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$N = 4$ supersymmetric Yang Mills scattering amplitudes at high energies: the Regge cut contribution

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Abstract

We further investigate, in the planar limit of $N = 4$ supersymmetric Yang Mills theories, the high energy Regge behavior of six-point MHV scattering amplitudes. In particular, for the new Regge cut contribution found in our previous paper, we compute in the leading logarithmic approximation (LLA) the energy spectrum of the BFKL equation in the color octet channel, and we calculate explicitly the two loop corrections to the discontinuities of the amplitudes for the transitions $2 \to 4$ and $3 \to 3$. We find an explicit solution of the BFKL equation for the octet channel for arbitrary momentum transfers and investigate the intercepts of the Regge singularities in this channel. As an important result we find that the universal collinear and infrared singularities of the BDS formula are not affected by this Regge-cut contribution. Any improvement of the BDS formula should reproduce this cut to all orders in the coupling.

1 Introduction

In a recent work [1] we have investigated the high energy Regge behavior of MHV scattering amplitudes in the planar limit of $N = 4$ supersymmetric Yang Mills Theories, and we have found that, for n-point amplitudes with $n > 5$ beyond the one loop approximation, the simple factorizing structure of the Bern-Dixon-Smirnov (BDS) conjecture [2] is not valid. In detail, it was shown that for the cases of the transitions $2 \to 4$ and $3 \to 3$ their factorized form is violated by Regge cut contributions which satisfy the BFKL equation [3] in the color octet channel. These terms are obtained from specific single energy discontinuities, and in the scattering amplitudes they become visible in particular physical kinematic regions only. In the one loop approximation, these terms are correctly contained in the BDS formula, but in higher orders they cannot be cast into the simple exponential form conjectured by Bern et al.

In this paper we further investigate these Regge cut contributions. We study the BFKL equation in the color octet state, and we compute the two-loop expressions for the $2 \to 4$ and $3 \to 3$ amplitudes. In particular, we show that the collinear and infrared divergences of the BDS formula are not affected by the Regge cut contributions.

The paper is organized as follows. In section 2 we briefly review the derivation of the factorization-breaking contributions, and we write down the expression for the Regge-cut contribution, using the calculus of complex momenta. Sections 3 - 5 are devoted to the detailed investigation of this Regge-cut contribution: we first (section 3) study the structure of the infrared singularities, we then (section 4) compute the two loop expressions for the cut contributions, and finally in section 5 we obtain the explicit solution of the BFKL equation for the octet channel. In the final section we present conclusions and further strategies. Solutions of the BFKL equation for the forward case are presented in an appendix.

2 The Regge cut contribution: review and representation in terms of complex momenta

In our previous paper we have studied the high energy Regge behavior of scattering amplitudes of $N = 4$ supersymmetric Yang Mills theories. In the leading logarithmic approximation (LLA) we can make use of the QCD calculations since the supersymmetric partners of quarks and gluons do not contribute (in this limit t channel exchanges with the highest spin dominate). We now summarize the main results of [1]. For n-point amplitudes with $n > 4$, the high energy scattering amplitudes can be written as sums of separate pieces (named 'analytic representation' or 'dispersion representation'). This decomposition reflects the analytic structure required by the Steinmann relations [4]. The different terms appearing in this representation can be computed from single energy discontinuities ('imaginary parts') or multiple energy discontinuities. To be definite, we consider the $2 \to 4$ and the $3 \to 3$ scattering amplitudes, illustrated in Figs. 1 and 2. For the $2 \to 4$ scattering we are interested in the kinematic limit (double

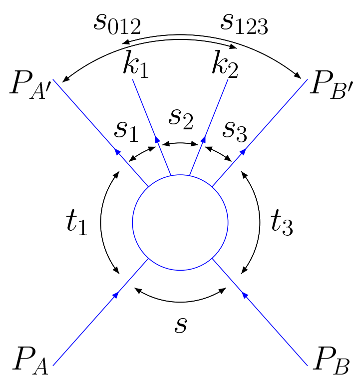

Figure 1: Kinematics of the $2 \to 4$ amplitude

Regge limit)

$$s \gg s_1, s_2, s_3 \gg t_1, t_2, t_3, \tag{1}$$

whereas the $3 \to 3$ scattering process will be studied in the limit

$$s \gg s_{13}, s_{02} \gg s_1, s_3, t_2' = (p_A - p_{A'} - k_2)^2 \gg t_1, t_2, t_3. \tag{2}$$

The analytic decompositions are illustrated in Figs. 3 and 4.

In the physical region where all energies are positive, there are, both for the $2 \to 4$ and for the $3 \to 3$ case, substantial cancellations of the Regge cut contributions between these five terms: in the sum their imaginary parts cancel, and the amplitude takes the well-known factorized Regge form. In other physical regions, however, where some energies are positive and others are negative, the cancellations are less complete, and pieces become visible which do not show up in the region of only positive energies. For the $2 \to 4$ case, the physical region of interest is

$$s > 0, \; s_2 > 0, \qquad s_1 < 0, \; s_3 < 0, \; s_{012} < 0 \;, s_{123} < 0. \tag{3}$$

Here non-vanishing discontinuities are only in s and s_2, and both of them contain a new term which violates the simple factorizing form. It contains a Regge-cut structure which is described in terms of the color octet BFKL equation. We illustrate these discontinuities in Fig. 5.

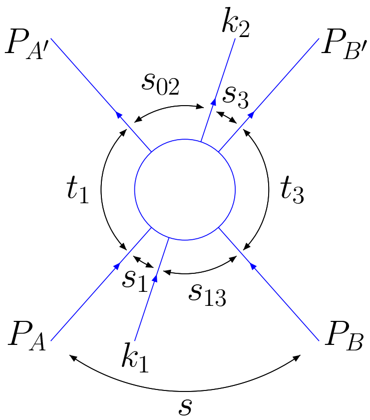

Figure 2: Kinematics of the $3 \to 3$ amplitude

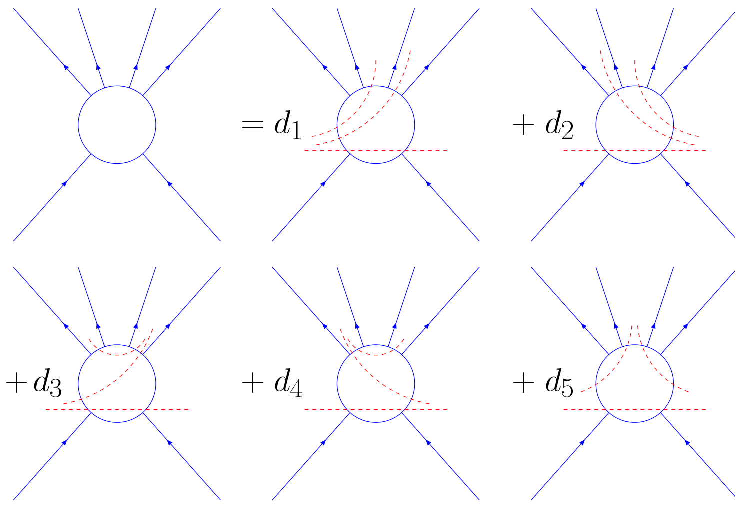

Figure 3: Analytic representation of the amplitude $M_{2\to 4}$

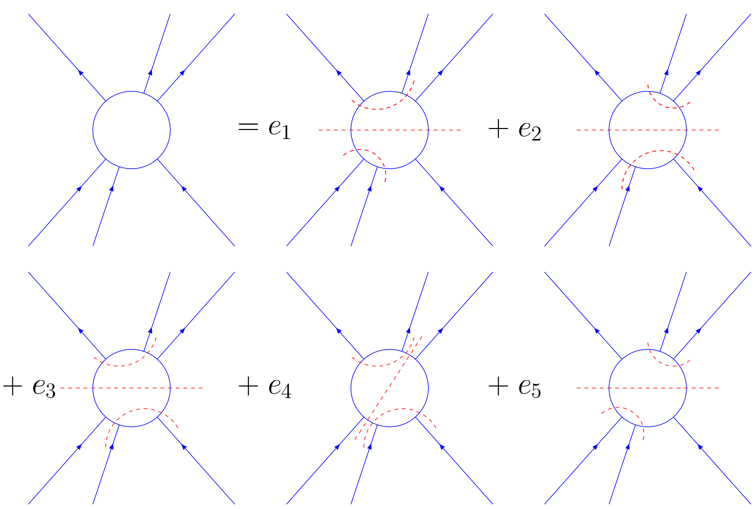

Figure 4: Analytic representation of the amplitude $M_{3\to 3}$

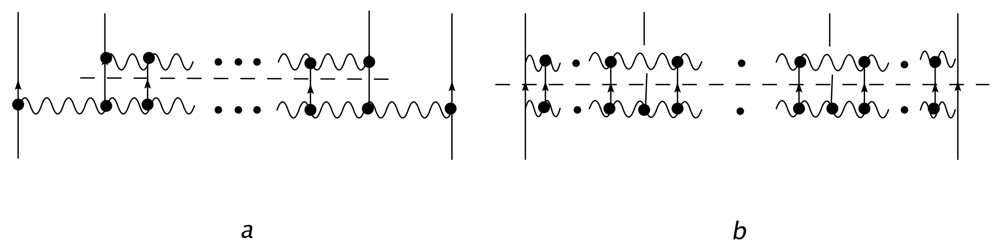

Figure 5: (a) the s_2 discontinuity for the $2 \to 4$ amplitude; (b) the s discontinuity for the $2 \to 4$ amplitude

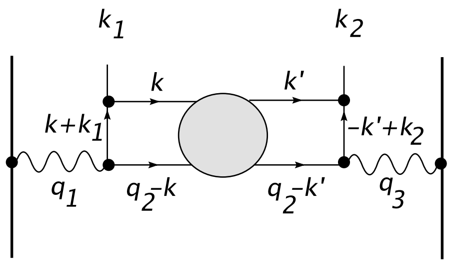

Figure 6: The s_2 discontinuity for the $2 \to 4$ amplitude. The big blob denotes the BFKL gluon Green's function in colour octet state including the reggeization of the lines with momentum $k, q_2 - k, k'$ and $q_2 - k'$. The dots indicate the production vertices in eqs. (7) and (13). This is equivalent to Fig. 5 (a) indicating the momentum notation.

For the $3 \to 3$ scattering amplitude the corresponding region is

$$s > 0,\ t_2' > 0, \qquad s_1 < 0,\ s_3 < 0,\ s_{13} < 0\ , s_{02} < 0. \tag{4}$$

The non-vanishing discontinuities belong to s and $t_2' > 0$, and they, again, contain the Regge cut pieces.

In [1] we have compared these results with the expression given by Bern et al. Whereas for the $2 \to 2$ and $2 \to 3$ amplitudes the QCD results are in full agreement with the BDS formula, the $2 \to 4$ and $3 \to 3$ BDS amplitudes are correct only in the one loop approximation. For two or more loops, the Regge cut piece cannot be reproduced by the BDS expression. As explained above, this implies that the BDS formula (in LLA) still gives the correct result in the physical region where all energies are positive, but it fails (beyond one loop) in the regions (3) and (4).

In the following we shall investigate these Regge cut pieces in more detail. Rather than returning to the five terms illustrated in Figs. 3 and 4, we directly present an explicit Feynman diagram calculation of the single energy discontinuities in s_2 and s for the $2 \to 4$ amplitude and in t_2' and s for the $3 \to 3$ amplitude. Let us begin with the s_2 discontinuity illustrated in Fig. 6. Here the blob in the center denotes the BFKL Green's function in the color octet channel which sums the s-channel emissions in the center of Fig. 5a, and on both sides we have to convolute this Green's function with the 'impact factors' Φ_1 and Φ_2. Introducing complex momenta

$$k = k_x + ik_y,\ k^* = k_x - ik_y \tag{5}$$

and making use of the expression for the vertex describing the production of a gluon with definite helicity (cf. eq.(6) of [1]):

$$C_\mu(\boldsymbol{q}_2, \boldsymbol{q}_1) e_\mu^*(\boldsymbol{k}_1) = \sqrt{2}\frac{q_2^* q_1}{k_1^*} \tag{6}$$

we obtain for the production vertex to the left of the Green's function

$$\sqrt{2}\frac{q_1(q_2 - k)^*}{(k + k_1)^*}. \tag{7}$$

Here we have used that, in Fig. 6, the gluon with momentum $\boldsymbol{k} + \boldsymbol{k}_1$ is on shell (we consider the discontinuity in s_2), and at the upper vertex where the gluon with momentum $\boldsymbol{k}$ is attached the outgoing gluon helicity is conserved. Since the scattering amplitude $T_{2\to n}$ for the case of the maximal helicity violation (MHV) can be written as [2]

$$T_{2\to n} = T_{2\to n}^{Born} \cdot M_{2\to n}, \tag{8}$$

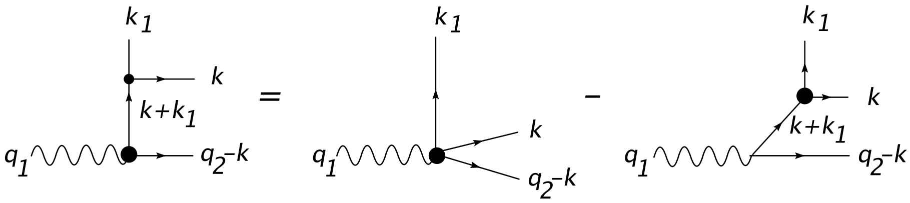

Figure 7: Decomposition of the production vertex in eq.(7). The dots denote the effective production vertex in eq.(6).

we will, throughout our paper, consider the factor $M_{2\to n}$ only. We, therefore, separate the production vertex of the Born approximation and rewrite (7) as:

$$\sqrt{2}\frac{q_1 q_2^*}{k_1^*}\,\Phi_1(\boldsymbol{k},\boldsymbol{q}_2,\boldsymbol{q}_1) \tag{9}$$

with the impact factor:

$$\begin{aligned}\Phi_1(\boldsymbol{k},\boldsymbol{q}_2,\boldsymbol{q}_1) &= \frac{k_1^*(q_2-k)^*}{q_2^*(k+k_1)^*}\\ &= 1-\frac{k^* q_1^*}{q_2^*(k+k_1)^*}.\end{aligned} \tag{10}$$

In the following we shall work with this impact factor.

In order to make contact with [1] we should note that, with the result in the second line of eq. (10), the production vertex in (7) can be written as a sum of two terms of the form

$$\sqrt{2}\frac{q_1(q_2-k)^*}{(k+k_1)^*} = \sqrt{2}\frac{q_1 q_2^*}{k_1^*} - \frac{q_1^2}{(k_1+k)^2}\sqrt{2}\frac{(k_1+k)k^*}{k_1^*}. \tag{11}$$

We illustrate this structure in Fig. 7. The first term is 'local', *i.e.* it has no further dependence on the internal momenta, whereas the second one is 'nonlocal'.

Similarly, on the right side of the Green's function in Fig. 6 we have

$$\sqrt{2}\frac{(q_2-k')q_3^*}{k_2-k'} = \sqrt{2}\frac{q_2 q_3^*}{k_2}\,\Phi_2(\boldsymbol{k}',\boldsymbol{q}_2,\boldsymbol{q}_3) \tag{12}$$

with the impact factor

$$\begin{aligned}\Phi_2(\boldsymbol{k}',\boldsymbol{q}_2,\boldsymbol{q}_3) &= \frac{k_2(k'-q_2)}{q_2(k'-k_2)}\\ &= 1-\frac{k' q_3}{(k'-k_2)q_2}\end{aligned} \tag{13}$$

and

$$\sqrt{2}\frac{(q_2-k')q_3^*}{k_2-k'} = \sqrt{2}\frac{q_2 q_3^*}{k_2} + \frac{q_3^2 k'}{(k_2-k')^2}\sqrt{2}\frac{(k_2-k')^*}{k_2}. \tag{14}$$

The discontinuity in s_2, to an arbitrary loop accuracy, of the amplitude $M_{2\to 4}$ in LLA then has the form:

$$\frac{1}{\pi}\Im_{s_2} M_{2\to 4} = \int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d\omega}{2\pi i}\left(\frac{s_2}{\mu^2}\right)^{\omega} f_2(\omega)\,, \tag{15}$$

where the t_2-channel partial wave is

$$f_2(\omega) = \hat{\alpha}_\epsilon\,\boldsymbol{q}_2^2\int d^{2-2\epsilon}k\,d^{2-2\epsilon}k'\,\Phi_1(\boldsymbol{k},\boldsymbol{q}_2,\boldsymbol{q}_1)\,G_\omega^{(8_A)}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}_2)\,\Phi_2(\boldsymbol{k}',\boldsymbol{q}_2,\boldsymbol{q}_3) \tag{16}$$

and

$$\hat{\alpha}_\epsilon = \frac{\alpha_s N_c \mu^{2\epsilon}}{(2\pi)^{2-2\epsilon}} \quad , a = \frac{\alpha_s N_c}{2\pi}\left(4\pi e^{-\gamma}\right)^\epsilon . \tag{17}$$

The overall factor $\boldsymbol{q}_2^2$ in front of the integral in (16) takes into account that the Born approximation of the amplitude contains the pole $1/|q_2|^2$ but in our calculations we are interested in the scattering amplitude $M_{2\to 4}$ with the Born factor being removed (cf. 8). The Green's function $G_\omega^{(8_A)}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}_2)$ satisfies the BFKL equation for the color octet channel (putting $\epsilon = 0$):

$$\omega G_\omega^{(8_A)}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}_2) = \frac{\delta^{(2)}(\boldsymbol{k}-\boldsymbol{k}')}{\boldsymbol{k}^2(\boldsymbol{k}-\boldsymbol{q}_2)^2} + \frac{1}{\boldsymbol{k}^2(\boldsymbol{k}-\boldsymbol{q}_2)^2}\left(K^{(8_A)}\otimes G_\omega^{(8_A)}\right)(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}_2), \tag{18}$$

where $K^{(8_A)}$ denotes the BFKL kernel in the color octet channel, containing both real emission and the gluon trajectory, and the convolution symbol stands for $\otimes = \int \frac{d^2k}{(2\pi)^3}$. Using complex momenta the kernel can be written in the form:

$$K^{(8_A)}(k,k';q_2) = \delta^{(2)}(k-k')\left(\omega(-|k|^2)+\omega(-|q_2-k|^2)\right) + \frac{a}{2}\frac{k^*(q_2-k)k'(q_2-k')^* + c.c.}{|k-k'|^2}, \tag{19}$$

where the gluon trajectory is

$$\omega(-k^2) = a\left(\frac{1}{\epsilon} - \ln\frac{k^2}{\mu^2}\right). \tag{20}$$

In contrast to the color singlet BFKL kernel, the color octet kernel is not infrared finite and needs to be dimensionally regularized. It is convenient to separate the singular pieces by writing the octet kernel as

$$\begin{aligned} K^{(8_A)}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q}_2) &= \delta^{(2)}(\boldsymbol{k}-\boldsymbol{k}')\left[\omega(-\boldsymbol{q}_2^2) + \frac{1}{2}\left(\omega(-\boldsymbol{k}^2)+\omega(-(\boldsymbol{q}_2-\boldsymbol{k})^2) - 2\omega(-\boldsymbol{q}_2^2)\right)\right] \\ &\quad + \frac{1}{2}K^{(1)}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q}_2) \\ &= \delta^{(2)}(\boldsymbol{k}-\boldsymbol{k}')\left[\omega(-\boldsymbol{q}_2^2) - \frac{a}{2}\ln\frac{\boldsymbol{k}^2(\boldsymbol{q}_2-\boldsymbol{k})^2}{\boldsymbol{q}_2^2\boldsymbol{q}_2^2}\right] + \frac{1}{2}K^{(1)}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q}_2). \end{aligned} \tag{21}$$

In this expression, $K^{(1)}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q}_2)$ denotes the color singlet BFKL kernel, and infrared singularities are contained in the trajectory function $\omega(-\boldsymbol{q}_2^2)$.

Inserting this form of the octet kernel into (15), the discontinuity takes the form:

$$\frac{1}{\pi}\Im_{s_2} M_{2\to 4} = s_2^{\omega(t_2)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\omega}{2\pi i}\left(\frac{s_2}{\mu^2}\right)^\omega \widetilde{f}_2(\omega) \tag{22}$$

where the reduced partial wave $\widetilde{f}_2(\omega)$ is given by

$$\widetilde{f}_2(\omega) = \hat{\alpha}_\epsilon\, \boldsymbol{q}_2^2 \int d^{2-2\epsilon}k\, d^{2-2\epsilon}k'\, \Phi_1(\boldsymbol{k},\boldsymbol{q}_2,\boldsymbol{q}_1)\, \widetilde{G}_\omega(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}_2)\, \Phi_2(\boldsymbol{k}',\boldsymbol{q}_2,\boldsymbol{q}_3). \tag{23}$$

The Green's function $\widetilde{G}_\omega(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}_2)$ satisfies the BFKL equation (18) with the reduced kernel

$$\tilde{K}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q}_2) = -\delta^{(2)}(\boldsymbol{k}-\boldsymbol{k}')\frac{a}{2}\ln\frac{\boldsymbol{k}^2(\boldsymbol{q}_2-\boldsymbol{k})^2}{\boldsymbol{q}_2^2\boldsymbol{q}_2^2} + \frac{1}{2}K^{(1)}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q}_2). \tag{24}$$

From the explicit form of the function $\widetilde{f}_2(\omega)$ and of the impact factors Φ_i one sees that there are potential divergences only for $|\boldsymbol{k}| \sim |\boldsymbol{k}'| \to 0$ (and not for $|\boldsymbol{q}_2-\boldsymbol{k}| \sim |\boldsymbol{q}_2-\boldsymbol{k}'| \to 0$). The one loop contribution to the partial wave, $\tilde{f}_2^{(0)}$, takes the form

$$\begin{aligned}\omega \widetilde{f}_2^{(0)}(\omega) &= \hat{\alpha}_\epsilon\, \boldsymbol{q}_2^2 \int d^{2-2\epsilon}k\, \Phi_1(\boldsymbol{k},\boldsymbol{q}_2,\boldsymbol{q}_1)\, \frac{1}{\boldsymbol{k}^2(\boldsymbol{q}_2-\boldsymbol{k})^2} \Phi_2(\boldsymbol{k},\boldsymbol{q}_2,\boldsymbol{q}_3) \\ &= \frac{a}{2}\left(\ln \frac{\boldsymbol{k}_1^2\boldsymbol{k}_2^2}{(\boldsymbol{k}_1+\boldsymbol{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right). \end{aligned} \tag{25}$$

In our previous paper [1] we isolated the term which violates the BDS factorization ansatz. This term, named V_{cut}, is contained in (16): in the impact factors Φ_1 and Φ_2 one retains only the nonlocal pieces (cf. (11) and Fig. 7), and one subtracts the Regge pole contribution. The one loop contribution was given in eqs. (94) and (95) of [1]. In the normalization of (16) it reads:

$$\hat{\alpha}_\epsilon\, \boldsymbol{q}_2^2 \int d^{2-2\epsilon}k\, \frac{k^* q_1^*}{q_2^*(k+k_1)^*}\, \frac{1}{\boldsymbol{k}^2(\boldsymbol{q}_2-\boldsymbol{k})^2}\, \frac{k q_3}{q_2(k-k_2)} = \frac{a}{2}\left(\ln \frac{\boldsymbol{q}_1^2\boldsymbol{q}_3^2}{(\boldsymbol{k}_1+\boldsymbol{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right) \tag{26}$$

and it was shown to coincide (apart from an overall factor) with the phase factor C in eq.(75) of [1]. In this paper we address the discontinuity in s_2, for which we do not need to split the impact factors into local and nonlocal pieces. However, for the discontinuity in s we will come back to the result (26).

In the following sections we will study the reduced partial wave (23) and the reduced kernel (24) in some detail. First we will investigate the infrared properties and show that the infrared divergence is contained only in the one-loop approximation (25), *i.e.* the reduced kernel is infrared finite and introduces no further divergences. This implies that the divergent term $\sim 1/\epsilon$ is not renormalized. We will then compute explicitly the two loop approximation to the reduced partial wave. Finally, we will return to the reduced kernel and calculate its eigenfunctions and eigenvalues.

3 Infrared properties and eigenvalues of the octet kernel

In this section we concentrate on the infrared properties of the octet kernel. By investigating the most singular part of the reduced partial wave (23), we find the exact expression for the eigenvalues, and we prove that the infrared singularity of the reduced partial wave coincides with the $1/\epsilon$ pole of the one loop approximation. The exact solution of the color octet BFKL equation will be derived in section 5, and it allows to find a closed expression for the reduced partial wave.

The starting point of our further discussion is eq.(23). During this section we will denote the transverse momenta by $\boldsymbol{p}, \boldsymbol{p}'$ and $p = p_x + ip_y, p^* = p_x - ip_y$. The Green's function $\widetilde{G}_\omega$ satisfies the 'renormalized' equation

$$\omega \widetilde{G}_\omega(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q}) = \frac{1}{\boldsymbol{p}^2(\boldsymbol{q}-\boldsymbol{p})^2}\, \delta^2(\boldsymbol{p}-\boldsymbol{p}') - a\, \widetilde{H}\, \widetilde{G}_\omega(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q}_2)\,, \tag{27}$$

where

$$\widetilde{H} = \ln \frac{|p|^2|q-p|^2}{|q|^2} + \frac{1/p}{q^*-p^*}\frac{\ln|\rho|^2}{2}\, p(q-p)^* + \frac{1/p^*}{q-p}\frac{\ln|\rho|^2}{2}\, p^*(q-p) + 2\gamma\,. \tag{28}$$

As we stated before, the most interesting region is the infrared divergent region $|p| \sim |p'| \ll |q|$. In this asymmetric kinematic it is possible to find eigenvalues and eigenfunctions of the reduced kernel. First, the expression (23) for $\tilde{f}_2$ is simplified:

$$\widetilde{f}_2(\omega) = \hat{\alpha}_\epsilon \int d^2p\, d^2p'\, \widetilde{g}_\omega(\vec{p},\vec{p}'), \tag{29}$$

and $\widetilde{g}_\omega$ satisfies the equation

$$\omega \widetilde{g}_\omega(\vec{p},\vec{p}') = \frac{1}{|p|^2}\, \delta^2(p-p') - a\, \widetilde{H}\, \widetilde{g}_\omega(\vec{p},\vec{p}')\,. \tag{30}$$

Here

$$\widetilde{H} = \ln|p|^2 + \frac{1}{p}\frac{\ln|\rho|^2}{2}\,p + \frac{1}{p^*}\frac{\ln|\rho|^2}{2}\,p^* + 2\gamma \tag{31}$$

and $\gamma = -\psi(1)$ is the Euler constant.

The Hamiltonian for the octet quantum numbers has the property of the holomorphic separability

$$\widetilde{H} = \widetilde{h}_8 + \widetilde{h}_8^*\,,\ \widetilde{h}_8 = \ln p + \frac{\ln\rho}{2} + \frac{1}{p}\frac{\ln\rho}{2}\,p + \gamma\,. \tag{32}$$

The holomorphic Hamiltonian h_8 is slightly different from the corresponding Hamiltonian for the singlet case

$$h = \frac{h_P}{2} = \ln p + \frac{1}{p}\,(\ln\rho)\,p + \gamma\,. \tag{33}$$

The difference is also in the normalization conditions for the wave functions in these two cases

$$||\Psi||_8^2 = \int d^2p\,\Psi^*|p|^2\Psi\,,\ ||\Psi||_{BFKL}^2 = \int d^2p\,\Psi^*|p|^4\Psi\,. \tag{34}$$

The eigenfunctions belonging to the principal series of the unitary representations of the Möbius group in the holomorphic subspace have the different form

$$\Psi_8^{(m)} = p^{-3/2+m}\,,\ \Psi_{BFKL}^{(m)} = p^{-2+m}\,,\ m = \frac{1}{2} + i\nu + \frac{n}{2}\,. \tag{35}$$

The eigenvalue of the total Hamiltonian for the octet case is given by

$$E_8^{(m,\widetilde{m})} = \epsilon_8^{(m)} + \epsilon_8^{(\widetilde{m})}\,,\ \epsilon_8^{(m)} = \frac{1}{2}\psi\left(\frac{3}{2}-m\right) + \frac{1}{2}\psi\left(\frac{1}{2}+m\right) - \psi(1)\,. \tag{36}$$

To verify this result we act on the wave function $f(k)$ with amputated propagators with the Hamiltonian regularized by a mass parameter μ^2

$$\begin{aligned}
\widetilde{H}f &= \ln\frac{|k|^2}{\mu^2}\,|k|^{2i\nu}\left(\frac{k}{k^*}\right)^{n/2} - \int\frac{d^2k'}{2\pi|k'|^2}\,\frac{kk'^* + k^*k'}{(|k-k'|^2+\mu^2)}\,|k'|^{2i\nu}\left(\frac{k'}{k'^*}\right)^{\frac{n}{2}} \\
&= \ln\frac{|k|^2}{\mu^2}\,|k|^{2i\nu}\left(\frac{k}{k^*}\right)^{\frac{n}{2}} - \int_0^1 dx\int\frac{d^2k'\,(1-x)^{-i\nu+\frac{n}{2}}\,(1-i\nu+\frac{n}{2})\,(kk'^*+k^*k')\,k'^n}{2\pi\,(|k'-xk|^2 + x(1-x)|k^2| + x\mu^2)^{2-i\nu+\frac{n}{2}}} \\
&= \left(\ln\frac{|k|^2}{\mu^2} - \frac{\frac{n}{2}}{\nu^2+\frac{n^2}{4}} - \int_0^1\frac{dx\,(1-x)^{-i\nu+\frac{n}{2}}\,x^{i\nu+\frac{n}{2}}}{\left(1-x+\frac{\mu^2}{|k|^2}\right)^{1-i\nu+\frac{n}{2}}}\right)|k|^{2i\nu}\left(\frac{k}{k^*}\right)^{n/2}\,.
\end{aligned} \tag{37}$$

One immediately sees that the result is finite when μ^2 is taken to zero.

From this expression we can obtain the eigenvalue

$$\begin{aligned}
E_{\nu n} &= \frac{1}{2}\left(\frac{1}{i\nu-\frac{|n|}{2}} - \frac{1}{i\nu+\frac{|n|}{2}}\right) + \psi(1+i\nu+|n|/2) + \psi(1-i\nu+|n|/2) - 2\psi(1) \\
&= \Re\,\psi(1+i\nu+n/2) + \Re\,\psi(1+i\nu-n/2) - 2\psi(1)\,.
\end{aligned} \tag{38}$$

In particular, for $n = 0, 1$, we have

$$\begin{aligned}
E_{\nu 0} &= 2\Re\,\psi(1+i\nu) - 2\psi(1), \\
E_{\nu 1} &= \frac{1}{2}\frac{1}{\nu^2+\frac{1}{4}} + 2\Re\,\psi(\frac{1}{2}+i\nu) - 2\psi(1)\,.
\end{aligned} \tag{39}$$

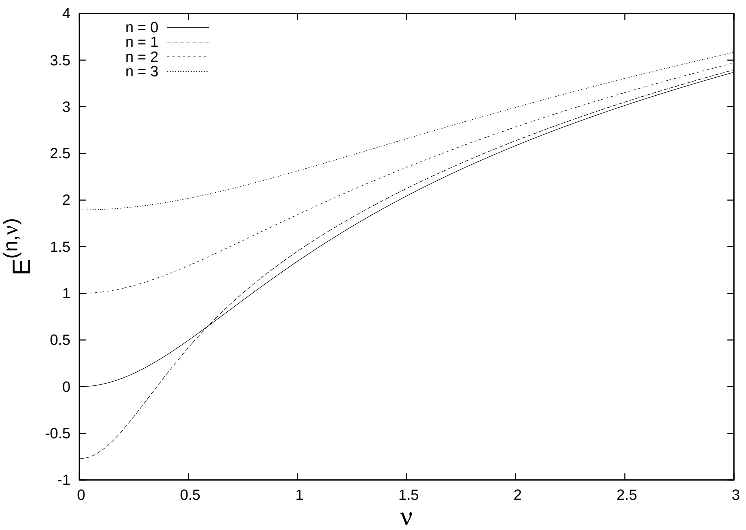

Figure 8: Spectrum of the lowest eigenvalues

The corresponding lowest energies are $E^{(0)} = 0$ and $E^{(1)} = 2 - 4\ln 2 < 0$. In Fig. 8 we show the ν-dependent eigenvalues for different values of n. Thus, the ground state energy corresponds to $|n| = 1$, as it was in the case of the colorless Odderon state [5]. In section 5 we will re-derive this spectrum by solving the eigenvalue problem exactly.

The solution of the equation for the Green's function can be found with the use of the completeness condition for the eigenfunctions

$$\tilde{g}_\omega(\vec{p},\vec{p}') = \frac{1}{|p|^2|p'|^2}\,\frac{1}{\pi}\sum_{n=-\infty}^{\infty} e^{i\,n\,(\phi-\phi')}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\left(\frac{|p|^2}{|p'|^2}\right)^{i\nu}\frac{1}{\omega-\omega(\nu,n)}\,. \tag{40}$$

Here

$$\omega(\nu,n) = -\frac{g^2\,N_c}{8\pi^2}\,E_{\nu n} = \frac{g^2\,N_c}{8\pi^2}\,\left(2\psi(1) - \Re\,\psi\left(1+i\nu+\frac{n}{2}\right) - \Re\psi\left(1+i\nu-\frac{n}{2}\right)\right)\,. \tag{41}$$

Inserting this expression into (29) one sees that $\tilde{f}_2$, in this approximation, reduces to the $1/\epsilon$-pole of the one-loop expression: the reduced color octet BFKL Green's function only leads to finite corrections to the one-loop result and introduces no further infrared divergences. Therefore, the divergent contribution is

$$\frac{1}{\pi}\Im_{s_2}M_{2\to4}|_{div} = -\frac{a}{\epsilon}\,s_2^{\omega(t_2)+\omega_n(0,0)} = -\frac{a}{\epsilon}\,s_2^{\omega(t_2)}\,, \tag{42}$$

where $\omega(0,0)$ is the leading eigenvalue of the 'reduced' octet BFKL kernel discussed before (note that, in this 'infrared' approximation, the impact factors are equal to 1, and the solution belonging to the eigenvalue $\omega(0,1)$ does not contribute). Thus, the divergent contribution $\sim 1/\epsilon$ is not renormalized: its asymptotic behavior corresponds to the usual gluon Regge pole. The reason for this is that the collinear and infrared divergences are factorized and that the BDS ansatz is valid in the one-loop approximation.

4 Contributions at two loops in the perturbative expansion

Let us return to the reduced partial wave $\tilde{f}_2(\omega)$ in eq.(23) and compute the first terms in the perturbative expansion. The one loop approximation has already been given in (25). As mentioned before, it contains the infrared singularity coming from the region $|k| \to 0$. In this section we iterate the integral equation for the Green's function (27) inside the partial wave and, using the calculus of complex momenta, compute the two loop expression. Starting, in Fig. 6, from the impact factor on the right, given in eq.(13), we have the following expression for the first iteration (using the reduced color octet BFKL Hamiltonian in eq.(24)):

$$\begin{aligned}\widetilde{H}\Phi_2 &= -a\,\ln\frac{|k|^2|q_2-k|^2}{|q_2|^2\,\mu^2}\,\Phi_2(\boldsymbol{k},\boldsymbol{q}_2,\boldsymbol{q}_3)\\ &+ a\int\frac{d^2k'}{2\pi}\,\frac{kk'^*(q_2^*-k^*)(q_2-k')+k^*k'(q_2-k)(q_2^*-k'^*)}{(|k-k'|^2+\mu^2)\,|k'|^2\,|q_2-k'|^2}\,\Phi_2(\boldsymbol{k}',\boldsymbol{q}_2,\boldsymbol{q}_3)\,. \qquad (43)\end{aligned}$$

Here μ^2 plays the rôle of an intermediate infrared cut-off which will be removed at the end of our calculations. The result of integration can be written in the form

$$\widetilde{H}\Phi_2 = \frac{k_2}{q_2}\,\frac{1}{2}\,\chi(\boldsymbol{k})\,, \qquad (44)$$

where

$$\chi(\boldsymbol{k}) = -a\left(\frac{q_2-k}{k_2-k}\,\ln\frac{|k|^2|q_2-k|^2|k_2|^2}{|q_2|^4|k-k_2|^2} + \frac{q_2}{k_2}\,\ln\frac{|q_2|^2}{|k|^2} + \frac{(q_2-k_2)k}{k_2(k_2-k)}\,\ln\frac{|q_2-k_2|^2}{|k_2-k|^2}\right)\,. \qquad (45)$$

Next we perform the integration over k with the impact factor on the right hand side in eq.(10), and we obtain for the two-loop approximation of the imaginary part in the s_2-channel in (22):

$$\begin{aligned}A_{s_2} &= \int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d\omega}{2\pi i}\left(\frac{s_2}{\mu^2}\right)^{\omega}\,\widetilde{f}_2(\omega)\\ &= -\frac{\pi}{2}\boldsymbol{q}_2^2\,\frac{k_2}{q_2}\,a\,\ln s_2\int\frac{d^2k}{2\pi\,|k|^2|q_2-k|^2}\,\Phi_1(\boldsymbol{k},\boldsymbol{q}_2,\boldsymbol{q}_1)\,\chi(\boldsymbol{k})\,. \qquad (46)\end{aligned}$$

With the use of complex number algebra this two-loop expression can be reduced to the form

$$A_{s_2} = -\frac{\pi}{2}a^2\,\ln s_2\int\frac{d^2k}{2\pi}\,\rho(\boldsymbol{k})\,, \qquad (47)$$

where

$$\begin{aligned}\rho(\boldsymbol{k}) &= \left(\frac{1}{|k|^2}-\frac{1}{k(k^*+k_1^*)}\right)\ln\frac{|q_2-k|^2|k_2|^2}{|q_2|^2|k-k_2|^2}\\ &+ \left(\frac{1}{k-q_2}\frac{1}{k^*}-\frac{1}{k-q_2}\frac{1}{k^*+k_1^*}\right)\ln\frac{|q_2-k_2|^2|k|^2}{|q_2|^2|k-k_2|^2}\\ &+ \left(\frac{1}{k-k_2}\frac{1}{k^*+k_1^*}-\frac{1}{k-k_2}\frac{1}{k^*}\right)\ln\frac{|q_2-k|^2|k|^2|q_2-k_2|^2|k_2|^2}{|q_2|^4|k-k_2|^4}\,. \qquad (48)\end{aligned}$$

One can easily verify that the ultraviolet divergences cancel. Also, in agreement with the previous sections, the divergence at $k=0$ is absent. The above integrals over k can be expressed (with the shift $k\to k+c$) in terms of the following expression:

$$f(\boldsymbol{a},\boldsymbol{b}) \equiv \int\frac{d^2k}{\pi(k-a)(k^*-b^*)}\,\ln|k|^2\,. \qquad (49)$$

To regularize the ultraviolet divergence we introduce the cut-off

$$|k|^2 < \Lambda^2\,, \qquad (50)$$

which at the end cancels in the expression for A_{s_2}. One can then write f in the form

$$f(\boldsymbol{a},\boldsymbol{b}) = \frac{\ln^2\Lambda^2}{2} + f_r(\boldsymbol{a},\boldsymbol{b}) \tag{51}$$

and use further the regularized value f_r because $\ln^2\Lambda$ is canceled in the final result. To calculate this function we take derivatives in the complex coordinates a^* and b

$$\frac{\partial}{\partial a^*} f = -\frac{1}{a^*-b^*}\ln|a|^2\,,\ \frac{\partial}{\partial b} f = -\frac{1}{b-a}\ln|b|^2\,. \tag{52}$$

After integrating these expressions we obtain

$$\begin{aligned} f_r(\boldsymbol{a},\boldsymbol{b}) &= -\int_0^1 \frac{dx}{x-\frac{b^*}{a^*}}\ln x - \int_0^1 \frac{dy}{y-\frac{a}{b}}\ln y \\ &+ \ln|a|^2\,\ln\frac{b^*}{a^*-b^*} + \ln|b|^2\,\ln\frac{a}{b-a} - \frac{1}{2}\ln^2(a\,b^*)\,. \end{aligned} \tag{53}$$

The last term was obtained as an integration constant: it can be determined from the conditions that it must depend upon a and $b*$, and the full function f should depend on the invariants

$$|a|^2 = a\,a^*\,,\ |b|^2 = b\,b^*\,,\ a\,b^* = \boldsymbol{a}\boldsymbol{b} - i[\boldsymbol{a},\boldsymbol{b}]_3\,,\ a^*\,b = \frac{|a|^2|b|^2}{a\,b^*}\,. \tag{54}$$

Moreover, from dimensional considerations it follows that it contains the term $\frac{1}{2}\ln^2 s$, where the invariant s has the dimension of Λ^2.

The function f_r can be expressed in terms of the Spence's function (dilogarithm)

$$\begin{aligned} f_r(\boldsymbol{a},\boldsymbol{b}) &= -Li_2\left(\frac{a^*}{b^*}\right) - Li_2\left(\frac{b}{a}\right) - \frac{1}{2}\ln^2(a\,b^*) \\ &+ \ln|a|^2\,\ln\frac{b^*}{a^*-b^*} + \ln|b|^2\,\ln\frac{a}{b-a}\,, \end{aligned} \tag{55}$$

where

$$Li_2(z) = -\int_0^z \frac{\ln(1-t)}{t}\,dt\,. \tag{56}$$

Note that the above expression for f has the following properties

$$f(-\boldsymbol{a},-\boldsymbol{b}) = f(\boldsymbol{a},\boldsymbol{b})\,,\ f^*(\boldsymbol{a},\boldsymbol{b}) = f(\boldsymbol{b},\boldsymbol{a})\,. \tag{57}$$

In some particular cases it can be simplified. For example,

$$f_r(0,\boldsymbol{b}) = \int \frac{d^2k}{\pi k\,(k^*-b^*)}\,\ln|k|^2 = -\frac{1}{2}\,\ln^2(|b|^2)\,. \tag{58}$$

We shall use also the values of the integrals

$$\int \frac{d^2k}{\pi|k|^2}\,\ln\frac{|k-c|^2}{|c|^2} = \frac{1}{2}\,\ln^2\frac{\Lambda^2}{|c|^2}\,,\ \int \frac{d^2k}{\pi(k-a)(k-b^*)} = \ln\frac{\Lambda^2}{|a-b|^2}\,. \tag{59}$$

With these results we can calculate the two-loop contribution to the imaginary part of the amplitude in the s_2-channel:

$$\begin{aligned} -\frac{4}{a^2\ln s_2}\frac{A_{s_2}}{\pi} &= \ln\frac{|k_1|^2}{|q_2|^2}\,\ln\frac{|k_2|^2}{|q_2|^2} - f_r(-\boldsymbol{q}_2,-\boldsymbol{q}_2-\boldsymbol{k}_1) + f_r(-\boldsymbol{k}_2,-\boldsymbol{k}_1-\boldsymbol{k}_2) \\ &+\ln\frac{|q_2+k_1|^2}{|q_2|^2}\,\ln\frac{|q_2-k_2|^2}{|q_2|^2} + f_r(\boldsymbol{q}_2,0) - f_r(\boldsymbol{q}_2,-\boldsymbol{k}_1) + f_r(\boldsymbol{q}_2-\boldsymbol{k}_2,-\boldsymbol{k}_1-\boldsymbol{k}_2) \\ &-f_r(\boldsymbol{q}_2-\boldsymbol{k}_2,-\boldsymbol{k}_2) + \ln\frac{|k_2|^2}{|k_1+k_2|^2}\,\ln\frac{|q_2-k_2|^2|k_2|^2}{|q_2|^4} + f_r(\boldsymbol{k}_2,-\boldsymbol{k}_1) - f_r(\boldsymbol{k}_2,0) \\ &+f_r(\boldsymbol{k}_2-\boldsymbol{q}_2,-\boldsymbol{k}_1-\boldsymbol{q}_2) - f_r(\boldsymbol{k}_2-\boldsymbol{q}_2,-\boldsymbol{q}_2) - 2f_r(0,-\boldsymbol{k}_1-\boldsymbol{k}_2) + 2f_r(0,-\boldsymbol{k}_2)\,. \end{aligned} \tag{60}$$

Using the following properties of dilogarithms

$$
\begin{aligned}
Li_2\left(\frac{1}{z}\right) &= -Li_2(z) - \frac{1}{2}\ln^2(-z) - \zeta_2\,, \\
Li_2\,(1-z) &= -Li_2(z) - \ln(1-z)\ln z + \zeta_2\,, \\
Li_2\left(\frac{z}{1-z}\right) &= -Li_2(z) - \frac{1}{2}\ln^2(1-z)\,, \\
Li_2\left(\frac{1}{1-z}\right) &= Li_2(z) + \ln(1-z)\ln(-z) - \frac{1}{2}\ln^2(1-z) + \zeta_2\,, \\
Li_2\left(\frac{z-1}{z}\right) &= Li_2(z) + \ln(1-z)\ln z - \frac{1}{2}\ln^2 z - \zeta_2\,,
\end{aligned} \tag{61}
$$

we can simplify the following sums entering in $2A_{s_2}$

$$
\begin{aligned}
-f_r(\boldsymbol{q}_2,\boldsymbol{q}_2+\boldsymbol{k}_1) - f_r(\boldsymbol{q}_2,-\boldsymbol{k}_1) &= \ln|k_1|^2\ln|q_1|^2 - 2\zeta_2\,, \\
f_r(\boldsymbol{k}_2,\boldsymbol{k}_1+\boldsymbol{k}_2) + f_r(\boldsymbol{k}_2,-\boldsymbol{k}_1) &= -\ln|k_1|^2\ln|q_1-q_3|^2 + 2\zeta_2\,, \\
f_r(\boldsymbol{q}_3,-\boldsymbol{k}_1-\boldsymbol{k}_2) + f_r(\boldsymbol{q}_3,\boldsymbol{q}_1) &= -\ln|q_1|^2\ln|q_1-q_3|^2 + 2\zeta_2\,, \\
-f_r(\boldsymbol{q}_3,-\boldsymbol{k}_2) - f_r(\boldsymbol{q}_3,\boldsymbol{q}_2) &= \ln|k_2|^2\ln|q_2|^2 - 2\zeta_2\,.
\end{aligned} \tag{62}
$$

The final result for A_{s_2} can be written in the very simple form

$$
A_{s_2} = -\pi\frac{a^2}{4}\ln s_2 \ln\frac{|q_1-q_3|^2|q_2|^2}{|q_1|^2|k_2|^2}\ln\frac{|q_1-q_3|^2|q_2|^2}{|q_3|^2|k_1|^2}\,. \tag{63}
$$

It is symmetric with respect to the simultaneous substitutions

$$
\boldsymbol{k}_1 \leftrightarrow \boldsymbol{k}_2\,,\ \boldsymbol{q}_1 \leftrightarrow -\boldsymbol{q}_3\,. \tag{64}
$$

In a similar way we can calculate the discontinuity in the s-channel. Starting, in Fig. 5b, from the gluon ladders in the t_1 and the t_3 channels, we invoke the bootstrap equation. This equation allows us to write, instead of the gluon ladders, simple Regge pole exchanges. The resulting s-discontinuity has the same form as the s_2 discontinuity with the impact factors Φ_1 and Φ_2 being replaced by

$$
\widetilde{\Phi}_1 = \frac{k^*}{k^*+k_1^*}\frac{q_1^*}{q_2^*}\,,\ \widetilde{\Phi}_2 = \frac{q_3}{q_2}\frac{k'}{k'-k_2}\,. \tag{65}
$$

One easily verifies that these modified impact factors coincide with the nonlocal pieces of Φ_1 and Φ_2 in eqs.(10) and (13). We also note that A_s can be obtained from A_{s_2} by substituting

$$
k_1 \leftrightarrow -q_1\,,\ k_2 \leftrightarrow q_3, \tag{66}
$$

and by changing, inside Fig.6, the integration variables $k \to q_2 - k$, $k' \to q_2 - k'$. In fact, in the two loop approximation, A_s coincides with A_{s_2}:

$$
A_s = -\pi\frac{a^2}{4}\ln s_2 \ln\frac{|q_1-q_3|^2|q_2|^2}{|q_1|^2|k_2|^2}\ln\frac{|q_1-q_3|^2|q_2|^2}{|q_3|^2|k_1|^2} \tag{67}
$$

due to the energy-momentum conservation

$$
\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{q}_1 - \boldsymbol{q}_3\,. \tag{68}
$$

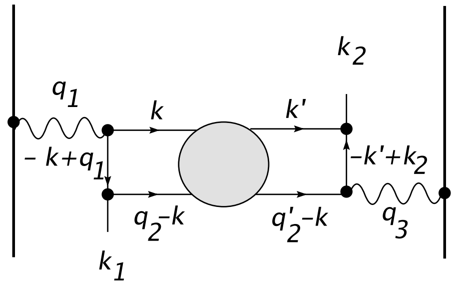

Figure 9: The t_2' discontinuity for the $3 \to 3$ amplitude.

Let us now consider the non-factorisable contribution for the scattering amplitude $3 \to 3$ (Fig. 2). In this case we, again, have the imaginary parts in t_2' and s-channel. For the imaginary part in the t_2'-channel (Fig. 9) we have, on the left side, a slightly modified impact factor, $\widehat{\Phi}_1$: the two corresponding impact factors are

$$\widehat{\Phi}_1(\boldsymbol{k},\boldsymbol{q}_2,\boldsymbol{q}_1) = -\frac{k^*}{k^*-q_1^*}\,\frac{k_1^*}{q_2^*}\,,\ \ \Phi_2(k) = \frac{k'-q_2}{k'-k_2}\,\frac{k_2}{q_2} \tag{69}$$

and, therefore, the infrared divergence at $k=0$ is absent. For completeness, we first list the one loop results. The one loop result for the partial wave (analogous to (25)) is:

$$\hat{\alpha}_\epsilon \boldsymbol{q}_2^2 \int d^{2-2\epsilon}k\ \ \widehat{\Phi}_1(\boldsymbol{k},\boldsymbol{q}_2,\boldsymbol{q}_1)\frac{1}{k^2(q_2-k)^2}\Phi_2(\boldsymbol{k},\boldsymbol{q}_2,\boldsymbol{q}_3) = \frac{a}{2}\ln\frac{q_2^2(q_1+q_3-q_2)^2}{q_1^2q_3^2}. \tag{70}$$

Note that, in contrast to the $2 \to 4$ amplitude, there is no infrared divergence. Next we quote the term analogous to (26) which is obtained by retaining, in the impact factors, only the 'nonlocal' pieces:

$$\hat{\alpha}_\epsilon \boldsymbol{q}_2^2 \int d^{2-2\epsilon}k\ \ \frac{q_1^*(q_2-k)^*}{q_2^*(q_1-k)^*}\frac{1}{k^2(q_2-k)^2}\frac{kq_3}{q_2(k-q_2)} = \frac{a}{2}\ln\frac{q_2^2(q_1+q_3-q_2)^2}{k_1^2k_2^2}. \tag{71}$$

It coincides (up to an overall factor) with the exponent of C' in [1] (eq.(80)). In contrast to the $2 \to 4$ amplitude, this one loop result, again, is infrared finite.

For the two loop calculation we proceed in the same way as for the $2 \to 4$ case. Using our previous results for the function $\chi(\boldsymbol{k})$ we obtain

$$A_{t_2'} = \boldsymbol{q}_2^2\ a\ \ \ln t_2' \int \frac{d^2k}{2\pi\,|k|^2|q_2-k|^2}\,\frac{k_2}{q_2}\,\widehat{\Phi}_1(\boldsymbol{k},\boldsymbol{q}_2,\boldsymbol{q}_1)\,\chi(\boldsymbol{k})\,. \tag{72}$$

With the use of complex number algebra it is possible to transform this expression into the form

$$A_{t_2'} = -\frac{\pi}{2}a^2\ \ \ln t_2' \int \frac{d^2k}{2\pi}\,\widetilde{\rho}(\boldsymbol{k})\,, \tag{73}$$

where

$$\begin{aligned}
\widetilde{\rho}(\boldsymbol{k}) \ &= \ \left(\frac{1}{k(k^*-q_2^*)} - \frac{1}{k(k^*-q_1^*)}\right)\ln\frac{|q_2-k|^2|k_2|^2}{|q_2|^2|k-k_2|^2} \\
&+ \ \left(\frac{1}{|k-q_2|^2} - \frac{1}{k-q_2}\,\frac{1}{k^*-q_1^*}\right)\ln\frac{|q_2-k_2|^2|k|^2}{|q_2|^2|k-k_2|^2} \\
&+ \ \left(\frac{1}{k-k_2}\,\frac{1}{k^*-q_2^*} - \frac{1}{k-k_2}\,\frac{1}{k^*-q_1^*}\right)\ln\frac{|q_2|^4|k-k_2|^4}{|k|^2|k-q_2|^2|k_2|^2|q_2-k_2|^2}\,.
\end{aligned} \tag{74}$$

The integral over k can be expressed in terms of the function $f_r(\boldsymbol{a},\boldsymbol{b})$ introduced above:

$$\frac{2}{a^2\,\ln t_2'}A_{t_2'} = f_r(-\boldsymbol{k}_2,-\boldsymbol{k}_2+\boldsymbol{q}_1)-f_r(-\boldsymbol{q}_2,-\boldsymbol{q}_2+\boldsymbol{q}_1)-\ln\frac{|q_2|^2}{|q_1|^2}\ln\frac{|k_2|^2}{|q_2|^2}+f_r(-\boldsymbol{q}_2,0)$$
$$-f_r(-\boldsymbol{k}_2,\boldsymbol{q}_3)+f_r(\boldsymbol{q}_2-\boldsymbol{k}_2,-\boldsymbol{k}_2+\boldsymbol{q}_1)-f_r(\boldsymbol{q}_2,+\boldsymbol{q}_1)+\ln\frac{q_3^2}{q_2^2}\ln\frac{q_2^2}{k_1^2}$$
$$-\ln\frac{k_2^2q_3^2}{q_2^4}\ln\frac{(k_2-q_1)^2}{q_3^2}-2f_r(0,\boldsymbol{q}_1-\boldsymbol{k}_2)$$
$$+2f_r(0,\boldsymbol{q}_2-\boldsymbol{k}_2)+f_r(\boldsymbol{k}_2,\boldsymbol{q}_1)-f_r(\boldsymbol{k}_2,\boldsymbol{q}_2)+f_r(\boldsymbol{k}_2-\boldsymbol{q}_2,-\boldsymbol{q}_2+\boldsymbol{q}_1)-f_r(\boldsymbol{k}_2-\boldsymbol{q}_2,0)\,. \quad (75)$$

With the use of the identities for the sums of the functions $f(\boldsymbol{a},\boldsymbol{b})$ listed in (62), we can significantly simplify $A_{t_2'}$:

$$A_{t_2'} = -\frac{\pi}{4}a^2\,\ln t_2'\,\ln\frac{|q_2-q_1-q_3|^2|q_2|^2}{|k_1|^2|k_2|^2}\,\ln\frac{|q_2-q_1-q_3|^2|q_2|^2}{|q_3|^2|q_1|^2}\,. \quad (76)$$

Thus, $A_{t_2'}$ is different from A_{s_2} and A_s by the substitution $\boldsymbol{q}_1\leftrightarrow-\boldsymbol{k}_1$. In fact, one can also verify that the same result is obtained for the imaginary part in s for the $3\to3$ transitions.

$$A_s^{3\to3} = -\frac{\pi}{4}a^2\,\ln t_2'\,\ln\frac{|q_2-q_1-q_3|^2|q_2|^2}{|k_1|^2|k_2|^2}\,\ln\frac{|q_2-q_1-q_3|^2|q_2|^2}{|q_3|^2|q_1|^2}\,. \quad (77)$$

As indicated before, all the two loop results are infrared finite and, hence, do not affect the infrared singularities in the BDS formula. In the next section we find the explicit solution at all loops.

5 Solution of the BFKL equation in the octet channel

In this section we solve the eigenvalue problem for the reduced color octet kernel and derive all-order expression for the $2\to4$ and $3\to3$ amplitudes in the leading-log approximation. For the eigenvalue problem it is convenient to return to the symmetric notations of the momenta $p_1=p$, $p_2=q-p$ and write the homogeneous BFKL equation for the wave function f with the removed propagators in the octet channel as follows

$$E\,f(\vec{p}_1,\vec{p}_2)=\widetilde{H}\,f(\vec{p}_1,\vec{p}_2)\,, \quad (78)$$

where $\widetilde{H}$ has the holomorphic separability property

$$\widetilde{H}=\widetilde{h}+\widetilde{h}^*\,,\ \widetilde{h}=\ln\frac{p_1\,p_2}{q}+\frac{1}{2}\left(p_1\ln\rho_{12}\,\frac{1}{p_1}+p_2\ln\rho_{12}\,\frac{1}{p_2}\right)+\gamma\,. \quad (79)$$

With the use of the relations (see [6])

$$\ln(z^2\partial)=\ln z+\frac{1}{2}\left(\psi(z\partial)+\psi(-z\partial+1)\right)\,,\ \ln(\partial)=-\ln z+\frac{1}{2}\left(\psi(z\partial+1)+\psi(-z\partial)\right) \quad (80)$$

one can transform the holomorphic Hamiltonian to the form

$$\widetilde{h}=-\ln q+\frac{1}{2}\left(\ln\left(p_1^2\rho_{12}\right)+\ln\left(p_2^2\rho_{12}\right)\right)+\gamma\,. \quad (81)$$

By introducing the conjugated variables

$$y=\frac{p_1}{p_2}\,,\ \partial=\frac{\partial}{\partial y}=-i\,\frac{p_2^2}{q}\,\rho_{12}\,, \quad (82)$$

$\widetilde{h}$ can be simplified as follows

$$\widetilde{h}=\frac{1}{2}\left(\ln(y^2\partial)+\ln\partial\right)+\gamma=\frac{1}{2}\left(\psi(y\partial)+\psi(y\partial+1)\right)+\gamma, \quad (83)$$

where we neglected pure imaginary terms which cancel in $\widetilde{H}$.

Thus, the solution of the homogeneous BFKL equation in the momentum space can be found in the form

$$f_{\nu n}(\vec{k},\vec{q}) = \left(\frac{k}{q-k}\right)^{i\nu+\frac{n}{2}} \left(\frac{k^*}{q^*-k^*}\right)^{i\nu-\frac{n}{2}} . \tag{84}$$

The corresponding energies were calculated above

$$E_{\nu n} = \frac{1}{2}\left[\psi\left(i\nu+\frac{n}{2}\right)+\psi\left(-i\nu-\frac{n}{2}\right)+\psi\left(i\nu-\frac{n}{2}\right)+\psi\left(-i\nu+\frac{n}{2}\right)\right]-2\psi(1)\,. \tag{85}$$

The orthogonality condition for the above wave functions is

$$\int \frac{d^2k}{\pi|k|^2|q-k|^2}\, f^*_{\nu' n'}(\vec{k},\vec{q})\, f_{\nu n}(\vec{k},\vec{q}) = 2\pi\delta(\nu'-\nu)\,\delta_{n',n}\,. \tag{86}$$

Their completeness condition can be written as follows

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} d\nu\, f^*_{\nu n}(\vec{k}',\vec{q}')\, f_{\nu n}(\vec{k},\vec{q}) = 2\pi^2\delta^2(k'-k)\,\frac{|k|^2||q-k|^2}{|q|^2}\,. \tag{87}$$

Therefore the Green's function for the t-channel partial waves is

$$G_\omega(\vec{k},\vec{k}';\vec{q}) = \frac{1}{2\pi^2}\,\frac{|q|^2}{|k|^2||q-k|^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} d\nu\,\frac{f^*_{\nu n}(\vec{k}',\vec{q}')\, f_{\nu n}(\vec{k},\vec{q})}{\omega-\omega(\nu,n)}\,, \tag{88}$$

where

$$\omega(\nu,n) = -\frac{g^2N_c}{8\pi^2}\,E_{\nu n}\,. \tag{89}$$

With these results we can find explicit expressions for the s_2-discontinuity of the $2\to 4$ scattering amplitude and for the t_2'-discontinuity of the $3\to 3$ scattering amplitude. Starting from eq.(23), we have to convolute the octet channel Green's function with the corresponding impact factors. Returning to Fig. 6 and to the notation of section 2 we have to calculate the integral

$$\chi_2 = \int \frac{d^2k'}{2\pi}\,\frac{|q_2|^2}{|k'|^2|q_2-k'|^2}\left(\frac{q_2-k'}{k'}\right)^{i\nu+\frac{n}{2}}\left(\frac{q_2^*-k'^*}{k'^*}\right)^{i\nu-\frac{n}{2}}\frac{k_2(k'-q_2)}{(k'-k_2)q_2}. \tag{90}$$

The simplest way to calculate χ_2 is its differentiation in k_2^* with the subsequent integration, which gives

$$\chi_2 = -\frac{1}{2}\frac{1}{\left(i\nu-\frac{n}{2}\right)}\left(\frac{q_3^*}{k_2^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3}{k_2}\right)^{i\nu+\frac{n}{2}}. \tag{91}$$

In a similar way the integral over k gives

$$\chi_1 = \frac{1}{2}\frac{1}{\left(i\nu+\frac{n}{2}\right)}\left(-\frac{q_1}{k_1}\right)^{-i\nu-\frac{n}{2}}\left(-\frac{q_1^*}{k_1^*}\right)^{-i\nu+\frac{n}{2}}. \tag{92}$$

As a result, the imaginary part of the production amplitude in s_2 for the transition $2\to 4$ takes the form

$$\frac{1}{\pi}\Im_{s_2} M_{2\to 4} = \frac{a}{4\pi}s_2^{\omega(t_2)}\sum_{n=-\infty}^{\infty}(-1)^n Reg_{s_2}\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^*k_1^*}{k_2^*q_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3k_1}{k_2q_1}\right)^{i\nu+\frac{n}{2}} s_2^{\omega(\nu,n)}\,. \tag{93}$$

where the regularization refers to the divergence at $\nu=0, n=0.$ which appears only in in the one loop approximation. In appendix B we compute the one and two loop results (obtained from expanding $s_2^{\omega(\nu,n)} = 1+\ln s_2\omega(\nu,n)$), and verify the agreement with (25) and (63):

$$\frac{1}{\pi}\Im_{s_2} M_{2\to 4} = \frac{a}{2}\,s_2^{\omega(t_2)}\left(\ln\frac{|k_1|^2|k_2|^2}{|k_1+k_2|^2\mu^2}-\frac{1}{\epsilon}-\frac{a}{2}\ln s_2\,\ln\frac{|k_1+k_2|^2|q_2|^2}{|k_2|^2|q_1|^2}\,\ln\frac{|k_1+k_2|^2|q_2|^2}{|k_1|^2|q_3|^2}\right). \tag{94}$$

In an analogous way we compute the discontinuity in s. In (90) we replace the impact factor Φ_2 by $\widetilde{\Phi}_2$ (and similarly for Φ_1 in (92)), and proceed in the same way as before. The result can be written in the form

$$\frac{1}{\pi}\Im_s M_{2\to 4} = \frac{a}{4\pi}\, s_2^{\omega(t_2)} \sum_{n=-\infty}^{\infty}(-1)^n Reg_s \int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}} \left(\frac{q_3^* k_1^*}{k_2^* q_1^*}\right)^{i\nu-\frac{n}{2}} \left(\frac{q_3 k_1}{k_2 q_1}\right)^{i\nu+\frac{n}{2}} s_2^{\omega(\nu,n)}\,. \tag{95}$$

with the regularization prescription Reg_s for the singularity at $\nu = 0$, $n = 0$ which, again, applies to the one loop approximation and takes care of the difference between the discontinuities in s_2 and s.

As a result, the production amplitude $2 \to 4$ in the multi-Regge kinematics with $s, s_2 > 0$ and $s_1, s_3 < 0$ in the leading approximation can be written as follows

$$A_{2\to 4} = A_{2\to 4}^{BDS}(1 + i\Delta_{2\to 4})\,, \tag{96}$$

where

$$\Delta_{2\to 4} = \frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n \int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}} \left(\frac{q_3^* k_1^*}{k_2^* q_1^*}\right)^{i\nu-\frac{n}{2}} \left(\frac{q_3 k_1}{k_2 q_1}\right)^{i\nu+\frac{n}{2}} (s_2^{\omega(\nu,n)} - 1)\,. \tag{97}$$

has no infrared singularities. We mention that in the region $s, s_2 < 0$ and $s_1, s_3 > 0$ the scattering amplitude has the similar form

$$A_{2\to 4} = A_{2\to 4}^{BDS}(1 - i\Delta_{2\to 4})\,. \tag{98}$$

We emphasize that the correction $\Delta_{2\to 4}$ does not contribute outside these physical regions.

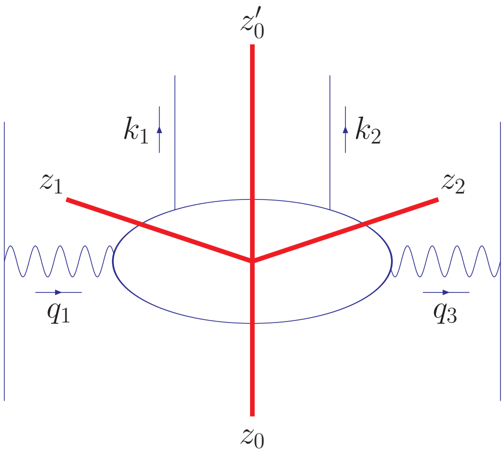

Figure 10: Dual variables for $A_{2\to 4}$.

It is noteworthy that if we perform the duality transformation shown in Fig. 10 (cf. Ref. [7])

$$q_1 \to z_{01}\,,\ k_1 \to z_{0'1}\,,\ q_3 \to z_{02}\,,\ k_2 \to z_{20'}\,, \tag{99}$$

and introduce 'coordinate' variables z_i, we see that our results for the imaginary parts depend on the anharmonic ratio

$$x = \frac{z_{02} z_{0'1}}{z_{0'2} z_{01}}\,. \tag{100}$$

The reason why the BFKL equation in the octet channel can be solved is its invariance under Möbius transformations in these z_i variables.

It is interesting to note that the correction to the BDS formula in our kinematics can be written in terms of four dimensional anharmonic ratios [8,9]. In particular, in second order of perturbation theory we can write

$$
\begin{aligned}
i\Delta^{(2)}_{2\to4} &= -2i\pi\frac{a^2}{4}\ln s_2 \ln\frac{|k_1+k_2|^2|q_2|^2}{|k_2|^2|q_1|^2}\ln\frac{|k_1+k_2|^2|q_2|^2}{|k_1|^2|q_3|^2} \\
&= \frac{a^2}{4}Li_2(1-\Phi)\ln\frac{(1-\Phi)}{\Phi_2}\ln\frac{(1-\Phi)}{\Phi_1}+\ldots
\end{aligned}
\qquad (101)
$$

where the dots indicate corrections beyond the leading logarithmic accuracy, and we have used the notation

$$
\Phi=\frac{ss_2}{s_{012}s_{123}},\ \Phi_1=\frac{s_1t_3}{s_{012}t_2},\ \Phi_2=\frac{s_3t_1}{s_{123}t_2}. \qquad (102)
$$

An analogous result holds for the $3\to3$ amplitudes (for details see Appendix B). The discontinuity in t_2' of the scattering amplitude $3\to3$ in the multi-Regge kinematics with $s,t_2'>0$ and $s_1,s_3<0$ in the leading approximation is given by

$$
\frac{1}{\pi}\Im_{t_2'}M_{3\to3}=\frac{a}{4\pi}\,t_2'^{\,\omega(t_2)}\sum_{n=-\infty}^{\infty}(-1)^n Reg_{t_2'}\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^*q_1^*}{k_2^*k_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3q_1}{k_2k_1}\right)^{i\nu+\frac{n}{2}}t_2'^{\,\omega(\nu,n)}\,, \qquad (103)
$$

(where, in this case, the regularized integral over ν for $n=0$ and $a=0$ does not contain any $1/\epsilon$ divergence), and the $3\to3$ amplitude takes the form

$$
A_{3\to3}=A^{BDS}_{3\to3}(1+i\Delta_{3\to3})\,, \qquad (104)
$$

where

$$
\Delta_{3\to3}=\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^*q_1^*}{k_2^*k_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3q_1}{k_2k_1}\right)^{i\nu+\frac{n}{2}}(t_2'^{\,\omega(\nu,n)}-1)\,. \qquad (105)
$$

In the region $s,t_2'<0$ and $s_1,s_3>0$ we can write

$$
A_{3\to3}=A^{BDS}_{3\to3}(1-i\Delta_{3\to3})\,. \qquad (106)
$$

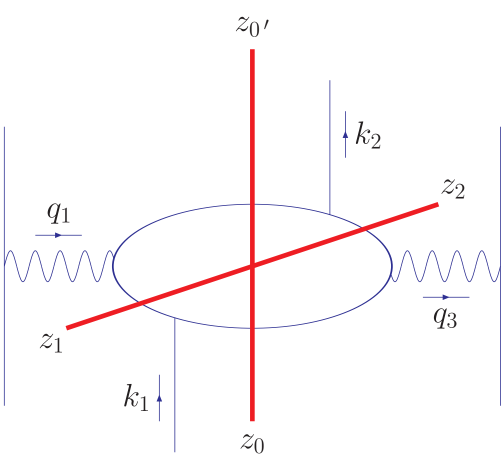

Figure 11: Dual variables for $A_{3\to3}$.

Similarly to the $2 \to 4$ scattering amplitude, if we perform the duality transformation (see Fig. 11)

$$q_1 \to z_{10'}\,,\ k_1 \to z_{01}\,,\ q_3 \to z_{02}\,,\ k_2 \to z_{20'}\,, \tag{107}$$

the imaginary parts of the $3 \to 3$ scattering amplitude in the t_2' and s channels depend on the same anharmonic ratio (100). Again, the corrections to the BDS formula can be expressed in terms of four dimensional anharmonic ratios.

From these results for the $2 \to 4$ and for the $3 \to 3$ amplitudes we conclude that the infrared structure of the inelastic amplitudes is given correctly by the BDS expression, whereas the finite factors are correct only in the one loop approximation.

6 Conclusions

In this paper we studied, in the leading logarithmic approximation, the cut contribution which, in our previous paper, was found to violate the simple Regge factorization of the BDS formula. As a main result we have verified that the factorization of universal infrared singularities is not affected, *i.e.* the violation of the BDS formula is in the finite pieces. We have computed the energy spectrum of the color octet BFKL Hamiltonian, and we have concluded that the infrared divergent gluon trajectory can be separated from the finite remainder of the BFKL Green's function. We have explicitly computed the two loop approximation of the Regge cut piece. The integral equation for the wave function of two reggeized gluons in the octet channel is solved explicitly and the intercepts of the Regge singularities are calculated.

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A Forward scattering with color octet exchange

In this appendix we relate the octet equation in the forward case to the BFKL equation for the singlet case. As it will be shown, its physical solution exists only if the gluon Regge trajectory is not expanded in a power series on ϵ.

The relevant integral equation to be satisfied by the gluon Green's function in the octet case reads

$$\omega G_\omega(\vec{p},\vec{p}\,',\vec{q}) = \frac{1}{|p|^2|q-p|^2}\,\delta^{2-2\epsilon}(p-p') - a\,H_\epsilon\,G_\omega(\vec{p},\vec{p}\,',\vec{q}_2)\,, \tag{A.1}$$

where

$$a = \frac{\alpha_s N_c}{2\pi}\left(4\pi e^{-\gamma}\right)^\epsilon, \tag{A.2}$$

and γ is the Euler's constant. From now on we shall use the notation $\vec{p}_1 \equiv \vec{p}$ and $\vec{p}_2 \equiv \vec{q} - \vec{p}$ for the transverse momenta of the two Reggeized gluons.

The BFKL Hamiltonian for the channel with the octet quantum numbers can be written in operator form as

$$H_\epsilon = \ln\frac{|p_1|^2}{\mu^2} - \frac{1}{\epsilon} + \ln|p_2|^2 + \frac{1}{p_1 p_2^*}\frac{\ln|\rho_{12}|^2}{2}p_1 p_2^* + \frac{1}{p_1^* p_2}\frac{\ln|\rho_{12}|^2}{2}p_1^* p_2 + 2\gamma\,, \tag{A.3}$$

where we have neglected terms of $\mathcal{O}(\epsilon)$ and introduced for the two Reggeized gluons the complex variables ρ_k, ρ_k^* ($\rho_{12} = \rho_1 - \rho_2$) and their canonically conjugated momenta p_k, p_k^*. The dependence on the divergence $1/\epsilon$ can be removed by the following shift in the parameter ω:

$$\omega \to \omega + \frac{a}{\epsilon}\,, \tag{A.4}$$

which leads to the appearance of a Sudakov-type infrared divergent factor in the amplitude $M_{2\to4}$, *i.e.*

$$M_{2\to 4} \to Z\, M_{2\to 4}\,,\;\; Z = \exp\left(\frac{a}{\epsilon}\,\ln\frac{s_2}{\mu^2}\right)\,. \tag{A.5}$$

We can then work with the renormalized Hamiltonian H removing the divergent term

$$H = H_\epsilon + \frac{1}{\epsilon}\,. \tag{A.6}$$

It is known that the BFKL equation in the color singlet channel is Möbius invariant in coordinate space. Its solutions are

$$E_{m,\widetilde{m}}(\vec\rho_{10},\vec\rho_{20}) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^m \left(\frac{\rho^*_{12}}{\rho^*_{10}\rho^*_{20}}\right)^{\widetilde{m}}\,, \tag{A.7}$$

where $m,\widetilde{m}$ are conformal weights

$$m = \frac{1}{2} + i\nu + \frac{n}{2}\,,\;\; \widetilde{m} = \frac{1}{2} + i\nu - \frac{n}{2} \tag{A.8}$$

and $\vec\rho_0$ is the Pomeron coordinate. The expression (A.7) corresponds to the three-point Green's function with non-amputated legs. The scalar product of two of these functions is defined by

$$< E_{m,\widetilde{m}}||E_{m',\widetilde{m}'} >= \int d^2\rho_1\, d^2\rho_2\, E_{m,\widetilde{m}}(\vec\rho_{10},\vec\rho_{20})\Delta_1\,\Delta_2 E^*_{m',\widetilde{m'}}(\vec\rho_{10'},\vec\rho_{20'})\,, \tag{A.9}$$

where Δ_k are the corresponding Laplace operators.

A.1 The solution for the octet case at $q_2 = 0$

The solution in momentum space for $q_2 = 0$ in the color singlet case can be obtained by using the Fourier transform of the singlet solution (A.7), *i.e.*

$$f_{m,\widetilde{m}}(\vec p) = \int d^2\rho_{12}\, d^2\rho_0 e^{i\vec p\vec\rho_{12}}\, E_{m,\widetilde{m}}(\vec\rho_{10},\vec\rho_{20}) \sim p^{m-2}\,(p^*)^{\widetilde{m}-2}\,. \tag{A.10}$$

It is convenient to introduce the new function $\phi_{\nu,n}(\vec p)$ as follows

$$f_{m,\widetilde{m}}(\vec p) = |p|^{-3}\phi_{\nu,n}(\vec p)\,, \tag{A.11}$$

with normalization

$$\phi_{\nu,n}(\vec p) = \left(\frac{|p|^2}{\mu^2}\right)^{i\nu} e^{i\alpha n}\,,\;\; p = |p|e^{i\alpha}\,. \tag{A.12}$$

These functions satisfy the following orthonormality and completeness properties

$$\int \frac{d^2p}{|p|^2}\,\phi_{\nu,n}(\vec p)\,\phi^*_{\nu',n'}(\vec p) \;=\; 2\pi^2\,\delta(\nu-\nu')\,\delta_{n,n'}\,, \tag{A.13}$$

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} d\nu\,\phi_{\nu,n}(\vec p)\,\phi^*_{\nu,n}(\vec p\,') \;=\; 4\pi^2\,\delta(\ln|p|^2 - \ln|p'|^2)\,\delta(\alpha-\alpha')\,. \tag{A.14}$$

The homogeneous BFKL equation for the octet case can be written in the form

$$\omega\psi = -aH\psi\,,\;\; H = \ln\frac{|p|^2}{\mu^2} + \frac{1}{2}H_0\,, \tag{A.15}$$

where H_0 is the Hamiltonian for the singlet case. Its solution can be constructed in terms of the linear combination of the functions $\phi_{\nu,n}(\vec{p})$

$$\psi_{\omega,n}(\vec{p}) = e^{i\alpha n} \int_{-\infty}^{\infty} d\nu \left(\frac{|p|^2}{\mu^2} \right)^{i\nu} a_{\omega,n}(\nu)\,. \tag{A.16}$$

Taking into account (A.15) the function $a_{\omega,n}(\nu)$ should satisfy the equation

$$\omega a_{\omega,n}(\nu) = -a \left[i \frac{\partial}{\partial \nu} + \psi\left(\frac{1}{2} + i\nu + \frac{|n|}{2} \right) + \psi\left(\frac{1}{2} - i\nu + \frac{|n|}{2} \right) + 2\gamma \right] a_{\omega,n}(\nu). \tag{A.17}$$

Its solution is

$$a_{\omega,n}(\nu) = \exp\left[i\nu \left(\frac{\omega}{a} + 2\gamma \right) \right] \frac{\Gamma\left(\frac{1}{2} + i\nu + \frac{|n|}{2} \right)}{\Gamma\left(\frac{1}{2} - i\nu + \frac{|n|}{2} \right)}\,. \tag{A.18}$$

For $a_{\omega,n}(\nu)$ we have the following normalization

$$\int_{-\infty}^{\infty} d\nu\, a_{\omega,n}(\nu)\, a^*_{\omega',n}(\nu) = 2\pi a \delta(\omega - \omega') \tag{A.19}$$

and completeness conditions

$$\int_{-\infty}^{\infty} d\omega\, a_{\omega,n}(\nu)\, a^*_{\omega,n}(\nu') = 2\pi a \delta(\nu - \nu')\,. \tag{A.20}$$

Therefore the completeness relation for the eigenfunctions $\psi_{\omega,n}(\vec{p})$ has the form

$$\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \psi_{\omega,n}(\vec{p})\, \psi^*_{\omega,n}(\vec{p'}) = a\,(2\pi)^2\, \delta(\ln|p|^2 - \ln|p'|^2)\, \delta(\alpha - \alpha')\,. \tag{A.21}$$

Using these expressions, we can solve the inhomogeneous equation for the Green's function $g_\omega(\vec{p}, \vec{p}', 0)$

$$\omega g_\omega(\vec{p}, \vec{p}', 0) = (2\pi)^2\, \frac{a}{2}\, |p|^2\, \delta^2(p - p') - a\, H\, g_\omega(\vec{p}, \vec{p}', 0) \tag{A.22}$$

in terms of a superposition of eigenfunctions of the homogeneous equation, *i.e.*

$$g_\omega(\vec{p}, \vec{p}', 0) = \sum_{n=-\infty}^{\infty} P \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, \frac{\psi_{\omega',n}(\vec{p})\, \psi^*_{\omega',n}(\vec{p'})}{\omega - \omega'}\,, \tag{A.23}$$

where P means that the integral over ω is taken with the principal value prescription.

In terms of this Green's function the t_2-channel partial wave can be written as

$$f_2(\omega) = \frac{Z}{4\pi} \int_0^{\infty} \frac{d|k|^2}{|k|^3} \int_0^{2\pi} \frac{d\alpha\, k^* k_1^*}{k^* + k_1^*} \int_0^{\infty} \frac{d|k'|^2}{|k'|^3} \int_0^{2\pi} \frac{d\alpha'\, k' k_2}{k_2 - k'}\, g_\omega(\vec{k}, \vec{k}', 0)\,, \tag{A.24}$$

where Z is the divergent factor discussed in (A.5). We can now write $f_2(\omega)$ in a different form using the explicit expression for $g_\omega(\vec{k}, \vec{k}', 0)$:

$$f_2(\omega) = \frac{Z}{4\pi} \sum_{n=-\infty}^{\infty} P \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i(\omega - \omega')}\, b_{\omega' n}(\vec{k}) \tag{A.25}$$

where

$$b_{\omega n}(\vec{k}_1) = \int_{-\infty}^{\infty} d\nu\, K_{\nu n}(\vec{k}_1)\, a_{\omega,n}(\nu)\,, \tag{A.26}$$

$$\widetilde{b}_{\omega n}(\vec{k}_2) = \int_{-\infty}^{\infty} d\nu\, \widetilde{K}_{\nu n}(\vec{k}_2)\, a^*_{\omega,n}(\nu)\,. \tag{A.27}$$

The functions K and $\widetilde{K}$ read

$$K_{\nu n}(\vec{k}_1) = \int_0^\infty \frac{d|k|^2\, k_1^*}{|k|^3} \frac{|k|^{2i\nu}}{\mu^{2i\nu}} \int_0^{2\pi} \frac{d\alpha\, k^* e^{i\alpha n}}{k^* + k_1^*} = \frac{(-1)^{n-1} 2\pi\, \Phi_{\nu,|n|}(\vec{k}_1)}{\frac{|n|-1}{2} + i\nu}\,, \tag{A.28}$$

$$\widetilde{K}_{\nu n}(\vec{k}_2) = \int_0^\infty \frac{d|k'|^2\, k_2}{|k'|^3} \frac{|k'|^{-2i\nu}}{\mu^{-2i\nu}} \int_0^{2\pi} \frac{d\alpha'\, k' e^{-i\alpha' n}}{k_2 - k'} = \frac{(-2\pi)\, \Phi^*_{\nu,|n|}(\vec{k}_2)}{\frac{|n|-1}{2} - i\nu}\,, \tag{A.29}$$

where

$$\Phi_{\nu,|n|}(\vec{k}_1) = \left(\frac{|k_1|^2}{\mu^2}\right)^{i\nu} e^{i\alpha_1(|n|-1)}. \tag{A.30}$$

Thus, one can obtain the following simple representation for the functions b and $\widetilde{b}$

$$b_{\omega n}(\vec{k}_1) = -2\pi(-1)^n\, e^{i\alpha_1(|n|-1)} c_{\omega n}(\vec{k}_1)\,, \tag{A.31}$$

$$\widetilde{b}_{\omega n}(\vec{k}_2) = -2\pi\, e^{-i\alpha_2(|n|-1)} c^*_{\omega n}(\vec{k}_1)\,, \tag{A.32}$$

where

$$c_{\omega n}(\vec{k}_1) = \int_{-\infty}^{\infty} d\nu\, e^{i\nu\left(\frac{\omega}{a} + 2\gamma + \ln\frac{|k_1|^2}{\mu^2}\right)} \frac{\Gamma(\frac{|n|-1}{2} + i\nu)}{\Gamma(\frac{|n|+1}{2} - i\nu)}\,. \tag{A.33}$$

The final result for the imaginary part of the amplitude in the variable s_2 reads

$$\frac{\Im_{s_2} M_{2\to 4}}{\pi} = Z\pi \sum_{n=-\infty}^{\infty} (-1)^n e^{i\alpha_{12}(|n|-1)} \int_{-\infty}^{\infty} d\omega \left(\frac{s_2}{\mu^2}\right)^{\omega} c_{\omega n}(\vec{k}_1)\, c^*_{\omega n}(\vec{k}_2)\,, \tag{A.34}$$

where $\alpha_{12} = \alpha_1 - \alpha_2$. There is an ambiguity in the integration over ν at $\nu = 0$ for $|n| = 1$, but at that point $c_{\omega n}(\vec{k}_1)$ does not depend on ω.

A.2 Spectrum quantization

It is important to note that in the region $\vec{p} \to 0$ we can not use the simplest form for the Regge trajectory in the Born approximation and we should write the exact expression instead:

$$\ln|p|^2 - \frac{1}{\epsilon} \to E_g(|p|) = -\frac{1}{\epsilon}\left(\frac{|p|^2}{\mu^2}\right)^{-\epsilon}\,. \tag{A.35}$$

The reason for this is that the solution $\psi_{\omega,n}(\vec{p})$ has a good behavior only for large $|p|$, *i.e.*

$$\lim_{|p|\to\infty} \psi_{\omega,n}(\vec{p}) \sim e^{i\alpha n} \left(\frac{|p|^2}{\mu^2}\right)^{-\frac{1+|n|}{2}}\,. \tag{A.36}$$

In the region of small $|p|$ its asymptotics is given by the saddle point contribution in the integral over ν and is not stable for the simplified expression $E_g(|p|)$ at $\epsilon \to 0$. The position of this saddle point ν is defined by the solution of the BFKL equation in the classical approximation

$$\omega = -a\left[E_g(|p|) + \frac{1}{\epsilon} + \psi\left(\frac{1}{2} + i\nu + \frac{|n|}{2}\right) + \psi\left(\frac{1}{2} - i\nu + \frac{|n|}{2}\right) + 2\gamma\right]\,. \tag{A.37}$$

In this expression we have used the exact expression $E_g(|p|)$ for the gluon energy. Let us indicate that, due to the symmetry of $E_0(\nu)$ under the substitution $\nu \to -\nu$, there are two solutions of this equation related by this symmetry and the semiclassical expression for $\psi_{\omega,n}(\vec{p})$ oscillates in this region.

The intercept Δ of the corresponding singularity in the $j-1$-plane of the t-channel corresponds to the values $\nu=0, n=0$ of the Möbius parameters (for $\epsilon<0$)

$$\Delta = -a \min_{|p|} \left(E_g(|p|) - 4\ln 2\right) = \frac{\omega_P}{2}\,,\ \omega_P = \frac{g^2}{\pi^2} N_c \ln 2\,, \tag{A.38}$$

where ω_P is the intercept of the BFKL Pomeron.

Let us solve the Schrödinger equation for the wave function with the modified expression for the Regge trajectory analytically. For this purpose we shall use the representation where the coordinate is

$$x = \ln \frac{|p|^2}{\mu^2}\,. \tag{A.39}$$

The Schrödinger equation has the form

$$E_0(\nu,n)\Psi_{\nu,n}(x) = \left(-\frac{e^{-\epsilon x}}{\epsilon} + \frac{H_0}{2}\right)\Psi_{\nu,n}(x)\,, \tag{A.40}$$

where $\epsilon \to -0$ and $E_0(\nu,n)$ is the total energy at $x\to-\infty$

$$E_0(\nu,n) = \psi\left(\frac{1}{2} + i\nu + \frac{|n|}{2}\right) + \psi\left(\frac{1}{2} - i\nu + \frac{|n|}{2}\right) + 2\gamma\,. \tag{A.41}$$

At $x\to-\infty$ the potential energy goes to zero and we can search for two solutions of this equation of the form

$$\Psi^{\pm}_{\nu,n}(x) = e^{\pm i\nu x}\sum_{r=0}^{\infty} C_r^{\pm}(\nu)\, e^{-\epsilon\, r\, x}\,, \tag{A.42}$$

where $C_r^{\pm}(\nu)$ satisfies the recurrence relation

$$E_0(\nu,n)\, C_r^{\pm}(\nu) = -\frac{1}{\epsilon}\, C_{r-1}^{\pm}(\nu) + E_0(\nu \pm i\, r, n)\, C_r^{\pm}(\nu). \tag{A.43}$$

Therefore one can write the following expression for the coefficients $C_r^{\pm}(\nu)$

$$C_r^{\pm}(\nu) = \left(\frac{-1}{\epsilon}\right)^r \prod_{t=1}^{r} \frac{1}{E_0(\nu,n) - E_0(\nu \pm i\, t, n)}\,. \tag{A.44}$$

In principle, the expansion in (A.42) with these coefficients is convergent for all values of x and therefore we could find at least numerically the linear combination of Ψ^+ and Ψ^- for which the wave function decreases at $x\to\infty$.

We consider now the case of small ν, where we can use the diffusion approximation for E_0:

$$E_0(\nu,n) = 2\,\psi\left(\frac{1}{2} + \frac{|n|}{2}\right) + 2\gamma - \psi''\left(\frac{1}{2} + \frac{|n|}{2}\right)\frac{\nu^2}{2}\,. \tag{A.45}$$

In particular for $n=0$ we have

$$E_0(\nu,n) = -4\ln 2 + 14\,\zeta(3)\,\nu^2\,. \tag{A.46}$$

In this case one obtains

$$C_r^{\pm}(\nu) = \left(\frac{-1}{14\,\zeta(3)\,\epsilon}\right)^r \prod_{t=1}^{r}\frac{1}{t(t\pm 2i\nu)} = \frac{(-14\,\zeta(3)\,\epsilon)^{-r}\Gamma(1\pm 2i\nu)}{\Gamma(r+1)\,\Gamma(r+1\pm 2i\nu)}\,. \tag{A.47}$$

As a result, we can express $\Psi^{\pm}$ with an appropriate normalization constant in terms of the Bessel function with imaginary argument (for $\epsilon < 0$)

$$\Psi^{\pm}_{\nu,0}(x) = I_{\pm 2i\nu}\left(\sqrt{\frac{2\,\exp(-\epsilon x)}{-7\zeta(3)\epsilon}}\right) . \tag{A.48}$$

The solution, which has the good asymptotic behavior at $x \to \infty$

$$\Psi_{\nu,0}(x) \sim e^{-\sqrt{\frac{2\,\exp(-\epsilon x)}{(-7\zeta(3)\epsilon)}}} , \tag{A.49}$$

is

$$\Psi_{\nu,0}(x) = K_{2i\nu}\left(\sqrt{\frac{2\,\exp(-\epsilon x)}{-7\zeta(3)\epsilon}}\right) , \tag{A.50}$$

where

$$K_{2i\nu}(z) = \frac{\pi}{2}\,\frac{I_{-2i\nu}(z) - I_{2i\nu}(z)}{\sin(2\pi i\nu)} . \tag{A.51}$$

B Calculation of the one and two loop contributions

In this appendix we calculate, starting from (93), the one and two loop approximations. For the one loop approximation we compute the integral:

$$M = \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + n^2/4} \beta^{\,i\nu} \alpha^{n/2} \,, \tag{B.1}$$

where

$$\alpha = \frac{q_3 k_1 q_1^* k_2^*}{q_3^* k_1^* q_1 k_2} \tag{B.2}$$

and

$$\beta = \frac{|q_3|^2 |k_1|^2}{|k_2|^2 |q_1|^2}. \tag{B.3}$$

We begin with the terms $n \neq 0$ and integrate over ν:

$$\begin{aligned} M_{n\neq 0} = \; &= \theta(\beta - 1) \left(\sum_{n=1}^{\infty} \frac{2\pi}{n} (-1)^n \beta^{-n/2} \left(\alpha^{n/2} + \alpha^{-n/2} \right) \right) \\ &\quad + \theta(1-\beta) \left(\sum_{n=1}^{\infty} \frac{2\pi}{n} (-1)^n \beta^{n/2} \left(\alpha^{n/2} + \alpha^{-n/2} \right) \right) \\ &= -\theta(\beta - 1) 2\pi \left(\ln \left((1 + \sqrt{\alpha}/\sqrt{\beta})(1 + 1/\sqrt{\alpha\beta}) \right) \right) \\ &\quad -\theta(1-\beta) 2\pi \left(\ln \left((1 + \sqrt{\beta}/\sqrt{\alpha})](1 + \sqrt{\alpha\beta}) \right) \right) \end{aligned} \tag{B.4}$$

Using (B.2) and (B.3) we obtain:

$$M_{n\neq 0} = -2\pi \left(\ln \frac{|k_1 + k_2|^2 |q_2|^2}{|k_1 k_2 q_1 q_3|} - \frac{1}{2} |\ln \beta| \right) \tag{B.5}$$

For the term $n = 0$ the divergence at $\nu = 0$ needs to be regularized. In order to reproduce the one loop result (25) derived in dimensional regularization we need

$$Reg_{s_2} \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2} \left| \frac{q_3 k_1}{k_2 q_1} \right|^{2i\nu} = 2\pi \left(-\frac{1}{\epsilon} + \ln \frac{|q_2|^2}{\mu^2} - \ln \left| \frac{q_3 q_1}{k_1 k_2} \right| - \left| \ln \frac{|q_3||k_1|}{|k_2||q_1|} \right| \right) . \tag{B.6}$$

The sum of this contribution for $n = 0$ and $M_{n\neq 0}$ is (apart from the overall factor) in agreement with the one loop result in (25).

Next let us consider the two-loop contribution. We need to calculate the following integral:

$$R = \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu \; E_{\nu n}}{\nu^2 + n^2/4} \beta^{\,i\nu} \alpha^{n/2} \tag{B.7}$$

where according to eq. (38) and (85)

$$E_{\nu n} = -\frac{|n|}{\nu^2 + \frac{n^2}{4}} + \sum_{k=0}^{\infty} \left(\frac{2}{k+1} - \frac{1}{k+1+i\nu+|n|/2} - \frac{1}{k+1-i\nu+|n|/2} \right) . \tag{B.8}$$

The integral over ν can be calculated by residues, and we take into account that the contributions from the poles $\nu = \pm i|n|/2$ exist only for $n \neq 0$. As for the other poles, they give contributions also at $n = 0$. Thus, we obtain

$$
\begin{aligned}
R \;=\;& \pi\theta(\beta-1)\sum_{n=1}^{\infty}(-1)^n(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\,\beta^{-\frac{n}{2}}\left(-\frac{\ln\beta}{n}-\frac{2}{n^2}+\frac{2}{n}\sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+1+n}\right)\right)\\
&+\pi\theta(1-\beta)\sum_{n=1}^{\infty}(-1)^n(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\,\beta^{\frac{n}{2}}\left(\frac{\ln\beta}{n}-\frac{2}{n^2}+\frac{2}{n}\sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+1+n}\right)\right)\\
&+2\pi\theta(\beta-1)\sum_{n=-\infty}^{\infty}(-1)^n\alpha^{\frac{n}{2}}\sum_{k=0}^{\infty}\beta^{-(k+1+\frac{|n|}{2})}\,\frac{1}{(k+1)(k+1+|n|)}\\
&+2\pi\theta(1-\beta)\sum_{n=-\infty}^{\infty}(-1)^n\alpha^{\frac{n}{2}}\sum_{k=0}^{\infty}\beta^{k+1+\frac{|n|}{2}}\,\frac{1}{(k+1)(k+1+|n|)}\\
=\;& -\pi\ln\beta\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\,\left(\theta(\beta-1)\,\beta^{-\frac{n}{2}}-\theta(1-\beta)\beta^{\frac{n}{2}}\right)\\
&+2\pi\sum_{n=1}^{\infty}(-1)^n(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\left(-\frac{1}{n^2}+\frac{1}{n}\sum_{r=1}^{n}\frac{1}{r}\right)\left(\theta(\beta-1)\,\beta^{-\frac{n}{2}}+\theta(1-\beta)\beta^{\frac{n}{2}}\right)\\
&+2\pi\theta(\beta-1)\left(\sum_{n=1}^{\infty}(-1)^n\left(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}}\right)\sum_{m=1+\frac{n}{2}}^{\infty}\frac{\beta^{-m}}{(m-\frac{n}{2})(m+\frac{n}{2})}+\sum_{m=1}^{\infty}\frac{\beta^{-m}}{m^2}\right)\\
=\;& -\pi\ln\beta\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\,\left(\theta(\beta-1)\,\beta^{-\frac{n}{2}}-\theta(1-\beta)\beta^{\frac{n}{2}}\right)\\
&\pi\theta(\beta-1)\left(\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\alpha^{\frac{n}{2}}\beta^{-\frac{n}{2}}\right)^2+\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\alpha^{-\frac{n}{2}}\beta^{-\frac{n}{2}}\right)^2\right)\\
&+\pi\theta(1-\beta)\left(\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\alpha^{\frac{n}{2}}\beta^{\frac{n}{2}}\right)^2+\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\alpha^{-\frac{n}{2}}\beta^{\frac{n}{2}}\right)^2\right)\\
&+2\pi\theta(\beta-1)\sum_{n_1=1}^{\infty}\frac{(-1)^{n_1}}{n_1}\alpha^{\frac{n_1}{2}}\beta^{-\frac{n_1}{2}}\sum_{n_2=1}^{\infty}\frac{(-1)^{n_2}}{n_2}\alpha^{-\frac{n_2}{2}}\beta^{-\frac{n_2}{2}}\\
&+2\pi\theta(1-\beta)\sum_{n_1=1}^{\infty}\frac{(-1)^{n_1}}{n_1}\alpha^{\frac{n_1}{2}}\beta^{\frac{n_1}{2}}\sum_{n_2=1}^{\infty}\frac{(-1)^{n_2}}{n_2}\alpha^{-\frac{n_2}{2}}\beta^{\frac{n_2}{2}}\,. \qquad \text{(B.9)}
\end{aligned}
$$

In obtaining two last contributions we passed to the new summation variables $n_1 = m + n/2$ and $n_2 = m - n/2$. These transformations give a possibility to write the total result for R in the following simple form

$$
\begin{aligned}
R &= \pi\theta(\beta-1)\left(-\ln\beta\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\beta^{-\frac{n}{2}}+\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\beta^{-\frac{n}{2}}\right)^2\right)\\
&+\pi\theta(1-\beta)\left(\ln\beta\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\beta^{\frac{n}{2}}+\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\beta^{\frac{n}{2}}\right)^2\right)\\
&= \pi\theta(\beta-1)\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\beta^{-\frac{n}{2}}\right)\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\beta^{-\frac{n}{2}}-\ln\beta\right)\\
&+\pi\theta(1-\beta)\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\beta^{\frac{n}{2}}\right)\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}})\beta^{\frac{n}{2}}+\ln\beta\right)\\
&= \pi\theta(\beta-1)\ln\left((1+\sqrt{\alpha}/\sqrt{\beta})(1+1/\sqrt{\alpha\beta})\right)\left(\ln\left((1+\sqrt{\alpha}/\sqrt{\beta})(1+1/\sqrt{\alpha\beta})\right)+\ln\beta\right)\\
&+\pi\theta(1-\beta)\ln\left((1+\sqrt{\beta}/\sqrt{\alpha})(1+\sqrt{\alpha\beta})\right)\left(\ln\left((1+\sqrt{\beta}/\sqrt{\alpha})(1+\sqrt{\beta\alpha})\right)-\ln\beta\right).
\end{aligned}
\tag{B.10}
$$

Using finally the above expression for $M_{n\neq 0}$ we obtain

$$
R=\pi\ln\frac{|k_1+k_2|^2|q_2|^2}{|k_2|^2|q_1|^2}\,\ln\frac{|k_1+k_2|^2|q_2|^2}{|k_1|^2|q_3|^2}\,. \tag{B.11}
$$

Combination of the one and two loop results leads to:

$$
\begin{aligned}
\frac{1}{\pi}\Im_{s_2}M_{2\to 4} &= \frac{a}{4\pi}\,s_2^{\omega(t_2)}\sum_{n=-\infty}^{\infty}(-1)^n Reg_{s_2}\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^*k_1^*}{k_2^*q_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3k_1}{k_2q_1}\right)^{i\nu+\frac{n}{2}}s_2^{\omega(\nu,n)}\\
&= \frac{a}{2}\,s_2^{\omega(t_2)}\left(\ln\frac{|k_1|^2|k_2|^2}{|k_1+k_2|^2\mu^2}-\frac{1}{\epsilon}-\frac{a}{2}\ln s_2\,\ln\frac{|k_1+k_2|^2|q_2|^2}{|k_2|^2|q_1|^2}\,\ln\frac{|k_1+k_2|^2|q_2|^2}{|k_1|^2|q_3|^2}+\mathcal{O}(a^2)\right).
\end{aligned}
\tag{B.12}
$$

Indeed, the second term of the expansion in a coincides with the result (63), obtained by an independent calculation.

In an analogous way the imaginary part in s can be written in the form

$$
\frac{1}{\pi}\Im_{s}M_{2\to 4} = \frac{a}{4\pi}\,s_2^{\omega(t_2)}\sum_{n=-\infty}^{\infty}(-1)^n Reg_{s}\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^*k_1^*}{k_2^*q_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3k_1}{k_2q_1}\right)^{i\nu+\frac{n}{2}}s_2^{\omega(\nu,n)}\,, \tag{B.13}
$$

where

$$
Reg_s\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2}\left|\frac{q_3k_1}{k_2q_1}\right|^{2i\nu}=2\pi\left(-\frac{1}{\epsilon}+\ln\frac{|q_2|^2}{\mu^2}+\ln\left|\frac{q_3q_1}{k_1k_2}\right|-\left|\ln\left|\frac{q_3k_1}{q_1k_2}\right|\right|\right)\,. \tag{B.14}
$$

It corresponds to the following expansion in a

$$
\begin{aligned}
&\frac{1}{\pi}\Im_{s}M_{2\to 4} =\\
&\frac{a}{2}\,s_2^{\omega(t_2)}\left(\ln\frac{|q_1|^2|q_3|^2}{|k_1+k_2|^2\mu^2}-\frac{1}{\epsilon}-\frac{a}{2}\ln s_2\,\ln\frac{|k_1+k_2|^2|q_2|^2}{|k_2|^2|q_1|^2}\,\ln\frac{|k_1+k_2|^2|q_2|^2}{|k_1|^2|q_3|^2}+\mathcal{O}(a^2)\right).
\end{aligned}
\tag{B.15}
$$

For the imaginary part of the amplitude $M_{3\to 3}$ in the variable t_2' we obtain the similar result

$$
\frac{1}{\pi}\Im_{t_2'}M_{3\to 3} = \frac{a}{4\pi}\,t_2'^{\,\omega(t_2)}\sum_{n=-\infty}^{\infty}(-1)^n Reg_{t_2'}\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^*q_1^*}{k_2^*k_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3q_1}{k_2k_1}\right)^{i\nu+\frac{n}{2}}t_2'^{\,\omega(\nu,n)}\,, \tag{B.16}
$$

where in this case the regularized integral over ν for $n=0$ and $a=0$ does not contain any $1/\epsilon$ divergence

$$
Reg_{t_2'}\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2}\left|\frac{q_3q_1}{k_2k_1}\right|^{2i\nu}=2\pi\left(\ln\left|\frac{q_3q_1}{k_1k_2}\right|-\left|\ln\left|\frac{q_3q_1}{k_1k_2}\right|\right|\right)\,. \tag{B.17}
$$

It gives the following a-expansion of $\Im_{t_2'}\,M_{3\to3}$

$$
\frac{1}{\pi}\Im_{t_2'}\,M_{3\to3}=
$$

$$
\frac{a}{2}\,t_2'^{\,\omega(t_2)}\left(\ln\frac{|q_1|^2|q_3|^2}{|q_1+q_3-q_2|^2|q_2|^2}-\frac{a}{2}\ln t_2'\,\ln\frac{|q_1+q_3-q^2|^2|q_2|^2}{|k_2|^2|k_1|^2}\,\ln\frac{|q_1+q_3-q_2|^2|q_2|^2}{|q_1|^2|q_3|^2}+\mathcal{O}(a^2)\right)\,. \tag{B.18}
$$

Analogously we find the the imaginary part of the amplitude $M_{3\to3}$ in the variable s

$$
\frac{1}{\pi}\Im_s\,M_{3\to3}=\frac{a}{4\pi}\,t_2'^{\omega(t_2)}\sum_{n=-\infty}^{\infty}(-1)^nReg_s\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^*q_1^*}{k_2^*k_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3q_1}{k_2k_1}\right)^{i\nu+\frac{n}{2}}t_2'^{\omega(\nu,n)}\,, \tag{B.19}
$$

where

$$
Reg_s\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2}\left|\frac{q_3q_1}{k_2k_1}\right|^{2i\nu}=-2\pi\left(-\ln\left|\frac{q_3q_1}{k_1k_2}\right|-\left|\ln\left|\frac{q_3q_1}{k_1k_2}\right|\right|\right)\,. \tag{B.20}
$$

The expansion in a beyond the one loop approximation coincides with that of $\Im_{t_2'}\,M_{3\to3}$:

$$
\frac{1}{\pi}\Im_s\,M_{3\to3}=
$$

$$
\frac{a}{2}\,t_2'^{\,\omega(t_2)}\left(\ln\frac{|k_1|^2|k_2|^2}{|q_1+q_3-q_2|^2|q_2|^2}-\frac{a}{2}\ln t_2'\,\ln\frac{|q_1+q_3-q^2|^2|q_2|^2}{|k_2|^2|k_1|^2}\,\ln\frac{|q_1+q_3-q_2|^2|q_2|^2}{|q_1|^2|q_3|^2}+\mathcal{O}(a^2)\right)\,. \tag{B.21}
$$

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