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Boundary Correlators in Supergroup WZNW Models

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Abstract

We investigate correlation functions for maximally symmetric boundary conditions in the WZNW model on GL(1|1). Special attention is payed to volume filling branes. Generalizing earlier ideas for the bulk sector, we set up a Kac-Wakimoto-like formalism for the boundary model. This first order formalism is then used to calculate bulk-boundary 2-point functions and the boundary 3-point functions of the model. The note ends with a few comments on correlation functions of atypical fields, point-like branes and generalizations to other supergroups.

Contents

1 Introduction

Sigma models on supergroups and their cosets are an interesting subject of current research. They occur in a number of very different problems ranging from string theory to disordered electron systems. In addition to such concrete applications, conformal field theories with target space supersymmetry may also be studied for their structural and mathematical properties. They provide examples of non-unitary models, many of which have vanishing or negative central charge. Moreover, their correlation functions often possess logarithmic singularities. As shown in [1], both properties are intimately related to features of the supergroup geometry.

The simplest non-trivial model to consider is the WZNW model on the supergroup GL(1|1). Studies of this field theory go back to the work of Rozanski and Saleur [2, 3]. These early investigations of the GL(1|1) WZNW model stimulated much further work on the emerging topic of logarithmic conformal field theory (see e.g. [4, 5] for a review). A few years back, the GL(1|1) WZNW model was revisited in [1] from a geometric rather than algebraic perspective. Based on the harmonic analysis of the supergroup GL(1|1), a proposal was formulated for the exact spectrum of the field theory. Furthermore, efficient computational tools were developed to calculate correlation functions of tachyon vertex operators. Finally, the consistency of the proposed spectrum was demonstrated explicitly.

The work [1] was restricted to the GL(1|1) WZNW model on the sphere, i.e. neither boundaries nor higher genus surfaces were included. Subsequent work [6] extended part of the bulk analysis to the boundary sector. In particular, the geometric interpretation of maximally symmetric boundary conditions was unravelled. This led to several proposals for the spectra of boundary operators in the corresponding boundary conformal field theories. These were tested partially through the so-called modular bootstrap. Correlation functions with non-trivial insertions of bulk and boundary operators were not computed in [6]. We are now aiming to close this gap, at least for one type of boundary conditions.

There are several motivations to determine boundary correlation functions in supergroup WZNW models. To begin with, the conjectured boundary spectra in [6] contained information that cannot be probed through the modular bootstrap alone. In particular, certain boundary correlation functions were predicted to contain logarithmic singularities. Below we shall be able to verify such features of the boundary conformal field theory. Moreover, 2-dimensional boundary field theories are intimately related with quantization theory (see e.g. [7, 8, 9, 10] and references therein). While the GL(1|1) WZNW model itself is a bit too simple to accommodate for interesting supersymmetric extensions of non-commutative geometry, the methods we shall develop below possess generalizations to cases with a curved bosonic base. The latter provide a much richer geometric framework, with further links to representation theory of affine algebras and the quantization of Lie superalgebras. Finally, let us also mention possible applications to the study of branes and open strings in superspaces, and in particular to AdS backgrounds.

To be a bit more specific about the results we are going to obtain, we recall from [6] that there are two different families of maximally symmetric boundary conditions in the GL(1|1) WZNW model. Geometrically, the first set consists of D-branes that are point-like localized in the bosonic base. They extend into both fermionic directions, unless they are placed along very special lines in the base manifold. The second set of boundary conditions contains a single object: a volume filling brane that extends in all bosonic and fermionic directions. We called this brane *twisted* because it is associated with the only non-trivial gluing automorphism of the current algebra. In [6], some simple amplitudes for the point-like D-branes have been computed. On the other hand, the methods of [6] were not sufficient to obtain non-trivial amplitudes for the volume filling brane.

In this work we shall extend some of the techniques from [1] to compute correlation functions of bulk and boundary operators for the volume filling brane. The main results include explicit formulas (4.2,4.7,4.9) for the bulk-boundary 2-point function and (4.16-4.19) for the boundary 3-point functions. The information they contain is equivalent to the bulk-boundary and the boundary operator product expansion, respectively. Our results provide a complete solution of the boundary theory for the volume filling brane. We shall also determine a non-trivial annulus amplitude.

In order to obtain these results we set up a first order formalism for the volume filling brane. It is obtained by adding an appropriate square root of the bulk interaction term along the boundary of the world-sheet. As in other theories containing fermions, taking the square root forces us to introduce an auxiliary fermion along the boundary. All this will be explained in great detail in section 2. A perturbative expansion for correlators of the boundary conformal field theory is set up in section 3. It is employed in Section 4 to solve explicitly the boundary GL(1|1) WZNW model with twisted boundary conditions. Section 5 contains an alternative approach to computing amplitudes that involve only special (atypical) fields/states of the theory. It is used to prove that the GL(1|1) WZNW contains a special subsector whose correlation functions are independent of the level k. The second approach is finally employed to compute a particular annulus amplitude for the volume filling brane. The latter provides a nice test for the boundary state that was proposed in [6]. We conclude with a list of open problems, mostly related to the point-like branes for GL(1|1) and extensions to higher supergroups.

2 Volume filling brane: The classical action

Our aim in this section is to discuss the classical description of volume filling branes in the GL(1|1) WZNW model. To begin with, we spell out the standard action of the

WZNW model with so-called twisted boundary conditions. Their geometric interpretation as volume filling branes with a non-zero B-field is recalled briefly. In order to set up a successful computation scheme for the quantum theory later on, we shall need a different formulation of the theory. As in the bulk theory, computations of correlations functions require a Kac-Wakimoto like representation of the model [1]. Finding such a first order formalism for the boundary theory is not entirely straightforward. We shall see that it requires introducing an additional fermionic boundary field.

2.1 The boundary WZNW model

Following our earlier work on WZNW models for type I supergroups, we parametrize the supergroup GL(1|1) through a Gauss-like decomposition of the form

$$g \;=\; e^{i\eta_-\psi^-}\, e^{ixE+iyN}\, e^{i\eta_+\psi^+}$$

where E, N and $\psi^\pm$ denote bosonic and fermionic generators of gl(1|1), respectively. In the WZNW model, the two even coordinates x, y become bosonic fields X, Y and similarly, two fermionic fields $c_\pm$ come with the odd coordinates $\eta_\pm$. Let us now consider a boundary WZNW model with the action

$$\begin{aligned} S_{\text{WZNW}}(X,Y,c_\pm) \;&=\; -\frac{k}{4\pi i}\int_\Sigma d^2z\ \left(\partial X\bar\partial Y + \partial Y\bar\partial X + 2e^{iY}\partial c_+\bar\partial c_-\right) + \\ &\quad + \frac{k}{8\pi i}\int du\ e^{iY}(c_+ + c_-)\partial_u(c_+ + c_-)\ , \end{aligned} \tag{2.1}$$

where u parametrizes the boundary of the upper half plane. Variation of the action leads to the usual bulk equations of motion along with the following set of boundary conditions

$$\begin{aligned} \partial_v Y \;=\; 0 \quad , \quad 2\partial_v X \;&=\; e^{iY}(c_+ + c_-)\,\partial_u(c_+ + c_-)\ , \\ \pm 2\partial_v c_\pm \;&=\; 2i\partial_u c_\mp - (c_- + c_+)\,\partial_u Y\ . \end{aligned} \tag{2.2}$$

Here, we have used the derivatives $\partial_u = \partial + \bar\partial$ and $\partial_v = i(\partial - \bar\partial)$ along and perpendicular to the boundary. The equations (2.2) imply Neumann boundary conditions for all four fields of our theory, i.e. we are dealing with a volume filling brane. Since the normal derivatives of the fields X and $c_\pm$ do not vanish, our brane comes equipped with a B-field. A more detailed discussion of the brane's geometry can be found in our recent paper [6].

In order to see that our boundary conditions preserve the full chiral symmetry, we recall that the holomorphic currents of the GL(1|1) WZNW model take the form

$$\begin{aligned} J^E &= ik\partial Y \quad , & J^N &= ik\partial X - kc_-\partial c_+ e^{iY} \ , \\ J^- &= -ke^{iY}\partial c_+ \quad , & J^+ &= k\partial c_- + ikc_-\partial Y \ , \end{aligned}$$

and similarly for the anti-holomorphic currents,

$$\begin{aligned} \bar{J}^E &= -ik\bar{\partial} Y \quad , & \bar{J}^N &= -ik\bar{\partial} X + k\bar{\partial} c_-\, c_+ e^{iY} \ , \\ \bar{J}^+ &= -ke^{iY}\bar{\partial} c_- \quad , & \bar{J}^- &= k\bar{\partial} c_+ + ikc_+\bar{\partial} Y \ . \end{aligned}$$

If we plug the boundary conditions (2.2) into these expressions for chiral currents, we obtain the gluing condition $J^X(z) = \Omega\bar{J}^X(\bar{z})$ for $X = E, N, \pm$ and all along the boundary at $z = \bar{z}$. Here, the relevant gluing automorphism Ω is obtained by lifting the automorphism

$$\Omega(E) \ = \ -E, \quad \Omega(N) \ = \ -N, \quad \Omega(\psi^+) \ = \ -\psi^-, \quad \Omega(\psi^-) \ = \ \psi^+ \tag{2.3}$$

from the finite dimensional superalgebra gl(1|1) to the full affine symmetry. In [6] we called these gluing conditions *twisted* and showed that there is a unique brane corresponding to this particular choice of Ω.

2.2 First order formulation

Computations of bulk and boundary correlators in the presence of twisted D-branes shall be performed in a first order formalism. In the bulk, it is well-known how this works [1]. There, the bulk action is built of a free field theory involving two additional fermionic auxiliary fields $b_\pm$ of weight $\Delta(b_\pm) = 1$ along with the original fields X, Y and $c_\pm$,

$$\begin{aligned} S^{\text{bulk}}_{0;\text{cl}}[X,Y,c_\pm,b_\pm] \ = \ & -\frac{k}{4\pi i}\int_\Sigma d^2z \ \left(\partial X\bar{\partial} Y + \partial Y\bar{\partial} X\right) \\ & -\frac{1}{2\pi i}\int_\Sigma d^2z \ \left(b_+\partial c_+ + b_-\bar{\partial} c_-\right) \ . \end{aligned} \tag{2.4}$$

We placed a subscript 'cl' on the actin to distinguish it from the action we shall use in our path integral computations later on. If the following bulk marginal interaction term is added to the free field theory,

$$S^{\text{bulk}}_{int}[X,Y,c_\pm,b_\pm] \ = \ -\frac{1}{2k\pi i}\int_\Sigma d^2z \ e^{-iY} b_- b_+ \tag{2.5}$$

the equations of motion for $b_\pm$ read $b_- = k\partial c_+ \exp iY$ and $b_+ = -k\bar{\partial} c_- \exp iY$ so that we recover the bulk WZNW-model upon insertion into the first order action. In extending this treatment to the boundary sector, we are tempted to add the "square root" of the bulk interaction as a boundary term. This is indeed what happens for the closely related AdS_2 branes in AdS_3 [11]. Here, however, it cannot possibly be the right answer, at least not without a proper notion of what we mean by taking the square root. In fact, the naive square root of $b_- b_+ \exp(-iY)$ is something like $b_\pm \exp(-iY/2)$, i.e. a fermionic operator. It makes no sense to add such an object to the bulk theory. In order to take a bosonic square root of the bulk interaction, we introduce a new fermionic boundary field C of weight $\Delta(C) = 0$ and add the following terms to the bulk theory,

$$S_0^{\mathrm{bdy}}[X, Y, c_\pm, b_\pm, C] \;=\; \frac{1}{8\pi i}\int du \;\; (kC\partial_u C + 4(c_+ + c_-)b_+) \tag{2.6}$$

$$S_{\mathrm{int}}^{\mathrm{bdy}}[X, Y, c_\pm, b_\pm, C] \;=\; -\frac{1}{2\pi i}\int du \; e^{-iY/2} b_+ C \; . \tag{2.7}$$

The idea to involve an additional fermionic boundary field in the action of supersymmetric brane configurations is not new. It was initially proposed in [12] and has been put to use more recently [13, 14] in the context of matrix factorizations. Our boundary action resembles the one Hosomichi employed to treat branes in $N = 2$ super Liouville theory [15]. The full gl(1|1) boundary theory now takes the form

$$S[X, Y, c_\pm, b_\pm, C] \;=\; S_{0,\mathrm{cl}}^{\mathrm{bulk}} + S_0^{\mathrm{bdy}} + S_{\mathrm{int}}^{\mathrm{bulk}} + S_{\mathrm{int}}^{\mathrm{bdy}} \;=\; S_{0,\mathrm{cl}} + S_{\mathrm{int}} \tag{2.8}$$

where

$$\begin{aligned} S_{0,\mathrm{cl}} \;&=\; -\frac{k}{4\pi i}\int_\Sigma d^2z \;\; \big(\partial X \bar{\partial} Y + \partial Y \bar{\partial} X\big) \\ &\quad - \frac{1}{2\pi i}\int_\Sigma d^2z \;\; \big(c_+ \partial b_+ + c_- \bar{\partial} b_-\big) + \frac{1}{8\pi i}\int du \; kC\partial_u C \; , \\ S_{\mathrm{int}} \;&=\; -\frac{1}{2k\pi i}\int_\Sigma d^2z \; e^{-iY} b_- b_+ \;\; - \frac{1}{2\pi i}\int du \; e^{-iY/2} b_+ C \;\; . \end{aligned} \tag{2.9}$$

Here, we have performed a partial integration on the kinetic term for the bc-system, thereby absorbing the contribution $b_+(c_- + c_+)$ from the boundary action. This is similar to the case of AdS_2 branes in AdS_3 [11]. In order to complete the description of the classical action, we add the following Dirichlet boundary condition for the fields $b_\pm$,

$$b_+(z) + b_-(\bar{z}) \;=\; 0 \quad \text{ for } \quad z \;=\; \bar{z} \; . \tag{2.10}$$

If the action is varied with this boundary condition, we recover the boundary equations of motion (2.2). More precisely, we obtain four equations among boundary fields. Two of these can be used to determine the boundary fields C and $b_+ = -b_-$ through X, Y and $c_\pm$,

$$C \;=\; e^{iY/2}\,(c_+ + c_-) \quad , \quad \pm 2b_\pm \;=\; k\, e^{iY/2}\partial_u C \; . \tag{2.11}$$

The four equations among boundary fields along with the bulk equations motion for $b_\pm$ imply the eqs. (2.2). We leave the details of this simple computation to the reader.

We have now set up a first order formalism for the twisted brane on GL(1|1). Let us stress again that is was necessary to introduce an additional fermionic field C on the boundary of the world-sheet. Above we have motivated this new degree of freedom by our desire to take a bosonic square root of the bulk interactions. But there is another, more geometric, way to argue for the additional field C. We mentioned before that the first order formalism for the GL(1|1) WZNW model is very similar to that for the Euclidean AdS_3, only that the bosonic coordinates $\gamma, \bar{\gamma}$ of the latter are replaced by fermionic ones. The first order formalism for AdS_2 branes in AdS_3 was set up in [11] and it describes a brane that is localized along a 1-dimensional subspace of the $\gamma\bar{\gamma}$ plane. Correspondingly, only a single γ zero mode remains after imposing the boundary conditions. The brane on GL(1|1) we are attempting to describe, however, is volume filling and therefore it extends in both fermionic directions. Therefore, we need two independent fermionic zero modes. These are provided by the zero modes of the three fields $c_\pm$ and C. Note that these fields are related by equation (2.11).

3 Volume filling branes: The quantum theory

Our next step is to develop a computational scheme for correlation functions in the boundary WZNW model with twisted boundary conditions. We shall use the first order formulation of section 2.2 as our starting point and consider the full WZNW model as a deformation of a free field theory involving the fields $X, Y, c_\pm, b_\pm$ and the fermionic boundary field C. This free field theory will be described in more detail in the first subsection. The definition of vertex operators and their correlation functions in the WZNW model is the subject of subsection 3.2.

3.1 The free theory and its correlation functions

Our strategy is to employ the first order formulation we set up in the previous section. In order to do so, we have to add a few comments on the measures we are using in the path integral treatment. To begin with, the supergroup invariant measure of the WZNW model is given by

$$d\mu_{\text{WZNW}} \;\sim\; \mathcal{D}X\mathcal{D}Y\mathcal{D}(e^{iY/2}c_-)\mathcal{D}(e^{iY/2}c_+)\ . \tag{3.1}$$

This gets multiplied with $\mathcal{D}b_+\mathcal{D}b_-\mathcal{D}C$ when we pass to the first order formalism. But in the following we would like to employ the standard free field measure

$$d\mu_{\text{free}} \;\sim\; \mathcal{D}X\mathcal{D}Y\mathcal{D}c_-\mathcal{D}c_+\ .$$

The two measures are related by a Jacobian of the form (see e.g. [16] for similar computations)

$$\begin{aligned} d\mu_{\text{WZNW}} \;&=\; \left(\text{sdet}(G^{ab}e^{iY}\partial_a e^{-iY}\partial_b)\right)^{-1}\ d\mu_{\text{free}} \\ &=\; e^{\frac{1}{8\pi}\int dudv\ \sqrt{G}(-G^{ab}\partial_a\ Y\partial_b Y+i\mathcal{R}Y)+\frac{1}{8\pi}\int du\ i\sqrt{G}\mathcal{K}Y}\ d\mu_{\text{free}}. \end{aligned} \tag{3.2}$$

Here, G_{ab} is the metric on the world-sheet, $\mathcal{R} = \partial_a\partial^a \log G$ and $\mathcal{K} = \frac{1}{2i}\partial_v \log G$ are its Gaussian and geodesic curvature, respectively. These two quantities feature in the Gauss-Bonnet theorem for surfaces with boundary,

$$\frac{1}{4\pi}\int_\Sigma dudv\ \sqrt{G}\mathcal{R} + \frac{1}{4\pi}\int du\ \sqrt{G}\mathcal{K} \;=\; \chi(\Sigma) \;=\; 1\ , \tag{3.3}$$

where $\chi(\Sigma) = 1$ is the Euler characteristic of the disc. We can now pass to the upper half plane again where all curvature is concentrated at infinity. The effect of the curvature terms in the WZNW measure is to insert a background charge $Q_Y = \chi(\Sigma)/2 = 1/2$ for the field Y at infinity. In addition, the measure (3.2) also contains a term that is quadratic in Y. We simply add this to the free part of our action, i.e. we define

$$\begin{aligned} S_0 \;=\; &-\frac{1}{4\pi i}\int_\Sigma d^2z\ \left(k\,\partial X\bar{\partial}Y + k\,\partial Y\bar{\partial}X - \partial Y\bar{\partial}Y\right) \\ &-\frac{1}{2\pi i}\int_\Sigma d^2z\ \left(c_+\partial b_+ + c_-\bar{\partial}b_-\right) + \frac{1}{8\pi i}\int du\ kC\partial_u C\ , \end{aligned} \tag{3.4}$$

Note, that the new term in the actions modifies the formula for the current J^N by adding an additional ∂Y and similarly for the anti-holomorphic partner.

In our path integral we now integrate with the free field theory measure $d\mu_{\rm free}$ over all fields subject to the boundary condition $b_+ + b_- = 0$. Configurations for the other fields are not constrained in the path integral. In the free quantum field theory, they satisfy the linear ("Neumann") boundary conditions

$$\begin{aligned} \partial_v Y &= 0 \quad , \qquad \partial_v X \;=\; 0 \;, \\ \partial_u C &= 0 \quad , \quad c_+ + c_- \;=\; 0 \;. \end{aligned} \tag{3.5}$$

These equations are satisfied in all correlation functions or, equivalently, as operator equations on the state space of the free field theory. Note that, according to the last equation, the zero modes of c_+ and c_- coincide in our free boundary theory. The necessary second fermionic zero mode is exactly what is provided by the field C.

Arbitrary correlation functions in the free field theory can now easily be computed with the help of Wick's theorem. All we need to use is the following list of operator product expansions

$$\begin{aligned} X(z,\bar z)Y(z,\bar z) \;&\sim \frac{1}{k}\ln|z-w|^2 + \frac{1}{k}\ln|z-\bar w|^2 \\ c_-(z)b_-(w) \;\sim\; \frac{1}{w-z} \qquad & c_+(\bar z)b_+(\bar w) \;\sim\; \frac{1}{\bar w-\bar z} \\ c_-(z)b_+(\bar w) \;\sim\; \frac{1}{z-\bar w} \qquad & c_+(\bar z)b_-(w) \;\sim\; \frac{1}{\bar z-w} \\ C(v)C(u) \;&\sim\; \frac{2\pi i}{k}\mathrm{sign}(v-u) \quad . \end{aligned} \tag{3.6}$$

Let us remark that a non-vanishing correlation function in the free field theory requires that the fields c outnumber the insertions of b by one. Furthermore, C must be inserted an odd number of times. We also recall that there is a non-vanishing background charge $Q_Y = 1/2$ for the field Y. On the disk, the corresponding U(1) charges of all tachyon vertex operators must add up to $Q_Y\chi(\Sigma) = 1/2$ in order for the correlator to be non-zero. These rules imply that the 1-point function of the bulk identity field vanishes. In order to normalize the vacuum expectation value, we require that

$$\langle\, (c_-(z) - c_+(\bar z))\; C(u)\, e^{ieX(z,\bar z)+inY(z,\bar z)}\, \rangle_0 \;=\; \delta(e)\delta(n-1/2)\;. \tag{3.7}$$

Note that the product of fields in brackets is the simplest expression that meets all our requirements: The $\mathrm{U}(1)_Y$ charge of the tachyon vertex operators is $m = 1/2$, we inserted one $c_\pm$ and no field $b_\pm$ and multiplied with a single C in order to make the total insertion bosonic again.

3.2 Correlation functions in boundary WZNW model

Now that we have learned how to perform computations in the free field theory described by the action (3.4), we would like to add our interaction term

$$S_{\text{int}} \;=\; -\frac{1}{2k\pi i}\int_\Sigma d^2z\; e^{-iY}b_-b_+ \;-\; \frac{1}{2\pi i}\int du\; e^{-iY/2}b_+C\;. \tag{3.8}$$

The idea is to calculate correlators of the full boundary WZNW model perturbatively, i.e. by expanding the exponential of the interaction in a power series. Even though there is a priori an infinite number of terms to be considered, only finitely many contribute to our perturbative expansion. This is very similar to what has been observed in the bulk model [1].

Before we can spell out precise formulas for the quantities we want to compute, we need to explain how to associate free field theory vertex operators to the fields of the interacting WZNW model. The latter are in one-to-one correspondence with functions on the supergroup GL(1|1) and they may be characterized by their behavior with respect to global gl(1|1) transformations. We shall first recall from [1] how this works for bulk fields.

Let us begin by collecting a few basic facts about the space of functions on the supergroup GL(1|1) [1]. As for any other group or supergroup, Ł2 carries two graded-commuting actions of the Lie superalgebra gl(1|1). These are generated by the following right and left invariant vector fields

$$\begin{aligned} &R_E = i\partial_x\;, \quad R_N = i\partial_y + \eta_-\partial_-\;, \quad R_+ = -e^{-iy}\partial_+ - i\eta_-\partial_x\;, \quad R_- = -\partial_-\;,\\ &L_E = -i\partial_x\;, \quad L_N = -i\partial_y - \eta_+\partial_+\;, \quad L_- = e^{-iy}\partial_- - i\eta_+\partial_x\;, \quad L_+ = \partial_+\;. \end{aligned} \tag{3.9}$$

A typical irreducible multiplet for gl(1|1) is 2-dimensional. Hence, typical irreducible multiplets of the combined left and right action are spanned by four functions in the supergroup. As in [1] we shall combine these functions into a 2×2 matrix of the form

$$\varphi_{\langle -e,-n+1\rangle} \;=\; e^{iex+iny}\begin{pmatrix} 1 & \eta_- \\ \eta_+ & e^{-1}e^{-iy}+\eta_+\eta_- \end{pmatrix} \tag{3.10}$$

The rows span the typical irreducibles $\langle -e,-n+1\rangle$ of the right regular action. Columns transform in the representations $\langle e,n\rangle$ of the left regular action. Note that $\varphi_{\langle e,n\rangle}$ is only well defined for $e\neq 0$, i.e. in the typical sector of the minisuperspace theory.

Following [1], the bulk vertex operators in the free field theory are modelled after the matrices $\varphi_{\langle e,n\rangle}$. More precisely, let us introduce typical bulk operators through

$$V_{\langle -e,-n+1\rangle}(z,\bar{z}) \;=\; e^{ieX+inY}\begin{pmatrix} 1 & c_- \\ c_+ & c_+c_- \end{pmatrix} \tag{3.11}$$

Since the weight of the fermionic fields $c_\pm$ vanishes, all four fields in this matrix possess the same conformal dimension,

$$\Delta_{(e,n)} \;=\; \frac{e}{2k}(2n-1+\frac{e}{k})\;. \tag{3.12}$$

Note that one of the terms in the lower left corner of the minisuperspace matrix $\varphi_{\langle e,n\rangle}$ has no analogue on the vertex operator $V_{\langle -e,-n+1\rangle}$. We consider this term as 'subleading'. It is reconstructed when we build correlation functions of the interacting WZNW model (see [1] and [17] for more details).

Let us now repeat the previous analysis for the boundary fields. Since our twisted brane is volume filling, the relevant space of minisuperspace wave functions is again the space Ł2 of all functions on the supergroup GL(1|1). But this time, it comes equipped with a different action of the Lie superalgebra gl(1|1). In fact, minisuperspace wave functions as well as boundary vertex operators are now distinguished by their transformation under a single twisted adjoint action $\text{ad}^\Omega_X = R_X + L^\Omega_X$ of GL(1|1) on Ł2. Explicitly, the generators of gl(1|1) transformations are given by

$$\begin{aligned} \text{ad}^\Omega_E &= 2i\partial_x \quad &,\quad \text{ad}^\Omega_N &= 2i\partial_y + \eta_+\partial_+ + \eta_-\partial_-\;, \\ \text{ad}^\Omega_- &= \partial_+ - \partial_- \quad &,\quad \text{ad}^\Omega_+ &= -e^{-iy}(\partial_- + \partial_+) + i(\eta_+ - \eta_-)\partial_x\;. \end{aligned} \tag{3.13}$$

Under the twisted adjoint action of gl(1|1) on Ł2, each typical multiplet appears with two-fold multiplicity [6]. Once more, we propose to assemble the corresponding four functions into a 2×2 matrix of the form

$$\psi_{\langle -2e,-2n+1\rangle} \;=\; e^{iex+iny}\begin{pmatrix} 1 & \eta_+ - \eta_- \\ \eta & 2e^{-1}e^{-iy/2} + (\eta_+ - \eta_-)\eta \end{pmatrix} \tag{3.14}$$

where we introduced the shorthand $\eta = e^{iy/2}(\eta_- + \eta_+)$. The reader is invited to check that the two rows of this matrix each span the 2-dimensional typical irreducible $\langle -2e,-2n+1\rangle$ under the twisted adjoint action (3.13) of the superalgebra gl(1|1).

Boundary vertex operators are modelled after the matrices $\psi_{\langle -2e,-2n+1\rangle}$ more or less in the same way as in the case of bulk fields,

$$U_{\langle -2e,-2n+1\rangle}(u) \;=\; e^{ieX+inY}\begin{pmatrix} 1 & c_+ - c_- \\ C & (c_+ - c_-)C \end{pmatrix}\;. \tag{3.15}$$

Again, we dropped the y-dependent term in the lower right corner of the matrix (3.14). Eventually, we will see how this term is recovered in boundary correlation functions. The main new aspect of the prescription (3.15), however, concerns the appearance of the fermionic boundary field C that we inserted in place of the function η. This substitution is motivated by the classical equation of motion (2.11).

After this preparation we are able to spell out how correlation functions of bulk and boundary fields can be computed for the interacting WZNW model. More precisely, we define,

$$\begin{aligned}\left\langle \prod_{\nu=1}^{m} \Phi_{\langle e_\nu, n_\nu\rangle}(z_\nu, \bar{z}_\nu) \prod_{\mu=1}^{m'} \Psi_{\langle e_\mu, n_\mu\rangle}(u_\mu) \right\rangle \ = \\ \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left\langle (S_{\rm int})^s \prod_{\nu=1}^{m} V_{\langle e_\nu, n_\nu\rangle}(z_\nu, \bar{z}_\nu) \prod_{\mu=1}^{m'} U_{\langle e_\mu, n_\mu\rangle}(u_\mu) \right\rangle_0 .\end{aligned} \tag{3.16}$$

Here, $S_{\rm int}$ is the interaction (3.8) and all correlation functions on the right side are to be computed in the free field theory (3.4). The relevant vertex operators V and U were introduced in equations (3.11) and (3.15) above. For later use we also note that bosonic correlators can be determined by means of the following standard formula,

$$\begin{aligned}\left\langle \prod_{\nu=1}^{m} V_{(e_\nu, n_\nu)}(z_\nu, \bar{z}_\nu) \prod_{\lambda=1}^{m'} V_{(e_\lambda, n_\lambda)}(u_\lambda) \right\rangle = \delta(\textstyle\sum_{\nu=1}^{m} n_\nu + \sum_{\lambda=1}^{m'} n_\lambda + \frac{1}{2})\delta(\sum_{\nu=1}^{m} e_\nu + \sum_{\lambda=1}^{m'} e_\lambda) \\ \times \prod_{\nu>\mu} |z_\nu - z_\mu|^{-2\alpha_{\nu\mu}} \prod_{\nu>\mu} |z_\nu - \bar{z}_\mu|^{-2\alpha_{\nu\mu}} \prod_{\nu,\lambda} |z_\nu - u_\lambda|^{-4\alpha_{\nu\lambda}} \prod_{\lambda>\kappa} |u_\lambda - u_\kappa|^{-4\alpha_{\kappa\lambda}}\end{aligned} \tag{3.17}$$

$$\text{where} \qquad \alpha_{\nu\mu} \ = \ -n_\nu \frac{e_\mu}{k} - n_\mu \frac{e_\nu}{k} - \frac{e_\nu e_\mu}{k^2}$$

and $V_{(e_\nu, n_\nu)} = \exp(ieX + inY)$ are bosonic vertex operators. As in the bulk theory it is easy to see that the all expansions (3.16) truncate after a finite number of terms. In fact, the inserted bulk and boundary vertex operators on the right hand side of eq. (3.16) contain at most $2m + m'$ fermionic fields $c_\pm$. Since each interaction term from $S_{\rm int}$ contributes at least one insertion of $b_\pm$, we conclude that terms with $s \geq 2m + m'$ vanish.

4 Solution of the boundary WZNW model

A boundary conformal field theory is uniquely characterized by the bulk-boundary and the boundary operator product expansions. We shall now employ the perturbative calculational scheme we developed in the previous section in order to determine these data. After a short warm-up with the discussion of bulk 1-point functions, we determine the bulk-boundary 2-point function in the second subsection. The 3-point function of boundary fields is addressed in subsection 4.3.

4.1 Bulk 1-point function

The bulk 1-point function is the simplest non-vanishing quantity in a boundary conformal field theory. It contains the same information as the boundary state. For volume filling branes, the boundary state was determined in our previous work [6]. Our first aim now is to reproduce our old result through our new perturbative expansion.

The 1-point function of a typical bulk field $\Phi_{\langle e,n\rangle}$ is computed by inserting a single vertex operator (3.11) into the expansion (3.16). Since bulk vertex operators contain at most two fields c, the only non-zero terms can come from $s = 0, 1$. The term with $s = 0$ contains no insertion of the interaction and it vanishes identically. So, let us see what happens for $s = 1$. In this case, only the insertion of the boundary interaction can contribute. The results is

$$\begin{aligned}\langle\Phi_{\langle e,n\rangle}(z,\bar{z})\rangle &= \frac{i}{2\pi}\int du\ \langle e^{-iY(u)/2}b_+(u)C(u)V_{\langle e,n\rangle}(z,\bar{z})\rangle \\ &= E^1_1\delta(e)\delta(n-1)\frac{1}{4\pi i}\int du\ \left(\frac{1}{u-\bar{z}}-\frac{1}{u-z}\right) \;=\; \int d\mu\ \varphi_{\langle e,n\rangle}\ .\end{aligned}$$

Here, E^1_1 is the elementary matrix which has zeroes everywhere except in the lower right corner. Note that the only field with non-vanishing 1-point function has conformal weight $\Delta = 0$. Hence, there is no dependence on the insertion point $(z, \bar{z})$. In the last line we have expressed the numerical result as an integral of the matrix valued function (3.10) over the supergroup GL(1|1). The integration is performed with the Haar measure

$$d\mu \;=\; 2^{-1}e^{-iy}dxdyd\eta_+d\eta_- \quad . \tag{4.1}$$

Since the Haar measure is gl(1|1) invariant, the integral of $\varphi_{\langle e,n\rangle}$ is an intertwiner from $\langle e,n\rangle \otimes \langle e,n\rangle$ to the trivial representation. This proves that the expectation value we computed has the desired transformation behavior.

4.2 Bulk-boundary 2-point function

Now we want to compute the full bulk-boundary 2-point function. It is quite useful to determine the general form of this 2-point function first before we enter the detailed calculations. Let us suppose for a moment that our calculations were guaranteed to give a gl(1|1) covariant answer. Then it is clear that the bulk-boundary 2-point function can be written as

$$\langle \Psi_{\langle 2e',2n'\rangle}(0)\, \Phi_{\langle -e,-n+1\rangle}(iy,-iy)\rangle \;=\; \sum_{\nu=0,1} C_\nu(e) \frac{\big\langle \psi_{\langle 2e',2n'\rangle}\, \varphi_{\langle -e,-n+1\rangle}\big\rangle_\nu}{|y|^{2\Delta_\nu}} \tag{4.2}$$

$$\text{where} \qquad \Delta_0 \;=\; \frac{2e}{k}\big(2n-1+\frac{e}{k}\big) \qquad \text{and} \qquad \Delta_1 \;=\; \frac{2e}{k}\big(2n-\frac{1}{2}+\frac{e}{k}\big)\ . \tag{4.3}$$

The structure constants $C_\nu(e)$ are not determined by the gl(1|1) symmetry. We will calculate them perturbatively below (see eqs. (4.7) and (4.9) below). The expressions in the numerator on the right hand side are certain gl(1|1) intertwiners which are defined by

$$\langle \psi_{\langle 2e',2n'\rangle}\, \varphi_{\langle -e,-n+1\rangle}\rangle = \int d\mu\, \psi_{\langle 2e',2n'\rangle}\, \phi_{\langle -e,-n+1\rangle} =: \sum_{\nu=0,1} \langle \psi_{\langle 2e',2n'\rangle}\, \varphi_{\langle -e,-n+1\rangle}\rangle_\nu \tag{4.4}$$

$$\text{where} \qquad \langle \psi_{\langle 2e',2n'\rangle}\, \varphi_{\langle -e,-n+1\rangle}\rangle_\nu \;=\; \delta(e-e')\delta(n-n'-\nu/2)\; G_\nu \tag{4.5}$$

is the part of the full integral that contains the factor $\delta(n-n'-\nu/2)$. Understanding the previous formulas requires some input from the representation theory of gl(1|1) (see e.g. [1] for all necessary details). Let us start with the matrix $\varphi_{\langle -e,-n+1\rangle}$. Under the twisted adjoint action of gl(1|1) this multiplet transforms in the tensor product

$$\langle -e,-n+1\rangle \otimes \langle -e,-n+1\rangle \;=\; \langle -2e,-2n+2\rangle \oplus \langle -2e,-2n+1\rangle \ \ .$$

Hence, there exist only two matrices $\psi_{\langle 2e',2n'\rangle}$ for which the integral (4.4) does not vanish. These are the matrices $\psi_{\langle 2e,2n\rangle}$ and $\psi_{\langle 2e,2n-1\rangle}$. The two non-vanishing terms are used to define the the symbols (4.5). A similar analysis can now be repeated for the fields in the WZNW model. We conclude immediately, that the 2-point function can only have two contributions. By gl(1|1) symmetry, these must be proportional to the intertwiners (4.5). The gl(1|1) symmetry, however, does not fix an overall constant C_ν that can depend on the parameters of the fields. Finally, the exponents Δ_ν are simply determined by the conformal dimensions of bulk and boundary fields. Let us point out that the entire discussion leading to the expression (4.2) is based on the global gl(1|1) symmetry. Since we have not yet shown that our perturbative computations respect the action of gl(1|1) it will be important to verify that the form of the 2-point function comes out right.

In our perturbative computation, there are at most three fields $c_\pm$ inserted and hence we only have to determine the expansion terms for $s = 0, 1, 2$. Contributions to the $\nu = 0$ term in the 2-point function (4.2), i.e. to the correlator with the boundary field $\Psi_{\langle 2e,2n\rangle}$, can only come from $s = 0$. In fact, insertions of an interaction term - bulk or boundary - would violate the conservation of Y-charge. Computation without any insertion of an interaction are easily performed, e.g.

$$\langle U^{11}_{\langle 2e',2n'\rangle}(0) V^{00}_{\langle -e,-n+1\rangle}(iy,-iy)\rangle \;=\; -\delta(n-n')\,\delta(e-e')|y|^{-4e/k(2n-1/2+e/k)} \tag{4.6}$$

Here, we have introduced the notation $U^{\epsilon'\epsilon}$ and $V^{\epsilon'\epsilon}$ for matrix elements. The field $U^{11}_{\langle 2e',2n'\rangle}$, for example, denotes the lower right corner etc. The computation of the associated integral (4.5) with $\nu = 0$ is equally simple and allows us to read off that

$$C_0(e,n) \;=\; 1 \quad . \tag{4.7}$$

Let us note that there are other combinations of bulk and boundary fields that can have a non-zero 2-point function without any insertion of interactions. In all those cases one may repeat the above calculation to find the same coefficient $C_0 = 1$, in agreement with gl(1|1) symmetry.

Next we would like to address the coefficient C_1 in the expression (4.2). Y-charge conservation implies that its only contributions are associated with a single insertion of the boundary interaction. This time, the computations are slightly more involved. As an example we treat the following 2-point function

$$\begin{aligned}
\langle U^{00}_{\langle 2e',2n'\rangle}(0) V^{11}_{\langle -e,-n+1\rangle}(iy,-iy)\, S^{\rm bdy}_{\rm int}\rangle \;=&\\
=&\; -\frac{\delta(n-n'-\frac{1}{2})\delta(e-e')}{|y|^{4\frac{e}{k}(2n-1+\frac{e}{k})}}\frac{y}{2\pi}\int du\;\frac{|u|^{2\alpha}}{|u^2+y^2|^{\alpha+1}}\\
=&\; -\frac{\delta(n-n'-\frac{1}{2})\delta(e-e')}{|y|^{4\frac{e}{k}(2n-1+\frac{e}{k})}}\frac{1}{2\pi}\int du\;|1+u^2|^{-\alpha-1}\\
=&\; -\frac{\delta(n-n'-\frac{1}{2})\delta(e-e')}{2|y|^{4\frac{e}{k}(2n-1+\frac{e}{k})}}2^{-2\alpha}\frac{\Gamma(2\frac{e}{k}+1)}{\Gamma^2(\frac{e}{k}+1)}\\
=&\; -\frac{\delta(n-n'-\frac{1}{2})\delta(e-e')}{2|y|^{4\frac{e}{k}(2n-1+\frac{e}{k})}}\frac{\Gamma(\frac{e}{k}+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{e}{k}+1)}
\end{aligned} \tag{4.8}$$

The second step is the substitution $u \to y/u$, then we can apply (A.8) which is a special case of the integral formula in [11]. The last step is the Euler doubling formula of the Gamma function. Comparison with the associated contribution to the minisuperspace integral (4.4) gives

$$C_1(e) \;=\; \frac{\Gamma(e/k+1/2)}{\sqrt{\pi}\Gamma(e/k+1)} \ . \tag{4.9}$$

Once more, one can perform similar computations with a single insertion of a boundary interaction for other pairs of bulk and boundary fields. All these calculations lead to the same result for C_1, as predicted by gl(1|1) covariance.

At this point, we have computed all the data we were interested in. But there are more contributions to the perturbative expansion of the bulk-boundary 2-point function. As we stated above, non-vanishing contributions arise from $s = 0, s = 1$ and $s = 2$. We have completely determined the $s = 0$ term. At $s = 1$, however, our attention so far was restricted to the boundary interaction. The other term with a single bulk insertion can also contribute since it contains a product of only two $b_\pm$. Similarly, at $s = 2$, two insertions of the boundary interaction can lead to a non-vanishing result. Products of bulk and boundary interactions or two bulk interactions, on the other hand, involve too many fields $b_\pm$ and vanish by simple zero mode counting. Hence, we are left with two more terms to calculate, those arising from a product of two boundary interactions $S^{\rm bdy}_{\rm int}$ and from a single bulk interaction $S^{\rm bulk}_{\rm int}$. Y-charge conservation implies that the additional terms involve a factor $\delta(n-n'-1)$. Such a term, if present, would be inconsistent with the global gl(1|1) symmetry. Our task therefore is to show that the sum of the two aforementioned contributions vanishes.

Let us begin with the computation of the term that arises from a single insertion of the bulk interaction,

$$\begin{aligned}
&\langle U^{11}_{\langle 2e,2n-2\rangle}(0)V^{11}_{\langle -e,-n+1\rangle}(iy,-iy)\ S^{\rm bulk}_{\rm int}\rangle \ \sim \\
&\qquad \sim\ y^{-2\frac{e}{k}(4n-3+2\frac{e}{k})}\,\frac{y^3}{k\pi}\int_{UHP} d^2z\ |z^2+y^2|^{-2(\frac{e}{k}+1)}|z^2|^{2\frac{e}{k}-1}(z-\bar z) \\
&\qquad =\ -\,y^{-2\frac{e}{k}(4n-3+2\frac{e}{k})}\,\frac{1}{e\sqrt{\pi}}\,\frac{\Gamma(2e/k+1/2)}{\Gamma(2\frac{e}{k}+1)}
\end{aligned} \tag{4.10}$$

We have been a bit sloppy here by setting the parameters the parameters $2e' = 2e$ and $2n'-2$ to the values at which the expectation value has a non-vanishing contribution. Strictly speaking, this quantity is divergent, but the divergence is an overall (volume) factor $\delta(0)$ which we suppressed consistently. In the first equality we simply inserted the relevant free field correlator. After the substitution $z \to y/z$, the integral over the insertion point u of the boundary interaction can be evaluated using an integral formula from [11] (see also (A.7)). Finally, the answer is simplified by means of Euler's doubling formula for Gamma functions.

Next we turn to the contributions coming from two boundary interactions. Since the corresponding free field correlator is slightly more involved in this case, we state an expression for the fermionic contribution before going into the actual computation,

$$\begin{aligned} &\langle b_+(u_1)C(u_1)b_+(u_2)C(u_2)(c_+ - c_-)(0)C(0)c_+(-iy)c_-(iy)\rangle^{\mathrm{F}} \ = \\ &\qquad = \frac{-4\pi y^3(u_2-u_1)}{u_1u_2(u_1^2+y^2)(u_2^2+y^2)}\Big[\mathrm{sign}(u_2-u_1) - \mathrm{sign}(u_2) + \mathrm{sign}(u_1)\Big] \ . \end{aligned} \tag{4.11}$$

This result is inserted to compute

$$\begin{aligned} &\langle U^{11}_{\langle 2e,2n-2\rangle}(0)V^{11}_{\langle -e,-n+1\rangle}(iy,-iy)\ \left(S^{\mathrm{bdy}}_{\mathrm{int}}\right)^2\rangle \ \sim \\ &\quad = y^{-2\frac{e}{k}(4n-3+2\frac{e}{k})}\frac{y^3}{\pi k}\int du_1du_2\ |u_1^2+y^2|^{-e/k-1}|u_2^2+y^2|^{-\frac{e}{k}-1}|u_1^2|^{\frac{e}{k}-1} \\ &\qquad |u_2^2|^{\frac{e}{k}-1}(u_2-u_1)\Big[\mathrm{sign}(u_2-u_1)-\mathrm{sign}(u_2)+\mathrm{sign}(u_1)\Big] \\ &\quad = y^{-2\frac{e}{k}(4n-3+2\frac{e}{k})}\frac{1}{\pi k}\int dx_1dx_2\ |x_1^2+1|^{-\frac{e}{k}-1}|x_2^2+1|^{-\frac{e}{k}-1}|x_1-x_2| \\ &\quad = y^{-2\frac{e}{k}(4n-3+2\frac{e}{k})}\frac{2}{e\sqrt{\pi}}\frac{\Gamma(2\frac{e}{k}+\frac{1}{2})}{\Gamma(2\frac{e}{k}+1)} \end{aligned} \tag{4.12}$$

The integral in the fourth line is again evaluated with a special case of the integral formula of Fateev and Ribault (A.9). Putting the results of eqs. (4.10) and (4.12) together we arrive at

$$\langle U^{11}_{\langle 2e',2n'\rangle}(0)V^{11}_{\langle -e,-n+1\rangle}(iy,-iy)\left(S^{\mathrm{bulk}}_{\mathrm{int}} + \frac{1}{2!}\left(S^{\mathrm{bdy}}_{\mathrm{int}}\right)^2\right)\rangle \ = \ 0 \ , \tag{4.13}$$

in agreement with gl(1|1) covariance of the 2-point function. Thereby, we have now established the formula (4.2) through our perturbative computations.

Before we leave the subject of bulk boundary 2-point functions, we would like to make a few comments on the cases when e/k is an integer multiple of $1/2$. Consider inserting a bulk vertex operator with e momentum $e = -mk - k/2 - k\varepsilon$ and sending ε to zero. In the limit, the second term of eq. (4.2) develops a logarithmic singularity,

$$
\begin{aligned}
C_1(-mk-k/2-k\epsilon)|y|^{-\Delta_1} &= \frac{(-1)^m}{m!\Gamma(-m+1/2)|y|^{2\Delta}}(\mathcal{Z}+\tilde{\Delta}\ln|y|+o(\epsilon)) \\
\text{where} \qquad \mathcal{Z} &= \frac{1}{\epsilon}+\Psi(-m)-\Psi(-m+1/2)\ , \\
\Delta &= -(2m+1)(2n-m-1)\ .
\end{aligned}
\tag{4.14}
$$

and $\tilde{\Delta}=4n-4m-3$. Here, Ψ is the usual Di-gamma function. The form of our bulk-boundary 2-point function (4.14) resembles a similar expression in [18]. A link between boundary correlation functions of symplectic fermions and the corresponding correlators in the GL(1|1) WZNW model may be established following ideas in [19].

4.3 Boundary 3-point functions

The second object of interest for us is the boundary 3-point function. Before we get there, we have to turn our attention to an important detail that we glossed over in the previous subsection. We recall that our 2×2 matrices $\Psi_{\langle e,n\rangle}, e\neq k\mathbb{Z}$, of boundary fields contain two irreducible multiplets $\langle e,n\rangle$ under the unbroken global gl(1|1) symmetry. These two multiplets have opposite fermion number, i.e. the state with lower eigenvalue of N is bosonic for one of them and fermionic for the other. In general, the two multiplets are allowed to have different couplings to the other fields in the theory. When we studied bulk-boundary 2-point function, only one of the two multiplets from each of the 2×2 matrices $\Psi_{\langle 2e,2n\rangle}$ and $\Psi_{\langle 2e,2n-1\rangle}$ could have a non-vanishing overlap with the bulk field $\Phi_{\langle -e,-n+1\rangle}$, simply because of fermion number conservation. Hence, the bulk-boundary 2-point functions were parametrized by two non-vanishing structure constants $C_\nu(e)$ rather than four. For boundary 3-point functions, however, the distinction becomes important. Consequently, we introduce the symbols

$$
\begin{aligned}
U^0_{\langle -2e,-2n+1\rangle}(u) &= e^{ieX+inY}\,(\,1\ ,\ c_+-c_-\,) \\
U^1_{\langle -2e,-2n+1\rangle}(u) &= e^{ieX+inY}\,(\,C,(c_+-c_-)\,C\,)
\end{aligned}
\tag{4.15}
$$

for the first and second row of the matrix (3.15). The same notation is used for the rows of the matrices ψ of functions and Ψ of boundary fields.

Let us now begin with the 3-point function of three fields from the first multiplet Ψ^0. These acquire contributions exclusively from a single insertion of the boundary interaction.

A non-vanishing correlator requires that the parameters e_i of the three fields sum up to $\tilde{e} = e_1 + e_2 + e_3 = 0$ and similarly that $\tilde{n} = n_1 + n_2 + n_3 = 1$. Using the integral formulas from Appendix A, the 3-point function of fields Ψ^0 in the regime $0 < x < 1$ is found to be

$$\begin{aligned}\langle \Psi^{0\epsilon_1}_{\langle -2e_1, -2n_1+1\rangle}(0)\Psi^{0\epsilon_2}_{\langle -2e_2, -2n_2+1\rangle}(1)\Psi^{0\epsilon_2}_{\langle -2e_3, -2n_3+1\rangle}(x)\rangle \; &= \delta(\tilde{e})\,\delta(\tilde{n}-1)\,\delta(\tilde{\epsilon}-2)\times \\ \times\; x^{2\Delta_{13}}(1-x)^{2\Delta_{23}}\; \frac{\pi}{i}&\frac{s(\alpha_1)+s(\alpha_2)+s(\alpha_3)}{s(\alpha_1)s(\alpha_2)s(\alpha_3)\Gamma(\alpha_1+\epsilon_1)\Gamma(\alpha_2+\epsilon_2)\Gamma(\alpha_3+\epsilon_3)}\end{aligned} \tag{4.16}$$

where we defined the parameters α_i by $\alpha_i = 2e_i/k$ and introduced the short-hands $s(z)$ and $\tilde{\epsilon}$ for $s(z) = \sin(\pi z)$ and $\tilde{\epsilon} = \sum \epsilon_i$. The conformal weights are given by

$$\Delta_{ij} \;=\; (n_i - 1/2)\alpha_j + (n_j - 1/2)\alpha_i + \alpha_i\alpha_j \;.$$

In the limit $k \to \infty$ the function $s(\alpha_i)$ can be approximated by $s(\alpha) \sim 2\pi e_i/k$ and the entire 3-point function is seen to vanish due to the conservation of e momentum. This is consistent with the minisuperspace theory. In fact, the corresponding integral of functions on our brane is easily seen to vanish,

$$\langle \psi^{0\epsilon_1}_{\langle -2e_1, -2n_1+1\rangle}\psi^{0\epsilon_2}_{\langle -2e_2, -2n_2+1\rangle}\psi^{0\epsilon_2}_{\langle -2e_3, -2n_3+1\rangle}\rangle \;=\; 0\;.$$

This is so because integration with the Haar measure needs a product of two different fermionic zero modes in order to give a non-zero result. Our functions ψ^0, however, only contain the zero mode $\eta_+ - \eta_-$.

Let us now move on to discuss the 3-point in the case where a single field from the second multiplet Ψ^1 is inserted. Contributions to such correlators arise only from the leading term $s = 0$ of the perturbation series (see below). The result is therefore straightforward to write down

$$\begin{aligned}\langle \Psi^{0\epsilon_1}_{\langle -2e_1, -2n_1+1\rangle}(0)\Psi^{0\epsilon_2}_{\langle -2e_2, -2n_2+1\rangle}(1)\Psi^{1\epsilon_3}_{\langle -2e_3, -2n_3+1\rangle}(x)\rangle \;&= \\ =\; \delta(\tilde{e})\,\delta(\tilde{n}-1/2)\,\delta(\tilde{\epsilon}-1)\; x^{2\Delta_{13}}(1-x)^{2\Delta_{23}}\;.\end{aligned} \tag{4.17}$$

This coupling in independent of the level k and it matches the minisuperspace answer which is non-zero because the multiplet ψ^1 contains both fermionic zero modes.

The most interesting 3-point coupling appears when we insert two fields from the second multiplet Ψ^1. Once more, non-vanishing terms can only arise from the insertion of a single boundary interaction. They can be worked out with the help of integral formulas in Appendix A,

$$\langle \Psi^{0\epsilon_1}_{\langle -2e_1,-2n_1+1\rangle}(0)\Psi^{1\epsilon_2}_{\langle -2e_2,-2n_2+1\rangle}(1)\Psi^{1\epsilon_3}_{\langle -2e_3,-2n_3+1\rangle}(x)\rangle \;=\; \delta(\tilde{e})\,\delta(\tilde{n}-1)\,\delta(\tilde{\epsilon}-2)\;\times$$
$$\times\;\frac{2\pi^2 i}{k}\,x^{2\Delta_{13}}(1-x)^{2\Delta_{23}}\;\frac{s(\alpha_1)-s(\alpha_2)-s(\alpha_3)}{s(\alpha_1)s(\alpha_2)s(\alpha_3)\Gamma(\alpha_1+\epsilon_1)\Gamma(\alpha_2+\epsilon_2)\Gamma(\alpha_3+\epsilon_3)}\quad . \tag{4.18}$$

Note that the factor $\sim 1/k$ in the first term of the second row is necessary in order for the whole expression to scale to a finite value as we send the level k to infinity. The expression that arises in this limit can be checked easily in the minisuperspace theory.

There remains one more case to consider, namely the 3-point function for three fields from the second multiplet Ψ^1. It is given by

$$\langle \Psi^{1\epsilon_1}_{\langle -2e_1,-2n_1+1\rangle}(0)\Psi^{1\epsilon_2}_{\langle -2e_2,-2n_2+1\rangle}(1)\Psi^{1\epsilon_3}_{\langle -2e_3,-2n_3+1\rangle}(x)\rangle \;=$$
$$=\;\delta(\tilde{e})\,\delta(\tilde{n}-1/2)\,\delta(\tilde{\epsilon}-1)\frac{2\pi}{k}\;x^{2\Delta_{13}}(1-x)^{2\Delta_{23}}\quad . \tag{4.19}$$

As in the previous formula (4.18), the result contains a factor $1/k$. Consequently, the 3-point coupling on the right hand side of eq. (4.19) vanishes at $k\sim\infty$, in agreement with the associated minisuperspace computation.

The last result (4.19) was obtained without any insertion of bulk or boundary interactions, though naively one might expect to see contributions from one bulk or two boundary insertions. A similar comment applies to the second case (4.17) above. It is indeed true that the insertion of $S^{bulk}_{\rm int}$ or $(S^{\rm bdy}_{\rm int})^2$ both lead to non-vanishing expressions. But, as in the case of the bulk boundary 2-point functions, their sum vanishes, i.e.

$$\langle\, U^{\epsilon'_1\epsilon_1}_{\langle e_1,n_1\rangle}(0)\,U^{\epsilon'_2\epsilon_2}_{\langle e_2,n_2\rangle}(1)\,U^{\epsilon'_3\epsilon'_3}_{\langle e_3,n_3\rangle}(u)\,\left(S^{\rm bulk}_{\rm int}+\frac{1}{2!}\left(S^{\rm bdy}_{\rm int}\right)^2\right)\rangle \;=\; 0\;.$$

The result is trivially fulfilled for $\tilde{\epsilon}'=0,2$. It requires rather elaborate computations when $\tilde{\epsilon}'=1,3$. These can be performed with the help of the integral formulas (A.3-A.5) we list in Appendix A.

Before closing this section we would like to add two more comments. The first one concerns the logarithmic singularities that appear in the 3-point functions whenever one of the parameters $2e_i$ is an integer multiple of k. If we consider joining two open strings with e momentum $e_1=e-\varepsilon/2$ and $e_2=-e-\varepsilon/2$, for example, and send ε to zero, we obtain

$$\langle\Psi^{00}_{\langle -2e+\varepsilon,-2n_1+1\rangle}(0)\Psi^{11}_{\langle 2e+\varepsilon,-2n_2+1\rangle}(1)\Psi^{11}_{\langle -2\varepsilon,-2n_3+1\rangle}(u)\rangle \sim$$

$$\sim u^{2\Delta}(1-u)^{-2\Delta}\,\delta(\tilde{n}-1)\,\big(\mathcal{Z}+\mathcal{R}(\alpha)+A_{23}\ln|1-u|+A_{13}\ln|u|+o(\varepsilon)\big)$$

$$\text{where}\quad \mathcal{Z} = \frac{1}{\varepsilon}+\frac{4\varepsilon\gamma}{k} \quad , \quad \mathcal{R}(\alpha) = -2\pi\frac{1+c(\alpha)}{ks(\alpha)} \tag{4.20}$$

$$A_{13} = \frac{1}{k}(2n_1-n_3-1/2+2\alpha) \quad , \quad A_{23} = \frac{1}{k}(2n_2-n_3-1/2-2\alpha)$$

and $\Delta = \alpha(n_3-1/2)$. The function $c(\alpha)$ stands for $c(\alpha)=\cos(\pi\alpha)$ and γ is the Euler-Mascheroni constant. In the limit $\varepsilon\to 0$, the constant $\mathcal{Z}$ diverges. This divergency can be regularized by adding to Ψ^{11} an appropriate field from the socle of the involved atypical multiplet. In the following, we shall assume that $\mathcal{Z}$ has been set to zero.

Our final comment deals with an interesting quantum symmetry of the boundary 3-point functions. As in the bulk sector [1], the boundary 3-point function is periodic under shifts of the e-momentum, in the following sense,

$$\langle\Psi^{\epsilon_1\epsilon_1'}_{\langle -2e_1,-2n_1+1\rangle}(0)\Psi^{\epsilon_2\epsilon_2'}_{\langle -2e_2,-2n_2+1\rangle}(1)\Psi^{\epsilon_3\epsilon_3'}_{\langle -2e_3,-2n_3+1\rangle}(x)\rangle =$$

$$(1-u)^{2n_3-1}u^{1-2n_3}\langle\Psi^{\epsilon_2\epsilon_2'}_{\langle -2e_1+k,-2n_1\rangle}(1)\Psi^{\epsilon_1\epsilon_1'}_{\langle -2e_2-k,-2n_2+2\rangle}(0)\Psi^{\epsilon_3\epsilon_3'}_{\langle -2e_3,-2n_3+1\rangle}(x)\rangle \quad .$$

Further shifts by multiples of $\pm k$ can also be considered, but necessarily involve inserting descendants of the tachyon vertex operators. Our observation proves that the boundary GL(1|1) model for volume filling branes possesses spectral flow symmetry. Shifts by integer multiples of the level k are a symmetry of the affine representation theory. In principle, this symmetry could be broken by the boundary structure constants. The previous formula asserts that, like in the bulk sector, the boundary operator product expansions preserve the spectral flow symmetry. The same is true for the bulk-boundary operator product expansions.

5 Correlation functions involving atypical fields

Throughout the last few sections we have learned how to compute correlation functions of bulk and boundary tachyon vertex operators for a volume filling brane in the GL(1|1) WZNW model. We now want to add a few comments on a particular set of correlation functions that are essentially not effected by the interaction and hence can be derived without cumbersome calculations. These will include a non-vanishing annulus amplitude. We shall use the latter to perform a highly non-trivial test on the proposed boundary state of volume filling branes [6].

5.1 Correlators for special atypical fields

In the previous sections we developed a first order formalism for computations of correlation functions in the GL(1|1) WZNW model. Very special correlators, however, can also be computed in the original formulation. To begin with, let us explain the main idea at the example of bulk correlators. We recall that the bulk action of the GL(1|1) model is given by

$$S_{\text{bulk}} \;=\; -\frac{k}{4\pi i}\int_\Sigma d^2z\;\left(\partial X\bar\partial Y+\partial Y\bar\partial X+2e^{iY}\partial c_+\bar\partial c_-\right) \tag{5.1}$$

The path integral is evaluated with the gl(1|1) invariant measure (3.1) on the space of fields. A glance at the interaction term of the WZNW model and the measure suggests to introduce the new coordinates $\chi_\pm = e^{iY/2}c_\pm$. After this substitution, the path integral measure is the canonical one,

$$d\mu_{\text{WZW}} \;\sim\; \mathcal{D}X\mathcal{D}Y\mathcal{D}\chi_-\mathcal{D}\chi_+\;. \tag{5.2}$$

Our bulk action $S_{\text{bulk}} = S_0 + Q$, on the other hand, splits naturally into a free field theory S_0 and an interaction term Q where

$$\begin{aligned} S_0 \;&=\; -\frac{k}{4\pi i}\int_\Sigma d^2z\;\left(\partial X\bar\partial Y+\partial Y\bar\partial X+2\partial\chi_+\bar\partial\chi_-\right)\\ Q \;&=\; \frac{k}{4\pi i}\int_\Sigma d^2z\;\left(i\chi_+\bar\partial\chi_-\partial Y+i\partial\chi_+\chi_-\bar\partial Y+\chi_+\chi_-\partial Y\bar\partial Y\right)\;. \end{aligned} \tag{5.3}$$

Due to the complicated form of Q, treating the WZNW model as a perturbation by the interaction terms in Q is not too useful for most practical computations. Under very special circumstances, however, the split into S_0 and Q allows for a very interesting conclusion. Observe that each term in the interaction Q contains at least one derivative ∂Y or $\bar\partial Y$. In our free field theory S_0, the only non-vanishing contractions involving derivatives of Y are those with the field X. Hence, we can simply ignore the presence of Q for all correlation functions of tachyon vertex operators that do not involve X. In other words, correlation functions of fields without any X-dependence are given by their free field theory expressions! This had already been observed in the results of [1]. Our split of the action in S_0 and Q provides a rather simple and general explanation. Let us stress again that this split is not helpful for any other computation involving more generic typical fields.

It is clear that all this is not restricted to the bulk theory. In fact, we can use the same substitution for the boundary terms of the action (2.1),

$$S_{\partial 0} \ = \frac{k}{8\pi i}\int_\Sigma du \ (\chi_+ + \chi_-)\partial_u(\chi_+ + \chi_-) \ . \tag{5.4}$$

Since $S_{\partial 0}$ is quadratic in the fields $\chi_\pm$, it gets added to the free bulk action S_0, i.e. we now work with a free field theory on the upper half plane whose action is given by $S_0 + S_{\partial 0}$. There is no additional boundary contribution to the bulk interaction Q. In the free theory, the fields $\chi_\pm$ satisfy Neumann gluing conditions of the following simple form,

$$\partial\chi_\pm(z,\bar{z}) \ = \ \mp\bar{\partial}\chi_\mp(z,\bar{z}) \qquad \text{for} \qquad z \ = \ \bar{z} \ . \tag{5.5}$$

The gluing condition implies that fermions of the free boundary theory are contracted as follows,

$$\begin{aligned} \chi_-(z,\bar{z})\chi_+(w,\bar{w}) \ &\sim \ \frac{1}{k}\ln|z-w|^2 \ , \\ \chi_\pm(z,\bar{z})\chi_\pm(w,\bar{w}) \ &\sim \ \frac{1}{k}\ln(\bar{z}-w) - \frac{1}{k}\ln(\bar{w}-z) \ . \end{aligned} \tag{5.6}$$

The bosonic fields X, Y also obey simple Neumann boundary conditions so that the evaluation of correlators in the free field theory $S_0 + S_{\partial 0}$ is straightforward. Taking the interaction Q into account is a difficult task unless none of the vertex operators in the correlation function contain the field X. If all field are X independent, then the correlator is simply given by the free field theory formula, just as in the bulk theory above.

One may apply the observation in the previous paragraph to the evaluation of boundary 3-point functions of three atypical fields for the volume filling brane. Note that we did not spell out a formula for this particular correlator before. In principle, it can be computed in the first order formalism, but the corresponding calculation requires some care. Our new approach allows to write down the result right away. We shall discuss another interesting application of our new approach to atypical correlation functions in the next subsection. Let us mention in passing that we expect similar results to hold for the completely atypical sectors in all $GL(N|N)$ and $PSL(N|N)$ WZNW models. This will be discussed in more detail elsewhere.

5.2 Twisted boundary state and modular bootstrap

In our previous paper [6], we proposed a formula for a boundary state of volume filling brane on GL(1|1). The usual annulus amplitude for this boundary state was trivially zero, in agreement with the observation that open string states are perfectly paired. In fact, as we have mentioned at various places throughout this note, for each multiplet $\langle e, n\rangle$ of boundary fields there exists one with opposite parity. Contributions of such pairs to the boundary partition function cancel each other, leading to a vanishing boundary partition function.

In order to construct a non-trivial quantity on the annulus, we need to insert some fermionic zero modes, see e.g. [20] for similar tests in the simpler bc ghost system. Previously, we have not been able to compute such quantities in the GL(1|1) WZNW model. We can now fill this gap! Let us anticipate that only atypical bulk fields couple to the volume filling brane. Hence, if we insert fermionic zero modes through some atypical bulk field, the entire amplitude is built from atypical terms and should be computable through a simple free field formalism, as explained in the previous subsection. Let us see now how the details of this calculation work out.

To begin with, let us review the construction of the boundary state $|\Omega\rangle$ for the volume filling brane. With the help of our free field realization, the formula becomes quite explicit. We shall start from the boundary state $|\Omega\rangle_0$ of the free theory. This state clearly factorizes into a product of a bosonic $|\Omega, B\rangle_0$ and a fermionic $|\Omega, F\rangle_0$ contribution. The latter two obey the following gluing conditions

$$(X_n + \bar{X}_{-n})\,|\,\Omega, B\rangle_0 \;=\; (Y_n + \bar{Y}_{-n})\,|\,\Omega, B\rangle_0 \;=\; 0 \tag{5.7}$$

and

$$(\chi^{\pm}_n \mp \bar{\chi}^{\mp}_{-n})\,|\,\Omega, F\rangle_0 \;=\; 0 \quad . \tag{5.8}$$

Here, X_n and $\bar{X}_n$ are the modes of the currents $i\sqrt{k}\partial X$ and $i\sqrt{k}\bar{\partial} X$ etc. Up to normalization, there exists a unique solution for these linear constraints. For the bosonic and the fermionic sector, they are given by the following coherent states,

$$|\,\Omega, B\rangle_0 \;=\; \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}(Y_{-n}\bar{X}_{-n} + X_{-n}\bar{Y}_{-n}\right)\,|0,0\rangle_B \tag{5.9}$$

$$|\,\Omega, F\rangle_0 \;=\; \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}(\chi^{+}_{-n}\bar{\chi}^{+}_{-n} - \chi^{-}_{-n}\bar{\chi}^{-}_{-n}\right)\,|0,0\rangle_F \quad . \tag{5.10}$$

Here, $|0,0\rangle$ denote the vacua in the bosonic and the fermionic theory. The product of the two components is the boundary state of the free field theory, before the interaction is taken into account. We now include the effects of the interaction by multiplying the free boundary state with the exponential of the interaction Q,

$$|\,\Omega\rangle \;=\; \mathcal{N} e^{Q}\,|\,\Omega\rangle_0 \;=\; \mathcal{N}\,\left(\sum_{n=0}^{\infty}\frac{Q^n}{n!}\right)\,|\,\Omega,B\rangle_0\times\,|\,\Omega,F\rangle_0\;, \tag{5.11}$$

where $\mathcal{N}=\sqrt{\pi/2i}$ is a normalization constant. The operator Q is defined as in eq. (5.3), but with the integration restricted to the interior of the unit disc. It is possible to check that $\exp Q$ rotates the gluing conditions from the free field theory relations (5.7) and (5.8) to their interacting counterparts (see (2.2)). The dual boundary state is constructed analogously.

Our main aim now is to compute some non-vanishing overlap of the twisted boundary state $|\Omega\rangle$. This requires the insertion of the invariant bulk field $\Phi^{11}_{\langle 0,0\rangle}=\chi_-\chi_+$, i.e. we are going to study

$$Z_\Omega(q,z)\;:=\;\langle\Omega\,|\,\tilde{q}^{L_0^c}(-1)^{F^c}\,\tilde{z}^{N_0^c}\,\Phi^{11}_{\langle 0,0\rangle}\,|\,\Omega\rangle\;, \tag{5.12}$$

where $L_0^c=(L_0+\bar{L}_0)/2$ and $N_0^c=(N_0-\bar{N}_0)/2$ are obtained from the zero modes of the Virasoro field and the current N. The corresponding expressions are standard, see e.g. [1]. Our parameters $\tilde{q}$ and $\tilde{z}$ are defined in terms of μ,τ through $\tilde{q}=\exp(-2\pi i/\tau)$ and $\tilde{z}=\exp(2\pi i\mu/\tau)$. We are now going to argue that the computation of Z_Ω can be reduced to a simple calculation in free field theory, i.e.

$$\langle\Omega\,|\,\tilde{q}^{L_0^c}(-1)^{F^c}\,\tilde{z}^{N_0^c}\,\Phi^{11}_{\langle 0,0\rangle}\,|\,\Omega\rangle\;=\;\mathcal{N}^2\;{}_0\langle\Omega\,|\,\tilde{q}^{L_0^c}(-1)^{F^c}\,\tilde{z}^{N_0^c}\,\Phi^{11}_{\langle 0,0\rangle}\,|\,\Omega\rangle_0\;. \tag{5.13}$$

The reasoning goes as follows. In a first step we write the interacting boundary state as a product of the interaction term $\exp Q$ and the free boundary state $|\Omega\rangle_0$. Next we observe that all bosonic operators in between the two boundary states involve derivatives such as ∂X etc. Hence, we can use the gluing conditions (5.7) to express all these terms through Y_n and X_n. The modes $\bar{Y}_n$ and $\bar{X}_n$ of the anti-holomorphic derivatives only appear in the construction (5.9) of the free bosonic boundary state $|\Omega,B\rangle_0$. A non-vanishing term requires that the number of $\bar{X}_n$ equals the number of $\bar{Y}_{-n}$. But since the $\bar{X}_{-n}$ and $\bar{Y}_{-n}$ come paired with their holomorphic partners Y_{-n} and X_{-n} in the boundary state, the operator in between ${}_0\langle\Omega|$ and $|\Omega\rangle_0$ must have equal numbers for X_n and Y_n modes in order for the corresponding term not to vanish. In Q, all terms have an excess of Y modes. Since no term in L_0^c or N_0^c can compensate this through an excess of X-modes, we can safely replace $\exp Q$ by its zeroth order term, i.e. $\exp Q \sim 1$.

The computation of the overlap (5.13) in free field theory is straightforward. In a first step, the amplitude is split into a product of bosonic and fermionic terms. The bosonic contribution is the same as for extended branes in flat 2-dimensional space. The fermionic factor involves an insertion. Its evaluation is reminiscent of a similar calculation in [20]. We can express the result through a single character of the affine gl(1|1) algebra,

$$Z_\Omega(q,z) \;=\; \mathcal{N}^2 \;\; \hat{\chi}_{\mathcal{P}_0}(-1/\tau, \mu/\tau) \;=\; \frac{\pi}{k} \int dedn\, \frac{\hat{\chi}_{\langle e,n\rangle}(\tau,\mu)}{\sin(\pi e/k)}\ . \tag{5.14}$$

The affine characters $\hat{\chi}$ along with their behavior under modular transformations can be found in the Appendix A of [6]. In order to achieve proper normalization (see below) we have set $\mathcal{N}^2 = \pi/2i$. Since the spectrum of boundary operators on the volume filling brane is continuous, the result involves some open string spectral density function. From the result, this is read off as

$$\rho(e,n) \;=\; \rho(e) \;=\; \frac{\pi}{k\sin(\pi e/k)}\quad . \tag{5.15}$$

We would expect ρ to be encoded in the boundary 3-point function of $\Psi_{\langle e,n\rangle}$, $\Psi_{\langle -e,-n\rangle}$ with the special boundary field $\Psi^{11}_{\langle 0,0\rangle}$. One possible 3-point function that contains the required information is a particular case of our more general formula (4.20), i.e.

$$\begin{aligned} &\langle \Psi^{00}_{\langle e,n\rangle}(0)\Psi^{11}_{\langle -e,-n\rangle}(1)\Psi^{11}_{\langle 0,0\rangle}\rangle \;\sim \\ &\qquad\sim\; u^{2\Delta}(1-u)^{-2\Delta}\big(\mathcal{Z} + \mathcal{R}(-\pi e/k) + A_{23}\ln|1-u| + A_{13}\ln|u|\big)\quad . \end{aligned} \tag{5.16}$$

All quantities that appear on the right hand side were introduced in equation (4.20). The additive constant $\mathcal{Z}$ is not universal. It is naively infinite, but can be made finite by a proper regularization prescription. We use the universal term $\mathcal{R}$ to determine the spectral density

$$\frac{d}{de}\ln\mathcal{R}(-\pi e/k) \;=\; \frac{\pi}{k}\frac{d}{d\alpha}\ln\frac{1+c(\alpha)}{s(-\alpha)} \;=\; \frac{\pi}{k\sin(\pi e/k)} \;=\; \rho(e)\ . \tag{5.17}$$

Here, we have used that $\alpha = e/k$, as before. The result agrees with the expression (5.15) that was obtained through modular transformation of the overlap (5.13). Thereby, we have now been able to subject our formula (5.11) for the boundary state of the volume filling brane to a strong consistency check.

There is another somewhat weaker but still non-trivial test for the boundary state that arises from the minisuperspace limit of the boundary WZNW model. In fact, in the particle limit we find that

$$\mathrm{tr}(z^{\mathrm{ad}_N^\Omega}(-1)^F\psi^{11}_{\langle 0,0\rangle}) \;=\; \int dedn\, \frac{\chi_{\langle e,n\rangle}(z)}{e} \;=\; \lim_{k\to\infty} Z_\Omega(q,z)\ . \tag{5.18}$$

In the first step we simply evaluated the trace directly in the minisuperspace theory. We then observed in the second equality that the result coincides with the modular transform of the overlap (5.13) in the appropriate limit $k \to \infty$.

6 Conclusions and open problems

In this note we have solved the boundary theory for the volume filling brane on GL(1|1). We achieved this with the help of a Kac-Wakimoto-like representation of the boundary theory. The first order formalism we developed in section 2 is similar to the one used in [11] for AdS_2 branes in the Euclidean AdS_3. The main difference is that we were forced to introduce an additional fermion on the boundary. Such auxiliary boundary fermions are quite common in fermionic theories (see e.g. [12, 15] and references therein). With the help of our first order formalism we were then able to set up a perturbative calculational scheme for correlation functions of bulk and boundary fields. The main features of this scheme are similar to the pure bulk case [1]. In particular, for any given correlator, only a finite number of terms from the expansion can contribute. We computed the exact bulk-boundary 2-point functions and the boundary 3-point functions, thereby solving the boundary conformal field theory of volume filling branes on GL(1|1) explicitly. Finally, we proposed a second approach to correlation functions of atypical fields. It singles out a particular subsector of the bulk and boundary GL(1|1) WZNW model that is not affected at all by the interaction. Hence, within this subsector, all quantities agree with their free field theory counterparts. The insight was then put to use for a calculation of a particular non-vanishing annulus amplitude in section 5.2. Together with our previous results on boundary 3-point functions, we obtained a strong test for the boundary state of the volume filling brane in the GL(1|1) WZNW model.

There are several obvious extensions that should be worked out. To begin with, it would be interesting to set up an equally efficient framework to calculate correlation functions for the boundary theories of point-like localized branes. Unfortunately, we have not succeeded to calculate correlators from a finite number of contributions, as in the case of the volume filling brane. It is possible to develop a Kac-Wakimoto-like presentation for point-like branes using the boundary conditions of [20] for the bc system. But since the gluing conditions of [20] identify derivatives of c with $\bar{b}$ etc., zero mode counting does not furnish simple vanishing results. Therefore, an infinite number of terms can contribute to any given correlation function. On the other hand, the second approach of section 5 does generalize to point-like branes. Since the boundary spectrum on a single point-like brane is purely atypical, some interesting quantities can be computed. This applies in particular to the boundary 3-point functions on a single point-like brane. Correlation functions involving boundary condition changing fields or typical bulk fields, however, are not accessible along these lines.

It is certainly interesting to investigate how much of our program extends to higher supergroups. Encouraged by the recent developments on the bulk sector [21], it seems likely that most of our constructions may be generalized, at least to supergroups of type I. This includes the superconformal algebras psl(N|N) and many other interesting Lie superalgebras (see e.g. [22] for a complete list). We believe that in all these cases there exists one class of branes which can be solved through some appropriate square root of the bulk formalism. Taking the proper square root will certainly involve a larger number of fermionic boundary fields. Our second approach to atypical correlation functions may also be extended to higher supergroups and it provides interesting insights on the atypical subsector of the WZNW models. We plan to return to these issues in a forthcoming publication.

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A Some integral formulas

In this section, we provide a complete list of integral formulas needed for the computation of the correlation functions. As reference we use [23].

We start with the formulas needed in the computation of boundary three-point functions. First recall the integral representations of the hypergeometric function $F(\alpha,\beta;\gamma|x)$

$$
\begin{aligned}
&\int_1^\infty du\ |u|^{-\alpha}|u-1|^{-\beta}|u-x|^{-\gamma} \ = \\
&\qquad\qquad \frac{\Gamma(\alpha+\beta+\gamma-1)\Gamma(1-\beta)}{\Gamma(\alpha+\gamma)} F(\gamma,\alpha+\beta+\gamma-1;\alpha+\gamma \mid x) \\
&\int_0^x du\ |u|^{-\alpha}|u-1|^{-\beta}|u-x|^{-\gamma} \ = \\
&\qquad\qquad x^{1-\alpha-\gamma}\frac{\Gamma(1-\alpha)\Gamma(1-\gamma)}{\Gamma(2-\alpha-\gamma)} F(\beta,1-\alpha;2-\alpha-\gamma \mid x) \\
&\int_{-\infty}^0 du\ |u|^{-\alpha}|u-1|^{-\beta}|u-x|^{-\gamma} \ = \\
&\qquad\qquad \frac{\Gamma(\alpha+\beta+\gamma-1)\Gamma(1-\alpha)}{\Gamma(\beta+\gamma)} F(\gamma,\alpha+\beta+\gamma-1;\beta+\gamma \mid 1-x) \\
&\int_x^1 du\ |u|^{-\alpha}|u-1|^{-\beta}|u-x|^{-\gamma} \ = \\
&\qquad\qquad (1-x)^{1-\beta-\gamma}\frac{\Gamma(1-\beta)\Gamma(1-\gamma)}{\Gamma(2-\beta-\gamma)} F(\alpha,1-\beta;2-\beta-\gamma \mid 1-x)
\end{aligned} \tag{A.1}
$$

these integrals converge for $|x|<1$.

If only the first order boundary interaction contributes, we need the special case $\alpha+\beta+\gamma=2$ of the above integrals which can be expressed as

$$
\begin{aligned}
\int_{[-\infty,0]\,\cup\,[1,\infty]} du\ |u|^{-\alpha}|u-1|^{-\beta}|u-x|^{-\gamma} \ &= \ (1-x)^{\alpha-1}x^{\beta-1}\frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(\gamma)} \\
\int_{[0,x]} du\ |u|^{-\alpha}|u-1|^{-\beta}|u-x|^{-\gamma} \ &= \ (1-x)^{\alpha-1}x^{\beta-1}\frac{\Gamma(1-\alpha)\Gamma(1-\gamma)}{\Gamma(\beta)} \\
\int_{[x,1]} du\ |u|^{-\alpha}|u-1|^{-\beta}|u-x|^{-\gamma} \ &= \ (1-x)^{\alpha-1}x^{\beta-1}\frac{\Gamma(1-\beta)\Gamma(1-\gamma)}{\Gamma(\alpha)}\ .
\end{aligned} \tag{A.2}
$$

If the bulk interaction term contributes, we have to evaluate the following integral for

$\alpha+\beta+\gamma=0$

$$
\begin{aligned}
\int d^2z\, \frac{(z-\bar z)}{|z|^{2\alpha+2}|z-1|^{2\beta+2}|z-x|^{2\gamma+2}} &= \\
&= \frac{1}{\gamma x+\beta}\int d^2z\, \bar\partial\left(\frac{\bar z(\bar z-1)(\bar z-x)}{|z|^{2\alpha+2}|z-1|^{2\beta+2}|z-x|^{2\gamma+2}}\right)+ \\
&\quad -\frac{1}{\gamma x+\beta}\int d^2z\, \partial\left(\frac{z(z-1)(z-x)}{|z|^{2\alpha+2}|z-1|^{2\beta+2}|z-x|^{2\gamma+2}}\right) \\
&= -\frac{2}{\gamma x+\beta}\int du\, \frac{u(u-1)(u-x)}{|u|^{2\alpha+2}|u-1|^{2\beta+2}|u-x|^{2\gamma+2}} \\
&= -\frac{1}{\gamma(\gamma x+\beta)}\frac{d}{dx}\left(\int_{[-\infty,0]\,\cup\,[1,\infty]} du\, \frac{1}{|u|^{2\alpha+1}|u-1|^{2\beta+1}|u-x|^{2\gamma}}+\right. \\
&\quad \left. -\int_0^1 du\, \frac{1}{|u|^{2\alpha+1}|u-1|^{2\beta+1}|u-x|^{2\gamma}}\right) \\
&= -4(1-x)^{2\alpha-1}x^{2\beta-1}\Big(\frac{\Gamma(-2\alpha)\Gamma(-2\beta)}{\Gamma(2\gamma+1)}+\frac{\Gamma(-2\alpha)\Gamma(-2\gamma)}{\Gamma(2\beta+1)}+\frac{\Gamma(-2\beta)\Gamma(-2\gamma)}{\Gamma(2\alpha+1)}\Big)
\end{aligned} \tag{A.3}
$$

and if two boundary interactions contribute, we need (again $\alpha+\beta+\gamma=0$)

$$
\begin{aligned}
\int_{a_1}^{b_1} du_1 \int_{a_2}^{b_2} du_2\, \frac{|u_1-u_2|}{|u_1u_2|^{\alpha+1}|(u_1-1)(u_2-1)|^{\beta+1}|(u_1-x)(u_2-x)|^{\gamma+1}} &= \\
= x^{2\beta-1}(1-x)^{2\alpha-1}\int_{c_1}^{d_1} du_1 \int_{c_2}^{d_2} du_2\, \frac{|u_1-u_2|}{|(u_1-1)(u_2-1)|^{\beta+1}|u_1u_2|^{\gamma+1}}&,
\end{aligned} \tag{A.4}
$$

where $c_i=\frac{b_i^{-1}-x^{-1}}{1-x^{-1}}$ and $d_i=\frac{a_i^{-1}-x^{-1}}{1-x^{-1}}$. For these integrals one has to evaluate

$$
\begin{aligned}
\int_1^\infty du_1 \int_1^{u_1} du_2\, \frac{(u_1-u_2)}{|(u_1-1)(u_2-1)|^{\beta+1}|u_1u_2|^{\gamma+1}} &= 4\frac{\Gamma(-2\alpha)\Gamma(-2\beta)}{\Gamma(2\gamma+1)} \\
\int_0^1 du_1 \int_0^{u_1} du_2\, \frac{(u_1-u_2)}{|(u_1-1)(u_2-1)|^{\beta+1}|u_1u_2|^{\gamma+1}} &= 4\frac{\Gamma(-2\gamma)\Gamma(-2\beta)}{\Gamma(2\alpha+1)} \\
\int_{-\infty}^0 du_1 \int_{-\infty}^{u_1} du_2\, \frac{(u_1-u_2)}{|(u_1-1)(u_2-1)|^{\beta+1}|u_1u_2|^{\gamma+1}} &= 4\frac{\Gamma(-2\gamma)\Gamma(-2\alpha)}{\Gamma(2\beta+1)}
\end{aligned} \tag{A.5}
$$

where we used the following special form of the Gamma doubling formula

$$
\frac{\Gamma(1/2-\alpha)\Gamma(-\alpha)\Gamma(1/2-\beta)\Gamma(-\beta)}{\Gamma(1/2)\Gamma(\gamma+1/2)\Gamma(\gamma+1)} = 4\frac{\Gamma(-2\alpha)\Gamma(-2\beta)}{\Gamma(2\gamma+1)}\,. \tag{A.6}
$$

For the computation of bulk-boundary 2-point functions we use some special cases of an integral formula that can be found in the recent work of Fateev and Ribault [11]. In case of a single insertion of the bulk interaction we need

$$\int d^2z \, \frac{|z-\bar{z}|}{|1+z^2|^{2(\alpha+1)}} \;=\; -\,2i\pi^{3/2}2^{-4\alpha}\frac{\Gamma(2\alpha+1/2)\Gamma(2\alpha)}{\Gamma^2(\alpha+1)\Gamma^2(\alpha+1/2)}\;. \tag{A.7}$$

To treat the insertion of one boundary interaction we employ

$$\int du \; |1+u^2|^{-(\alpha+1)} \;=\pi 2^{-2\alpha}\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\;. \tag{A.8}$$

The insertion of boundary interactions may be evaluated by means of the following formula

$$\int du_1 du_2 \, \frac{|u_1-u_2|}{|1+u_1^2|^{1+\alpha}|1+u_2^2|^{1+\alpha}} \;=4\pi^{3/2}2^{-4\alpha}\frac{\Gamma(2\alpha+1/2)\Gamma(2\alpha)}{\Gamma^2(\alpha+1)\Gamma^2(\alpha+1/2)}\;. \tag{A.9}$$

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