angular one-form on M_5 for the w' circle fibration is Dw'. After completing the square in the metric on M_6 we obtain an expression for what will become the globally defined angular one-form corresponding to the v' circle fibration and it has the form

$$dv - A_{v'} D\psi - k(y) Dw' \tag{4.6}$$

for some smooth function $k(y)$ that can easily be determined. The connection one-form on M_5 for this circle fibration is thus $A_{v'} D\psi + k(y) Dw'$. This will turn out to be a local connection one-form on the same circle bundle as that for the connection one-form $A_{v'} D\psi$, since kDw' will be globally defined on M_5 (in particular, the corresponding curvature two-forms are in the same cohomology class on M_5). Below, for convenience, we will use the connection one-form $A_{v'} D\psi$.

The analysis now proceeds in an almost identical fashion as in the last sections, so we can be brief. We choose the period of the w' circle to be $2\pi l_{w'}$ so that $l_{w'}^{-1} A_{w'} D\psi$ is a connection on a $U(1)$ fibration. We demand that $(2\pi l_w)^{-1} d(A_w D\psi)$ has integer periods on B_4, as in (2.29), with primes on all w, for some integers p, q, now *not necessarily positive*. When $r + t = 0$ we have $q = 0$, while when $r - t = 0$ we have $p + q = 0$ and these cases require a separate analysis which we will return to later. We thus continue here with $r \neq \pm t$ and conclude that[14]

$$\begin{aligned} \beta &= \frac{t-r}{t+r} \frac{q}{p+q} \\ l_{w'} &= \frac{(2-Z)(r+t)}{2q(1-Z)(1-\beta^2)\Delta} \end{aligned} \tag{4.7}$$

The topology of M_5 is again $S^3 \times S^2$. For the generator of $H^3(M_5, \mathbb{Z})$ we can use the primed version of (2.45).

We now turn to the v' circle fibration over M_5 to give M_6. We let v' be a periodic coordinate with period $2\pi l_{v'}$, and the connection one-form is given by $l_{v'}^{-1} A_{v'} D\psi$. To

[14] Note that if we choose $t = \beta(Z-2)/(Z-1)$, $r = -\beta Z/(Z-1)$, $h = (Z-2)/2(Z-1)$ and $s = Z(Z-2)/4Q\beta(Z-1)$, then we have $w' = z$, $v' = u^1$, where z, u^1 are the coordinates that we started with in (2.2). In this case equation (4.7) becomes $\beta = (1-Z)/(1+X)$ and $l_{w'} = 2(1+X)/q(X+Z)(2+X-Z)$, where $X = p/q$ and this agrees with the results in equation (4.22) of [27].

ensure that the circle fibration is well-defined and that $M_6 = S^3 \times S^3$ we impose the primed version of (2.46) to conclude that

$$l_{v'} = \frac{2qQ}{(r+t)\,(1-Z)} . \tag{4.8}$$

Now we determine the flux quantisation conditions. The electric flux quantisation condition (3.1) fixes the period of u^2 as before:

$$n_1 = \frac{L^6}{l_s^6 g_s} \frac{Q}{8\pi^2 \beta(1-\beta^2)^2} \Delta u^2 \ . \tag{4.9}$$

For the magnetic flux quantisation, we follow the same procedure as before, by introducing explicit coordinate patches and considering integrals on submanifolds of $\hat{M}_6 = M_6/\mathbb{Z}_{(p+q)q}$. By following the same steps as in section 3.2 we find that

$$\begin{aligned}
\frac{1}{(2\pi l_s)^2 g_s} \int_{V_2} F^{(3)} &= \frac{L^2}{l_s^2 g_s} \frac{1}{q} \frac{1}{\beta^2 - 1} \equiv -\frac{M_2}{q} \\
\frac{1}{(2\pi l_s)^2 g_s} \int_{V_1} F^{(3)} &= \frac{L^2}{l_s^2 g_s} \frac{1}{q} \frac{r+t}{r-t} \frac{1}{\beta\,(1-\beta^2)} \equiv -\frac{M_1}{p+q}
\end{aligned} \tag{4.10}$$

for integers M_2, M_1. Consistency implies that we must have

$$\frac{M_2}{M_1} = \beta^2 = \frac{(r-t)^2}{(r+t)^2} \frac{q^2}{(p+q)^2}, \tag{4.11}$$

which implies that $(r+t)^2/(r-t)^2$ must be rational, and that the length scale is fixed by

$$\frac{L^2}{l_s^2 g_s} = (1-\beta^2) M_2 = \frac{(M_1 - M_2) M_2}{M_1} \tag{4.12}$$

The central charge can now be calculated, and we find that it can be expressed as

$$c = 6 n_1 \frac{(M_1 - M_2) M_2}{M_1} \ . \tag{4.13}$$

In particular we note that, in addition to Q, there is also no dependence on the parameters r, s, t and h. We note that the only restrictions on these parameters is (4.2), (4.11) with β given in (4.7) satisfying $0 < \beta < 1$. We have thus constructed large continuous families of solutions that are dual to SCFTs. Note that, in general, the solutions of this section are not exactly marginal deformations of those in section 3: for example, in section 3 we saw that the magnetic three-form flux quantum numbers were constrained to be of the form (3.36), whereas here there is no such constraint.

When $r = -t$:

When $r = -t$, we have $A_{w'}(y_1) = A_{w'}(y_2)$ and hence in considering the w' circle fibration over B_4 to construct M_5 we find that the period over $C_2 = \Sigma_f$ vanishes, $q = 0$. We choose $p = 1$, so that the period over $C_1 = \Sigma_2 - \Sigma_f$ is one, and hence $M_5 = S^3 \times S^2$, which implies that

$$l_{w'} = 2A_{w'}(y_2) = -\frac{t(Z-2)}{\beta(\beta^2-1)(Z-1)\Delta} \; . \tag{4.14}$$

At this stage, there is no restriction on the parameter β (apart from the usual $0 < \beta < 1$). We now find on M_5 that $E_1, E_2 \cong S^3$, and $[E_1] = [E_2]$ generate $H_3(M_5, \mathbb{Z})$. On the other hand, now $E_f \cong S^1 \times S^2$ (and hence there is a section of the w' circle fibration over Σ_f). The generator of $H^2(M_5, \mathbb{Z})$ is $\tau = \sigma_f$ *i.e.* $a = 1, b = 0$, in the notation of section 2.3.

In order to construct $M_6 = S^3 \times S^3$, we can again fix the period of the v' circle using Φ as in (2.46) and we find that

$$l_{v'} = A_{v'}(y_2) - A_{v'}(y_1) = -\frac{\beta Q}{(Z-1)t} \; . \tag{4.15}$$

It is now easier to find representatives of the two generators of $H_3(M_6, \mathbb{Z})$, and we won't have to consider a quotient of M_6 in order to impose the flux quantisation conditions. In particular, one generator of $H_3(M_6, \mathbb{Z})$, W_1, can be taken to be, as above, the section of the v' circle fibration over a desingularised version of $E_2 \cup -E_f$. For the other generator, W_2, we can take the v' circle bundle over the section $s(S)$ on M_5 where[15] $S = \Sigma_f$. We note that two other three-cycles W', W'' are obtained by considering a section of the v' circle fibration over E_1, E_2, respectively: we shall show that $[W'] = [W''] = [W_1] + [W_2]$.

We now introduce patches in exactly the same way as section 3.2. The analogue of (3.17) now reads $w'_{2N} = w'_{1N}$ and we explicitly see that the w' circle fibration is indeed trivial over Σ_f. To obtain the section $s[\Sigma_f]$ we can simply set $w'_{2N} = constant$.

Moving to M_6, we have the analogue of the connection one-forms as in (3.19), (3.20) with $a = 1$, $b = 0$, $p = 1$, $q = 0$. Equation (3.22) shows that the v' circle fibration is indeed trivial over E_2 and we can take a section to obtain the three-cycle W''. One can then obtain the integral of the three-form flux over W'' by using the connection one form A'_{2N}, and after a calculation we find

$$\frac{1}{(2\pi l_s)^2 g_s} \int_{W''} F^{(3)} = -\frac{L^2}{l_s^2 g_s} \frac{1}{\beta^2} \; . \tag{4.16}$$

[15]Before, when $[S] = q[\Sigma_2] + p[\Sigma_f]$, it was not clear how to take a smooth representative for S.

The v' circle fibration is also trivial over E_1. Indeed, after considering (3.22) we can see that the connection one-form

$$A'_{1S} + 2l_{v'}\frac{dw'_{1s}}{l_{w'}} \tag{4.17}$$

is a globally defined one-form on E_1. We can use this gauge to calculate the integral over W' and we find exactly the same result as for W''.

To calculate the integral of the flux over the three-cycle W_2, the v' circle bundle over the section $s(\Sigma_f)$, we just need to set $w'_{2N} = constant$ in the expression for the three-form and then integrate. We therefore impose

$$\frac{1}{(2\pi l_s)^2 g_s}\int_{W_2} F^{(3)} = \frac{L^2}{l_s^2 g_s}\frac{1}{(1-\beta^2)} = M_2 \ . \tag{4.18}$$

To carry out the flux integral over W_1, a section of the v' circle fibration over $E_2 \cup -E_f$, we define

$$\begin{aligned} \frac{1}{l_{v'}}\tilde{A}'_{2N} &= \frac{1}{l_{v'}}A'_{2N} + d\psi_N + \frac{dw_{2N}}{l_w} \\ \frac{1}{l_{v'}}\tilde{A}'_{1N} &= \frac{1}{l_{v'}}A'_{1N} + \frac{dw_{1N}}{l_w} \\ \frac{1}{l_{v'}}\tilde{A}'_{2S} &= \frac{1}{l_{v'}}A'_{2S} + \frac{dw_{2S}}{l_w} \ . \end{aligned} \tag{4.19}$$

Then $\tilde{A}'$ is a global one-form on $E_2 \cup -E_f$. To calculate the integral of flux over the section over the v' circle bundle over $E_2 \cup -E_f$ we use $\tilde{A}'_{2S}$ on E_2 and $\tilde{A}'_{1N}$ on E_f. We find

$$\frac{1}{(2\pi l_s)^2 g_s}\int_{W_1} F^{(3)} = -\frac{L^2}{l_s^2 g_s}\frac{1}{\beta^2(1-\beta^2)} = -M_1 \ . \tag{4.20}$$

Comparing (4.16) with (4.18) and (4.20), we can deduce the homology relation $[W''] = [W'] = [W_1] + [W_2]$, as mentioned above.

Consistency of (4.18) and (4.20) implies that the length scale of the solution is again as in (4.12) and that β^2 is rational

$$\beta^2 = \frac{M_2}{M_1} \ . \tag{4.21}$$

The electric flux quantisation condition is given again by (4.9) and the central charge takes the form (4.13).

When $r = t$:

When $r = t$, we have $A_{w'}(y_1) = -A_{w'}(y_2)$ and hence in considering the w' circle fibration over B_4 to construct M_5 we find that the period over $C_1 = \Sigma_2 - \Sigma_f$ vanishes,

$p+q=0$. We choose $q=1$ so that the period over $C_2=\Sigma_f$ is one, and hence $M_5=S^3\times S^2$, which implies that

$$l_{w'}=\frac{-t(Z-2)}{(\beta^2-1)(Z-1)\Delta} \tag{4.22}$$

with no restriction on the parameter β. We now find $E_1, E_2, E_f \cong S^3$, and $-[E_1]=[E_2]=[E_f]$ generate $H_3(M_5,\mathbb{Z})$. The generator of $H^2(M_5,\mathbb{Z})$ is $\tau=b\sigma_2+a\sigma_f$ with $b-a=1$.

In order to construct $M_6=S^3\times S^3$, we find that the period of the v' circle is

$$l_{v'}=A_{v'}(y_1)+A_{v'}(y_2)=-\frac{Q}{(Z-1)t}\ . \tag{4.23}$$

For the generators of $H_3(M_6,\mathbb{Z})$ we can take W_1 to be the v' circle fibration over the a representative of the section $s(S)$ of M_5, with $[S]=[\Sigma_2]-[\Sigma_f]$. For W_2 we take a section of the v' circle fibration over E_f. We note that we can also obtain three-cycles W', W'' which are obtained by considering sections of the v' circle fibration over E_1, E_2 respectively: we shall see that $-[W']=[W'']=[W_1]+[W_2]$.

We again introduce patches in exactly the same way as section 3.2. The connection one-forms are as in (3.19), (3.20) with $b-a=1$ and $q=-p=1$. By taking $a=0$, $b=1$, we see from (3.21) that A'_{2N} is a globally defined connection one-form on E_f. Calculating the flux integral we find that we should impose

$$\frac{1}{(2\pi l_s)^2 g_s}\int_{W_2}F^{(3)}=-\frac{L^2}{l_s^2 g_s}\frac{1}{(1-\beta^2)}=-M_2\ . \tag{4.24}$$

To integrate the flux integrals for W' one should take $b=2$, $a=1$ while for W'' we should take $b=0$, $a=-1$ and we find

$$-\frac{1}{(2\pi l_s)^2 g_s}\int_{W'}F^{(3)}=\frac{1}{(2\pi l_s)^2 g_s}\int_{W''}F^{(3)}=\frac{L^2}{l_s^2 g_s}\frac{1}{\beta^2}\ . \tag{4.25}$$

We now turn to the flux integral over W_1. For S we desingularise $\Sigma_2-\Sigma_f$ as in Figure 1. By making the gauge transformation $w'_{2N}\to w'_{2N}-l_{w'}d\psi_N$ in (3.15), we find that we obtain a globally defined connection one-form on $S\subset M_5$ and hence we can take a section. W_1 is obtained by considering the v' circle fibration over this section. Thus to calculate the flux integral, one should set $w'_{1N}=constant$ in Dw'_{1N} for the Σ_2 piece and $w'_{1S}=constant$ in Dw'_{1S} for the Σ_f piece. After doing this we find

$$\frac{1}{(2\pi l_s)^2 g_s}\int_{W_1}F^{(3)}=-\frac{L^2}{l_s^2 g_s}\frac{1}{\beta^2(1-\beta^2)}=-M_1\ . \tag{4.26}$$

We thus find the same conditions as for the $r=-t$ case above.

5 Final Comments

We have analysed in detail some local supersymmetric AdS_3 solutions of type IIB supergravity, first found in [27], that have non-vanishing dilaton and RR three-form flux. We have shown that the parameters can be chosen and coordinates identified in such a way that the solutions extend to give rich classes of globally defined solutions of the form $AdS_3 \times_w (S^3 \times S^3 \times S^1)$ with properly quantised flux. We have shown that the solutions depend on continuous parameters and are hence dual to continuous families of SCFTs in two spacetime dimensions with $(0, 2)$ supersymmetry.

Although the internal compact spaces are diffeomorphic to $S^3 \times S^3 \times S^1$, the diffeomorphisms are far from apparent in the local coordinates that the solutions are presented in. It seems unlikely to us that there is a simple change of coordinates that will make the topology more manifest. In this paper we used a number of techniques to illuminate various aspects of the topology which, in particular, allowed us to find a workable procedure to impose flux quantisation. It seems likely that our approach, or generalisations thereof, will be very useful in other contexts.

In section 4 we considered identifications on the coordinates after we made a general linear transformation on the v, w coordinates. It is worth pointing out that we could consider more general linear coordinate transformations that also involve the u^2 coordinate. This will lead to larger families of solutions that would be worth exploring. It seems possible that some of these solutions can be obtained as β-deformations using the techniques of [40]. In fact returning to the solutions in section 2 and 3, where we showed that Q was an exactly marginal deformation of the solutions with $Q = 0$, one might wonder if Q corresponds to a β-deformation. One way to see that it is not is to return to the local solutions as written down at the beginning of section 4 of [27], which are obtained after two T-dualities on the solutions we have discussed in this paper. In this duality frame only the metric and the self-dual five-form are non-trivial for any Q, and in particular the dilaton is constant. However, looking at equation (A.16) of [40] we see that the β-deformation activates a non-trivial dilaton and three-form.

It is an important outstanding issue to identify the dual $(0, 2)$ SCFTs for the solutions discussed here and in [25, 14, 26, 27]. In the duality frame that we have used in this paper, the amount of supersymmetry that is preserved combined with the fluxes that are active suggests that the dual SCFTs might arise on a D1-D5-brane system that is wrapped on a holomorphic four-cycle in a Calabi-Yau four-fold. While we remain hopeful that progress will be made in this direction, we note that the

SCFTs dual to the much simpler type IIB $AdS_3 \times S^3 \times S^3 \times S^1$ solutions of [34], which have $(4,4)$ supersymmetry, are still not well-understood, despite interesting progress [35, 36, 37, 38].

The AdS_3 solutions with $Q = 0$, that were analysed in [27], and with $Q \neq 0$ that we have discussed here, can be generalised further and we have presented some details in appendix C. It will be interesting to carry out a complete analysis of the conditions for regularity and flux quantisation conditions for these more general solutions.

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A $U(1)$ bundles over Lens spaces

In this section we briefly review the Lens spaces $S^3/\mathbb{Z}_q$, which appear throughout the main text, and also the construction of $U(1)$ principal bundles over these manifolds.

We construct $S^3/\mathbb{Z}_q$ as the total space of a $U(1)$ bundle over S^2 with Chern number q. Let θ, ϕ be standard coordinates on S^2, and cover the S^2 with two patches: V_N which excludes the south pole $\theta = \pi$, and V_S which excludes the north pole $\theta = 0$. We then consider the products $S^1 \times V_N$, $S^1 \times V_S$, and on each space define the one-forms

$$\begin{aligned} D\nu_N &= d\nu_N - \frac{q}{2}(1 - \cos\theta)d\phi \\ D\nu_S &= d\nu_S + \frac{q}{2}(1 + \cos\theta)d\phi \ . \end{aligned} \tag{A.1}$$

Here ν_N and ν_S are coordinates on the S^1s, each with period 2π. If we now glue the two patches together via

$$\nu_S - \nu_N = -q\phi \tag{A.2}$$

on the overlap then note that

$$D\nu = D\nu_N = D\nu_S \tag{A.3}$$

extends to a global one-form on the whole manifold, because the two one-forms agree on the overlap. This is a global connection form on the total space of the $U(1)$ principal bundle $\mathrm{p} : S^3/\mathbb{Z}_q \to S^2$ with $U(1)$ fibre parametrised by ν, and is sometimes also called the global angular form.

Now consider the connection form

$$\begin{aligned} A_N &= \frac{a}{2}(1-\cos\theta)d\phi \\ A_S &= -\frac{a}{2}(1+\cos\theta)d\phi \end{aligned} \tag{A.4}$$

on the base S^2. This has Chern number $a \in \mathbb{Z}$ over the base S^2. We denote the corresponding $U(1)$ principal bundle by P. We may pull back P to a $U(1)$ bundle p^*P over $S^3/\mathbb{Z}_q$. Pulling back the connection (A.4), on the overlap one finds

$$A_S - A_N = -ad\phi = \frac{a}{q}d(\nu_S - \nu_N) \ . \tag{A.5}$$

Note that $a\nu_S/q$ is a multi-valued $U(1)$ function on the patch $S^1 \times V_S$ unless $a/q \in \mathbb{Z}$. If $a/q \in \mathbb{Z}$ then in each patch we can define the new connection one-forms $A_S - ad\nu_S/q$ and $A_N - ad\nu_N/q$, and since they agree on the overlap, this defines a globally defined connection one-form and hence p^*P is trivial.

Thus p^*P is trivial if and only if $a \cong 0 \bmod q$. One sees this in a more abstract way by recalling that $U(1)$ principal bundles are classified by $H^2(S^3/\mathbb{Z}_q, \mathbb{Z}) \cong H_1(S^3/\mathbb{Z}_q, \mathbb{Z}) \cong \mathbb{Z}_q$. Thus $a \in \mathbb{Z}_q$ is precisely the Chern number of p^*P, and the latter bundle is torsion. Because of this, the topology cannot be measured by integrating the curvature of a connection A over a two-cycle – to see torsion classes using the connection is more subtle. This is explained in general in the paper [39]. The latter reference implies that the torsion first Chern class may be computed by picking a *flat* connection on p^*P, and then computing the log of the holonomy of this flat connection around the one-cycles that generate $H_1(S^3/\mathbb{Z}_q, \mathbb{Z})$. We may shift to a flat connection here by defining

$$\begin{aligned} A_S^{\text{flat}} &= A_S + \frac{a}{q}D\nu_S = \frac{a}{q}d\nu_S \\ A_N^{\text{flat}} &= A_N + \frac{a}{q}D\nu_N = \frac{a}{q}d\nu_N \end{aligned} \tag{A.6}$$

Here we have added a global one-form $(a/q)D\nu$ to the original connection – we are simply picking a different connection on the same bundle. Then $H_1(S^3/\mathbb{Z}_q, \mathbb{Z}) \cong \mathbb{Z}_q$ is generated by, for example, the ψ_N circle at $\theta = 0$. Thus the log of the holonomy is

$$i\int_{S^1} A_N^{\text{flat}} = \frac{2\pi i a}{q} \mod 2\pi i \ . \tag{A.7}$$

This implies that our connection above is a times the generator of $\mathbb{Z}_q$.

Finally, we make a comment about quotients. First note that quotienting the period of the $U(1)$ fibre coordinate of P by q is the same as taking the qth power of P. In particular, the $\mathbb{Z}_q$ quotient of the bundle p^*P over $S^3/\mathbb{Z}_q$ is then trivial. This follows simply because the connection on this bundle in the two patches is qA_S and qA_N, or after a gauge transformation $qA_S - ad\nu_S$ and $qA_N - ad\nu_N$, and from (A.5) we see that this is a globally defined connection one-form, and hence the bundle is trivial.

B More on the topology of M_5

Recall that, in the main text, M_6 is constructed as the total space of a circle bundle L over $M_5 \cong S^3 \times S^2$. Here $c_1(L) \in H^2(M_5, \mathbb{Z}) \cong \mathbb{Z}$ is the generator, so that $M_6 \cong S^3 \times S^3$. Although this is straightforward as stated, the issue is that we have infinitely many coordinate systems on M_5, labelled by the integers p and q, and the diffeomorphism $M_5 \cong S^3 \times S^2$ is not explicit for general p and q. For each p and q there are different naturally-defined three-submanifolds of M_5 – we are especially interested in three-submanifolds since we would like to quantise the RR three-form flux. In this appendix we consider these submanifolds in more detail, and in particular determine the topology of L restricted to them.

Consider restricting this circle bundle L over M_5 to one of the three-submanifolds of M_5: E_1, E_2 or E_f. For example, take $E_f \cong S^3/\mathbb{Z}_q$. Recall this is itself a circle bundle over $\Sigma_f \cong S^2$ with Chern class q. There is an inclusion map $i_f : E_f \hookrightarrow M_5$, and we can define a circle bundle L_f over E_f by pulling back

$$L_f \equiv i_f^* L \ . \tag{B.1}$$

Since E_f is a lens space, $E_f = S^3/\mathbb{Z}_q$, circle bundles over E_f are classified up to isomorphism by

$$c_1(L_f) \in H^2(E_f, \mathbb{Z}) \cong \mathbb{Z}_q \ . \tag{B.2}$$

To compute this Chern class, recall that $c_1(L) = \pi^*\tau$, where $\tau \in H^2(B_4, \mathbb{Z})$ was defined in (2.34). Hence to compute $c_1(L_f) = i_f^*\pi^*(\tau)$ we may instead first restrict τ to Σ_f, and then pull back using π^* the corresponding circle bundle to E_f. This is summarised by the following commutative square:

$$\begin{array}{ccc} H^2(M_5,\mathbb{Z}) & \stackrel{i_f^*}{\longrightarrow} & H^2(E_f,\mathbb{Z}) \\ \pi^*\uparrow & & \uparrow\pi^* \\ H^2(B_4,\mathbb{Z}) & \stackrel{\iota_f^*}{\longrightarrow} & H^2(\Sigma_f,\mathbb{Z}) \end{array} \quad . \tag{B.3}$$

Here we have denoted the embedding of Σ_f into B_4 by $\iota_f : \Sigma_f \to B_4$. Then $\iota_f^*\tau$ defines an integer class in $H^2(\Sigma_f,\mathbb{Z}) \cong \mathbb{Z}$. This in turn defines a circle bundle with Chern number a, using (2.34). Using the results in appendix A, lifting this circle bundle to E_f then gives a bundle with Chern number

$$a = c_1(L_f) \in H^2(E_f,\mathbb{Z}) \cong \mathbb{Z}_q \ . \tag{B.4}$$

Thus the bundle L restricted to E_f is trivialisable only if $a = 0 \bmod q$; in other words, if $a = mq$ for some integer m. But if this were the case, then we would have

$$(mp + b)q = 1 \ . \tag{B.5}$$

This is only possible if $q = \pm 1$. Thus we see that for general q it is not possible to take a section of L over E_f to obtain a three-submanifold of M_6.

One can do similar computations for the three-submanifolds E_1 and E_2, with similar conclusions. We have

$$L_1 \equiv i_1^* L \quad , \qquad c_1(L_1) = b - 2a \in H^2(E_1,\mathbb{Z}) \cong \mathbb{Z}_{p+2q} \tag{B.6}$$

$$L_2 \equiv i_2^* L \quad , \qquad c_1(L_2) = b \qquad \in H^2(E_2,\mathbb{Z}) \cong \mathbb{Z}_p \ . \tag{B.7}$$

Thus the corresponding bundles are trivial[16] if and only if $b = m_2 p$, $b - 2a = m_1(p + 2q)$, respectively, where $m_1, m_2 \in \mathbb{Z}$, which implies

$$\begin{aligned} p(a + qm_2) &= 1 \\ (p+2q)(a + m_1 q) &= 1 \end{aligned} \tag{B.8}$$

respectively. These equations imply in particular that $p = \pm 1$ and $(p + 2q) = \pm 1$.

We thus conclude that, for generic p and q, the circle bundle L restricted to E_1, E_2 and E_f is non-trivial, and thus we cannot globally take a section of L. This means that these natural three-submanifolds of M_5 cannot be used to construct natural three-submanifolds of M_6.

[16]This analysis assumes that p, $p + 2q$, q are non-zero.

C More general AdS_3 solutions

We first recall from [7], [27] the local data that is sufficient to construct supersymmetric AdS_3 solutions of type IIB supergravity with non-vanishing five-form flux and complex three-form flux G. We require a six-dimensional local Kähler metric ds^2_6 whose Ricci tensor satisfies[17]

$$\Box R - \frac{1}{2}R^2 + R^{ij}R_{ij} + \frac{2}{3}G^{ijk}G^*_{ijk} = 0 \tag{C.1}$$

and G must be a closed, primitive and $(1,2)$ three-form on the six-dimensional space. We refer to [7], [27] for details of how the full ten-dimensional solution is constructed from this data.

For the solutions that we have discussed in this paper, which we will now generalise, the local six-dimensional Kahler metric has the form

$$ds^2_6 = ds^2_4 + ds^2(T^2) \tag{C.2}$$

where $ds^2(T^2) = (du^1)^2 + (du^2)^2$ is the standard metric on a two-torus, ds^2_4 is a four-dimensional local Kähler metric, and

$$G = d\bar{u} \wedge W \tag{C.3}$$

where $u = u^1 + iu^2$ and W is a closed, primitive $(1,1)$-form on the four-dimensional Kähler space.

Inspired[18] by the six-dimensional Kähler metrics discussed in equation 5.10 of [26], we start with the ansatz for a four-dimensional Kähler metric given by

$$ds^2_4 = \frac{Y}{4F}dw^2 + \sum_{i=1}^{2}(w+q_i)\left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right) + \frac{F-1}{Y}\left(\sum_{i=1}^{2}\mu_i^2 d\phi_i\right)^2 \tag{C.4}$$

with

$$\sum_{i=1}^{2}\mu_i^2 = 1, \qquad Y = \sum_{i=1}^{2}\frac{\mu_i^2}{w+q_i} \tag{C.5}$$

and F an arbitrary function of w. To show that the metric is Kähler we introduce the orthonormal frame

$$\begin{aligned} e_i =& \frac{1}{2\sqrt{F}}\frac{\mu_i}{\sqrt{w+q_i}}dw + \sqrt{w+q_i}\,d\mu_i \\ \bar{e}_i =& \frac{\sqrt{F}-1}{Y}\frac{\mu_i}{\sqrt{w+q_i}}\sum_{j=1}^{2}\mu_j^2 d\phi_j + \sqrt{w+q_i}\,\mu_i\, d\phi_i \end{aligned} \tag{C.6}$$

[17] Changing the sign of the last term leads to type IIB bubble solutions, as explained in [27]. The construction in this appendix can be easily adapted to construct bubble solutions.

[18] One can consider the scaling $\mu_3 \to \epsilon\rho$, $q_3 \to 1/\epsilon^2$, $\lambda \to \lambda/\epsilon^2$ in equation 5.10 of [26] and then take $\epsilon \to 0$.

with

$$ds_4^2 = \sum_{i=1}^{2} \left(e_i \otimes e_i + \bar{e}_i \otimes \bar{e}_i\right). \tag{C.7}$$

The Kähler form can be written

$$\begin{aligned} J =& \frac{i}{2} \sum_{i=1}^{2} \left(e_i - i\bar{e}_i\right) \wedge \left(e_i + i\bar{e}_i\right) = -\sum_{i=1}^{2} e_i \wedge \bar{e}_i \\ =& -\frac{1}{2} dw \wedge \sum_{i=1}^{2} \mu_i^2 d\phi_i - \sum_{i=1}^{2} \left(w + q_i\right) \mu_i d\mu_i \wedge d\phi_i \end{aligned} \tag{C.8}$$

which is clearly closed for any choice of F.

The holomorphic $(2,0)$-form Ω is given by

$$\begin{aligned} \Omega =& \prod_{i=1}^{2} \left(e_i - i\bar{e}_i\right) \\ =& \sqrt{w+q_1}\sqrt{w+q_2} \left[\frac{Y}{2\sqrt{F}}\, dw \wedge d\theta - \sqrt{F} \cos\theta \sin\theta\, d\phi_1 \wedge d\phi_2\right] \\ & - i\sqrt{w+q_1}\sqrt{w+q_2} \frac{1}{2\sqrt{F}} \cos\theta \sin\theta\, dw \wedge \left(\frac{d\phi_2}{w+q_1} - \frac{d\phi_1}{w+q_2}\right) \\ & + i\sqrt{w+q_1}\sqrt{w+q_2}\sqrt{F}\, d\theta \wedge \left(\cos^2\theta\, d\phi_1 + \sin^2\theta\, d\phi_2\right) \end{aligned} \tag{C.9}$$

where we have introduced $\mu_1 = \cos\theta$, $\mu_2 = \sin\theta$, $0 < \theta < \frac{\pi}{2}$. A calculation now shows that

$$d\Omega = iP \wedge \Omega \tag{C.10}$$

with

$$\begin{aligned} P =& \frac{2\sqrt{F}}{Y\sqrt{w+q_1}\sqrt{w+q_2}} \partial_w \left(\sqrt{F}\sqrt{w+q_1}\sqrt{w+q_2}\right) \left(\cos^2\theta\, d\phi_1 + \sin^2\theta\, d\phi_2\right) \\ & + \frac{1}{Y} \cos 2\theta \left(\frac{d\phi_2}{w+q_1} - \frac{d\phi_1}{w+q_2}\right). \end{aligned} \tag{C.11}$$

From this we deduce that the complex structure is integrable, and thus we do indeed have a local Kähler metric with Ricci form given by dP. It is helpful to observe that we can also write

$$P = \partial_w \left[\left(F-1\right)\left(w+q_1\right)\left(w+q_2\right)\right] \frac{\sum_{i=1}^{2} \mu_i^2 d\phi_i}{Y\left(w+q_1\right)\left(w+q_2\right)} + d\phi_1 + d\phi_2. \tag{C.12}$$

We now construct a closed two-form W which satisfies

$$\Omega \wedge W = 0, \tag{C.13}$$

which is the condition for it to be a $(1,1)$-form, and also

$$J \wedge W = 0, \tag{C.14}$$

which is the condition for it to be a primitive two-form. We make the ansatz

$$W = d\left[f(w)\,\frac{\sum_{i=1}^{2}\mu_i^2 d\phi_i}{Y(w+q_1)(w+q_2)}\right] \tag{C.15}$$

which satisfies the first equation. The second equation reads

$$J \wedge W = -\frac{\partial_w f}{Y(w+q_1)(w+q_2)}\,J\wedge J = 0 \tag{C.16}$$

and so we take

$$W = Q\,d\left[\frac{\sum_{i=1}^{2}\mu_i^2 d\phi_i}{Y(w+q_1)(w+q_2)}\right] \tag{C.17}$$

where Q is a constant. The two-form W is anti-self dual and we note that

$$W^{ij}W_{ij} = \frac{16Q^2}{\left[Y(w+q_1)(w+q_2)\right]^4}. \tag{C.18}$$

Having fixed W, and hence the three-form flux G, we just need to fix the function F to obtain the Kähler metric ds_4^2 by solving (C.1) which reads

$$\Box R - \frac{1}{2}R^2 + R^{ij}R_{ij} + 4W^{ij}W_{ij} = 0. \tag{C.19}$$

We consider the ansatz

$$F = 1 + \lambda w^2 \prod_{i=1}^{2}\frac{1}{w+q_i} + \Lambda\prod_{i=1}^{2}\frac{1}{w+q_i}, \tag{C.20}$$

observing from (C.12) that the constant Λ does not enter the Ricci potential. A calculation shows that the Ricci scalar is given by

$$R = -\frac{8\lambda}{Y(w+q_1)(w+q_2)}. \tag{C.21}$$

and that (C.19) boils down to solving

$$\frac{\Lambda}{Y(w+q_1)(w+q_2)}\partial_w^2 R + W^{ij}W_{ij} = 0 \tag{C.22}$$

which implies that $\Lambda = \frac{Q^2}{\lambda}$.

In summary, supersymmetric AdS_3 solutions of type IIB supergrvaity can be constructed from the six-dimensional Kähler metric (C.2), with the four-dimensional Kähler metric given by

$$ds_4^2 = \frac{Y}{4F}dw^2 + \sum_{i=1}^{2}(w+q_i)\left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right) + \frac{F-1}{Y}\left(\sum_{i=1}^{2}\mu_i^2 d\phi_i\right)^2 \tag{C.23}$$

and

$$F = 1 + \left(\lambda w^2 + \frac{Q^2}{\lambda}\right)\frac{1}{(w+q_1)(w+q_2)}. \tag{C.24}$$

The three-form flux is given by (C.3) with the closed, primitive and $(1,1)$-form W given by

$$W = Q\, d\left[\frac{\sum_{i=1}^{2}\mu_i^2 d\phi_i}{Y(w+q_1)(w+q_2)}\right]. \tag{C.25}$$

Observe that when $q_1 = q_2 \equiv q$, the metric is precisely of the form found in [27] leading to the AdS_3 solutions that we have analysed in detail in this paper. To see this we let $w + q = 1/x$ and we also introduce Euler angles via

$$\begin{aligned} \mu_1 e^{i\phi_1} &= \cos\frac{\theta}{2}e^{i\frac{\psi+\phi}{2}} \\ \mu_2 e^{i\phi_2} &= \sin\frac{\theta}{2}e^{i\frac{\psi-\phi}{2}}. \end{aligned} \tag{C.26}$$

We then find that

$$ds_4^2 = \frac{dx^2}{4x^3U} + \frac{1}{4x}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{U}{4x}(d\psi + \cos\theta d\phi)^2 \tag{C.27}$$

with

$$U = 1 + \lambda(1-qx)^2 + \frac{Q^2}{\lambda}x^2 \tag{C.28}$$

which should be compared with equations C.1 and C.7 of [27]. Furthermore,

$$W = \frac{Q}{2}d[x(d\psi + \cos\theta d\phi)] \tag{C.29}$$

which should be compared with equation C.5 of [27]. When $q_1 = q_2$, the metric ds_4^2 has local isometry group $SU(2)\times U(1)$ and the metric is cohomogeneity one. In the more general solutions with $q_1 \neq q_2$ the local isometry group is $U(1)\times U(1)$ and the metric is cohomogeneity two.

It will be interesting to analyse these more general AdS_3 solutions with $q_1 \neq q_2$ in more detail. When $Q = 0$ the internal space will have topology $S^2 \times S^3 \times T^2$ and when $Q \neq 0$ it will have topology $S^3 \times S^3 \times S^1$. This can be shown using the techniques used in [29] and in this paper. When $Q \neq 0$, one will also need to check the flux quantisation conditions and this will require generalising the techniques that we have used in this paper. We leave this for the future.

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