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Orthopositronium lifetime: analytic results in $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^3 \ln \alpha)$

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Abstract

We present the $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^3 \ln \alpha)$ corrections to the total decay width of orthopositronium in closed analytic form, in terms of basic transcendental numbers, which can be evaluated numerically to arbitrary precision.

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Quantum electrodynamics (QED), the gauged quantum field theory of the electromagnetic interaction, has celebrated ground-breaking successes in the twentieth century. In fact, its multi-loop predictions for the anomalous magnetic moments of the electron and the muon were found to agree with highest-precision measurements within a few parts of 10^{-12} and 10^{-10}, respectively.

Another ultrapure laboratory for high-precision tests of QED is provided by positronium (Ps), the lightest known atom, being the electromagnetic bound state of the electron e^- and the positron e^+, which was discovered in the year 1951 [1]. In fact, thanks to the smallness of the electron mass m relative to typical hadronic mass scales, its theoretical description is not plagued by strong-interaction uncertainties and its properties, such as decay widths and energy levels, can be calculated perturbatively in non-relativistic QED (NRQED) [2], as expansions in Sommerfeld's fine-structure constant α, with very high precision.

Ps comes in two ground states, ${}^1S_0$ parapositronium (p-Ps) and ${}^3S_1$ orthopositronium (o-Ps), which decay to two and three photons, respectively. In this Letter, we are concerned with the lifetime of o-Ps, which has been the subject of a vast number of theoretical and experimental investigations. Its first measurement [3] was performed later in the year 1951 and agreed well with its lowest-order (LO) prediction of 1949 [4]. Its first precision measurement [5], of 1968, had to wait nine years to be compared with the first complete one-loop calculation [6], which came two decades after the analogous calculation for p-Ps [7] being considerably simpler owing to the two-body final state. In the year 1987, the Ann Arbor group [8] published a measurement that exceeded the best theoretical prediction available then by more than ten experimental standard deviations. This so-called o-Ps lifetime puzzle triggered an avalanche of both experimental and theoretical activities, which eventually resulted in what now appears to be the resolution of this puzzle. In fact, the 2003 measurements at Ann Arbor [9] and Tokyo [10],

$$\begin{aligned}\Gamma(\text{Ann Arbor}) &= 7.0404(10\text{ stat.})(8\text{ syst.})\ \mu s^{-1},\\ \Gamma(\text{Tokyo}) &= 7.0396(12\text{ stat.})(11\text{ syst.})\ \mu s^{-1},\end{aligned}\tag{1}$$

agree mutually and with the present theoretical prediction,

$$\Gamma(\text{theory}) = 7.039979(11)\ \mu s^{-1}.\tag{2}$$

The latter is evaluated from

$$\begin{aligned}\Gamma(\text{theory}) = \Gamma_0 \Bigg[& 1 + A\frac{\alpha}{\pi} + \frac{\alpha^2}{3}\ln\alpha + B\left(\frac{\alpha}{\pi}\right)^2 \\ & - \frac{3\alpha^3}{2\pi}\ln^2\alpha + C\frac{\alpha^3}{\pi}\ln\alpha\Bigg],\end{aligned}\tag{3}$$

where [4]

$$\Gamma_0 = \frac{2}{9}(\pi^2 - 9)\frac{m\alpha^6}{\pi}\tag{4}$$

is the LO result. The leading logarithmically enhanced $\mathcal{O}(\alpha^2 \ln \alpha)$ and $\mathcal{O}(\alpha^3 \ln^2 \alpha)$ terms were found in Refs. [11,12] and Ref. [13], respectively. The coefficients $A = -10.286606(10)$ [6,11,14,15,16], $B = 45.06(26)$ [15], and $C = -5.51702455(23)$ [17] are only available in numerical form so far. Comprehensive reviews of the present experimental and theoretical status of o-Ps may be found in Ref. [18].

Given the fundamental importance of Ps for atomic and particle physics, it is desirable to complete our knowledge of the QED prediction in Eq. (3). Since the theoretical uncertainty is presently dominated by the errors in the numerical evaluations of the coefficients A, B, and C, it is an urgent task to find them in analytical form, in terms of transcendental numbers, which can be evaluated with arbitrary precision. In this Letter, this is achieved for A and C. The case of B is beyond the scope of presently available technology, since it involves two-loop five-point functions to be integrated over a three-body phase space. The quest for an analytic expression for A is a topic of old vintage: about 25 years ago, some of the simpler contributions to A, due to self-energy and outer and inner vertex corrections, were obtained analytically [19], but further progress then soon came to a grinding halt. The sustained endeavor of the community to improve the numerical accuracy of A [6,11,14,15,16] is now finally brought to a termination.

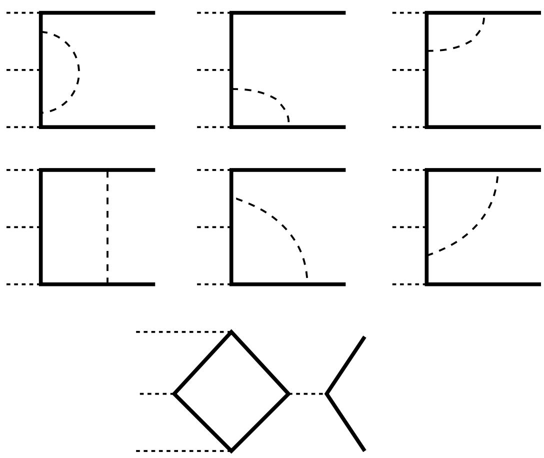

Figure 1: Feynman diagrams contributing to the total decay width of o-Ps at $\mathcal{O}(\alpha)$. Self-energy diagrams are not shown. Dashed and solid lines represent photons and electrons, respectively.

The $\mathcal{O}(\alpha)$ contribution in Eq. (3), $\Gamma_1 = \Gamma_0 A\alpha/\pi$, is due to the Feynman diagrams where a virtual photon is attached in all possible ways to the tree-level diagrams, with three real photons linked to an open electron line, and the electron box diagrams with an e^+e^- annihilation vertex connected to one of the photons being virtual (see Fig. 1). Taking the interference with the tree-level diagrams, imposing e^+e^- threshold kinematics, and performing the loop and angular integrations, one obtains the two-dimensional integral representation [16]

$$\Gamma_1 = \frac{m\alpha^7}{36\pi^2} \int_0^1 \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \frac{\mathrm{d}x_3}{x_3} \delta(2 - x_1 - x_2 - x_3) \times [F(x_1, x_3) + \mathrm{perm.}], \tag{5}$$

where x_i, with $0 \leq x_i \leq 1$, is the energy of photon i in the o-Ps rest frame normalized by its maximum value, the delta function ensures energy conservation, and *perm.* stands for the other five permutations of x_1, x_2, x_3. The function $F(x_1, x_3)$ is given by

$$F(x_1, x_3) = g_0(x_1, x_3) + \sum_{i=1}^{7} g_i(x_1, x_3) h_i(x_1, x_3), \tag{6}$$

where g_i are ratios of polynomials, which are listed in Eqs. (A5a)–(A5h) of Ref. [16], and

$$\begin{aligned}
h_1(x_1) &= \ln(2x_1), \quad h_2(x_1) = \sqrt{\frac{x_1}{\overline{x}_1}}\theta_1, \\
h_3(x_1) &= \frac{1}{2x_1}[\zeta_2 - \mathrm{Li}_2(1 - 2x_1)], \\
h_4(x_1) &= \frac{1}{4x_1}\left[3\zeta_2 - 2\theta_1^2\right], \quad h_5(x_1) = \frac{1}{2\overline{x}_1}\theta_1^2, \\
h_6(x_1, x_3) &= \frac{1}{\sqrt{x_1 \overline{x}_1 x_3 \overline{x}_3}}\left[\mathrm{Li}_2(r_A^+, \overline{\theta}_1) - \mathrm{Li}_2(r_A^-, \overline{\theta}_1)\right], \\
h_7(x_1, x_3) &= \frac{1}{2\sqrt{x_1 \overline{x}_1 x_3 \overline{x}_3}}\Big[2\,\mathrm{Li}_2(r_B^+, \theta_1) - 2\,\mathrm{Li}_2(r_B^-, \theta_1) \\
&\quad - \mathrm{Li}_2(r_C^+, 0) + \mathrm{Li}_2(r_C^-, 0)\Big],
\end{aligned} \tag{7}$$

with $\overline{x}_i = 1 - x_i$ and

$$\begin{aligned}
\theta_1 &= \arctan(\sqrt{\overline{x}_1/x_1}), \quad \overline{\theta}_1 = \arctan(\sqrt{x_1/\overline{x}_1}), \\
r_A^\pm &= \sqrt{\overline{x}_1}\left(1 \pm \sqrt{\frac{x_1 \overline{x}_3}{\overline{x}_1 x_3}}\right), \quad r_B^\pm = \sqrt{x_1}\left(1 \pm \sqrt{\frac{\overline{x}_1 \overline{x}_3}{x_1 x_3}}\right), \\
r_C^\pm &= r_B^\pm/\sqrt{x_1}.
\end{aligned} \tag{8}$$

Here, $\zeta_2 = \pi^2/6$ and

$$\mathrm{Li}_2(r, \theta) = -\frac{1}{2}\int_0^1 \frac{\mathrm{d}t}{t} \ln(1 - 2rt\cos\theta + r^2t^2) \tag{9}$$

is the real part of the dilogarithm [see line below Eq. (20)] of complex argument $z = re^{\mathrm{i}\theta}$ [20]. Since we are dealing here with a single-scale problem, Eq. (5) yields just one number.

Although Bose symmetry is manifest in Eq. (5), its evaluation is complicated by the fact that, for a given order of integration, individual permutations yield divergent integrals, which have to cancel in their combination. In order to avoid such a proliferation of terms, we introduce a regularization parameter, δ, in such a way that the symmetry unter $x_i \leftrightarrow x_j$ for any pair $i \neq j$ is retained. In this way, Eq. (5) collapses to

$$\Gamma_1 = \frac{m\alpha^7}{6\pi^2} \int\limits_{2\delta}^{1-\delta} \mathrm{d}x_1 \int\limits_{1-x_1+\delta}^{1-\delta} \frac{\mathrm{d}x_2}{x_1 x_2 x_3} F(x_1, x_3), \tag{10}$$

where $x_3 = 2 - x_1 - x_2$. Note that we may now exploit the freedom to choose any pair of variables x_i and x_j $(i \neq j)$ as the arguments of F and as the integration variables.

The analytical integration of Eq. (10) is rather tedious and requires a number of tricks to be conceived of. For lack of space, we can only outline here a few examples. Specifically, we consider the last two functions of Eq. (7), which are most complicated. Using Eq. (9) and after some manipulations, we obtain the following integral representation for $h_7(x_1, x_3)$:

$$h_7(x_1, x_3) = -\frac{1}{4} \int\limits_0^1 \frac{\mathrm{d}t}{\sqrt{t}(x_1 x_3 - \overline{x}_1 \overline{x}_3 t)} \left[\ln \frac{\overline{x}_1 x_3}{x_1 \overline{x}_3} \right.$$
$$\left. + 2\ln(x_3 + \overline{x}_3 t) - \ln t \right]. \tag{11}$$

Exploiting the $x_1 \leftrightarrow x_3$ symmetry of the coefficient $g_7(x_1, x_3)$ multiplying $h_7(x_1, x_3)$, this can be simplified as

$$h_7(x_1, x_3) = -\frac{1}{4} \int\limits_0^1 \frac{\mathrm{d}t}{\sqrt{t}(x_1 x_3 - \overline{x}_1 \overline{x}_3 t)} [2\ln(x_3 + \overline{x}_3 t)$$
$$- \ln t]. \tag{12}$$

At this point, it is useful to change the order of integrations. Observing that the logarithmic terms in Eq. (12) are x_1 independent, we first integrate over x_1 (for a similar approach, see Ref. [21]). In order to avoid the appearance of complicated functions in the intermediate results, the integration over t in Eq. (12) is performed last.

Analogously, $h_6(x_1, x_3)$ can be rewritten as

$$h_6(x_1, x_3) = -\frac{1}{2} \int\limits_0^1 \frac{\mathrm{d}t}{\sqrt{t}(\overline{x}_1 x_3 - x_1 \overline{x}_3 t)} [\ln x_1 - \ln x_3$$
$$+ \ln(x_3 + \overline{x}_3 t)], \tag{13}$$

in which the part proportional to $\ln x_1$ and the complementary part are first integrated over x_3 and x_1, respectively. The t integration is again performed last.

Let us now consider a typical integral that arises upon the first integration:

$$I = \int\limits_0^1 \frac{\mathrm{d}t}{t} \int\limits_0^1 \frac{\mathrm{d}x}{x} \ln[1 - 4t(1-t)(1-x)] \ln(1-x). \tag{14}$$

Direct integration over t or x would lead to rather complicated functions in the remaining variable. Instead, we Taylor expand the first logarithm using $\ln(1-x) = -\sum_{n=1}^{\infty} x^n/n$ to obtain

$$I = -\sum_{n=1}^{\infty} \frac{4^n}{n} \int_0^1 \frac{\mathrm{d}t}{t} [t(1-t)]^n \int_0^1 \frac{\mathrm{d}x}{x} (1-x)^n \ln(1-x). \tag{15}$$

Now the two integrals are separated and can be solved in terms of Euler's Gamma function, $\Gamma(x) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} t^{x-1}$. Using

$$\int_0^1 \frac{\mathrm{d}x}{x} (1-x)^n \ln(1-x) = -\psi'(n+1), \tag{16}$$

where $\psi(x) = \mathrm{d} \ln \Gamma(x)/\mathrm{d}x$ is the digamma function, we finally have

$$I = \sum_{n=1}^{\infty} \frac{4^n}{2n} \frac{\Gamma^2(n)}{\Gamma(2n)} \psi'(n+1). \tag{17}$$

Another class of typical integrals yields sums involving digamma functions of half-integer arguments, *e.g.*

$$\begin{aligned} J &= \int_0^1 \frac{\mathrm{d}t}{t} \int_0^1 \mathrm{d}x \frac{\ln[1+4t(1-t)(1-x)] \ln(1-x)}{x-2} \\ &= \sum_{n=1}^{\infty} \frac{(-4)^n}{8n} \frac{\Gamma^2(n)}{\Gamma(2n)} \left[\psi'\left(\frac{n+2}{2}\right) - \psi'\left(\frac{n+1}{2}\right) \right]. \end{aligned} \tag{18}$$

I and J belong to the class of so-called inverse central binomial sums [22,23], and methods for their summation are elaborated in Ref. [23]. With their help, I and J can be expressed in terms of known irrational constants, as

$$\begin{aligned} I &= -4\zeta_2 l_2^2 - \frac{l_2^4}{3} - 8\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{17}{2}\zeta_4, \\ J &= -\frac{3}{2}\zeta_2 l_2^2 + \frac{l_2^4}{4} - 3\zeta_2 l_2 l_r + l_2^2 l_r^2 + \frac{11}{12} l_2 l_r^3 + \frac{47}{288} l_r^4 \\ &\quad + 4 l_2 l_r \,\mathrm{Li}_2(r) + \frac{7}{6} l_r^2 \,\mathrm{Li}_2(r) - 6 l_2 \,\mathrm{Li}_3(-r) \\ &\quad - 2 l_r \,\mathrm{Li}_3(-r) + 5 l_2 \,\mathrm{Li}_3(r) + \frac{4}{3} l_r \,\mathrm{Li}_3(r) + 6\,\mathrm{Li}_4\left(\frac{1}{2}\right) \\ &\quad + 4\,\mathrm{Li}_4(-r) - 5\,\mathrm{Li}_4(r) - \frac{13}{3} l_r \,\mathrm{S}_{1,2}(r) + \frac{2}{3}\,\mathrm{S}_{1,2}(r^2) \\ &\quad - 4\,\mathrm{S}_{2,2}(-r) + 5\,\mathrm{S}_{2,2}(r) + \zeta_3 l_2 + \frac{19}{6}\zeta_3 l_r, \end{aligned} \tag{19}$$

where $r = (\sqrt{2}-1)/(\sqrt{2}+1)$, $l_x = \ln x$,

$$\mathrm{S}_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^1 \frac{\mathrm{d}t}{t} \ln^{n-1} t \ln^p(1-tx) \tag{20}$$

is the generalized poly-logarithm, $\mathrm{Li}_n(x) = \mathrm{S}_{n-1,1}(x)$ is the poly-logarithm of order n, and $\zeta_n = \zeta(n) = \mathrm{Li}_n(1)$, with $\zeta(x)$ being Riemann's zeta function [20,25].

Unfortunately, not all integrals can be computed so straightforwardly. In more complicated cases, the integrations are not separated after expansion into infinite series. We then rely on the PSLQ algorithm [24], which allows one to reconstruct the representation of a numerical result known to very high precision in terms of a linear combination of a set of irrational constants with rational coefficients, if that set is known beforehand. The experience gained with the explicit solution of the simpler integrals helps us to exhaust the relevant sets. In order for PSLQ to work in our applications, the numerical values of the integrals must be known up to typically 150 decimal figures.

After a laborious calculation, we obtain

$$
\begin{aligned}
\frac{2}{9}(\pi^2-9)A = {} & \frac{56}{27} - \frac{901}{216}\zeta_2 - \frac{11303}{192}\zeta_4 + \frac{19}{6}l_2 - \frac{2701}{108}\zeta_2 l_2 \\
& + \frac{253}{24}\zeta_2 l_2^2 + \frac{251}{144}l_2^4 + \frac{913}{64}\zeta_2 l_3^2 + \frac{83}{256}l_3^4 - \frac{21}{4}\zeta_2 l_2 l_r \\
& - \frac{49}{16}\zeta_2 l_r^2 + \frac{7}{16}l_2 l_r^3 + \frac{35}{384}l_r^4 + \frac{581}{16}\zeta_2 \mathrm{Li}_2\left(\frac{1}{3}\right) \\
& - \frac{21}{2}l_2 \mathrm{Li}_3(-r) - \frac{7}{2}l_r \mathrm{Li}_3(-r) + \frac{63}{4}l_2 \mathrm{Li}_3(r) \\
& + \frac{63}{8}l_r \mathrm{Li}_3(r) - \frac{249}{32}\mathrm{Li}_4\left(-\frac{1}{3}\right) + \frac{249}{16}\mathrm{Li}_4\left(\frac{1}{3}\right) \\
& + \frac{251}{6}\mathrm{Li}_4\left(\frac{1}{2}\right) + 7\,\mathrm{Li}_4(-r) - 7\mathrm{S}_{2,2}(-r) \\
& - \frac{63}{4}\mathrm{Li}_4(r) + \frac{63}{4}\mathrm{S}_{2,2}(r) + \frac{11449}{432}\zeta_3 - \frac{91}{6}\zeta_3 l_2 \\
& - \frac{35}{8}\zeta_3 l_r + \frac{1}{\sqrt{2}}\left[\frac{49}{2}\zeta_2 l_r - \frac{7}{72}l_r^3 - \frac{35}{6}l_r \mathrm{Li}_2(r)\right. \\
& \left. + \frac{35}{6}\mathrm{Li}_3(r) - \frac{175}{3}\mathrm{S}_{1,2}(r) + \frac{14}{3}\mathrm{S}_{1,2}(r^2) + \frac{119}{3}\zeta_3\right].
\end{aligned} \tag{21}
$$

The constant C in Eq. (3) is related to A through [17]

$$
C = \frac{A}{3} - \frac{229}{30} + 8l_2. \tag{22}
$$

From Eqs. (21) and (22), A and C can be numerically evaluated with arbitrary precision,

$$
\begin{aligned}
A &= -10.28661\,48086\,28262\,24015\,01692\,10991\,\ldots, \\
C &= -5.51702\,74917\,29858\,27137\,88660\,98665\,\ldots.
\end{aligned} \tag{23}
$$

These numbers agree with the best existing numerical evaluations [16,15] within the quoted errors.

In conclusion, we obtained the $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^3 \ln \alpha)$ corrections to the total decay width of o-Ps, *i.e.* the coefficients A and C in Eq. (3), respectively, in closed analytic form. Another important result is the appearance of new irrational constants in Eq. (21). These constants enlarge the class of the known constants in single-scale problems. The constant B in Eq. (3) still remains analytically unknown.

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