DESY 08-124

December 2008

Nonequilibrium Dynamics of Scalar Fields in a Thermal Bath

A. Anisimov, W. Buchmüller, M. Drewes, S. Mendizabal

Deutsches Elektronen-Synchrotron DESY, Hamburg, Germany

Abstract

We study the approach to equilibrium for a scalar field which is coupled to a large thermal bath. Our analysis of the initial value problem is based on Kadanoff-Baym equations which are shown to be equivalent to a stochastic Langevin equation. The interaction with the thermal bath generates a temperature-dependent spectral density, either through decay and inverse decay processes or via Landau damping. In equilibrium, energy density and pressure are determined by the Bose-Einstein distribution function evaluated at a complex quasi-particle pole. The time evolution of the statistical propagator is compared with solutions of the Boltzmann equations for particles as well as quasi-particles. The dependence on initial conditions and the range of validity of the Boltzmann approximation are determined.

1 Introduction

The current standard model of cosmology explains many features of our universe as the result of out-of-equilibrium processes during its very early high-temperature phase (cf. [1,2]). This includes the matter-antimatter asymmetry, i.e. the origin of matter, the production of dark matter, the formation of light elements and the decoupling of photons leading to the cosmic microwave background.

Many nonequilibrium processes in the early universe can be treated in the canonical way by means of Boltzmann equations (cf. [1]) with sufficient accuracy. In some cases, however, quantum effects play a crucial role. This applies in particular to baryogenesis, the generation of the matter-antimatter asymmetry. Here the CP asymmetry, which leads to the baryon asymmetry, is the result of a quantum interference. It is therefore important to go beyond the classical Boltzmann equations and to treat the entire baryogenesis process quantum mechanically.

An attractive baryogenesis scenario is leptogenesis [3, 4], where a quantitative understanding of the baryon asymmetry in terms of neutrino properties has been achieved [5]. In leptogenesis the out-of-equilibrium dynamics of a heavy Majorana neutrino, which is coupled to a large thermal bath of standard model particles, is the origin of the baryon asymmetry. Given the simplicity of this process, a full quantum mechanical treatment may be possible and some progress in this direction has already been made during the past years [6–8]. One important application is the study of flavor effects [9].

The treatment of nonequilibrium processes in quantum field theory is usually based either on Kadanoff-Baym equations and the Schwinger-Keldysh formalism [10–13] or on stochastic Langevin equations [14–17]. Both methods have been applied to various processes in particle physics and cosmology, including also electroweak baryogenesis [18]. In this paper we examine the connection between both approaches, which has been also considered in [19]. As we shall see, the Kadanoff-Baym equations and the Langevin equation are, in fact, equivalent for the case of a large thermal bath where backreaction effects can be neglected.

Boltzmann equations are first-order differential equations for number densities, which are local in time. They represent a valuable approximation for nonequilibrium processes in a dilute, weakly coupled gas. However, when the interactions between the quanta of the thermal plasma are strong, which is certainly the case in the presence of non-Abelian gauge interactions, the validity of the Boltzmann approximation is questionable. Correspondingly, the notion of number density becomes ambiguous, although several useful definitions have been suggested [13, 17].

In this paper we study the approach to equilibrium for a scalar field which is coupled to a thermal bath with many degrees of freedom such that backreaction effects can be neglected. We shall focus on the description of this nonequilibrium process in terms of Green's functions rather than number densities. This is analogous to studies of preheating after inflation based on the statistical propagator [13, 20]. As we shall see, the Kadanoff-Baym equations and the Langevin equation lead to identical results.

Knowing the exact solution of the initial value problem for the Green's function of the scalar field, we can systematically study the conditions for the validity of ordinary Boltzmann equations as well as Boltzmann equations for quasi-particles. At large times the scalar field reaches equilibrium. As we shall see, this state does not correspond to a gas of quasi-particles. There is an additional thermal 'vacuum' contribution which in principle can even lead to a negative pressure of low-momentum modes. The general solution of the Green's function also allows us to study the dependence of the equilibration on the initial conditions. This is an important problem in leptogenesis, because the baryon asymmetry can only be predicted in terms of neutrino properties when there is no dependence on the initial conditions [21].

To illustrate our results we consider a toy model of three scalars [17, 22, 23], one being much heavier than the other two. Two particles are in thermal equilibrium whereas the third one slowly approaches thermal equilibrium starting from zero initial abundance. Due to the interaction with the thermal bath this particle has a non-trivial spectral density, approximately described by a 'thermal mass' and a 'thermal width'. These are generated either by decays and inverse decays or by a process similar to Landau damping. Some aspects of this model have previously been studied based on the time evolution of a number density [17].

The paper is organized as follows. In Section 2 we define the various Green's functions in the Schwinger-Keldysh formalism and present a brief derivation of the Kadanoff-Baym equations. The theoretical framework leading to the Langevin equation is discussed in Section 3, following [17]. Section 4 deals with the solutions of the Kadanoff-Baym equations. Thermal equilibrium and the quasi-particle picture are discussed in Section 5, and a sytematic comparison with Boltzmann equations is made in Section 6. The results are illustrated for a thermal bath of scalars in Section 7. A brief summary and outlook is given in Section 8. Various properties of the spectral function are discussed in the Appendix.

2 The Schwinger-Keldysh formalism

Let us consider the nonequilibrium dynamics of a scalar field. In the Schwinger-Keldysh formalism the basic quantity is the Green's function defined on a contour C in the complex x^0-plane (cf. Figure 1),

$$\Delta_C(x_1, x_2) = \theta_C(x_1^0, x_2^0)\Delta^>(x_1, x_2) + \theta_C(x_2^0, x_1^0)\Delta^<(x_1, x_2) \ . \tag{2.1}$$

The θ-functions enforce path ordering along the contour C, and $\Delta^>$ and $\Delta^<$ are the correlation functions

$$\Delta^>(x_1, x_2) = \langle\Phi(x_1)\Phi(x_2)\rangle = \mathrm{Tr}(\rho\Phi(x_1)\Phi(x_2)) \ , \tag{2.2}$$

$$\Delta^<(x_1, x_2) = \langle\Phi(x_2)\Phi(x_1)\rangle = \mathrm{Tr}(\rho\Phi(x_2)\Phi(x_1)) \ , \tag{2.3}$$

where ρ is the density matrix of the system at some initial time t_i.

We consider the case that the field Φ is coupled to a thermal bath described by a self-energy Π. The Green's function Δ_C then satisfies the Schwinger-Dyson equation

$$(\Box_1 + m^2)\Delta_C(x_1, x_2) + \int_C d^4x'\Pi_C(x_1, x')\Delta_C(x', x_2) = -i\delta_C(x_1 - x_2) \ , \tag{2.4}$$

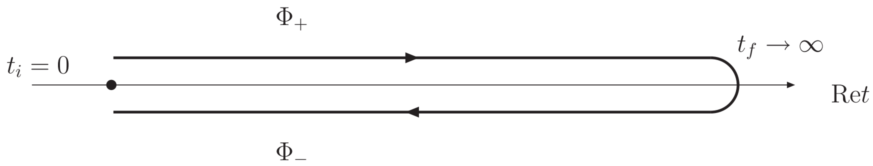

Figure 1: Path in the complex time plane for nonequilibrium Green's functions.

where $\Box_1 = (\partial^2/\partial x_1^2)$. Like the Green's function, also the self-energy can be decomposed as

$$\Pi_C(x_1, x_2) = \theta_C(x_1^0, x_2^0)\Pi^>(x_1, x_2) + \theta_C(x_2^0, x_1^0)\Pi^<(x_1, x_2) \ . \tag{2.5}$$

In the Schwinger-Dyson equation the time coordinates of Δ_C and Π_C can be on the upper or lower branch of the contour C, which we denote by the subscripts '+' and '−', respectively. Obviously, one has

$$\Delta_{-+}(x_1, x_2) = \Delta^>(x_1, x_2) \ , \quad \Delta_{+-}(x_1, x_2) = \Delta^<(x_1, x_2) \ , \tag{2.6}$$

$$\Pi_{-+}(x_1, x_2) = \Pi^>(x_1, x_2) \ , \quad \Pi_{+-}(x_1, x_2) = \Pi^<(x_1, x_2) \ , \tag{2.7}$$

whereas Δ_{++}, Π_{++} and Δ_{--}, Π_{--} are causal and anti-causal Green functions, respectively. From the Schwinger-Dyson equation (2.4) one obtains for the correlation functions $\Delta^<$ and $\Delta^>$,

$$(\Box_1 + m^2)\Delta^<(x_1, x_2) = \int d^4x' \left(-\Pi_{++}(x_1, x')\Delta^<(x', x_2) + \Pi^<(x_1, x')\Delta_{--}(x', x_2)\right) \ , \tag{2.8}$$

$$(\Box_1 + m^2)\Delta^>(x_1, x_2) = \int d^4x' \left(-\Pi^>(x_1, x')\Delta_{++}(x', x_2) + \Pi_{--}(x_1, x')\Delta^>(x', x_2)\right) \ , \tag{2.9}$$

where the relative sign in the integrands is due to the anti-causal time ordering on the lower branch of C.

It is convenient to also introduce retarded and advanced Green functions,

$$\begin{aligned}
\Delta^R(x_1, x_2) &= \theta(t_1 - t_2)(\Delta^>(x_1, x_2) - \Delta^<(x_1, x_2)) \\
&= \theta(t_1 - t_2)\langle[\phi(x_1), \phi(x_2)]\rangle \\
&= \Delta_{++}(x_1, x_2) - \Delta_{+-}(x_1, x_2) \\
&= \Delta_{-+}(x_1, x_2) - \Delta_{--}(x_1, x_2) \ ,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\Delta^A(x_1, x_2) &= -\theta(t_2 - t_1)(\Delta^>(x_1, x_2) - \Delta^<(x_1, x_2)) \\
&= -\theta(t_2 - t_1)\langle[\phi(x_1), \phi(x_2)]\rangle \\
&= \Delta_{++}(x_1, x_2) - \Delta_{-+}(x_1, x_2) \\
&= \Delta_{+-}(x_1, x_2) - \Delta_{--}(x_1, x_2) \ ,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\Pi^R(x_1, x_2) &= \theta(t_1 - t_2)(\Pi^>(x_1, x_2) - \Pi^<(x_1, x_2)) \\
&= \Pi_{++}(x_1, x_2) - \Pi_{+-}(x_1, x_2) \\
&= \Pi_{-+}(x_1, x_2) - \Pi_{--}(x_1, x_2) \ ,
\end{aligned} \tag{2.12}$$

$$\begin{aligned}\Pi^A(x_1,x_2) &= -\theta(t_2-t_1)(\Pi^>(x_1,x_2)-\Pi^<(x_1,x_2)) \\ &= \Pi_{++}(x_1,x_2)-\Pi_{-+}(x_1,x_2) \\ &= \Pi_{+-}(x_1,x_2)-\Pi_{--}(x_1,x_2)\ . \end{aligned} \tag{2.13}$$

From Eqs. (2.8) and (2.9) one obtains the Kadanoff-Baym equations for the correlation functions $\Delta^>$ and $\Delta^<$,

$$(\Box_1+m^2)\Delta^>(x_1,x_2) = -\int d^4x' \left(\Pi^>(x_1,x')\Delta^A(x',x_2)+\Pi^R(x_1,x')\Delta^>(x',x_2)\right)\ , \tag{2.14}$$

$$(\Box_1+m^2)\Delta^<(x_1,x_2) = -\int d^4x' \left(\Pi^<(x_1,x')\Delta^A(x',x_2)+\Pi^R(x_1,x')\Delta^<(x',x_2)\right)\ . \tag{2.15}$$

We now define the real symmetric and antisymmetric correlation functions

$$\Delta^+(x_1,x_2) = \frac{1}{2}\langle\{\Phi(x_1),\Phi(x_2)\}\rangle\ , \tag{2.16}$$

$$\Delta^-(x_1,x_2) = i\langle[\Phi(x_1),\Phi(x_2)]\rangle\ , \tag{2.17}$$

and self-energies

$$\Pi^+(x_1,x_2) = -\frac{i}{2}\left(\Pi^>(x_1,x_2)+\Pi^<(x_1,x_2)\right)\ , \tag{2.18}$$

$$\Pi^-(x_1,x_2) = \Pi^>(x_1,x_2)-\Pi^<(x_1,x_2)\ , \tag{2.19}$$

which also determine the retarded and advanced self-energies,

$$\Pi^R(x_1,x_2) = \theta(t_1-t_2)\Pi^-(x_1,x_2)\ , \quad \Pi^A(x_1,x_2) = -\theta(t_2-t_1)\Pi^-(x_1,x_2)\ . \tag{2.20}$$

Adding and subtracting the Kadanoff-Baym equations (2.14) and (2.15), one obtains from Eqs. (2.10)-(2.13) and (2.16)-(2.19) an homogeneous equation for Δ^- and an inhomogeneous equation for Δ^+,

$$(\Box_1+m^2)\Delta^-(x_1,x_2) = -\int d^3\mathbf{x}'\int_{t_2}^{t_1} dt'\Pi^-(x_1,x')\Delta^-(x',x_2)\ , \tag{2.21}$$

$$\begin{aligned}(\Box_1+m^2)\Delta^+(x_1,x_2) &= -\int d^3\mathbf{x}'\int_{t_i}^{t_1} dt'\Pi^-(x_1,x')\Delta^+(x',x_2) \\ &\quad +\int d^3\mathbf{x}'\int_{t_i}^{t_2} dt'\Pi^+(x_1,x')\Delta^-(x',x_2)\ . \end{aligned} \tag{2.22}$$

We shall refer to these as equations as the first and second Kadanoff-Baym equation. Δ^- and Δ^+ are known as spectral function and statistical propagator (cf. [13]). Together they determine the path ordered Green's function,

$$\Delta_C(x_1,x_2) = \Delta^+(x_1,x_2) - \frac{i}{2}\mathrm{sign}_C(x_1^0-x_2^0)\Delta^-(x_1,x_2)\ . \tag{2.23}$$

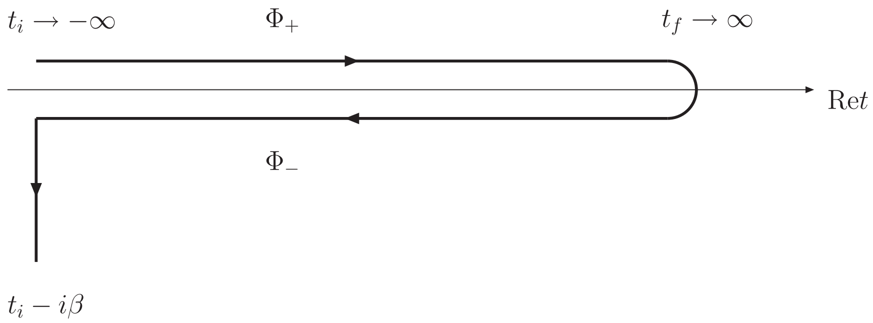

Figure 2: Path in the complex time plane for thermal Green's functions.

Δ^- carries information about the spectrum of the system and Δ^+ is related to occupation numbers of different modes.

Using microcausality and the canonical quantization condition for a real scalar field,

$$[\Phi(x_1), \Phi(x_2)]|_{t_1=t_2} = [\dot{\Phi}(x_1), \dot{\Phi}(x_2)]|_{t_1=t_2} = 0 \ , \tag{2.24}$$

$$[\Phi(x_1), \dot{\Phi}(x_2)]|_{t_1=t_2} = i\delta(\mathbf{x}_1 - \mathbf{x}_2) \ , \tag{2.25}$$

one obtains from the definitions (2.16) and (2.17)

$$\Delta^-(x_1, x_2)|_{t_1=t_2} = 0 \ , \tag{2.26}$$

$$\partial_{t_1}\Delta^-(x_1, x_2)|_{t_1=t_2} = -\partial_{t_2}\Delta^-(x_1, x_2)|_{t_1=t_2} = \delta(\mathbf{x}_1 - \mathbf{x}_2) \ , \tag{2.27}$$

$$\partial_{t_1}\partial_{t_2}\Delta^-(x_1, x_2)|_{t_1=t_2} = 0 \ . \tag{2.28}$$

In the following we shall restrict ourselves to systems with spatial translational invariance. In this case all two-point functions only depend on the difference of spatial coordinates, $\mathbf{x}_1 - \mathbf{x}_2$, and it is convenient to perform a Fourier transformation. The Green's functions $\Delta^{\pm}_{\mathbf{q}}(t_1, t_2)$ satisfy the two Kadanoff-Baym equations

$$(\partial_{t_1}^2 + \omega_{\mathbf{q}}^2)\Delta^-_{\mathbf{q}}(t_1, t_2) + \int_{t_2}^{t_1} dt' \Pi^-_{\mathbf{q}}(t_1, t')\Delta^-_{\mathbf{q}}(t', t_2) = 0 \ , \tag{2.29}$$

$$(\partial_{t_1}^2 + \omega_{\mathbf{q}}^2)\Delta^+_{\mathbf{q}}(t_1, t_2) + \int_{t_i}^{t_1} dt' \Pi^-_{\mathbf{q}}(t_1, t')\Delta^+_{\mathbf{q}}(t', t_2) = \int_{t_i}^{t_2} dt' \Pi^+_{\mathbf{q}}(t_1, t')\Delta^-_{\mathbf{q}}(t', t_2) \ , \tag{2.30}$$

where $\omega_{\mathbf{q}}^2 = \mathbf{q}^2 + m^2$. The initial conditions (2.26)-(2.28) for the spectral function become

$$\Delta^-_{\mathbf{q}}(t_1, t_2)|_{t_1=t_2} = 0 \ , \tag{2.31}$$

$$\partial_{t_1}\Delta^-_{\mathbf{q}}(t_1, t_2)|_{t_1=t_2} = -\partial_{t_2}\Delta^-_{\mathbf{q}}(t_1, t_2)|_{t_1=t_2} = 1 \ , \tag{2.32}$$

$$\partial_{t_1}\partial_{t_2}\Delta^-_{\mathbf{q}}(t_1, t_2)|_{t_1=t_2} = 0 \ . \tag{2.33}$$

For Green's functions in thermal equilibrium the density matrix in Eqs. (2.2), (2.3) is $\rho_{\mathrm{eq}} = \exp(-\beta H)$, where H is the Hamiltonian of the system, and $\beta = T^{-1}$ is the inverse temperature. Time coordinates of Green functions now lie on the contour shown in Figure 2, and one has invariance under time translations so that two-point functions only depend on the time difference $t_1 - t_2$. After a Fourier transformation, one obtains the KMS relations [12] for Green's functions and self-energies,

$$\Delta^+_{\mathbf{q}}(\omega) = -\frac{i}{2}\coth\left(\frac{\beta\omega}{2}\right)\Delta^-_{\mathbf{q}}(\omega) \ , \tag{2.34}$$

$$\Pi^+_{\mathbf{q}}(\omega) = -\frac{i}{2}\coth\left(\frac{\beta\omega}{2}\right)\Pi^-_{\mathbf{q}}(\omega) \ . \tag{2.35}$$

The Kadanoff-Baym equations describe the dynamics of a arbitrary nonequilibrium system. Depending on the self-energy and the initial conditions, the solutions will generally be complicated. An enormous simplification is achieved for a large medium such that the backreaction of the field Φ can be neglected. Furthermore, we assume that the medium is in thermal equilibrium and, therefore, the self energy of Φ is time-translation invariant,

$$\Pi_{\mathbf{q}}(t_1, t_2) = \Pi_{\mathbf{q}}(t_1 - t_2) \ . \tag{2.36}$$

In this case also the spectral function is time-translation invariant, as shown in Appendix A.1. With these simplifications, the Kadanoff-Baym equations become

$$(\partial^2_{t_1} + \omega^2_{\mathbf{q}})\Delta^-_{\mathbf{q}}(t_1 - t_2) = -\int_{t_2}^{t_1} dt' \Pi^-_{\mathbf{q}}(t_1 - t')\Delta^-_{\mathbf{q}}(t' - t_2) \ , \tag{2.37}$$

$$\begin{aligned}(\partial^2_{t_1} + \omega^2_{\mathbf{q}})\Delta^+_{\mathbf{q}}(t_1, t_2) = &\int_{t_i}^{t_2} dt' \Pi^+_{\mathbf{q}}(t_1 - t')\Delta^-_{\mathbf{q}}(t' - t_2) \\ &- \int_{t_i}^{t_1} dt' \Pi^-_{\mathbf{q}}(t_1 - t')\Delta^+_{\mathbf{q}}(t', t_2) \ . \end{aligned} \tag{2.38}$$

These equations will be solved in Section 4 for general initial conditions.

3 Stochastic Langevin equation

Nonequilibrium processes can also be studied by means of Langevin equations which describe the evolution of the field itself rather than the evolution of Green's functions (cf. [14–17]). Below we sketch a brief derivation of the Langevin equation describing a scalar field Φ coupled to a large thermal bath with bosonic and fermionic fields χ, following the discussion in [17]. We assume that the coupling is of the form $g\Phi\mathcal{O}[\chi]$ and neglect the backreaction of Φ on the thermal bath, which makes the problem solvable.

The starting point is the nonequilibrium generating functional [13, 17]

$$\mathcal{Z}[J_+, J_-] = \int D\Phi^+_{\text{in}} D\Phi^-_{\text{in}} \rho_{\text{in}}(\Phi^+_{\text{in}}; \Phi^-_{\text{in}}) \int \mathcal{D}\Phi_\pm \mathcal{D}\chi_\beta e^{iS[\Phi_\pm, \chi, J_\pm]} \ , \tag{3.1}$$

where the subscript 'in' stands for the initial condition. The action of the fields Φ and χ is given by

$$S[\Phi_\pm,\chi,J_\pm] = \int_{t_i}^{\infty} d^4x\,(\mathcal{L}_\Phi(\Phi_+) + g\Phi_+\mathcal{O}[\chi_+] + J_+\Phi_+ \\ -\mathcal{L}_\Phi(\Phi_-) - g\Phi_-\mathcal{O}[\chi_-] - J_-\Phi_-) + \int_{\mathcal{C}_\beta} d^4x\mathcal{L}_\chi(\chi)\ , \tag{3.2}$$

where $\mathcal{L}_\Phi$ is the Lagrangian of a free massive field,

$$\mathcal{L}_\Phi = \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{1}{2}m^2\Phi^2\ , \tag{3.3}$$

and ρ_{in} stands for the matrix elements of the initial density matrix,

$$\rho_{\mathrm{in}}(\Phi^+_{\mathrm{in}};\Phi'^-_{\mathrm{in}}) = \langle\Phi|\rho|\Phi'\rangle\ . \tag{3.4}$$

The field Φ lives on the Keldysh contour C shown in Figure 1. $\Phi_\pm(x)$ is the field with the time argument on the "forward"(C_+) and "backward"(C_-) part of this contour, respectively, satisfying the boundary conditions

$$\Phi_+(t_i,\mathbf{x}) = \Phi^+_{\mathrm{in}}(\mathbf{x})\ ,\quad \Phi_-(t_i,\mathbf{x}) = \Phi^-_{\mathrm{in}}(\mathbf{x})\ . \tag{3.5}$$

The fields χ are assumed to be in thermal equilibrium, corresponding to the contour $\mathcal{C}_\beta$ (Figure 2), which is possible since the backreaction of Φ on the thermal bath is neglected. In the following we shall choose as initial time $t_i = 0$.

It is convenient to perform a change of variables in the functional integral (3.1),

$$\Psi(x) = \frac{1}{2}\left(\Phi_+(x) + \Phi_-(x)\right)\ , \tag{3.6}$$

$$R(x) = \Phi_+(x) - \Phi_-(x)\ . \tag{3.7}$$

We are interested in the two-point function of Ψ, which couples to the source term $J = J_+ - J_-$. Integrating out the fields R and χ one finds [17],

$$\mathcal{Z}[J] = \int D\Psi_{\mathrm{in}}D\pi_{\mathrm{in}}\mathcal{W}(\Psi_{\mathrm{in}};\pi_{\mathrm{in}})\int \mathcal{D}\Psi\mathcal{D}\xi\mathcal{P}[\xi]e^{i\int d^4x J(x)\Psi(x)} \\ \times\delta\left[\ddot{\Psi}_{\mathbf{q}}(t) + \omega^2_{\mathbf{q}}\Psi_{\mathbf{q}}(t) + \int_0^t dt'\Pi^-_{\mathbf{q}}(t-t')\Psi_{\mathbf{q}}(t') - \xi_{\mathbf{q}}(t)\right]\ ; \tag{3.8}$$

here the measure $\mathcal{P}[\xi]$ is given by

$$\mathcal{P}[\xi] = \exp\left(\frac{1}{2}\int_0^\infty dt\int_0^\infty dt'\xi_{\mathbf{q}}(t)\Pi^+_{\mathbf{q}}(t-t')^{-1}\xi_{-\mathbf{q}}(t')\right]\ , \tag{3.9}$$

and $\xi_{\mathbf{q}}(t)$ is a *stochastic noise.* The Fourier transform $\Psi_{\mathbf{q}}(t)$ in (3.8) satisfies the initial conditions

$$\Psi_{\mathbf{q}}(0) = \Psi_{\mathbf{q},\mathrm{in}}\ ,\quad \dot{\Psi}_{\mathbf{q},\mathrm{in}}(0) = \pi_{\mathbf{q},\mathrm{in}}\ . \tag{3.10}$$

The function $\mathcal{W}(\Psi_{\rm in};\pi_{\rm in})$ is a functional Wigner transform of the initial density matrix,

$$\mathcal{W}(\Psi_{\rm in};\pi_{\rm in}) = \int DR_{\rm in} e^{-\int d^3x \pi_{\rm in}(\mathbf{x}) R_{\rm in}(\mathbf{x})} \rho_{\rm in}\left(\Psi_{\rm in} + \frac{R_{\rm in}}{2}; \Psi_{\rm in} - \frac{R_{\rm in}}{2}\right) . \tag{3.11}$$

For a pure vacuum state ρ is a product of the two delta functions $\delta(\Psi_{\rm in})$ and $\delta(\pi_{\rm in})$.

In order to obtain two-point correlators of the field Ψ one has to solve the classical *stochastic* Langevin equation,

$$\left(\partial_t^2 + \omega_{\mathbf{q}}^2\right)\Psi_{\mathbf{q}}(t) + \int_0^t dt' \Pi^-_{\mathbf{q}}(t-t')\Psi_{\mathbf{q}}(t') = \xi_{\mathbf{q}}(t) , \tag{3.12}$$

with the initial conditions (3.10). Since the backreaction of the field Φ is neglected, the only relevant correlation functions are

$$\langle \xi_{\mathbf{q}}(t)\rangle = 0 , \tag{3.13}$$
$$\langle \xi_{\mathbf{q}}(t)\xi_{\mathbf{q}'}(t')\rangle = -\Pi^+_{\mathbf{q}}(t-t')\delta(\mathbf{q}+\mathbf{q}') . \tag{3.14}$$

The solution of the Langevin equation is conveniently expressed in terms of an auxiliary function $f_{\mathbf{q}}(t)$ which is defined as solution of the homogeneous equation

$$\left(\partial_t^2 + \omega_{\mathbf{q}}^2\right) f_{\mathbf{q}}(t) + \int_0^t dt' \Pi^-_{\mathbf{q}}(t-t') f_{\mathbf{q}}(t') = 0 , \tag{3.15}$$

with the initial conditions

$$f_{\mathbf{q}}(0) = 0 , \quad \dot{f}_{\mathbf{q}}(0) = 1 . \tag{3.16}$$

One easily verifies that the solution of the Langevin equation is then given by

$$\Psi_{\mathbf{q}}(t) = \Psi_{\mathbf{q},\rm in}\dot{f}_{\mathbf{q}}(t) + \Pi_{\mathbf{q},\rm in} f_{\mathbf{q}}(t) + \int_0^t dt' f_{\mathbf{q}}(t-t')\xi_{\mathbf{q}}(t') . \tag{3.17}$$

Correlation functions of the scalar field can now be obtained by calculating the expectation values

$$\langle \Psi_{\mathbf{q}_1}(t_1)\ldots\Psi_{\mathbf{q}_n}(t_n)\rangle , \tag{3.18}$$

which involve the correlation functions of the stochastic noise and also an average over the initial conditions. For the simplest case, the two-point function, one has

$$\langle \Psi_{\mathbf{q}}(t_1)\Psi_{\mathbf{q}'}(t_2)\rangle \equiv g_{\mathbf{q}}(t_1,t_2)\delta(\mathbf{q}+\mathbf{q}') = g_{\mathbf{q}}(t_2,t_1)\delta(\mathbf{q}+\mathbf{q}') . \tag{3.19}$$

From the Langevin equation (3.12) one easily derives an analogous equation for the two-point function,

$$\left(\partial_t^2 + \omega_{\mathbf{q}}^2\right)\langle \Psi_{\mathbf{q}}(t_1)\Psi_{\mathbf{q}'}(t_2)\rangle + \int_0^{t_1} dt' \Pi^-_{\mathbf{q}}(t_1-t')\langle \Psi_{\mathbf{q}}(t')\Psi_{\mathbf{q}'}(t_2)\rangle \tag{3.20}$$
$$= \langle \xi_{\mathbf{q}}(t_1)\Psi_{\mathbf{q}'}(t_2)\rangle \tag{3.21}$$

$$= \delta(\mathbf{q}+\mathbf{q}') \int_0^{t_2} dt' \Pi^+_{\mathbf{q}}(t_1 - t') f_{\mathbf{q}}(t' - t_2) \ , \tag{3.22}$$

which implies

$$\left(\partial_t^2 + \omega_{\mathbf{q}}^2\right) g_{\mathbf{q}}(t_1, t_2) + \int_0^{t_1} dt' \Pi^-_{\mathbf{q}}(t_1 - t') g_{\mathbf{q}}(t', t_2) \tag{3.23}$$

$$= \int_0^{t_2} dt' \Pi^+_{\mathbf{q}}(t_1 - t') f_{\mathbf{q}}(t' - t_2) \ . \tag{3.24}$$

A solution of this equation can be directly obtained from the solution of the Langevin equation (3.12). In the case where the initial field and its time derivative vanish,

$$\langle \Psi_{\mathbf{q},\mathrm{in}} \rangle = \langle \dot{\Psi}_{\mathbf{q},\mathrm{in}} \rangle = 0 \ , \tag{3.25}$$

the relevant averages for the two-point function are

$$\langle \Psi_{\mathbf{q},\mathrm{in}} \Psi_{\mathbf{q},\mathrm{in}} \rangle = \delta(\mathbf{q}+\mathbf{q}') \alpha_{\mathbf{q}} \ , \tag{3.26}$$

$$\langle \dot{\Psi}_{\mathbf{q},\mathrm{in}} \dot{\Psi}_{\mathbf{q}',\mathrm{in}} \rangle = \delta(\mathbf{q}+\mathbf{q}') \beta_{\mathbf{q}} \ , \tag{3.27}$$

$$\langle \dot{\Psi}_{\mathbf{q},\mathrm{in}} \dot{\Psi}_{\mathbf{q},\mathrm{in}} \rangle = \delta(\mathbf{q}+\mathbf{q}') \gamma_{\mathbf{q}} \ . \tag{3.28}$$

Using the solution (3.17) and the correlations (3.14) one obtains the two-point function

$$g_{\mathbf{q}}(t_1, t_2) = \alpha_{\mathbf{q}} \dot{f}_{\mathbf{q}}(t_1) \dot{f}_{\mathbf{q}}(t_2) + \gamma_{\mathbf{q}} f(t_1) f(t_2) \tag{3.29}$$

$$+ \beta_{\mathbf{q}} \left(f_{\mathbf{q}}(t_1) \dot{f}_{\mathbf{q}}(t_2) + \dot{f}_{\mathbf{q}}(t_1) f_{\mathbf{q}}(t_2) \right) \tag{3.30}$$

$$+ \int_0^{t_1} dt' \int_0^{t_2} dt'' f_{\mathbf{q}}(t_1 - t') \Pi^+_{\mathbf{q}}(t' - t'') f_{\mathbf{q}}(t'' - t_2) \ . \tag{3.31}$$

In the following section we shall see that the auxiliary function $f_{\mathbf{q}}(t)$ and the two-point correlation function $g_{\mathbf{q}}(t_1, t_2)$ are precisely the spectral function and the statistical propagator of the field Φ, respectively.

4 Solving the Kadanoff-Baym equations

4.1 The equation for the spectral function

As proven in Appendix A.1, the spectral function is time translation invariant, i.e., it only depends on the time difference $y = t_1 - t_2$. Hence, the first Kadanoff-Baym equation (2.37) takes the form

$$\left(\partial_y^2 + \omega_{\mathbf{q}}^2\right) \Delta^-_{\mathbf{q}}(y) + \int_0^y dy' \Pi^-_{\mathbf{q}}(y - y') \Delta^-_{\mathbf{q}}(y') = 0 \ . \tag{4.1}$$

This equation can be solved by performing a Laplace transformation,

$$\tilde{\Delta}^-_{\mathbf{q}}(s) = \int_0^\infty dy e^{-sy} \Delta^-_{\mathbf{q}}(y) \ , \tag{4.2}$$

for which one obtains after a straightforward calculation

$$\tilde{\Delta}^-_{\mathbf{q}}(s) = \frac{\partial_y \Delta^-_{\mathbf{q}}(0) + s\Delta^-_{\mathbf{q}}(0)}{s^2 + \omega^2_{\mathbf{q}} + \tilde{\Pi}^R_{\mathbf{q}}(s)} \ , \tag{4.3}$$

with

$$\tilde{\Pi}^R_{\mathbf{q}}(s) = \int_0^\infty e^{-sy}\Pi^R_{\mathbf{q}}(y)dy = \int_0^\infty e^{-sy}\Pi^-_{\mathbf{q}}(y)dy = \tilde{\Pi}^-_{\mathbf{q}}(s) \ . \tag{4.4}$$

According to (4.3), the general solution of (4.1) depends on two parameters, the values of $\Delta^-_{\mathbf{q}}$ and $\partial_y \Delta^-_{\mathbf{q}}$ at $y = 0$. Using the inverse Laplace transform one finds

$$\Delta^-_{\mathbf{q}}(y) = \left(\partial_y \Delta^-_{\mathbf{q}}(0) + \Delta^-_{\mathbf{q}}(0)\partial_y\right) \int_{\mathcal{C}_B} \frac{ds}{2\pi i} \frac{e^{sy}}{s^2 + \omega^2_{\mathbf{q}} + \tilde{\Pi}^-_{\mathbf{q}}(s)} \ . \tag{4.5}$$

Here $\mathcal{C}_B$ is the Bromwich contour (see Figure 3): The part parallel to the imaginary axis is chosen such that all singularities of the integrand are to its left; the second part is the semicircle at infinity which closes the contour at $\mathrm{Re}(s) < 0$. Since the integrand of (4.5) has singularities only on the imaginary axis, the second part can be deformed to run parallel to the imaginary axis as well: $\mathcal{C}_B \rightarrow \int_{-i\infty+\epsilon}^{i\infty+\epsilon} + \int_{i\infty-\epsilon}^{-i\infty-\epsilon}$.

The spectral function $\Delta^-_{\mathbf{q}}(y)$ satisfies the boundary conditions (2.31) and (2.32), which implies

$$\Delta^-_{\mathbf{q}}(y) = \int_{\mathcal{C}_B} \frac{ds}{2\pi i} \frac{e^{sy}}{s^2 + \omega^2_{\mathbf{q}} + \tilde{\Pi}^-_{\mathbf{q}}(s)} \ . \tag{4.6}$$

This result can be further simplified by making use of the analytic properties of the self-energy $\tilde{\Pi}^-(s)$. On the real axis $\tilde{\Pi}^-(s)$ is real, while on the parts of the contour which are parallel to the imaginary axis one has

$$\tilde{\Pi}^-(i\omega \pm \epsilon) = \mathrm{Re}\Pi^R_{\mathbf{q}}(\omega) \pm i\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega) \ , \tag{4.7}$$

with

$$\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega) = \frac{1}{2i}\left(\Pi^R_{\mathbf{q}}(\omega + i\epsilon) - \Pi^R_{\mathbf{q}}(\omega - i\epsilon)\right) \ . \tag{4.8}$$

Hence, the expression (4.6) takes the form

$$\Delta^-_{\mathbf{q}}(y) = i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega y} \rho_{\mathbf{q}}(\omega) \ , \tag{4.9}$$

where the spectral function $\rho_{\mathbf{q}}(\omega)$ is given in terms of real and imaginary part of the self-energy $\Pi^R_{\mathbf{q}}(\omega)$,

$$\rho_{\mathbf{q}}(\omega) = \frac{-2\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega) + 2\omega\epsilon}{[\omega^2 - \omega^2_{\mathbf{q}} - \mathrm{Re}\Pi^R_{\mathbf{q}}(\omega)]^2 + [\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega) + \omega\epsilon]^2} = i\tilde{\Delta}^-_{\mathbf{q}}(i\omega) \ . \tag{4.10}$$

Note that $\mathrm{Im}\,\Pi^R_{\mathbf{q}}(\omega)$ and $\mathrm{Re}\,\Pi^R_{\mathbf{q}}(\omega)$ are odd and even functions, respectively, which implies that $\Delta_{\mathbf{q}}(y)$ is real. Further properties of this solution are discussed in Appendix A. Let us

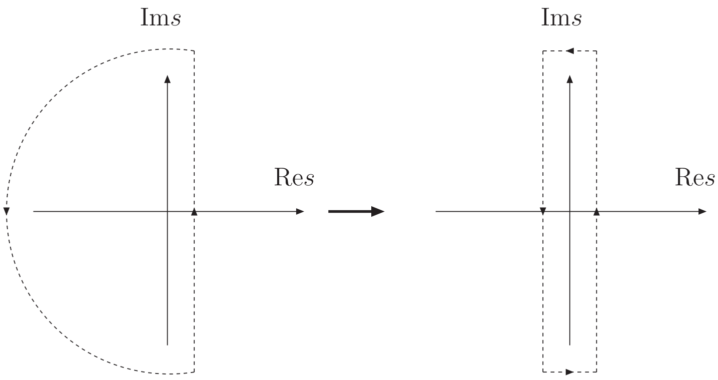

Figure 3: Bromwich contour

recall that the expression (4.10) is obtained after neglecting the backreaction of the field Φ on the thermal bath. This is the reason why the self-energy and the spectral function are time translation invariant.

The self-energy $\Pi^R_{\mathbf{q}}(\omega)$, and consequently the spectral function $\rho_{\mathbf{q}}(\omega)$, are divergent and have to be renormalized. This can be done by the usual mass and wave function renormalization at zero temperature. In (4.10) $\omega^2_{\mathbf{q}}$ is replaced by $\omega^2_{\mathbf{q}(0)} = m_0^2 + \mathbf{q}^2$, where m_0 is the bare mass of the field Φ. The difference between bare and renormalized mass squared is determined by requiring that at zero temperature the spectral function has a pole at $\omega^2_{\mathbf{q}} = m^2 + \mathbf{q}^2$,

$$\omega^2_{\mathbf{q}} - \omega^2_{\mathbf{q}(0)} - \mathrm{Re}\Pi^R_{\mathbf{q}}(\omega_{\mathbf{q}})|_{T=0} = 0 \ . \tag{4.11}$$

Expanding the self-energy around around $\omega_{\mathbf{q}}$, a further divergence can be absorbed in a wave function renormalization constant,

$$\mathrm{Re}\Pi^R_{\mathbf{q}}(\omega) = \mathrm{Re}\Pi^R_{\mathbf{q}}(\omega_{\mathbf{q}})|_{T=0} + \left(1 - Z^{-1}\right)\left(\omega^2 - \omega^2_{\mathbf{q}}\right) + \mathrm{Re}\hat{\Pi}^R_{\mathbf{q}}(\omega) \ , \tag{4.12}$$

where $\mathrm{Re}\hat{\Pi}^R_{\mathbf{q}}(\omega)$ is the finite part and

$$Z^{-1} = 1 - \frac{1}{2\omega_{\mathbf{q}}} \frac{\partial \mathrm{Re}\Pi^R_{\mathbf{q}}(\omega)}{\partial\omega}\Big|_{\omega=\omega_{\mathbf{q}},T=0} \ . \tag{4.13}$$

The spectral function (4.10) now takes the form

$$\rho_{\mathbf{q}}(\omega) = Z \frac{-2Z\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega) + 2\omega\epsilon}{\left(\omega^2 - \omega^2_{\mathbf{q}} - Z\mathrm{Re}\hat{\Pi}^R_{\mathbf{q}}(\omega)\right)^2 + \left(Z\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega) + \omega\epsilon\right)^2} \ . \tag{4.14}$$

Introducing the renormalized field operator $\Phi_r = \sqrt{Z}\Phi$, one obtains the renormalized spectral function $\rho^r_{\mathbf{q}}(\omega) = Z\rho_{\mathbf{q}}(\omega)$ in terms of the renormalized self-energy $\Pi^{R,r}_{\mathbf{q}}(\omega) = Z\hat{\Pi}^R_{\mathbf{q}}(\omega)$,

$$\rho^r_{\mathbf{q}}(\omega) = \frac{-2\mathrm{Im}\Pi^{R,r}_{\mathbf{q}}(\omega) + 2\omega\epsilon}{\left(\omega^2 - \omega^2_{\mathbf{q}} - \mathrm{Re}\Pi^{R,r}_{\mathbf{q}}(\omega)\right)^2 + \left(\mathrm{Im}\Pi^{R,r}_{\mathbf{q}}(\omega) + \omega\epsilon\right)^2} \ . \tag{4.15}$$

The divergencies of spectral function and statistical propagator can be removed in the same way by mass and wave function renormalization at zero temperature. In the following we shall drop the superscript 'r' to keep the notation simple.

The spectral function describes a quasi-particle resonance at finite temperature with energy $\Omega_{\mathbf{q}}$,

$$\Omega_{\mathbf{q}}^2 - \omega_{\mathbf{q}}^2 - \mathrm{Re}\Pi_{\mathbf{q}}^R(\Omega_{\mathbf{q}}) = 0, \quad \Omega_{\mathbf{q}}^2|_{T=0} = \omega_{\mathbf{q}}^2 \ , \tag{4.16}$$

and decay width

$$\Gamma_{\mathbf{q}} \simeq -\frac{1}{\Omega_{\mathbf{q}}}\mathrm{Im}\Pi_{\mathbf{q}}^R(\Omega_{\mathbf{q}}) \ . \tag{4.17}$$

For simplicity, we have neglected the effect of $\mathrm{Im}\Pi_{\mathbf{q}}^R$ on the quasi-particle energy. The correction $\delta\Omega_{\mathbf{q}} = \mathcal{O}(\Gamma_{\mathbf{q}}^2)$ is evaluated in Section 6.

In a free theory $\mathrm{Im}\,\Pi_{\mathbf{q}}^R(\omega) = 0$, and (4.15) is a representation of the δ-function. The spectral function (4.9) then oscillates without damping, i.e., there are no dissipative effects. Dissipation arises either from Φ decays and inverse decays or, similar to Landau damping, from scattering processes with particles in the plasma. Which of these mechanisms dominates the dissipative effects and therefore the equilibration process depends on the position of the quasi-particle pole relative to the masses of particles in the thermal bath. A specific example will be discussed in Section 7. For small width the spectral function is well approximated by the Breit-Wigner function. The relevant formulae are collected in Appendix A.5.

4.2 Solution for the statistical propagator

We are now ready to solve the second Kadanoff-Baym equation (2.38) for the statistical propagator, which for initial time $t_i = 0$ is given by

$$(\partial_{t_1}^2 + \omega_{\mathbf{q}}^2)\Delta_{\mathbf{q}}^+(t_1,t_2) + \int_0^{t_1} dt'\Pi_{\mathbf{q}}^-(t_1-t')\Delta_{\mathbf{q}}^+(t',t_2) = \zeta(t_1,t_2) \ , \tag{4.18}$$

with

$$\zeta(t_1,t_2) = \int_0^{t_2} dt'\Pi_{\mathbf{q}}^+(t_1-t')\Delta_{\mathbf{q}}^-(t'-t_2) \ . \tag{4.19}$$

One easily verifies that the solution can be expressed as

$$\Delta_{\mathbf{q}}^+(t_1,t_2) = \hat{\Delta}_{\mathbf{q}}^+(t_1,t_2) + \int_0^{t_1} dt'\Delta_{\mathbf{q}}^-(t_1-t')\zeta(t',t_2) \ , \tag{4.20}$$

where $\hat{\Delta}_{\mathbf{q}}^+(t_1,t_2)$ satisfies the homogeneous equation

$$(\partial_{t_1}^2 + \omega_{\mathbf{q}}^2)\hat{\Delta}_{\mathbf{q}}^+(t_1,t_2) + \int_0^{t_1} dt'\Pi_{\mathbf{q}}^-(t_1-t')\hat{\Delta}_{\mathbf{q}}^+(t',t_2) = 0 \ . \tag{4.21}$$

The homogeneous equation is identical to (4.1), with t_2 playing the role of a parameter. We can therefore read off the general solution from (4.5),

$$\hat{\Delta}_{\mathbf{q}}^+(t_1,t_2) = A_{\mathbf{q}}(t_2)\dot{\Delta}_{\mathbf{q}}^-(t_1) + B_{\mathbf{q}}(t_2)\Delta_{\mathbf{q}}^-(t_1) \ . \tag{4.22}$$

Using the symmetry $\hat{\Delta}^+_{\mathbf{q}}(t_1,t_2) = \hat{\Delta}^+_{\mathbf{q}}(t_2,t_1)$, one obtains

$$A_{\mathbf{q}}(t_2)\dot{\Delta}^-_{\mathbf{q}}(t_1) + B_{\mathbf{q}}(t_2)\Delta^-_{\mathbf{q}}(t_1) = A_{\mathbf{q}}(t_1)\dot{\Delta}^-_{\mathbf{q}}(t_2) + B_{\mathbf{q}}(t_1)\Delta^-_{\mathbf{q}}(t_2) \ . \tag{4.23}$$

Together with the boundary conditions (2.31)-(2.33), $\Delta^-_{\mathbf{q}}(0) = \ddot{\Delta}^-_{\mathbf{q}}(0) = 0$ and $\dot{\Delta}^-_{\mathbf{q}}(0) = 1$, this implies

$$A_{\mathbf{q}}(t) = A_{\mathbf{q}}(0)\dot{\Delta}^-_{\mathbf{q}}(t) + B_{\mathbf{q}}(0)\Delta^-_{\mathbf{q}}(t) \ , \quad B_{\mathbf{q}}(t) = \dot{A}_{\mathbf{q}}(0)\dot{\Delta}^-_{\mathbf{q}}(t) + \dot{B}_{\mathbf{q}}(0)\Delta^-_{\mathbf{q}}(t) \ . \tag{4.24}$$

Inserting $A_{\mathbf{q}}(t)$ and $B_{\mathbf{q}}(t)$ in (4.23) and using the symmetry of $\hat{\Delta}^+_{\mathbf{q}}(t_1,t_2)$, one finds $B_{\mathbf{q}}(0) = \dot{A}_{\mathbf{q}}(0)$. The initial state of the system is therefore characterized by three constants, which can be chosen as

$$\begin{aligned}
\Delta^+_{\mathbf{q},\mathrm{in}} &= \Delta^+_{\mathbf{q}}(t_1,t_2)|_{t_1=t_2=0} = A_{\mathbf{q}}(0) \ , & (4.25)\\
\dot{\Delta}^+_{\mathbf{q},\mathrm{in}} &= \partial_{t_1}\Delta^+_{\mathbf{q}}(t_1,t_2)|_{t_1=t_2=0} = \partial_{t_2}\Delta^+_{\mathbf{q}}(t_1,t_2)|_{t_1=t_2=0} = B_{\mathbf{q}}(0) = \dot{A}_{\mathbf{q}}(0) \ , & (4.26)\\
\ddot{\Delta}^+_{\mathbf{q},\mathrm{in}} &= \partial_{t_1}\partial_{t_2}\Delta^+_{\mathbf{q}}(t_1,t_2)|_{t_1=t_2=0} = \dot{B}_{\mathbf{q}}(0) \ . & (4.27)
\end{aligned}$$

From Eqs. (4.20), (4.22), (4.24) and the initial conditions (7.5)-(7.7) we now obtain the full solution for the statistical propagator,

$$\begin{aligned}
\Delta^+_{\mathbf{q}}(t_1,t_2) &= \Delta^+_{\mathbf{q},\mathrm{in}}\dot{\Delta}^-_{\mathbf{q}}(t_1)\dot{\Delta}^-_{\mathbf{q}}(t_2) + \ddot{\Delta}^+_{\mathbf{q},\mathrm{in}}\Delta^-_{\mathbf{q}}(t_1)\Delta^-_{\mathbf{q}}(t_2)\\
&+ \dot{\Delta}^+_{\mathbf{q};\mathrm{in}}\left(\dot{\Delta}^-_{\mathbf{q}}(t_1)\Delta^-_{\mathbf{q}}(t_2) + \Delta^-_{\mathbf{q}}(t_1)\dot{\Delta}^-_{\mathbf{q}}(t_2)\right)\\
&+ \Delta^+_{\mathbf{q},\mathrm{mem}}(t_1,t_2) \ ,
\end{aligned} \tag{4.28}$$

where

$$\Delta^+_{\mathbf{q},\mathrm{mem}}(t_1,t_2) = \int_0^{t_1} dt' \int_0^{t_2} dt'' \Delta^-_{\mathbf{q}}(t_1-t')\Pi^+_{\mathbf{q}}(t'-t'')\Delta^-_{\mathbf{q}}(t''-t_2) \ . \tag{4.29}$$

This contribution to the statistical propagator, which is independent of the initial conditions, is often referred to as *memory integral*. It can be expressed in the form

$$\Delta^+_{\mathbf{q},\mathrm{mem}}(t_1,t_2) = -\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t_1-t_2)}\mathcal{H}^*_{\mathbf{q}}(t_1,\omega)\mathcal{H}_{\mathbf{q}}(t_2,\omega)\Pi^+_{\mathbf{q}}(\omega) \ , \tag{4.30}$$

where [17]

$$\mathcal{H}_{\mathbf{q}}(t,\omega) = \int_0^t d\tau e^{-i\omega\tau}\Delta^-_{\mathbf{q}}(\tau) \ . \tag{4.31}$$

The expression (4.30) will be the basis of our numerical analysis in Section 7.

5 Thermal equilibrium and quasi-particles

Let us now verify that the solution (4.28) for the statistical propagator approaches thermal equilibrium at late times. This means that the quantity

$$\Delta^+_{\mathbf{q}}(t,\omega) = \int_{-2t}^{2t} dy e^{i\omega y}\Delta^+_{\mathbf{q}}\left(t+\frac{y}{2}, t-\frac{y}{2}\right) \ , \tag{5.1}$$

which becomes a Fourier transform for $t \to \infty$, satisfies the KMS condition asymptotically,

$$\Delta^+_{\mathbf{q}}(\infty, \omega) = -\frac{i}{2} \coth\left(\frac{\beta\omega}{2}\right) \Delta^-_{\mathbf{q}}(\omega) \ . \tag{5.2}$$

For late times only the memory integral is relevant, since $\Delta^-_{\mathbf{q}}(t)$ and $\dot{\Delta}^-_{\mathbf{q}}(t)$ fall off exponentially for $t \gg 1/\Gamma$. One then obtains

$$\Delta^+_{\mathbf{q}}(\infty, \omega) = \Delta^+_{\mathbf{q},\mathrm{mem}}(\infty, \omega) = -|\mathcal{H}_{\mathbf{q}}(\infty, \omega)|^2 \Pi^+_{\mathbf{q}}(\omega) \ . \tag{5.3}$$

The quantity $\mathcal{H}_{\mathbf{q}}(\infty, \omega)$ is the Laplace transform of the spectral function,

$$\begin{aligned} \mathcal{H}_{\mathbf{q}}(\infty, \omega) &= \int_0^\infty d\tau e^{-i(\omega - i\epsilon)\tau} \Delta^-_{\mathbf{q}}(\tau) \\ &= \tilde{\Delta}^-_{\mathbf{q}}(i\omega + \epsilon) \\ &= \frac{1}{s^2 + \omega_q^2 + \tilde{\Pi}_{\mathbf{q}}(s)}\Big|_{s=i\omega+\epsilon} \\ &= -\frac{1}{\omega^2 - \omega_q^2 - \mathrm{Re}\Pi^R_{\mathbf{q}}(\omega) - i\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega)} \ , \end{aligned} \tag{5.4}$$

which yields

$$\begin{aligned} |\mathcal{H}_{\mathbf{q}}(\infty, \omega)|^2 &= \frac{1}{(\omega^2 - \omega_{\mathbf{q}}^2 - \mathrm{Re}\Pi^R_{\mathbf{q}}(\omega))^2 + (\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega))^2} \\ &= -\frac{\rho_{\mathbf{q}}(\omega)}{2\,\mathrm{Im}\,\Pi^R_{\mathbf{q}}(\omega)} \ . \end{aligned} \tag{5.5}$$

Inserting this expression into (5.3), using the KMS condition for the self-energy and (A.30),

$$\Pi^-_{\mathbf{q}}(\omega) = 2i\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega) \ ,$$

one obtains (cf. (4.9),(4.10)),

$$\begin{aligned} \Delta^+_{\mathbf{q}}(\infty, \omega) &= -\coth\left(\frac{\beta\omega}{2}\right) \frac{\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega)}{(\omega^2 - \omega_{\mathbf{q}}^2 - \mathrm{Re}\Pi^R_{\mathbf{q}}(\omega))^2 + (\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega))^2} \\ &= -\frac{i}{2} \coth\left(\frac{\beta\omega}{2}\right) \Delta^-_{\mathbf{q}}(\omega) \ . \end{aligned} \tag{5.6}$$

Hence, our solution for the statistical propagator indeed fulfills the KMS condition (2.34) in the limit $t \to \infty$, which proves that the system reaches thermal equilibrium. For a specific example the approach to equilibrium will be studied numerically in Section 7.

It is instructive to evaluate the statistical propagator in thermal equilibrium at equal times, i.e., $y = t_1 - t_2 = 0$,

$$\Delta^+_{\mathbf{q}}\big|_{y=0} = \frac{1}{2} \int_{-\infty}^\infty \frac{d\omega}{2\pi} \coth\left(\frac{\beta\omega}{2}\right) \rho_{\mathbf{q}}(\omega) \ . \tag{5.7}$$

For a free field one has

$$\rho_{\mathbf{q}}(\omega) = 2\pi \mathrm{sign}(\omega)\delta(\omega^2 - \omega_{\mathbf{q}}^2) \ , \tag{5.8}$$

which yields the well know result

$$\Delta^+_{\mathbf{q}}|_{y=0} = \frac{1}{\omega_{\mathbf{q}}}\left(\frac{1}{2} + n_{\mathrm{B}}(\omega_{\mathbf{q}})\right) \ , \tag{5.9}$$

with the temperature dependent Bose-Einstein distribution function

$$n_{\mathrm{B}}(\omega_{\mathbf{q}}) = \frac{1}{e^{\beta\omega_{\mathbf{q}}} - 1} \ . \tag{5.10}$$

Generically, the interaction with the thermal bath changes the energy $\omega_{\mathbf{q}}$ of a free particle to a temperature dependent complex energy $\hat{\Omega}_{\mathbf{q}}$ which appears as a pole of the spectral function $\rho_{\mathbf{q}}(\omega)$ and the integrand of (5.7). The spectral function then has two poles in the upper plane, $\hat{\Omega}_{\mathbf{q}}$ and $-\hat{\Omega}^*_{\mathbf{q}}$, which are determined by the condition

$$\hat{\Omega}_{\mathbf{q}} - \left(\omega_{\mathbf{q}}^2 + \Pi^R_{\mathbf{q}}\left(\hat{\Omega}_{\mathbf{q}}\right)\right)^{1/2} = 0 \ . \tag{5.11}$$

Assuming that the integral can be closed in the upper half-plane, one obtains for the statistical propagator in equilibrium,[1]

$$\Delta^+_{\mathbf{q}}|_{y=0} = \mathrm{Re}\left(\frac{1}{\hat{\Omega}_{\mathbf{q}}}\left(\frac{1}{2} + n_{\mathrm{B}}(\hat{\Omega}_{\mathbf{q}})\right)\right) \ . \tag{5.12}$$

Compared to (5.8), the Bose-Einstein distribution function has been replaced by the complex distribution function $n_{\mathrm{B}}(\hat{\Omega}_{\mathbf{q}})$.

At high temperatures, where $\beta\omega_{\mathbf{q}} \ll 1$, the Bose-Einstein distribution has a well-known infrared divergence,

$$n_{\mathrm{B}}(\omega_{\mathbf{q}}) \simeq \frac{1}{\beta\omega_{\mathbf{q}}} \gg 1 \ . \tag{5.13}$$

For quasi-particles, where $\omega_{\mathbf{q}}$ is replaced by $\hat{\Omega}_{\mathbf{q}} = \Omega_{\mathbf{q}} + i\Gamma_{\mathbf{q}}/2$, this divergence is cut off by the finite width,

$$|n_{\mathrm{B}}(\hat{\Omega}_{\mathbf{q}})| \simeq \frac{1}{|\beta(\Omega_{\mathbf{q}} + \frac{i}{2}\Gamma_{\mathbf{q}})|} \leq \frac{2}{\beta\Gamma_{\mathbf{q}}} \ , \tag{5.14}$$

which remains finite even if the real part $\Omega_{\mathbf{q}}$ vanishes.

Comparison of equations (5.9) and (5.12) suggests that in thermal equilibrium the Φ particles may form a gas of quasi-particles. This question can be clarified by evaluating energy density and pressure of the Φ particles. Since the expectation value of Φ vanishes, one obtains from the energy momentum tensor[2]

$$T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \eta_{\mu\nu}L \tag{5.15}$$

[1] Here we restrict ourselves to the case where there are no additional poles.

[2] We use the convention diag $(\eta_{\mu\nu}) = (1, -1, -1, -1)$.

for the contribution of a mode with momentum $\mathbf{q}$ to energy density and pressure,

$$\epsilon_{\mathbf{q}} = \langle T_{00} \rangle|_{\mathbf{q}} = \frac{1}{2} \langle \dot{\Phi}^2 + (\vec{\nabla}\Phi)^2 + m^2\Phi^2 \rangle|_{\mathbf{q}} \ , \tag{5.16}$$

$$p_{\mathbf{q}} = \langle T_{ii} \rangle|_{\mathbf{q}} = \langle \frac{1}{3} (\nabla\Phi)^2 + \frac{1}{2} (\dot{\Phi}^2 - (\nabla\Phi)^2 - m^2\Phi^2) \rangle|_{\mathbf{q}} \ . \tag{5.17}$$

This yields for the energy density

$$\begin{aligned} \epsilon_{\mathbf{q}}(\infty) &= \frac{1}{2} \left(\partial_{t_1}\partial_{t_2} + \omega_{\mathbf{q}}^2\right) \Delta_{\mathbf{q}}^+(t_1, t_2)\big|_{t_1=t_2=\infty} \\ &= \frac{1}{2} \left(\Omega_{\mathbf{q}}^2 + \omega_{\mathbf{q}}^2\right) \frac{1}{\Omega_{\mathbf{q}}} \left(\frac{1}{2} + n_{\mathrm{B}}(\Omega_{\mathbf{q}}) \right) \ , \end{aligned} \tag{5.18}$$

and for the pressure

$$\begin{aligned} p_{\mathbf{q}}(\infty) &= \left(\frac{1}{3} \mathbf{q}^2 + \frac{1}{2} \left(\partial_{t_1}\partial_{t_2} - \omega_{\mathbf{q}}^2\right) \right) \Delta_{\mathbf{q}}^+(t_1, t_2)\big|_{t_1=t_2=\infty} \\ &= \left(\frac{1}{3} \mathbf{q}^2 + \frac{1}{2} \left(\Omega_{\mathbf{q}}^2 - \omega_{\mathbf{q}}^2\right) \right) \frac{1}{\Omega_{\mathbf{q}}} \left(\frac{1}{2} + n_{\mathrm{B}}(\Omega_{\mathbf{q}}) \right) \ , \end{aligned} \tag{5.19}$$

where, for simplicity, we have neglected the quasi-particle width.

In summary, the energy momentum tensor in thermal equilibrium can be expressed as sum of a quasi-particle gas contribution and a temperature dependent 'vacuum' term,

$$\langle T_{\mu\nu} \rangle|_{\mathbf{q}} = u_\mu u_\nu \left(\epsilon_{\mathbf{q}}^{\mathrm{QP}} + p_{\mathbf{q}}^{\mathrm{QP}} \right) - \eta_{\mu\nu} p_{\mathbf{q}}^{\mathrm{QP}} + \eta_{\mu\nu} \kappa_{\mathbf{q}}^{\mathrm{VAC}} \ . \tag{5.20}$$

Here $u^\mu = (1, \vec{0})$ is the 4-velocity of the thermal bath, and

$$\epsilon_{\mathbf{q}}^{\mathrm{QP}} = \Omega_{\mathbf{q}} \left(\frac{1}{2} + n_{\mathrm{B}}(\Omega_{\mathbf{q}}) \right) \ , \tag{5.21}$$

$$p_{\mathbf{q}}^{\mathrm{QP}} = \frac{1}{3} \frac{\mathbf{q}^2}{\Omega_{\mathbf{q}}} \left(\frac{1}{2} + n_{\mathrm{B}}(\Omega_{\mathbf{q}}) \right) \ , \tag{5.22}$$

$$\kappa_{\mathbf{q}}^{\mathrm{VAC}} = \frac{\omega_{\mathbf{q}}^2 - \Omega_{\mathbf{q}}^2}{2\Omega_{\mathbf{q}}} \left(\frac{1}{2} + n_{\mathrm{B}}(\Omega_{\mathbf{q}}) \right) \ . \tag{5.23}$$

Energy density and pressure of the quasi-particle gas agree with the corresponding expressions for a free gas, with the energy $\omega_{\mathbf{q}}$ of a free particle replaced by the quasi-particle energy $\Omega_{\mathbf{q}}$. The 'vacuum contribution' $\kappa_{\mathbf{q}}^{\mathrm{VAC}}$ vanishes for $\Omega_{\mathbf{q}} = \omega_{\mathbf{q}}$. For large thermal effects, i.e. $\Omega_{\mathbf{q}} \gg \omega_{\mathbf{q}}$ or $\Omega_{\mathbf{q}} \ll \omega_{\mathbf{q}}$, the equation of state differs significantly from the one of a free gas. Note that for $\Omega_{\mathbf{q}}^2 < \omega_{\mathbf{q}}^2$, the pressure can even become negative!

6 Comparison with Boltzmann equations

The time evolution of nonequilibrium systems is usually studied by means of Boltzmann equations for particle number densities. However, this notion does not have a well defined physical meaning in a nonequilibrium process. For a dilute, weakly coupled gas the number density of 'free particles' may be a good approximation, and in some cases the the effect of a medium can be taken into account by considering quasi-particles. In general, however, one has to study the time evolution of Green's functions, in particular if quantum interferences are important.

In order to determine the range of validity of the Boltzmann approximation, we shall consider in this section the time evolution of an observable, the energy density. The exact expression can be obtained from the statistical propagator, and approximations are given by solutions of Boltzmann equations. In this way, the description of the nonequilibrium process by means of Green's functions on the one hand, and Boltzmann equations on the other hand, can be directly compared, based on the same observable.

Consider the Boltzmann equation for a dilute gas of Φ particles. The competition between a gain and a loss term determines the change of the particle number density [22],

$$\partial_t n_{\mathbf{q}}(t) = (1 + n_{\mathbf{q}}(t))\gamma_{\mathbf{q}}^{<} - n_{\mathbf{q}}(t)\gamma_{\mathbf{q}}^{>} \ , \tag{6.1}$$

where production and decay rates satisfy the KMS relation and are obtained from the self-energy of the field Φ,

$$\gamma_{\mathbf{q}}^{>} = e^{-\beta\omega_{\mathbf{q}}}\gamma_{\mathbf{q}}^{<} \equiv n_{\mathrm{B}}(\omega_{\mathbf{q}})\gamma_{\mathbf{q}} \ , \tag{6.2}$$

$$\gamma_{\mathbf{q}} = -\frac{\mathrm{Im}\Pi_{\mathbf{q}}^{R}(\omega_{\mathbf{q}})}{\omega_{\mathbf{q}}} \ . \tag{6.3}$$

Using these relations, the Boltzmann equation (6.1) can be written in the form

$$\partial_t n_{\mathbf{q}}(t) = -\gamma_{\mathbf{q}}(n_{\mathbf{q}}(t) - n_{\mathrm{B}}(\omega_{\mathbf{q}})) \ , \tag{6.4}$$

with the obvious solution

$$n_{\mathbf{q}}(t) = n_{\mathrm{B}}(\omega_{\mathbf{q}}) + (n_{\mathbf{q}}(0) - n_{\mathrm{B}}(\omega_{\mathbf{q}}))\, e^{-\gamma_{\mathbf{q}}t} \ . \tag{6.5}$$

For comparison with the Kadanoff-Baym equations we now consider instead of the number density the energy density of a mode with momentum $\mathbf{q}$, normalized to the energy of a single quantum,

$$\hat{\epsilon}_{\mathbf{q}}(t) \equiv \frac{\epsilon_{\mathbf{q}}(t)}{\omega_{\mathbf{q}}} = \frac{1}{2} + n_{\mathbf{q}}(t) \ . \tag{6.6}$$

The deviation from the equilibrium density,

$$\hat{\epsilon}_{\mathbf{q}}(t) = \hat{\epsilon}_{\mathbf{q}}^{\mathrm{free}} + \delta\hat{\epsilon}_{\mathbf{q}}(t) \ , \tag{6.7}$$

with

$$\hat{\epsilon}_{\mathbf{q}}(\infty) \equiv \hat{\epsilon}_{\mathbf{q}}^{\mathrm{free}} = \frac{1}{2} + n_{\mathrm{B}}(\omega_{\mathbf{q}}) \ , \tag{6.8}$$

satisfies the differential equation

$$(\partial_t + \gamma_{\mathbf{q}})\delta\hat{\epsilon}_{\mathbf{q}}(t) = 0 \ . \tag{6.9}$$

The modification of the spectral function in a thermal bath (cf. (4.10)) suggests to replace the equilibrium value and the evolution equation for the energy density by the expressions

$$\hat{\epsilon}_{\mathbf{q}}(t) = \hat{\epsilon}^{\mathrm{QP}}_{\mathbf{q}} + \delta\hat{\epsilon}_{\mathbf{q}}(t) \ , \quad \hat{\epsilon}^{\mathrm{QP}}_{\mathbf{q}} = \frac{1}{2} + n_{\mathrm{B}}(\Omega_{\mathbf{q}}) \ , \tag{6.10}$$

and

$$(\partial_t + \Gamma_{\mathbf{q}})\delta\hat{\epsilon}_{\mathbf{q}}(t) = 0 \ , \tag{6.11}$$

where the quasi-particle width is given by (cf. Appendix A.5)

$$\Gamma_{\mathbf{q}} = -Z_{\mathbf{q}}\frac{\mathrm{Im}\Pi^R_{\mathbf{q}}(\Omega_{\mathbf{q}})}{\Omega_{\mathbf{q}}} \ , \quad Z^{-1}_{\mathbf{q}} = 1 - \frac{1}{2\Omega_{\mathbf{q}}}\frac{\partial}{\partial\omega}\mathrm{Re}\Pi^R_{\mathbf{q}}(\omega)\big|_{\Omega_{\mathbf{q}}} \ . \tag{6.12}$$

As long as the interaction of the field Φ with the thermal bath can be treated perturbatively, the difference between solutions of the two Boltzmann equations for particles and quasi-particles, respectively, should be small. When the quasi-particle width becomes large, however, the use of first-order differential equations, which are local in time, becomes clearly questionable.

As discussed in Section 5, the exact time dependence of the energy density can be directly obtained from the statistical propagator,

$$\hat{\epsilon}_{\mathbf{q}}(t) = \frac{1}{2\omega_{\mathbf{q}}}\left(\partial_{t_1}\partial_{t_2} + \omega^2_{\mathbf{q}}\right)\Delta^+_{\mathbf{q}}(t_1,t_2)\big|_{t_1=t_2=t} \ ,$$

which satisfies the Kadanoff-Baym equation (2.15),

$$(\partial^2_{t_1} + \omega^2_{\mathbf{q}})\Delta^+_{\mathbf{q}}(t_1,t_2) + \int_0^{t_1} dt'\Pi^-_{\mathbf{q}}(t_1-t')\Delta^+_{\mathbf{q}}(t',t_2) = \int_0^{t_2} dt'\Pi^+_{\mathbf{q}}(t_1-t')\Delta^-_{\mathbf{q}}(t'-t_2) \ . \tag{6.13}$$

For large times, $t \gg 1/\Gamma_{\mathbf{q}}$, the dependence on the initial values at $t_i = 0$ can be neglected, and one obtains

$$(\partial^2_{t_1} + \omega^2_{\mathbf{q}})\Delta^+_{\mathbf{q}}(t_1,t_2) + \int_{-\infty}^{\infty} dt' \left(\Pi^R_{\mathbf{q}}(t_1-t')\Delta^+_{\mathbf{q}}(t',t_2) + i\Pi^+_{\mathbf{q}}(t_1-t')\Delta^A_{\mathbf{q}}(t'-t_2)\right) = 0 \ . \tag{6.14}$$

Changing time variables,

$$t = \frac{t_1+t_2}{2} \ , \quad y = t_1 - t_2 \ , \quad \Delta^+_{\mathbf{q}}(t;y) \equiv \Delta^+_{\mathbf{q}}(t_1,t_2) \ , \tag{6.15}$$

and expanding,

$$\Delta^+_{\mathbf{q}}\left(\frac{t'+t_2}{2}; t'-t_2\right) = \Delta^+_{\mathbf{q}}(t;t'-t_2) + \frac{t'-t_1}{2}\partial_t\Delta^+_{\mathbf{q}}(t;t'-t_2) + \dots \ , \tag{6.16}$$

one finds for the Fourier transforms with respect to the time differences,

$$\left(\frac{1}{4}\partial^2_t - i\omega\partial_t - \omega^2 + \omega^2_{\mathbf{q}}\right)\Delta^+_{\mathbf{q}}(t;\omega)$$

$$= \quad -\Pi^R_{\mathbf{q}}(\omega)\Delta^+_{\mathbf{q}}(t;\omega) - i\Pi^+_{\mathbf{q}}(\omega)\Delta^A_{\mathbf{q}}(t;\omega) - \frac{i}{2}\frac{\partial \Pi^R_{\mathbf{q}}(\omega)}{\partial\omega}\frac{\partial \Delta^+_{\mathbf{q}}(t;\omega)}{\partial t} \ . \tag{6.17}$$

Using the relations (A.17) - (A.31), one obtains from the real and the imaginary part of this complex equation two equations for the real quantity $\Delta^+_{\mathbf{q}}(t,\omega)$,

$$\left(\frac{1}{4}\partial_t^2 - \omega^2 + \omega^2_{\mathbf{q}}\right)\Delta^+_{\mathbf{q}}(t,\omega) = -\mathrm{Re}\Pi^R_{\mathbf{q}}(\omega)\Delta^+_{\mathbf{q}}(t,\omega) + \Pi^+_{\mathbf{q}}(\omega)\mathrm{Im}\Delta^A_{\mathbf{q}}(t,\omega)$$
$$+ \frac{1}{2}\frac{\partial \mathrm{Im}\Pi^R_{\mathbf{q}}(\omega)}{\partial\omega}\frac{\partial \Delta^+_{\mathbf{q}}(t,\omega)}{\partial t} + \dots \ , \tag{6.18}$$
$$\omega\frac{\partial}{\partial t}\Delta^+_{\mathbf{q}}(t,\omega) = \mathrm{Im}\Pi^R_{\mathbf{q}}(\omega)\Delta^+_{\mathbf{q}}(t,\omega) + \Pi^+_{\mathbf{q}}(\omega)\mathrm{Re}\Delta^A_{\mathbf{q}}(t,\omega)$$
$$+ \frac{1}{2}\frac{\partial \mathrm{Re}\Pi^R_{\mathbf{q}}(\omega)}{\partial\omega}\frac{\partial \Delta^+_{\mathbf{q}}(t,\omega)}{\partial t} + \dots \ , \tag{6.19}$$

where the dots indicate neglected higher-order terms.

Consider now an expansion around the equilibrium solution,

$$\Delta^+_{\mathbf{q}}(t,\omega) = \Delta^+_{\mathbf{q}}(\omega) + \delta\Delta^+_{\mathbf{q}}(t,\omega) \ . \tag{6.20}$$

From equation (6.19) one reads off

$$\mathrm{Im}\,\Pi^R_{\mathbf{q}}(\omega)\Delta^+_{\mathbf{q}}(\omega) + \Pi^+_{\mathbf{q}}(\omega)\,\mathrm{Re}\,\Delta^A_{\mathbf{q}}(\omega) = 0 \ , \tag{6.21}$$

which is satisfied because of (A.30), (A.22) and the KMS conditions (2.34) and (2.35).

The first equation (6.18) yields for the equilibrium solution,

$$\left(\omega^2 - \omega^2_{\mathbf{q}} - \mathrm{Re}\Pi^R_{\mathbf{q}}(\omega)\right)\Delta^+_{\mathbf{q}}(\omega) = -\Pi^+_{\mathbf{q}}(\omega)\,\mathrm{Im}\,\Delta^A_{\mathbf{q}}(\omega) \ . \tag{6.22}$$

In the zero-width limit, this equation is fulfilled for

$$\omega = \Omega_{\mathbf{q}} = \sqrt{\omega^2_{\mathbf{q}} + \mathrm{Re}\,\Pi^R_{\mathbf{q}}(\Omega_{\mathbf{q}})} \ . \tag{6.23}$$

The finite width leads to a correction,

$$\omega = \Omega_{\mathbf{q}} + \delta\Omega_{\mathbf{q}} \ . \tag{6.24}$$

Expanding (6.22) in $\delta\Omega_{\mathbf{q}}$, one obtains to leading order

$$2\Omega_{\mathbf{q}}\delta\Omega_{\mathbf{q}}\Delta^+_{\mathbf{q}}(\Omega_{\mathbf{q}}) + \Pi^+(\Omega_{\mathbf{q}})\,\mathrm{Im}\,\Delta^A_{\mathbf{q}}(\Omega_{\mathbf{q}}) = 0 \ , \tag{6.25}$$

which implies

$$\delta\Omega_{\mathbf{q}} = -\frac{\Gamma_{\mathbf{q}}(\Omega_{\mathbf{q}})}{2}\frac{\mathrm{Im}\,\Delta^A_{\mathbf{q}}(\Omega_{\mathbf{q}})}{\mathrm{Re}\,\Delta^A_{\mathbf{q}}(\Omega_{\mathbf{q}})} \ . \tag{6.26}$$

We can use the free spectral function,

$$\Delta^-_{\mathbf{q}}(\omega) = 2\pi\mathrm{sign}(\omega)\delta(\omega^2 - \Omega^2_{\mathbf{q}}) \ , \tag{6.27}$$

to evaluate $\mathrm{Im}\,\Delta^A_{\mathbf{q}}(\Omega_{\mathbf{q}})$ to leading order in $\Gamma_{\mathbf{q}}$,

$$\mathrm{Im}\,\Delta^A(\Omega_{\mathbf{q}}) = -\frac{1}{2\pi}\mathcal{P}\int \frac{\rho(\omega')}{\omega' - \Omega_q} d\omega' = \frac{1}{4\Omega^2_{\mathbf{q}}} \ . \tag{6.28}$$

Using (6.26), (A.21), (4.10) and (A.41) we finally obtain

$$\delta\Omega_{\mathbf{q}} = \frac{1}{8}\frac{\Gamma^2_{\mathbf{q}}}{\Omega_{\mathbf{q}}} \ . \tag{6.29}$$

Hence, for $\Gamma_{\mathbf{q}} \ll \Omega_{\mathbf{q}}$, the leading term in the derivative expansion indeed implies $\omega = \Omega_{\mathbf{q}}$. If finite width effects are not negligible, however, off-shell effects become important and the derivative expansion becomes unreliable.

Inserting $\omega = \Omega_{\mathbf{q}}$ in the first-order differential equation (6.19), one obtains for the departure from equilibrium of the statistical propagator,

$$\left(\left(1 - \frac{1}{2\Omega_{\mathbf{q}}}\frac{\partial}{\partial\omega}\mathrm{Re}\Pi^R_{\mathbf{q}}(\omega)\big|_{\Omega_{\mathbf{q}}}\right)\frac{\partial}{\partial t} - \frac{1}{\Omega_{\mathbf{q}}}\mathrm{Im}\Pi^R_{\mathbf{q}}(\Omega_{\mathbf{q}})\right)\delta\Delta^+_{\mathbf{q}}(t;\Omega_{\mathbf{q}}) = 0 \ . \tag{6.30}$$

Hence, $\partial_t\delta\Delta^+_{\mathbf{q}}(t;\Omega_{\mathbf{q}}) = \mathcal{O}(\mathrm{Im}\Pi^R_{\mathbf{q}})$, and to this order Eq. (6.18) is also satisfied.

We can now evaluate the energy density

$$\begin{aligned}\hat{\epsilon}_{\mathbf{q}}(t) &= \frac{1}{2\omega_{\mathbf{q}}}\left(\partial_{t_1}\partial_{t_2} + \omega^2_{\mathbf{q}}\right)\Delta^+_{\mathbf{q}}(t_1,t_2)\big|_{t_1=t_2=t} \\ &= \frac{1}{2\omega_{\mathbf{q}}}\int_{-\infty}^{\infty} d\omega\left(\frac{1}{4}\partial_t^2 + \omega^2 + \omega^2_{\mathbf{q}}\right)\Delta^+_{\mathbf{q}}(t;\omega) \ ,\end{aligned} \tag{6.31}$$

which approaches the equilibrium value

$$\hat{\epsilon}_{\mathbf{q}}(\infty) \equiv \hat{\epsilon}^{\mathrm{full}}_{\mathbf{q}} = \frac{\Omega^2_{\mathbf{q}} + \omega^2_{\mathbf{q}}}{2\omega_{\mathbf{q}}\Omega_{\mathbf{q}}}\left(\frac{1}{2} + n_{\mathrm{B}}(\Omega_{\mathbf{q}})\right) \ . \tag{6.32}$$

As already discussed in the previous section, the true equilibrium value of the energy density does not correspond to a gas of quasi-particles,

$$\hat{\epsilon}^{\mathrm{full}}_{\mathbf{q}} \neq \hat{\epsilon}^{\mathrm{QP}}_{\mathbf{q}} \ . \tag{6.33}$$

From Eqs. (6.30), (A.41) and (A.42) one obtains for the deviation from the equilibrium value,

$$\left(\partial_t + \Gamma_{\mathbf{q}}\right)\hat{\epsilon}_{\mathbf{q}}(t) = 0 \ , \tag{6.34}$$

which is identical to the Boltzmann equation (6.11) for quasi-particles.

In summary, we have obtained the following conditions under which the approach to equilibrium can be described by Boltzmann equations. For a dilute, weakly coupled gas the ordinary Boltzmann equation for a number density is sufficient, which approaches the Bose-Einstein distribution for a gas of free particles. When interactions with a thermal

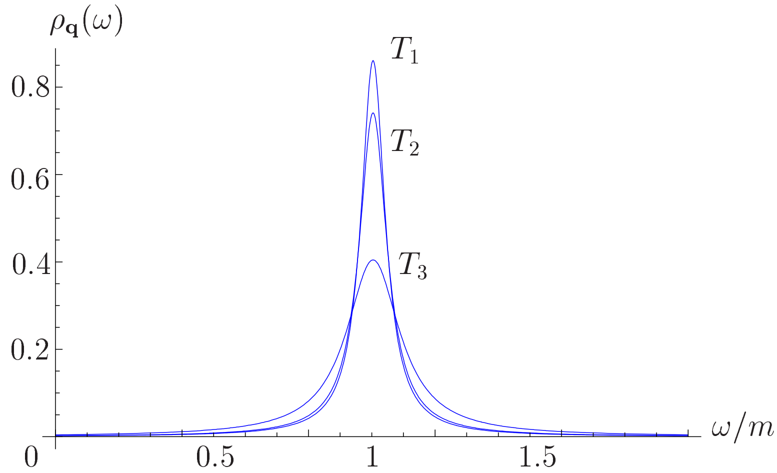

Figure 4: Spectral function $\rho_{\mathbf{q}}(\omega)$ for $\mathbf{q} = 0$; case (a) with masses $m_1 = m_2 = 0.2m$ and temperatures $T_1 = 0.1m$, $T_2 = 0.2m$, $T_3 = 0.5m$.

bath significantly change the spectral function, a Boltzmann equation for quasi-particles describes the approach to equilibrium as long as the quasi-particle width can be neglected. However, the equilibrium value of the energy density is different from the one for a gas of quasi-particles. Finally, when the width cannot be neglected and off-shell effects become significant, a linear evolution equation of 'Boltzmann type', which is local in time, is no longer adequate. Instead, the dynamics is non-local in time, and one has to solve Kadanoff-Baym equations.

7 A thermal bath of scalars

So far we have performed a very general analysis, and the only approximation has been to neglect the backreaction of the field Φ on the thermal bath. Furthermore, we have restricted our discussion to the case that Φ is linearly coupled to the bath via an interaction term $g\Phi\mathcal{O}(\chi)$ (cf. (3.2)). In general, χ represents an arbitrary number of bosonic or fermionic fields with arbitrary couplings including gauge interactions. In order to illustrate the results of the previous sections, we now consider a toy model (cf. [17, 22]), where the quanta of two massive scalar fields represent the thermal bath. The full Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2 + \sum_{i=1}^{2}\left(\frac{1}{2}\partial_\mu\chi_i\partial^\mu\chi_i - \frac{1}{2}m_i^2\chi_i^2\right) + g\Phi\chi_1\chi_2 + \mathcal{L}_{\chi\mathrm{int}}\,. \tag{7.1}$$

Note that the coupling g has the dimension of mass. In the following we shall neglect self-interaction of the χ fields and use free thermal propagators for simplicity.

We consider two cases: (a) $m \gg m_1, m_2$ and (b) $m_2 \gg m, m_1$. In the first case, dissipation is dominated by Φ decays and inverse decays, $\Phi \leftrightarrow \chi_1\chi_2$, whereas in the second one χ_2 decays and inverse decays, $\chi_2 \leftrightarrow \Phi\chi_1$, are most important.

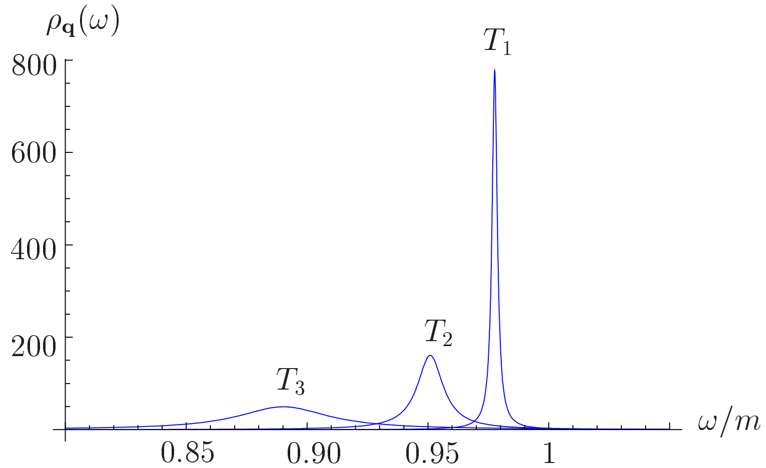

Figure 5: Spectral function $\rho_{\mathbf{q}}(\omega)$ for $\mathbf{q} = 0$; case (b) with masses $m_1 = m$, $m_2 = 5m$ and temperatures $T_1 = m$, $T_2 = 2m$, $T_3 = 5m$.

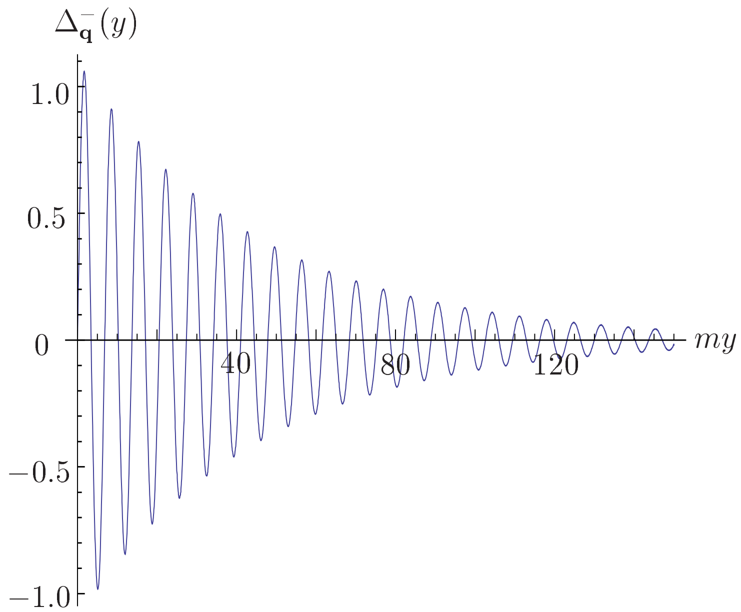

Figure 6: Spectral function $\Delta^-_{\mathbf{q}}(y)$ for $\mathbf{q} = 0$; case (b) with masses $m_1 = m$, $m_2 = 5m$ and $T = 10m$.

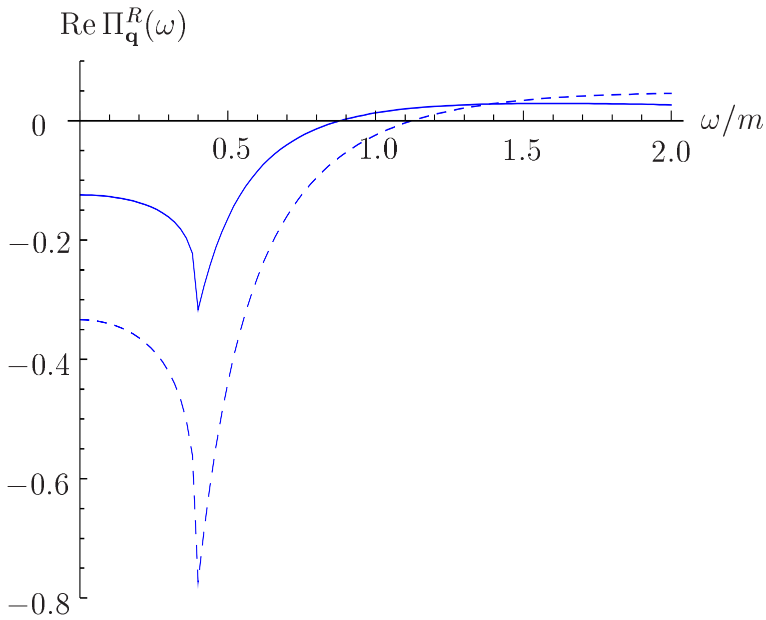

Figure 7: Real part of the self-energy $\Pi^R_{\mathbf{q}}(\omega)$ for $\mathbf{q}=0$; case (a) with masses $m_1 = m_2 = 0.2m$ and temperatures $T_1 = 0.5m$ (solid) and $T_2 = m$ (dashed).

In both cases the imaginary part of the self-energy is known analytically [17, 23]. The relevant formulae are collected in Appendix A.3. For $m \gg m_1, m_2$, the decay width of Φ at zero temperature is given by

$$\Gamma = \frac{1}{16\pi}\left(\frac{g}{m}\right)^2 m \ . \tag{7.2}$$

To illustrate thermal effects we shall use a rather large coupling which corresponds to $\Gamma/m = 0.1$.

The spectral function $\rho_{\mathbf{q}}(\omega)$ (cf. (4.15)) is shown in Figures 4 and 5 for the two mass patterns (a) and (b), respectively. In case (a), $\Pi^R_{\mathbf{q}}$ has an imaginary part at zero temperature. The width is large, and already at small temperatures the quasi-particle profile becomes broad. On the contrary, in case (b) the zero-temperature width is zero and the finite-temperature width is small. Hence, the quasi-particle profile becomes broad only at much larger temperatures. The spectral function $\Delta^-_{\mathbf{q}}(y)$ is the Fourier transform of $i\rho_{\mathbf{q}}(\omega)$. As Figure 6 illustrates, it approximately represents a damped oscillation with frequency $\Omega_{\mathbf{q}}$ and damping rate $\Gamma_{\mathbf{q}}$.

It is interesting that thermal corrections can increase or decrease the particle mass m. Whether the quasi-particle peak moves to the right or to the left depends on the position of the zero-temperature pole relative to the branch cuts, and it also depends on the temperature. This can be seen by considering the real part of the self-energy, which is displayed for two different temperatures in Fig. 7 for case (a). For the smaller temperature one has $\mathrm{Re}\,\Pi^R_{\mathbf{q}=\mathbf{0}}(m) > 0$, whereas $\mathrm{Re}\,\Pi^R_{\mathbf{q}=\mathbf{0}}(m) < 0$ holds for the larger temperature, which corresponds to a shift of the particle mass to the right and to the left, respectively.

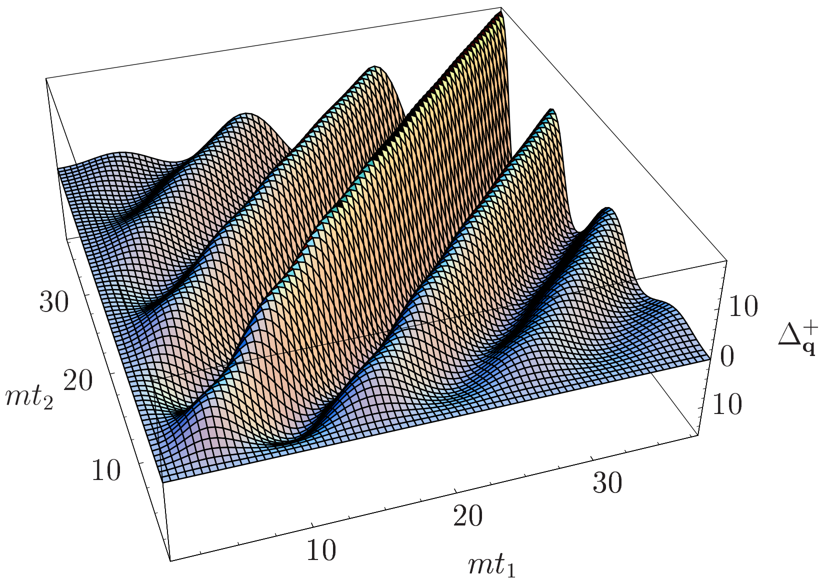

Figure 8: Statistical propagator $\Delta^+_{\mathbf{q}}(t_1, t_2)$ for $\mathbf{q} = 0$; case (b) with masses $m_1 = m$, $m_2 = 5m$ and $T = 10m$.

The statistical propagator $\Delta^+_{\mathbf{q}}(t_1, t_2)$ depends on the initial conditions. The most general gaussian initial density matrix has five free parameters (cf. [13]). We consider the simplest case of a free field density matrix and vanishing mean values Φ and $\dot{\Phi}$, which implies for each momentum mode,

$$\Phi_{\mathbf{q},\mathrm{in}} = 0 \ , \tag{7.3}$$

$$\dot{\Phi}_{\mathbf{q},\mathrm{in}} = 0 \ , \tag{7.4}$$

$$\Delta^+_{\mathbf{q},\mathrm{in}} = \Delta^+_{\mathbf{q}}(t_1, t_2)|_{t_1=t_2=0} = \frac{1}{\omega_{\mathbf{q}}}\left(\frac{1}{2} + n_{\mathbf{q}}\right) \ , \tag{7.5}$$

$$\dot{\Delta}^+_{\mathbf{q},\mathrm{in}} = \partial_{t_1}\Delta^+_{\mathbf{q}}(t_1, t_2)|_{t_1=t_2=0} = \partial_{t_2}\Delta^+_{\mathbf{q}}(t_1, t_2)|_{t_1=t_2=0} = 0 \ , \tag{7.6}$$

$$\ddot{\Delta}^+_{\mathbf{q},\mathrm{in}} = \partial_{t_1}\partial_{t_2}\Delta^+_{\mathbf{q}}(t_1, t_2)|_{t_1=t_2=0} = \omega_{\mathbf{q}}\left(\frac{1}{2} + n_{\mathbf{q}}\right) \ . \tag{7.7}$$

The initial state of the system is now characterized by only one parameter $n_{\mathbf{q}}$ which corresponds to an initial number density for a free field.

The general solution (4.28) for the statistical propagator $\Delta^+_{\mathbf{q}}(t_1, t_2)$ is shown in Figure 8. For fixed $t = (t_1+t_2)/2$ one sees damped oscillations in $y = t_1 - t_2$. The amplitude increases with increasing time t, as illustrated by Figure 9. For fixed $y = t_1 - t_2$ one observes the approach to equilibrium with increasing $t = (t_1+t_2)/2$. For large times the departure from

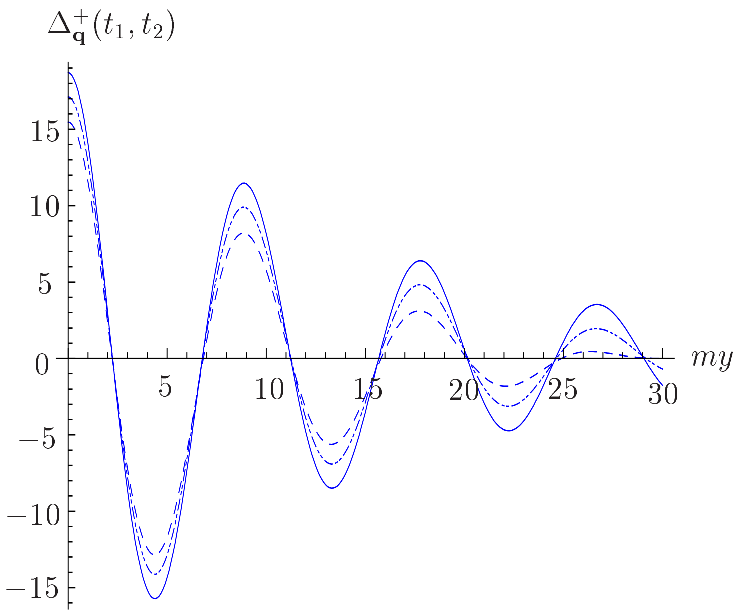

Figure 9: Statistical propagator $\Delta^+_{\mathbf{q}}(t_1, t_2)$ as function of $y = t_1 - t_2$ for $\mathbf{q} = 0$; case (b) with $m_1 = m$, $m_2 = 5m$, $T = 10m$ and three values of $t = (t_1 + t_2)/2$: $mt = 15$ (dashed line), $mt = 20$ (dotted-dashed), $mt = 60$ (solid).

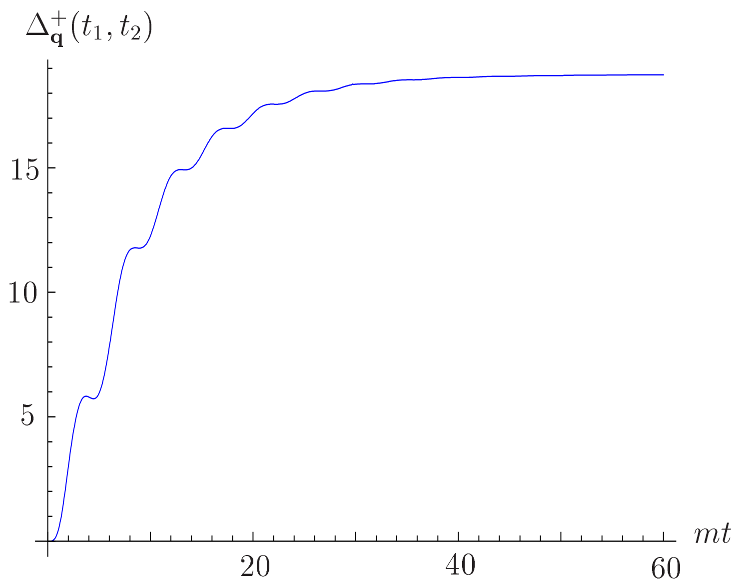

Figure 10: Statistical propagator $\Delta^+_{\mathbf{q}}(t_1, t_2)$ as function of $t = (t_1 + t_2)/2$ for $y = 0$, $\mathbf{q} = 0$; case (b) with masses $m_1 = m$, $m_2 = 5m$ and $T = 10m$.

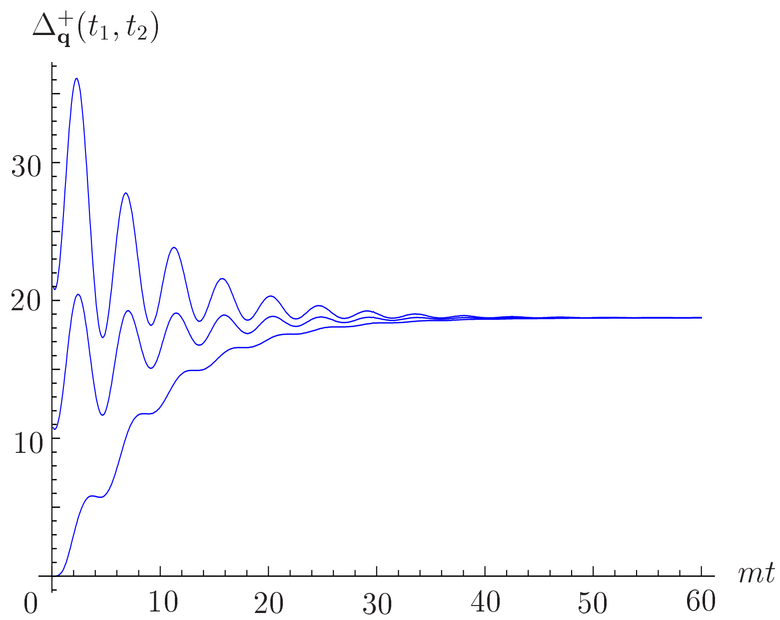

Figure 11: Statistical propagator $\Delta^+_{\mathbf{q}}(t_1, t_2)$ with $\mathbf{q} = 0$ as function of $t = (t_1 + t_2)/2$ for $y = 0$ and different initial conditions; case (b) with masses $\mathbf{q} = 0$, $m_1 = m$, $m_2 = 5m$ and $T = 10m$.

equilibrium is described by a first-order differential equation, and it decreases exponentially. At small times the evolution is governed by a second-order differential equation, which leads to the oscillations visible in Figure 10. The independence of the equilibrium solution from the initial conditions is illustrated by Figure 11. The memory of the initial conditions is lost at times $t > 1/\Gamma$.

Finally, it is important to recall that the equilibrium value of the energy differs from the one obtained in the Boltzmann approximation. This is illustrated in Figure 12 where the different contributions to the energy are compared as functions of temperature. For the chosen parameters the particle and quasi-particle energies are indistinguishable. The 'vacuum contribution' is positive, which means that the total energy is larger than the particle/quasi-particle one. The reason is that for the chosen parameters thermal corrections decrease the particle mass. For other parameter choices the 'vacuum contribution' can have opposite sign.

8 Conclusions and outlook

We have studied the approach to equilibrium for a real scalar field coupled to a large thermal bath. We have computed the exact two-point functions, the spectral function and the statistical propagator, for arbitrary initial conditions. This is possible for a thermal bath with many degrees of freedom such that the backreaction of the scalar field can be neglected.

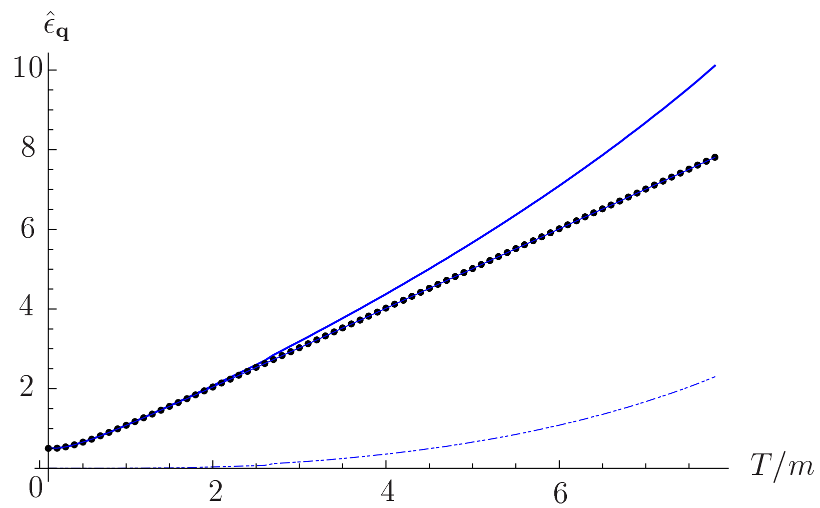

Figure 12: Energy density $\hat{\epsilon}_{\mathbf{q}} = \epsilon_{\mathbf{q}}/\omega_{\mathbf{q}}$ as function of temperature for $\mathbf{q} = 0$; case (b) with masses $m_1 = m$, $m_2 = 5m$: total energy density (solid), particle and quasi-particle energy densities (dotted), and 'vacuum' energy density (dashed).

The self-energy representing the thermal bath is time-translation invariant. We have shown that this is also the case for the spectral function, whereas the statistical propagator depends on two time coordinates, t_1 and t_2, and also the time t_i where the initial conditions are specified.

We have obtained the two-point functions by solving the Kadanoff-Baym equations, which turned out to be equivalent to solving a stochastic Langevin equation. As expected, the relaxation time is determined by the imaginary part of the self-energy, i.e., a 'quasi-particle width' Γ. For $t > 1/\Gamma$, the statistical propagator becomes independent of the initial conditions. It is then given by a memory integral which depends on the real and imaginary part of the self-energy.

As long as thermal corrections are small, the approach to equilibrium is well described by the ordinary Boltzmann equation, which is a local, first-order differential equation in time for the particle number density. However, in the case of large thermal corrections the notion of number density becomes ambiguous, and it is important to consider the second-order Kadanoff-Baym equations for the two-point functions rather than a Boltzmann equation. Still, as long as the quasi-particle decay width is small compared to the quasi-particle energy, a Boltzmann equation for quasi-particles describes the approach to equilibrium to good approximation. For large decay width the dynamics becomes nonlocal in time and the Boltzmann approximation breaks down.

It is interesting to study the contribution of the thermalized scalar field to energy density and pressure. For a free field these observables are determined by the Bose-Einstein distribution function. Interaction with the thermal bath can significantly modify energy density and pressure, and therefore the equation of state. The Bose-Einstein distribution as function of the complex quasi-particle pole is now the relevant quantity. Energy density and pressure differ from the expressions for a free gas of quasi-particles by a temperature-dependent 'vacuum term' which can become important at high temperatures.

We have illustrated these results for a toy model where the thermal bath consists of two massive scalar fields. We have considered two cases where equilibration takes place either via decays and inverse decays or via 'Landau damping'. In general, one has to study the Kadanoff-Baym equation for the statistical propagator which depends on two time coordinates as well as initial conditions. However, for large times, $t \gg 1/\Gamma$, the time evolution is well described by the Boltzmann equation for quasi-particles.

Our analysis has been motivated by the need of a full quantum mechanical description of leptogenesis. To achieve this, one has to consider correlation functions rather than number densities, although for parts of the calculation the use of Boltzmann equations will be sufficient. The heavy Majorana neutrino is very weakly coupled to the thermal bath. Hence, thermal corrections to its mass and width are small, and its approach to equilibrium is well described by Boltzmann equations. However, to study the dependence of the final baryon asymmetry on initial conditions it may be necessary to consider the statistical propagator, since leptogenesis takes place at $t_B \sim 1/\Gamma$. Furthermore, for lepton and Higgs fields, which have strong gauge interactions, finite-width effects can be important. At present it is unclear how accurately the leptogenesis process can be described based on a quasi-particle picture for the standard model particles which form the thermal bath. These questions are currently under investigation [24].

Acknowledgements

We would like to thank J. Berges, D. Bödeker, J. Schmidt and C. Wetterich for helpful discussions.

A The spectral function

A.1 Time-translation invariance

In this section we shall prove that the most general solution of the first Kadanoff-Baym equation is time-translation invariant. The starting point is Eq. (2.29) with the boundary conditions (2.31) - (2.33). Performing the change of variables $t_1 = t + y/2$, $t_2 = t - y/2$, Eq. (2.29) becomes

$$\left(\frac{1}{4}\partial_t^2 + \partial_t\partial_y + \partial_y^2 + \omega_{\mathbf{q}}^2\right)\Delta_{\mathbf{q}}^-(t;y) + \int_0^y dy'\Pi_{\mathbf{q}}^-(y-y')\Delta_{\mathbf{q}}^-(t';y') = 0 \ , \tag{A.1}$$

where $t' = t - (y-y')/2$ and $\omega_{\mathbf{q}}^2 = \mathbf{q}^2 + m^2$. Note that $\Delta_{\mathbf{q}}^-$ and $\Pi_{\mathbf{q}}^-$ only depend on $|\mathbf{q}|$ because of rotational invariance. Both functions are antisymmetric in y. The boundary conditions (2.31) - (2.33) read

$$\Delta_{\mathbf{q}}^-(t;0) = 0 \ , \tag{A.2}$$

$$\partial_t\Delta_{\mathbf{q}}^-(t;0) = 0 \ , \tag{A.3}$$

$$\partial_y\Delta_{\mathbf{q}}^-(t;y)|_{y=0} = 1 \ , \tag{A.4}$$

$$\left(\frac{1}{4}\partial_t^2 - \partial_y^2\right)\Delta_{\mathbf{q}}^-(t;y)|_{y=0} = 0 \ . \tag{A.5}$$

The condition (A.2) is automatically fulfilled because of the antisymmetry in y.

To prove that Δ^- is time-translation invariant we now perform an expansion in powers of Π^-,

$$\Delta_{\mathbf{q}}^- = \sum_{n=0}^{\infty}\Delta_{\mathbf{q}}^{(n)} \ , \quad \Delta_{\mathbf{q}}^{(n)} = \mathcal{O}(\Pi_{\mathbf{q}}^{(n)}) \ . \tag{A.6}$$

For $n = 0$ one has

$$\left(\frac{1}{4}\partial_t^2 + \partial_t\partial_y + \partial_y^2 + \omega_{\mathbf{q}}^2\right)\Delta_{\mathbf{q}}^{(0)}(t;y) = 0 \ . \tag{A.7}$$

Using the antisymmetry of $\Delta_{\mathbf{q}}^-$ in y, one obtains

$$\partial_t\partial_y\Delta_{\mathbf{q}}^{(0)}(t;y) = 0 \ , \tag{A.8}$$

which has the general solution

$$\Delta_{\mathbf{q}}^{(0)} = a_{\mathbf{q}}^{(0)}(t) + b_{\mathbf{q}}^{(0)}(y) \ . \tag{A.9}$$

Every solution of Eqs. (A.7) and (A.8) satisfies the boundary condition (A.5). The condition (A.3) implies

$$\partial_t\Delta_{\mathbf{q}}^{(0)}(t;0) = \partial_t a_{\mathbf{q}}^{(0)}(t) = 0 \ . \tag{A.10}$$

Hence, $a_{\mathbf{q}}^{(0)}$ is constant and $\Delta_{\mathbf{q}}^{(0)}$ only depends on y. Eq. (A.7) now becomes

$$\left(\partial_y^2 + \omega_{\mathbf{q}}^2\right)\Delta_{\mathbf{q}}^{(0)}(y) = 0 \ , \tag{A.11}$$

which has the antisymmetric solution

$$\Delta_{\mathbf{q}}^{(0)}(y) = c_{\mathbf{q}}^{(0)} \sin(\omega_{\mathbf{q}} y) \ . \tag{A.12}$$

For $n \neq 0$ one can use the recurrence relation

$$\left(\frac{1}{4}\partial_t^2 + \partial_t\partial_y + \partial_y^2 + \omega_{\mathbf{q}}^2\right) \Delta_{\mathbf{q}}^{(n+1)}(t;y) + \int_0^y dy' \Pi_{\mathbf{q}}^-(y-y')\Delta_{\mathbf{q}}^{(n)}(y') = 0 \ . \tag{A.13}$$

Using the antisymmetry of $\Pi_{\mathbf{q}}^-$ and $\Delta_{\mathbf{q}}^-$ in y, one again finds

$$\partial_t\partial_y\Delta_{\mathbf{q}}^{(n+1)}(t;y) = 0 \ . \tag{A.14}$$

Repeating the same steps as for $\Delta_{\mathbf{q}}^{(0)}$ yields the result that also $\Delta_{\mathbf{q}}^{(n+1)}$ is independent of t.

We conclude that the spectral function is the antisymmetric solution of the equation

$$\left(\partial_y^2 + \omega_{\mathbf{q}}^2\right) \Delta_{\mathbf{q}}^-(y) + \int_0^y dy' \Pi_{\mathbf{q}}^-(y-y')\Delta_{\mathbf{q}}^-(y') = 0 \ , \tag{A.15}$$

with the boundary condition

$$\partial_y\Delta_{\mathbf{q}}^-(y)|_{y=0} = 1 \ . \tag{A.16}$$

A.2 Conventions for propagators and self-energies

In thermal equilibrium the retarded and advanced propagators and self-energies only depend on the time difference $y = t_1 - t_2$. In the following we list several relations between their Fourier transforms, which are used in the different sections. In principle, these relations are all well know, but their specific form depends on the chosen conventions. All relations are not affected by the three-dimensional Fourier transform. We therefore drop the argument $\mathbf{q}$ or $\mathbf{x}$.

Propagators:

$$\Delta^-(\omega)^* = -\Delta^-(\omega) \ , \tag{A.17}$$

$$\Delta^+(\omega)^* = \Delta^+(\omega) \ , \tag{A.18}$$

$$\Delta^A(\omega) = \frac{i}{2}\Delta^-(\omega) - \mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\frac{\Delta^-(\omega')}{\omega'-\omega} \ , \tag{A.19}$$

$$\Delta^R(\omega) = -\frac{i}{2}\Delta^-(\omega) - \mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\frac{\Delta^-(\omega')}{\omega'-\omega} \ , \tag{A.20}$$

$$\mathrm{Re}\,\Delta^A(\omega) = -\mathrm{Re}\,\Delta^R(\omega) = \frac{i}{2}\Delta^-(\omega) \ , \tag{A.21}$$

$$\mathrm{Im}\,\Delta^A(\omega) = \mathrm{Im}\,\Delta^R(\omega) = -\mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i}\frac{\Delta^-(\omega')}{\omega'-\omega} \ , \tag{A.22}$$

$$\Delta^A(-\omega) = \Delta^R(\omega) \ . \tag{A.23}$$

(A.24)

Self-energies:

$$\begin{aligned}
\Pi^-(\omega)^* &= -\Pi^-(\omega) \ , && (A.25)\\
\Pi^+(\omega)^* &= \Pi^+(\omega) \ , && (A.26)\\
\Pi^A(\omega) &= -\frac{1}{2}\Pi^-(\omega) + \mathcal{P}\int \frac{d\omega'}{2\pi i}\frac{\Pi^-(\omega')}{\omega'-\omega} \ , && (A.27)\\
\Pi^R(\omega) &= \frac{1}{2}\Pi^-(\omega) + \mathcal{P}\int \frac{d\omega'}{2\pi i}\frac{\Pi^-(\omega')}{\omega'-\omega} \ , && (A.28)\\
\mathrm{Re}\,\Pi^A(\omega) &= \mathrm{Re}\,\Pi^R(\omega) = \mathcal{P}\int \frac{d\omega'}{2\pi i}\frac{\Pi^-(\omega')}{\omega'-\omega} \ , && (A.29)\\
\mathrm{Im}\,\Pi^A(\omega) &= -\,\mathrm{Im}\,\Pi^R(\omega) = \frac{i}{2}\Pi^-(\omega) \ , && (A.30)\\
\Pi^A(-\omega) &= \Pi^R(\omega) \ . && (A.31)
\end{aligned}$$

A.3 Scalar field model

The interaction with the thermal bath changes the spectral function of a free scalar particle,

$$\rho_{\mathbf{q}}(\omega) = 2\pi \mathrm{sign}(\omega)\delta(\omega^2 - \omega_{\mathbf{q}}^2) \ , \tag{A.32}$$

to the expression (4.15) which depends on real and imaginary part of the self-energy,

$$\rho_{\mathbf{q}}(\omega) = \frac{-2\mathrm{Im}\Pi_{\mathbf{q}}^R(\omega) + 2\omega\epsilon}{[\omega^2 - \omega_{\mathbf{q}}^2 - \mathrm{Re}\Pi_{\mathbf{q}}^R(\omega)]^2 + [\mathrm{Im}\Pi_{\mathbf{q}}^R(\omega) + \omega\epsilon]^2} \ . \tag{A.33}$$

We have computed the imaginary part of the self-energy in the scalar field model defined in Section 7, assuming free thermal propagators for the fields χ_1 and χ_2. The result agrees with [17]. One obtains ($q = (\omega_{\mathbf{q}}, \mathbf{q})$):

$$-\,\mathrm{Im}\,\Pi_{\mathbf{q}}^R(\omega) = \sigma_0(q) + \sigma_\beta^{(a)}(q) + \sigma_\beta^{(b)}(q) \ . \tag{A.34}$$

Here σ_0 is the zero-temperature contribution due to the decay process $\Phi \to \chi_1\chi_2$,

$$\begin{aligned}
\sigma_0(q) =& \frac{g^2}{16\pi q^2}\mathrm{sign}(\omega)\Theta(q^2-(m_1+m_2)^2)\\
&\times \left((q^2)^2 - 2q^2(m_1^2+m_2^2) + (m_1^2-m_2^2)^2\right)^{\frac{1}{2}} \ ,
\end{aligned} \tag{A.35}$$

$\sigma_\beta^{(a)}$ is the finite-temperature contribution from this process,

$$\sigma_\beta^{(a)}(q) = \frac{g^2}{16\pi|\mathbf{q}|\beta}\mathrm{sign}(\omega)\Theta(q^2-(m_1+m_2)^2)$$

$$\times \left(\ln \left(\frac{1 - e^{-\beta\omega_+}}{1 - e^{-\beta\omega_-}} \right) + (m_1 \leftrightarrow m_2) \right) , \tag{A.36}$$

and $\sigma_\beta^{(b)}(\mathbf{q})$ is the finite-temperature contribution from processes $\chi_i \to \chi_j \phi$,[3]

$$\begin{aligned} \sigma_\beta^{(b)}(q) =& \frac{g^2}{16\pi|\mathbf{q}|\beta} \mathrm{sign}(\omega)\Theta((m_1 - m_2)^2 - q^2) \\ &\times \left(\ln \left(\frac{1 - e^{-\beta|\omega_-|}}{1 - e^{-\beta|\omega_+|}} \right) + (m_1 \leftrightarrow m_2) \right) , \end{aligned} \tag{A.37}$$

where we have used the abbreviations

$$\omega_\pm = \frac{|\omega|}{2q^2}(q^2 + m_1^2 - m_2^2) \pm \frac{|\mathbf{q}|}{2|q^2|} \left((q^2 + m_1^2 - m_2^2)^2 - 4q^2 m_1^2 \right)^{\frac{1}{2}} . \tag{A.38}$$

The real part of the self-energy can be computed using the dispersion relation relation,

$$\mathrm{Re}\,\Pi_{\mathbf{q}}^R(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Im}\,\Pi_{\mathbf{q}}^R(\omega')}{\omega' - \omega} . \tag{A.39}$$

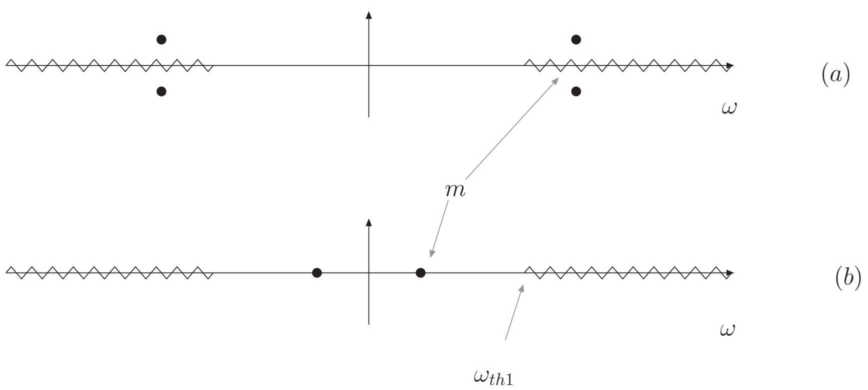

Figure 13: Poles and cuts of the spectral function $\rho(\omega)$ for $\mathbf{q} = 0$ at $T = 0$: (a) $m > m_1 + m_2$, and (b) $m < m_1 + m_2$.

Based on these expressions we can discuss the analytic structure of the spectral function. For a free field $\rho_{\mathbf{q}}(\omega)$ is given by (A.32) which has two poles at $\omega = \pm\omega_q$ in the complex ω-plane. The interaction of Φ with χ_1 and χ_2 does not modify these poles for $m < m_1 + m_2$, where Φ is stable at zero temperature. In addition there are branch cuts at the two-particle thresholds $|\omega| > \omega_{\mathrm{th1}} = \sqrt{\mathbf{q}^2 + (m_1 + m_2)^2}$ (see Fig. 13*b*). They correspond to virtual decays and inverse decays, $\Phi \leftrightarrow \chi\chi$. In the case $m > m_1 + m_2$ these processes can happen on-shell since $m > \omega_{\mathrm{th1}}$, and Φ becomes unstable. Now the spectral function has four poles in the complex ω-plane, whose real parts lie in the region of the branch cuts (see Fig. 13*a*). The imaginary parts of the poles correspond to the decay width of Φ.

[3] Note that we disagree with the discussion in [22] which implies the additional factor $\Theta(|m_1^2 - m_2^2| - q^2)$.

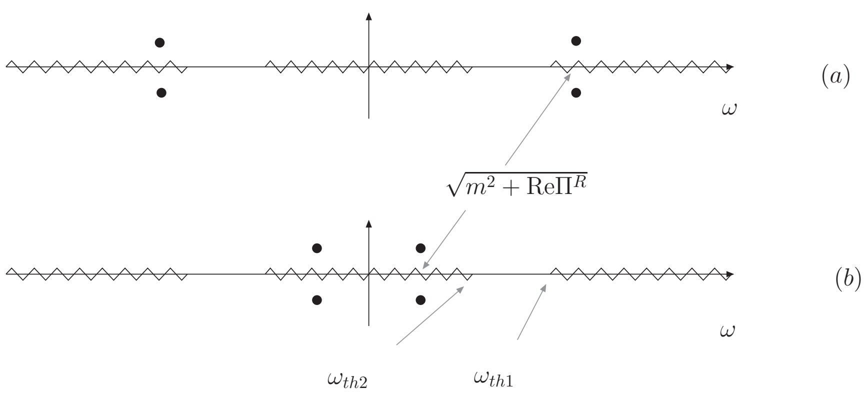

Figure 14: Poles and cuts of the spectral function $\rho(\omega)$ for $\mathbf{q} = 0$ at $T \neq 0$: (a) $m > m_1 + m_2$, and (b) $m < m_1 + m_2$.

The analytic structure of the spectral function at finite temperature is displayed in Fig. 14. The position of ω_{th1} is shifted due to thermal corrections from $\mathrm{Re}\,\Pi^R_{\mathbf{q}}$. Furthermore, a new branch cut appears in the region where $\sigma_b \neq 0$, i.e. for $|\omega| < \omega_{\mathrm{th2}} = \sqrt{\mathbf{q}^2 + (m_1 - m_2)^2}$. This is due to processes $\chi \leftrightarrow \phi\chi$ and corresponds to Landau damping of quasi-particles in the plasma. If the real part of the poles falls into the regions of one of the branch cuts, i.e. $|\omega| < \omega_{\mathrm{th2}}$ or $|\omega| > \omega_{\mathrm{th1}}$, they acquire an imaginary part which corresponds to the quasi-particle decay width (see Fig. 14). Qualitatively, this analytic structure is typical for interacting quantum field theories at finite temperature. In general, the spectral function can have additional singular contributions for $m < |\omega| < \omega_{\mathrm{th1}}$ corresponding to bound states. At finite temperature they are also dressed to quasi-particles.

A.4 Breit-Wigner approximation

In the regime of couplings and temperatures where $|\,\mathrm{Im}\,\Pi^R_{\mathbf{q}}(\Omega_{\mathbf{q}})| \ll \Omega^2_{\mathbf{q}}$, so that the quasi-particle picture holds, one can approximate the spectral function $\rho_{\mathbf{q}}(\omega)$ by a Breit-Wigner function. From the expression (4.15) one easily obtains

$$\rho_{\mathbf{q}}(\omega) \simeq \frac{Z_{\mathbf{q}}}{2\Omega_{\mathbf{q}}} \frac{\mathrm{sign}(\omega)\Gamma_{\mathbf{q}}}{\left(|\omega| - \Omega_{\mathbf{q}}\right)^2 + \frac{1}{4}\Gamma^2_{\mathbf{q}}} \,, \tag{A.40}$$

where $\Gamma_{\mathbf{q}}$ is the quasi-particle width

$$\Gamma_{\mathbf{q}} = -Z_{\mathbf{q}} \frac{\mathrm{Im}\Pi^R_{\mathbf{q}}(\Omega_{\mathbf{q}})}{\Omega_{\mathbf{q}}} \,, \tag{A.41}$$

with

$$Z_{\mathbf{q}} = \left(1 - \frac{1}{2\Omega_{\mathbf{q}}} \frac{\partial \mathrm{Re}\Pi^R_{\mathbf{q}}(\omega)}{\partial\omega}\Big|_{\omega=\Omega_{\mathbf{q}}}\right)^{-1} \,. \tag{A.42}$$

Contrary to the exact spectral function (4.15), the Breit-Wigner approximation (A.40) has no branch cuts. The integrals over ω are dominated by the regions around the quasi-particle poles where the two functions are very similar. For the Fourier transform, the spectral function in real time, one obtains

$$\Delta^-_{\mathbf{q}}(y) \simeq Z_{\mathbf{q}} \frac{\sin(\Omega_q y)}{\Omega_{\mathbf{q}}} e^{-\Gamma_q t/2} \ . \tag{A.43}$$

References

[1] E. W. Kolb and M. S. Turner, *The Early Universe*, Addison-Wesley, New York, 1990.

[2] V. Mukhanov, *Physical Foundations of Cosmology*, Cambridge University Press, Cambridge, 2005.

[3] M. Fukugita and T. Yanagida, Phys. Lett. B **174** (1986) 45.

[4] M. A. Luty, Phys. Rev. D **45** (1992) 455;
L. Covi, E. Roulet and F. Vissani, Phys. Lett. B **384** (1996) 169 [hep-ph/9605319];
W. Buchmuller and M. Plumacher, Phys. Lett. B **389** (1996) 73 [hep-ph/9608308].

[5] For reviews and references, see
W. Buchmuller, R. D. Peccei and T. Yanagida, Ann. Rev. Nucl. Part. Sci. **55** (2005) 311 [hep-ph/0502169];
S. Davidson, E. Nardi and Y. Nir, Phys. Rept. **466** (2008) 105 [arXiv:0802.2962].

[6] W. Buchmuller and S. Fredenhagen, Phys. Lett. B **483** (2000) 217 [hep-ph/0004145].

[7] M. Lindner and M. M. Muller, Phys. Rev. D **73** (2006) 125002 [hep-ph/0512147]; Phys. Rev. D **77**, 025027 (2008) [arXiv:0710.2917].

[8] A. De Simone and A. Riotto, JCAP **0708** (2007) 002 [hep-ph/0703175].

[9] R. Barbieri, P. Creminelli, A. Strumia and N. Tetradis, Nucl. Phys. B **575** (2000) 61 [hep-ph/9911315];
T. Endoh, T. Morozumi and Z. h. Xiong, Prog. Theor. Phys. **111** (2004) 123 [hep-ph/0308276];
A. Abada, S. Davidson, F. X. Josse-Michaux, M. Losada and A. Riotto, JCAP **0604** (2006) 004 [hep-ph/0601083];
E. Nardi, Y. Nir, E. Roulet and J. Racker, JHEP **0601** (2006) 164 [hep-ph/0601084];
S. Blanchet, P. Di Bari and G. G. Raffelt, JCAP **0703** (2007) 012 [hep-ph/0611337].

[10] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics*, Benjamin, New York, 1962.

[11] J. Schwinger, J. Math. Phys. **2** (1961) 407;
P. M. Bakschi and K. T. Mahanthappa, J. Math. Phys. **4** (1963) 1;
P. M. Bakschi and K. T. Mahanthappa, J. Math. Phys. **4** (1963) 12;
L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47** (1964) 1515 [Sov. Phys. JETP **20** (1965) 1018].

[12] M. Le Bellac, *Thermal Field Theory*, Cambridge University Press, Cambridge, 1996.

[13] J. Berges, *Introduction to Nonequilibrium Quantum Field Theory*, AIP Conf. Proc. **739** (2005) 3 [hep-ph/0409233].

[14] K. c. Chou, Z. b. Su, B. l. Hao and L. Yu, Phys. Rept. **118** (1985) 1.

[15] J. Zinn-Justin, *Quantum field theory and critical phenomena*, Int. Ser. Monogr. Phys. **85** (1993) 1.

[16] J. Yokoyama, Phys. Rev. D **70** (2004) 103511 [arXiv:hep-ph/0406072].

[17] D. Boyanovsky, K. Davey and C. M. Ho, Phys. Rev. D **71** (2005) 023523 [arXiv:hep-ph/0411042].

[18] K. Kainulainen, T. Prokopec, M. G. Schmidt and S. Weinstock, JHEP **0106** (2001) 031 [hep-ph/0105295]; Phys. Rev. D **66** (2002) 043502 [hep-ph/0202177].

[19] C. Greiner and S. Leupold, Annals Phys. **270** (1998) 328 [hep-ph/9802312].

[20] J. Berges, A. Rothkopf and J. Schmidt, Phys. Rev. Lett. **101** (2008) 041603 [arXiv:0803.0131].

[21] W. Buchmuller, P. Di Bari and M. Plumacher, Nucl. Phys. B **643** (2002) 367 [Erratum-ibid. B **793** (2008) 362] [hep-ph/0205349]; Nucl. Phys. B **665** (2003) 445 [hep-ph/0302092]; Annals Phys. **315** (2005) 305 [hep-ph/0401240].

[22] H. A. Weldon, Phys. Rev. D **28** (1983) 2007.

[23] M. Drewes, Thesis, DESY 2006 (unpublished)

[24] A. Anisimov, W. Buchmüller, M. Drewes and S. Mendizabal, in preparation.