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# Small Extra Dimensions from the Interplay of Gauge and Supersymmetry Breaking

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**Abstract**

Higher-dimensional theories provide a promising framework for unified extensions of the supersymmetric standard model. Compactifications to four dimensions often lead to $U(1)$ symmetries beyond the standard model gauge group, whose breaking scale is classically undetermined. Without supersymmetry breaking, this is also the case for the size of the compact dimensions. Fayet-Iliopoulos terms generically fix the scale $M$ of gauge symmetry breaking. The interplay with supersymmetry breaking can then stabilize the compact dimensions at a size $1/M$, much smaller than the inverse supersymmetry breaking scale $1/\mu$. We illustrate this mechanism with an $SO(10)$ model in six dimensions, compactified on an orbifold.

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# 1 Introduction

Higher-dimensional theories provide a promising framework for unified extensions of the supersymmetric standard model [1]. Interesting examples have been constructed in five and six dimensions compactified on orbifolds [2–7], which have many phenomenologically attractive features. During the past years it has become clear how to embed these orbifold GUTs into the heterotic string [8–10], separating the GUT scale from the string scale on anisotropic orbifolds [11]. A class of compactifications yielding supersymmetric standard models in four dimensions (4D) have been successfully constructed [12–14].

For a given orbifold compactification of the heterotic string, one can consider different orbifold GUT limits where one or two of the compact dimensions are larger than the other five or four, respectively [10]. One then obtains an effective five-dimensional (5D) or six-dimensional (6D) GUT field theory as intermediate step between the full string theory and the supersymmetric standard model. We shall focus on 6D field theories compactified on $T^2/\mathbb{Z}_2$ with two Wilson lines. These models have four fixed points where quantum corrections generically induce Fayet-Iliopoulos terms [15,16]. In the case of the heterotic string the magnitude of these local terms is $\mathcal{O}(M_{\rm GUT})$, which suggests that they may lead to a stabilization of the compact dimensions at $R \sim 1/M_{\rm GUT}$ [16].

Quantum corrections to the vacuum energy density, the Casimir energy, play a crucial role in the stabilization of compact dimensions [17]. Various aspects of the Casimir energy for 6D orbifolds have already been studied in [18–20]. Stabilization of the volume can be achieved by means of massive bulk fields, brane localized kinetic terms or bulk and brane cosmological terms [18]. Alternatively, the interplay of one- and two-loop contributions to the Casimir energy can lead to a stabilization at the length scale of higher-dimensional couplings [21]. In addition, fluxes and gaugino condensates play an important role [22, 23].

In this paper we consider orbifold GUTs, which generically have two mass scales: $M \sim M_{\rm GUT}$, the expectation value of bulk fields induced by local Fayet-Iliopoulos terms, and $\mu \ll M_{\rm GUT}$, the scale of soft supersymmetry breaking mass terms. As we shall see, the interplay of 'classical' and one-loop contributions to the vacuum energy density can stabilize the extra dimensions at small radii, $R \sim 1/M_{\rm GUT} \ll 1/\mu$ with bulk energy density $\mathcal{O}(\mu^2 M_{\rm GUT}^2)$. We shall illustrate this mechanism with an $SO(10)$ model in six dimensions [24] which together with gaugino mediation [25, 26] is known to lead to a successful phenomenology [27,28].

The paper is organized as follows. In Section 2 we briefly describe the relevant features of the 6D orbifold GUT model. The Casimir energies of scalar fields with different boundary conditions are discussed in Section 3. These results are used in Section 4 to evaluate the Casimir energy of the considered model. In Section 5 the stabilization mechanism is described. Appendices A and B deal with the mode expansion on $T^2/\mathbb{Z}_2^3$ and the evaluation of Casimir sums, respectively.

## 2 The Model

As an example, we consider a 6D $\mathcal{N} = 1$ $SO(10)$ gauge theory compactified on an orbifold $T^2/\mathbb{Z}_2^3$, corresponding to $T^2/\mathbb{Z}_2$ with two Wilson lines [24]. The model has four inequivalent fixed points ('branes') with the unbroken gauge groups $SO(10)$, the Pati-Salam group $G_{\text{PS}} = SU(4) \times SU(2) \times SU(2)$, the extended Georgi-Glashow group $G_{\text{GG}} = SU(5) \times U(1)_X$ and flipped $SU(5)$, $G_{\text{fl}} = SU(5)' \times U(1)'$, respectively. The intersection of these GUT groups yields the standard model group with an additional $U(1)$ factor, $G'_{\text{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$, as unbroken gauge symmetry below the compactification scale. At the fixed points only 4D $\mathcal{N} = 1$ supersymmetry remains unbroken. Gauge and supersymmetry breaking are realized by assigning different parities to the different components of the **45**-plet of $SO(10)$, which is a 6D $\mathcal{N} = 1$ vector multiplet containing 4D $\mathcal{N} = 1$ vector ($V$) and chiral ($\Sigma$) multiplets (cf. Table 1).

The model has three **16**-plets of matter fields, localized at the Pati-Salam, the Georgi-Glashow, and the flipped $SU(5)$ branes. Further, there are two **16**-plets, $\phi$ and $\phi^c$ and two **10**-plets, $H_5$ and $H_6$ of bulk matter fields. Their mixing with the brane fields yields the characteristic flavor structure of the model [24, 28].

The Higgs sector consists of two **16**-plets, $\Phi$ and $\Phi^c$, and four **10**-plets, $H_1, \ldots, H_4$, of bulk hypermultiplets. Each hypermultiplet contains two 4D $\mathcal{N} = 1$ chiral multiplets, the first of which we denote by the same symbol as the hypermultiplet. The Higgs multiplets have even $R$-charge and the matter fields have odd $R$-charge.

The hyperpermultiplets $H_1$ and $H_2$ contain the two Higgs doublets of the supersymmetric standard model as zero modes, whereas the zero modes of $H_3$ and $H_4$ are color triplets (cf. Table 2). The zero modes of the **16**-plets are singlets and color triplets,

$$\Phi : \ N^c,\ D^c\ ; \qquad \Phi^c : \ N,\ D\ . \tag{1}$$

The color triplets $D^c$ and $D$, together with the zero modes of $H_3$ and $H_4$, aquire masses through brane couplings.

Equal vacuum expectation values of $\Phi$ and $\Phi^c$ form a flat direction of the classical potential,

$$\langle \Phi \rangle = \langle N^c \rangle = \langle N \rangle = \langle \Phi^c \rangle\ . \tag{2}$$

Non-zero expectation values can be enforced by a brane superpotential term or by a Fayet-Iliopoulos term localized at the GG-brane where the $U(1)$ factor commutes with the standard model gauge group.

The expectation values (2) break $SO(10) \to SU(5)$, and therefore also the additional $U(1)_X$ symmetry, as is clear from the decomposition

$$\mathbf{16} \ \to \ \mathbf{10}_1 \oplus \bar{\mathbf{5}}_{-3} \oplus \mathbf{1}_5\ , \tag{3}$$

$$\overline{\mathbf{16}} \ \to \ \overline{\mathbf{10}}_{-1} \oplus \mathbf{5}_3 \oplus \mathbf{1}_{-5}\ , \tag{4}$$

| $G'_{\text{SM}}$ | $V$: $\mathbb{Z}_2$ | $\mathbb{Z}_2^{\text{GG}}$ | $\mathbb{Z}_2^{\text{PS}}$ | $\mathcal{M}^2_{m,n}$ | $\Sigma$: $\mathbb{Z}_2$ | $\mathbb{Z}_2^{\text{GG}}$ | $\mathbb{Z}_2^{\text{PS}}$ |
|---|---|---|---|---|---|---|---|
| $(\mathbf{8},\mathbf{1})_{0,0}$ | $+$ | $+$ | $+$ | $4\left(\frac{m^2}{R_1^2}+\frac{n^2}{R_2^2}\right)$ | $-$ | $-$ | $-$ |
| $(\mathbf{3},\mathbf{2})_{-\frac{5}{6},0}$ | $+$ | $+$ | $-$ | $4\left(\frac{m^2}{R_1^2}+\frac{(n+1/2)^2}{R_2^2}\right)$ | $-$ | $-$ | $+$ |
| $(\bar{\mathbf{3}},\mathbf{2})_{\frac{5}{6},0}$ | $+$ | $+$ | $-$ | $4\left(\frac{m^2}{R_1^2}+\frac{(n+1/2)^2}{R_2^2}\right)$ | $-$ | $-$ | $+$ |
| $(\mathbf{1},\mathbf{3})_{0,0}$ | $+$ | $+$ | $+$ | $4\left(\frac{m^2}{R_1^2}+\frac{n^2}{R_2^2}\right)$ | $-$ | $-$ | $-$ |
| $(\mathbf{1},\mathbf{1})_{0,0}$ | $+$ | $+$ | $+$ | $4\left(\frac{m^2}{R_1^2}+\frac{n^2}{R_2^2}\right)$ | $-$ | $-$ | $-$ |
| $(\mathbf{3},\mathbf{2})_{\frac{1}{6},4}$ | $+$ | $-$ | $-$ | $4\left(\frac{(m+1/2)^2}{R_1^2}+\frac{(n+1/2)^2}{R_2^2}\right)$ | $-$ | $+$ | $+$ |
| $(\bar{\mathbf{3}};\mathbf{1})_{-\frac{2}{3},4}$ | $+$ | $-$ | $+$ | $4\left(\frac{(m+1/2)^2}{R_1^2}+\frac{n^2}{R_2^2}\right)$ | $-$ | $+$ | $-$ |
| $(\mathbf{1},\mathbf{1})_{1,4}$ | $+$ | $-$ | $+$ | $4\left(\frac{(m+1/2)^2}{R_1^2}+\frac{n^2}{R_2^2}\right)$ | $-$ | $+$ | $-$ |
| $(\bar{\mathbf{3}},\mathbf{2})_{-\frac{1}{6},-4}$ | $+$ | $-$ | $-$ | $4\left(\frac{(m+1/2)^2}{R_1^2}+\frac{(n+1/2)^2}{R_2^2}\right)$ | $-$ | $+$ | $+$ |
| $(\mathbf{3},\mathbf{1})_{\frac{2}{3},-4}$ | $+$ | $-$ | $+$ | $4\left(\frac{(m+1/2)^2}{R_1^2}+\frac{n^2}{R_2^2}\right)$ | $-$ | $+$ | $-$ |
| $(\mathbf{1},\mathbf{1})_{-1,-4}$ | $+$ | $-$ | $+$ | $4\left(\frac{(m+1/2)^2}{R_1^2}+\frac{n^2}{R_2^2}\right)$ | $-$ | $+$ | $-$ |
| $(\mathbf{1},\mathbf{1})_{0,0}$ | $+$ | $+$ | $+$ | $4\left(\frac{m^2}{R_1^2}+\frac{n^2}{R_2^2}\right)$ | $-$ | $-$ | $-$ |

Table 1: Decomposition of the **45**-plet of $SO(10)$ into multiplets of the extended standard model gauge group $G'_{\text{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$ and corresponding parity assignments. For later convenience we also give the Kaluza-Klein masses $\mathcal{M}^2_{m,n}$.

where $\mathbf{1}_5$ and $\mathbf{1}_{-5}$ correspond to $N^c$ and $N$, respectively. The decomposition of the **45** vector multiplet reads

$$\mathbf{45} \to \mathbf{24}_0 \oplus \mathbf{10}_{-4} \oplus \overline{\mathbf{10}}_4 \oplus \mathbf{1}_0 \ . \tag{5}$$

The expectation values (2) generate for the **10**- and $\overline{\mathbf{10}}$-plets and the singlet in Eqs. (3)-(5) the bulk mass

$$M^2 \simeq g_6^2 \langle \Phi^c \rangle^2 \ , \tag{6}$$

where $g_6$ is the 6D gauge coupling. Hence, the fields $(\mathbf{3},\mathbf{2})_{\frac{1}{6},4}$, $(\bar{\mathbf{3}},\mathbf{1})_{-\frac{2}{3},4}$, $(\mathbf{1},\mathbf{1})_{1,4}$, $(\mathbf{1},\mathbf{1})_{0,0}$ and their complex conjugates contained in the vector multiplet as well as the corresponding fields in $\Phi$ and $\Phi^c$ obtain bulk masses from the Higgs mechanism in addition to their

| $SO(10)$ | **10** | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| SM′ | $(\mathbf{1},\mathbf{2})_{-\frac{1}{2},-2}$ | | $(\mathbf{1},\mathbf{2})_{\frac{1}{2},2}$ | | $(\bar{\mathbf{3}},\mathbf{1})_{\frac{1}{3},-2}$ | | $(\mathbf{3},\mathbf{1})_{-\frac{1}{3},2}$ | |
| | $H^c$ | | $H$ | | $G^c$ | | $G$ | |
| | $\mathbb{Z}_2^{\mathrm{PS}}$ | $\mathbb{Z}_2^{\mathrm{GG}}$ | $\mathbb{Z}_2^{\mathrm{PS}}$ | $\mathbb{Z}_2^{\mathrm{GG}}$ | $\mathbb{Z}_2^{\mathrm{PS}}$ | $\mathbb{Z}_2^{\mathrm{GG}}$ | $\mathbb{Z}_2^{\mathrm{PS}}$ | $\mathbb{Z}_2^{\mathrm{GG}}$ |
| $H_1$ | $+$ | $+$ | $+$ | $-$ | $-$ | $+$ | $-$ | $-$ |
| $H_2$ | $+$ | $-$ | $+$ | $+$ | $-$ | $-$ | $-$ | $+$ |
| $H_3$ | $-$ | $+$ | $-$ | $-$ | $+$ | $+$ | $+$ | $-$ |
| $H_4$ | $-$ | $-$ | $-$ | $+$ | $+$ | $-$ | $+$ | $+$ |
| $H_5$ | $-$ | $+$ | $-$ | $-$ | $+$ | $+$ | $+$ | $-$ |
| $H_6$ | $-$ | $-$ | $-$ | $+$ | $+$ | $-$ | $+$ | $+$ |

| $SO(10)$ | **16** | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| SM′ | $(\mathbf{3},\mathbf{2})_{\frac{1}{6},-1}$ | | $(\mathbf{1},\mathbf{2})_{-\frac{1}{2},3}$ | | $(\bar{\mathbf{3}},\mathbf{1})_{-\frac{2}{3},-1}$<br>$(\mathbf{1},\mathbf{1})_{1,-1}$ | | $(\bar{\mathbf{3}},\mathbf{1})_{\frac{1}{3},3}$<br>$(\mathbf{1},\mathbf{1})_{0,-5}$ | |
| | $Q$ | | $L$ | | $U^c, E^c$ | | $D^c, N^c$ | |
| | $\mathbb{Z}_2^{\mathrm{PS}}$ | $\mathbb{Z}_2^{\mathrm{GG}}$ | $\mathbb{Z}_2^{\mathrm{PS}}$ | $\mathbb{Z}_2^{\mathrm{GG}}$ | $\mathbb{Z}_2^{\mathrm{PS}}$ | $\mathbb{Z}_2^{\mathrm{GG}}$ | $\mathbb{Z}_2^{\mathrm{PS}}$ | $\mathbb{Z}_2^{\mathrm{GG}}$ |
| $\Phi$ | $-$ | $-$ | $-$ | $+$ | $+$ | $-$ | $+$ | $+$ |
| $\phi$ | $+$ | $-$ | $+$ | $+$ | $-$ | $-$ | $-$ | $+$ |

Table 2: Decomposition and parity assignments for the bulk **16**- and **10**-plets of $SO(10)$. The $\overline{\mathbf{16}}$-plets $\Phi^c, \phi^c$ have the same parities as $\Phi$ and $\phi$ and conjugate quantum numbers with respect to the extended standard model gauge group. Only fields with all parities positive remain in the low energy theory.

Kaluza-Klein masses. Since the spontaneous breaking of $SO(10)$ preserves 6D $\mathcal{N} = 1$ supersymmetry, one obtains an entire massive hypermultiplet for each set of quantum numbers.

Supersymmetry breaking is naturally incorporated via gaugino mediation [27]. The non-vanishing $F$-term of a brane field $S$ generates mass terms for vector- and hyper-multiplets. In the considered model, $S$ is localized at the $SO(10)$ preserving brane, which yields the same mass for all members of an $SO(10)$ multiplet. For the **45** vector multiplet and the **10** and **16** hypermultiplets of the Higgs sector one has

$$
\begin{aligned}
\Delta S \;=\; & \int \mathrm{d}^4x\mathrm{d}^2y\ \delta^2(y) \left\{ \int \mathrm{d}^2\theta \frac{1}{2\Lambda^3} S\mathrm{Tr}[W^\alpha W_\alpha] + \mathrm{h.c.} \right. \\
& + \int \mathrm{d}^4\theta \left( \frac{\lambda}{\Lambda^4} S^\dagger S \left( H_1^\dagger H_1 + H_2^\dagger H_2 \right) + \frac{\lambda'}{\Lambda^4} S^\dagger S \left( H_3^\dagger H_3 + H_4^\dagger H_4 \right) \right. \\
& \qquad \left. \left. + \frac{\lambda''}{\Lambda^4} S^\dagger S \left( \Phi^\dagger \Phi + \Phi^{c\dagger}\Phi^c \right) \right) \right\} \ . \qquad (7)
\end{aligned}
$$

Here $W^\alpha(V)$, $H_1, \dots, H_4$ and $\Phi, \Phi^c$ are the 4D $\mathcal{N} = 1$ multiplets contained in the 6D $\mathcal{N} = 1$ multiplets, which have positive parity at $y = 0$; $\Lambda$ is the UV cutoff of the model, which is much larger than the inverse size of the compact dimensions, $\Lambda \gg 1/\sqrt{V}$. For the zero modes, the corresponding gaugino and scalar masses are given by

$$
m_g = \frac{\mu}{\Lambda^2 V} \ , \quad m^2_{H_{1,2}} = -\frac{\lambda\mu^2}{\Lambda^2 V} \ , \quad m^2_{H_{3,4}} = -\frac{\lambda'\mu^2}{\Lambda^2 V} \ , \quad m^2_\Phi = -\frac{\lambda''\mu^2}{\Lambda^2 V} \ , \qquad (8)
$$

where $V = (2\pi)^2 R_1 R_2$ is the volume of the compact dimensions, and $\mu = F_S/\Lambda$. Note that the gaugino mass is stronger volume suppressed than the scalar masses.

# 3 The Casimir Energy

The zero-point energies of bulk fields depend on size and shape of the compact dimensions. Their sum, the Casimir energy, is a quantum contribution to the total energy density whose minimum determines the size of the compact dimensions in the lowest energy state, the vacuum. As long as supersymmetry is unbroken, the Casimir energy vanishes since bosonic and fermionic contributions compensate each other. In the following we shall evaluate the Casimir energy for the different boundary conditions which occur in $T^2/\mathbb{Z}_2^3$ orbifold compactifications.

## 3.1 Bulk, Brane and Kaluza-Klein Masses

Consider a real scalar field in 6D with bulk mass $M$ and brane mass $m$. As discussed in the previous section, in gaugino mediation $m$ is due to supersymmetry breaking on a brane whereas $M$ is generated by the Higgs mechanism in 6D. From the action

$$
S = \frac{1}{2} \int \mathrm{d}^4x\mathrm{d}^2y \phi(x,y) \left( -\partial_x^2 - \partial_y^2 + M^2 + \frac{\mu^2}{\Lambda^2}\delta^2(y) \right) \phi(x,y) \qquad (9)
$$

and the mode decomposition

$$
\phi(x,y) = \sum_i \phi_i(x)\xi_i(y) \ , \quad \int d^2y \xi_i(y)\xi_j(y) = \delta_{ij} \ , \qquad (10)
$$

one obtains

$$S = \frac{1}{2}\int \mathrm{d}^4x \left[\sum_i \phi_i(x)\left(-\partial_x^2 + M_i^2 + M^2\right)\phi_i(x) + \frac{\mu^2}{\Lambda^2}\sum_{ij}\phi_i(x)C_{ij}\phi_j(x)\right] , \qquad (11)$$

where $M_i$ are the Kaluza-Klein masses and

$$C_{ij} = \xi_i(0)\xi_j(0) . \qquad (12)$$

On the orbifold $T^2/\mathbb{Z}_2^3$, one has for all modes (cf. Appendix A),

$$\xi_i(0) = \sqrt{\frac{2}{V}} = \frac{1}{\sqrt{2\pi^2 R_1 R_2}} , \qquad (13)$$

except for the zero mode, where $\xi_0(y) = 1/\sqrt{V}$.

The one-loop contribution to the vacuum energy density depends on the Kaluza-Klein mass matrix $M_{KK}$, the universal mass $M$ and the brane mass matrix $C$,

$$V^{(1)} = \frac{1}{2}\ln\det\left(-\partial_x^2 + M_{\mathrm{KK}}^2 + M^2 + \frac{\mu^2}{\Lambda^2}C\right) . \qquad (14)$$

For small supersymmetry breaking, $\mu^2 \ll M_i^2 + M^2$, the effective potential can be expanded in powers of the small off-diagonal terms of the mass matrix,

$$\begin{aligned} V^{(1)} &= \frac{1}{2}\sum_i \ln\left(-\partial_x^2 + M_i^2 + M^2 + \frac{\mu^2}{\Lambda^2}C_{ii}\right) \\ &\quad + \frac{1}{2}\left(\frac{\mu^2}{\Lambda^2}\right)^2 \sum_{i\neq j}\frac{1}{(-\partial_x^2 + M_i^2 + M^2)}C_{ij}\frac{1}{(-\partial_x^2 + M_j^2 + M^2)}C_{ji} + \mathcal{O}(\mu^6) . \qquad (15)\end{aligned}$$

In the following we shall only keep the diagonal terms of $C$, which contribute to $V^{(1)}$ at leading order in $\mu^2$.

The Casimir energy of gauge fields and gauginos can be directly obtained from the Casimir energy of a real scalar field. After appropriate gauge fixing this essentially amounts to counting the physical degrees of freedom (cf. [18]). Thus, it is enough to perform the vacuum energy calculation for a real scalar field.

### 3.2 Casimir Energy of a Scalar Field

The geometry of the orbifold $T^2/\mathbb{Z}_2$ contains as free parameters the radii $R_1$ and $R_2$ of the torus. The Casimir energy of a scalar field on the orbifold is then given by the quantum corrections to the corresponding effective potential. At one-loop order, this is obtained by summing over the continuous and discrete spectrum corresponding to the four flat and two compact dimensions,

$$V_M = \frac{1}{2}\left[\sum\right]_{m,n} \int \frac{\mathrm{d}^4 k_E}{(2\pi)^4} \log\left(k_E^2 + \mathcal{M}_{m,n}^2 + M^2\right) \ , \tag{16}$$

with $[\sum]_{m,n}$ shorthand for the double sum and $\mathcal{M}_{m,n}^2$ denoting the Kaluza-Klein masses; the mass $M$ now stands for bulk and brane mass terms.

The Kaluza-Klein masses $\mathcal{M}_{m,n}^2$ depend on the possible boundary conditions on $T^2/\mathbb{Z}_2$ and can be read off from the mode expansion listed in Table 1. Generically they can be written as

$$\begin{aligned} \mathcal{M}_{m,n}^2 &= 4\left[\frac{(m+\alpha)^2}{R_1^2} + \frac{(n+\beta)^2}{R_2^2}\right] \\ &= \frac{4}{R_2^2}\left[e^2(m+\alpha)^2 + (n+\beta)^2\right] \ , \end{aligned} \tag{17}$$

where $(\alpha, \beta) = (0,0), (0,1/2), (1/2,0), (1/2,1/2)$ and $e^2 = R_2^2/R_1^2$. For simplicity, we restrict our discussion to 'rectangular tori'. The general case will be discussed elsewhere [29]. Clearly, the contributions for the different boundary conditions satisfy the relations,

$$\begin{aligned} V_M^{0,0}(R_1,R_2) = V_M^{0,0}(R_2,R_1) \ , \quad V_M^{1/2,1/2}(R_1,R_2) = V_M^{1/2,1/2}(R_2,R_1) \ , \\ V_M^{0,1/2}(R_1,R_2) = V_M^{1/2,0}(R_2,R_1) \ . \end{aligned} \tag{18}$$

The expression (16) for the Casimir energy is divergent. Following [18,30], we extract a finite piece using zeta function regularization,

$$V = -\left.\frac{\mathrm{d}\zeta(s)}{\mathrm{d}s}\right|_{s=0} \ , \tag{19}$$

where

$$\zeta(s) = \frac{1}{2}\left[\sum\right]_{m,n} \mu_r^{2s} \int \frac{\mathrm{d}^4 k_E}{(2\pi)^4} \left(k_E^2 + \frac{4}{R_2^2}\left[e^2(m+\alpha)^2 + (n+\beta)^2\right] + M^2\right)^{-s} \ . \tag{20}$$

Note that, as in dimensional regularization, a mass scale $\mu_r$ is introduced. The momentum integration can now be carried out and one obtains

$$\begin{aligned} \zeta(s) &= \frac{1}{2}\frac{1}{(2\pi)^4}\pi^2\frac{\Gamma(s-2)}{\Gamma(s)}\left[\sum\right]_{m,n} \mu_r^{2s}\left(\frac{4}{R_2^2}\left[e^2(m+\alpha)^2 + (n+\beta)^2\right] + M^2\right)^{2-s} \\ &= \frac{4^{2-s}}{32\pi^2 R_2^{4-2s}}\frac{\mu_r^{2s}}{(s-2)(s-1)}\left[\sum\right]_{m,n}\left(\left[e^2(m+\alpha)^2 + (n+\beta)^2\right] + \frac{R_2^2}{4}M^2\right)^{2-s} \ . \end{aligned} \tag{21}$$

The boundary conditions of fields on the orbifold $T^2/\mathbb{Z}_2^3$ are characterized by three parities. For positive (negative) parity the field is nonzero (zero) at the corresponding fixed point. For the Casimir energy only those chiral and vector multiplets are relevant which are nonzero at the fixed point where supersymmetry is broken. Hence one parity, chosen to be the first one, has to be positive. Inspection of the mode expansion in Appendix A shows that for the fields $\Phi_{+-\pm}$, corresponding to $(\alpha,\beta) = (1/2,0),(1/2,1/2)$, one has to perform the sum

$$\left[\sum\right]_{m,n} = \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} , \tag{22}$$

whereas the boundary conditions $\Phi_{++\pm}$, with $(\alpha,\beta) = (0,0),(0,1/2)$, requires the sum

$$\left[\sum\right]_{m,n} = \left[\delta_{0,m}\sum_{n=0}^{\infty} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\right] . \tag{23}$$

The two summations (22) and (23) are carried out in Appendix B. The result can be expressed in the following form, which is suitable for numerical analysis,

$$\begin{aligned}
V_M^{\alpha\beta}(R_1,R_2) = {} & \frac{M^6 R_1 R_2}{768\pi}\left(\frac{11}{12} - \log\left(\frac{M}{\mu_r}\right)\right) \\
& - \delta_{\alpha 0}\delta_{\beta 0}\frac{M^4}{64\pi^2}\left(\frac{3}{4} - \log\left(\frac{M}{\mu_r}\right)\right) \\
& - \frac{1}{8\pi^4}\frac{M^3 R_2}{R_1^2}\sum_{p=1}^{\infty}\frac{\cos(2\pi p\alpha)}{p^3}K_3(\pi p M R_1) \\
& - \frac{2}{\pi^4}\frac{1}{R_2^4}\sum_{p=1}^{\infty}\frac{\cos(2\pi p\beta)}{p^{5/2}}\sum_{m=0}^{\infty}\frac{1}{2^{\delta_{\alpha 0}\delta_{m0}}}\left(\tfrac{R_2}{R_1}\sqrt{(m+\alpha)^2 + \tfrac{M^2 R_1^2}{4}}\right)^{5/2} \\
& \qquad K_{5/2}\left(2\pi\, p\, \tfrac{R_2}{R_1}\sqrt{(m+\alpha)^2 + \tfrac{M^2 R_1^2}{4}}\right) .
\end{aligned} \tag{24}$$

We have checked numerically that this expression satisfies the symmetry relations (18). As a good approximation, where the symmetries are manifest, one can derive [29]

$$\begin{aligned}
V_M^{\alpha\beta}(R_1,R_2) = {} & \frac{M^6 R_1 R_2}{768\pi}\left(\frac{11}{12} - \log\left(\frac{M}{\mu_r}\right)\right) \\
& - \delta_{\alpha 0}\delta_{\beta 0}\frac{M^4}{64\pi^2}\left(\frac{3}{4} - \log\left(\frac{M}{\mu_r}\right)\right) \\
& - \frac{1}{8\pi^4}\frac{M^3 R_2}{R_1^2}\sum_{p=1}^{\infty}\frac{\cos(2\pi p\alpha)}{p^3}K_3(\pi p M R_1) \\
& - \frac{1}{8\pi^4}\frac{M^3 R_1}{R_2^2}\sum_{p=1}^{\infty}\frac{\cos(2\pi p\beta)}{p^3}K_3(\pi p M R_2) .
\end{aligned} \tag{25}$$

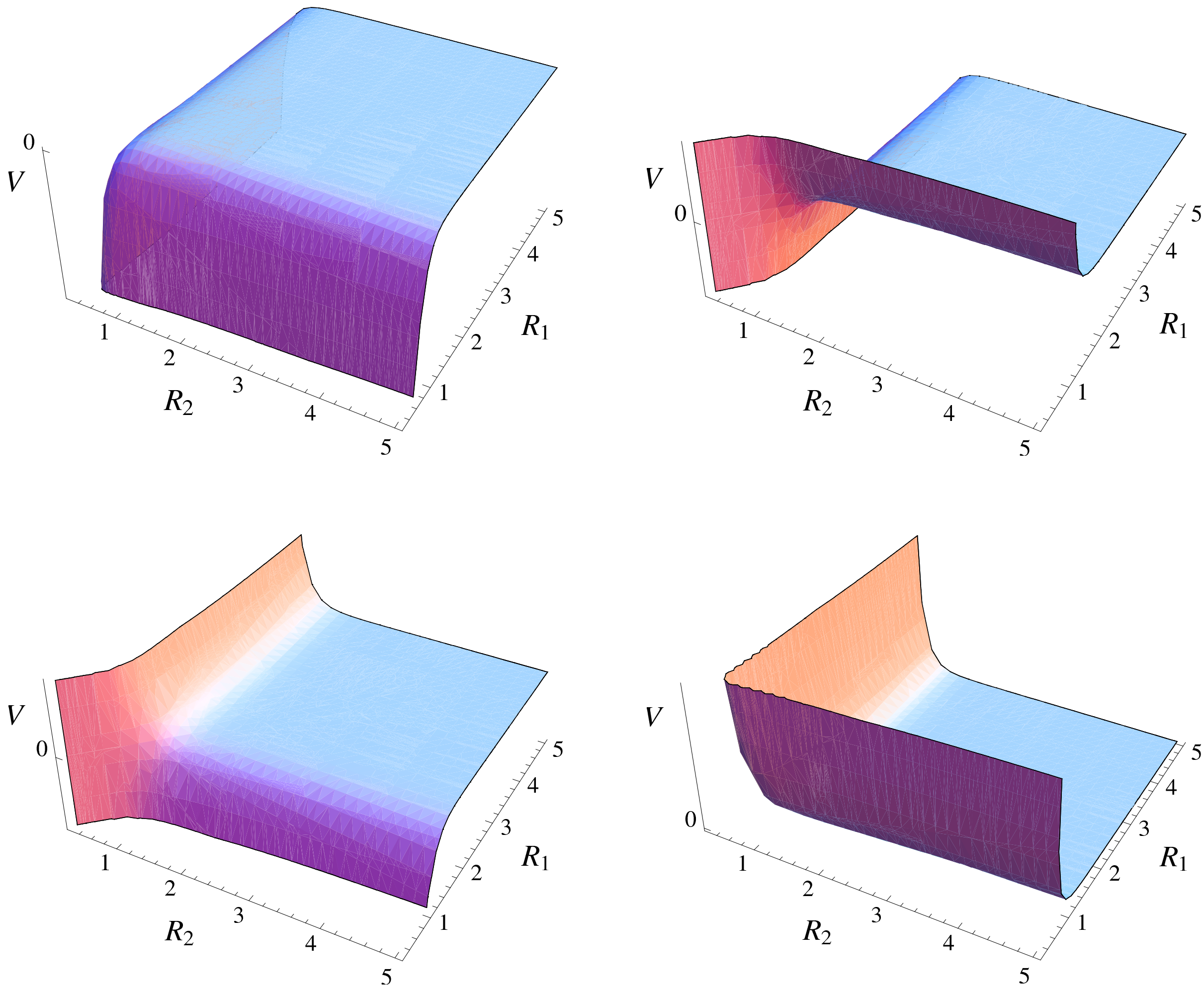


Figure 1: The four different contributions to the Casimir energy in units of the supersymmetric mass $M$. From top left to bottom right we have $V_M^{0,0}$, $V_M^{1/2,0}$, $V_M^{0,1/2}$ and $V_M^{1/2,1/2}$ as defined in the text.

The term $\propto \delta_{\alpha 0}\delta_{\beta 0}$, which is independent of $R_1$ and $R_2$, is precisely the contribution of the 'zero' mode in (21), with $\alpha = \beta = m = n = 0$.

The dependence of the first two terms in (24) on the regularization scale $\mu_r$ is a remnant of the subtraction of divergent bulk and brane cosmological terms, as in dimensional regularization [20]. The corresponding contributions to the anomalous dimensions of the 6D and 4D cosmological terms read

$$\gamma_6 = \mu_r \frac{\partial}{\partial \mu_r}\Lambda_6 = -\frac{M^6 R_1 R_2}{768\pi}\ , \quad \gamma_4 = \mu_r \frac{\partial}{\partial \mu_r}\Lambda_4 = \frac{M^4}{64\pi}\ . \tag{26}$$

The presence of these terms demonstrates that the renormalization of the divergent energy density (24) requires counter terms for the bulk and brane cosmological terms.

In general, the Casimir energy is a sum of the four possible terms,

$$V_M = AV_M^{0,0} + BV_M^{0,1/2} + CV_M^{1/2,0} + DV_M^{1/2,1/2}\ , \tag{27}$$

where the coefficients $A$,...,$D$ depend on the field content of the model and we have assumed equal masses for simplicity. The four functions $V_M^{0,0}$,...,$V_M^{1/2,1/2}$ are shown in Figure 1. For small $R_{1,2}$, $V_M^{0,0}$ is attractive and $V_M^{1/2,1/2}$ is repulsive, whereas the other two have mixed behavior.

In supersymmetric theories there is a cancellation between bosonic and fermionic contributions, and the expression (27) for the Casimir energy is replaced by

$$\begin{aligned} V &= a\left(V_{M'}^{0,0} - V_M^{0,0}\right) + b\left(V_{M'}^{0,1/2} - V_M^{0,1/2}\right) \\ &\quad + c\left(V_{M'}^{1/2,0} - V_M^{1/2,0}\right) + d\left(V_{M'}^{1/2,1/2} - V_M^{1/2,1/2}\right) , \end{aligned} \tag{28}$$

where $M' = \sqrt{M^2 + m^2}$, with supersymmetric mass $M$ and supersymmetry breaking mass $m$; the coefficients $a$,...,$d$ again depend on the field content of the model. Compared to the non-supersymmetric case (27), the behavior at small $R_{1,2}$ is inverted. For bulk vector- and hypermultiplets only the 4D $\mathcal{N} = 1$ vector and chiral multiplets are relevant, which couple to the brane where supersymmetry is broken.

The qualitative behavior of Figure 1 is easily understood by evaluating explicitly the Casimir energy (25) at small radii $R_1, R_2 \ll 1/M$. Expanding the Bessel function $K_3$ for small arguments and performing the summations over $p$, one obtains

$$V_M^{0,0}(R_1, R_2) = -\frac{1}{945\pi}\frac{R_2}{R_1^5}\left(1 - \frac{21}{16}M^2R_1^2 + \dots\right) + R_1 \leftrightarrow R_2 , \tag{29}$$

$$\begin{aligned} V_M^{0,1/2}(R_1, R_2) &= -\frac{1}{945\pi}\frac{R_2}{R_1^5}\left(1 - \frac{21}{16}M^2R_1^2 + \dots\right) \\ &\quad + \frac{31}{30240\pi}\frac{R_1}{R_2^5}\left(1 - \frac{147}{124}M^2R_2^2 + \dots\right) , \end{aligned} \tag{30}$$

$$\begin{aligned} V_M^{1/2,0}(R_1, R_2) &= \frac{31}{30240\pi}\frac{R_2}{R_1^5}\left(1 - \frac{147}{124}M^2R_1^2 + \dots\right) \\ &\quad - \frac{1}{945\pi}\frac{R_1}{R_2^5}\left(1 - \frac{21}{16}M^2R_2^2 + \dots\right) , \end{aligned} \tag{31}$$

$$V_M^{1/2,1/2}(R_1, R_2) = \frac{31}{30240\pi}\frac{R_2}{R_1^5}\left(1 - \frac{147}{124}M^2R_1^2 + \dots\right) + R_1 \leftrightarrow R_2 . \tag{32}$$

From these equations one immediately reads off the behavior of $V_M^{\alpha,\beta}$ at small radii. For $R_{1,2} \to 0$, with $R_1/R_2$ fixed, one obtains the behavior of the Casimir energy for 5D orbifolds. For supersymmetric models, the mass independent terms cancel, and with $M'^2 - M^2 = m^2$ the second terms in the expansion yield the inverted behavior at small $R_{1,2}$.

# 4 Casimir Energy of the Orbifold Model

Given the results of the previous section we can now easily evaluate the Casimir energy of the orbifold GUT model described in Section 2. At the branes, only 4D $\mathcal{N} = 1$ supersymmetry is preserved. A multiplet contributes to the Casimir energy if its bosonic and fermionic degrees of freedom have different masses. This only happens if its first $\mathbb{Z}_2$ parity is positive so that it can couple to the singlet $S$ at the $SO(10)$ brane, whose non-vanishing $F$-term breaks 4D $\mathcal{N} = 1$ supersymmetry spontaneously. Hence, from the 6D $\mathcal{N} = 1$ vector multiplet only $V$ contributes (cf. Table 1). Also for the hypermultiplets only one 4D $\mathcal{N} = 1$ chiral multiplet is relevant. The corresponding chiral multiplets with positive $\mathbb{Z}_2$ parity are listed in Table 2.

## 4.1 Contribution from the Vector Multiplet

The expectation values (2) break $SO(10)$ spontaneously to $SU(5)$. This generates the mass $M$ for the 21 vector multiplets of the coset $SO(10)/SU(5)$[1]. Since the Higgs mechanism preserves 6D $\mathcal{N} = 2$ supersymmetry, also 21 hypermultiplets become massive. In addition all gauginos acquire a supersymmetry breaking mass $m_g$.

From Tables 1 and 2 and from the mode decomposition we can now read off the total Casimir energy of the massive vector multiplet on $T^2/\mathbb{Z}_2^3$,

$$\begin{aligned} V_g &= 24\left(V^{0,0} - V^{0,0}_{m_g}\right) + 24\left(V^{0,1/2} - V^{0,1/2}_{m_g}\right) + 2\left(V^{0,0}_{M} - V^{0,0}_{M'}\right) \\ &\quad +16\left(V^{1/2,0}_{M} - V^{1/2,0}_{M'}\right) + 24\left(V^{1/2,1/2}_{M} - V^{1/2,1/2}_{M'}\right) , \end{aligned} \tag{33}$$

where $M' = \sqrt{M^2 + m_g^2}$. Using the expansion (29) and $m_g = \mu/(\Lambda^2 V)$ one finds at small radii,

$$V_g = -\frac{1}{48\pi}\frac{\mu^2}{\Lambda^4 V^2}\left(\frac{R_2}{R_1^3} + \dots\right) , \tag{34}$$

where the dots denote terms of relative order $\mathcal{O}(M_i R_{1,2})$, with $M_i = m_g, M, M'$, which have been neglected.

## 4.2 Contributions from Hypermultiplets

The contribution of hypermultiplets to the Casimir energy again depends on the symmetry breaking, i.e., the choice of parities. Consider the **10**-plets $H_{1,2}$ which contain the Higgs doublets as zero mode. From Table 2 one reads off,

$$\begin{aligned} V_H &= 8\left(V^{0,0}_{m_H} - V^{0,0}\right) + 8\left(V^{0,1/2}_{m_H} - V^{0,1/2}\right) \\ &\quad +12\left(V^{1/2,0}_{m_H} - V^{1/2,0}\right) + 12\left(V^{1/2,1/2}_{m_H} - V^{1/2,1/2}\right) , \end{aligned} \tag{35}$$

[1] We shall ignore the $\mathcal{O}(1)$ factors for the masses of different $SU(5)$ representations as they will not be important in the following discussion.

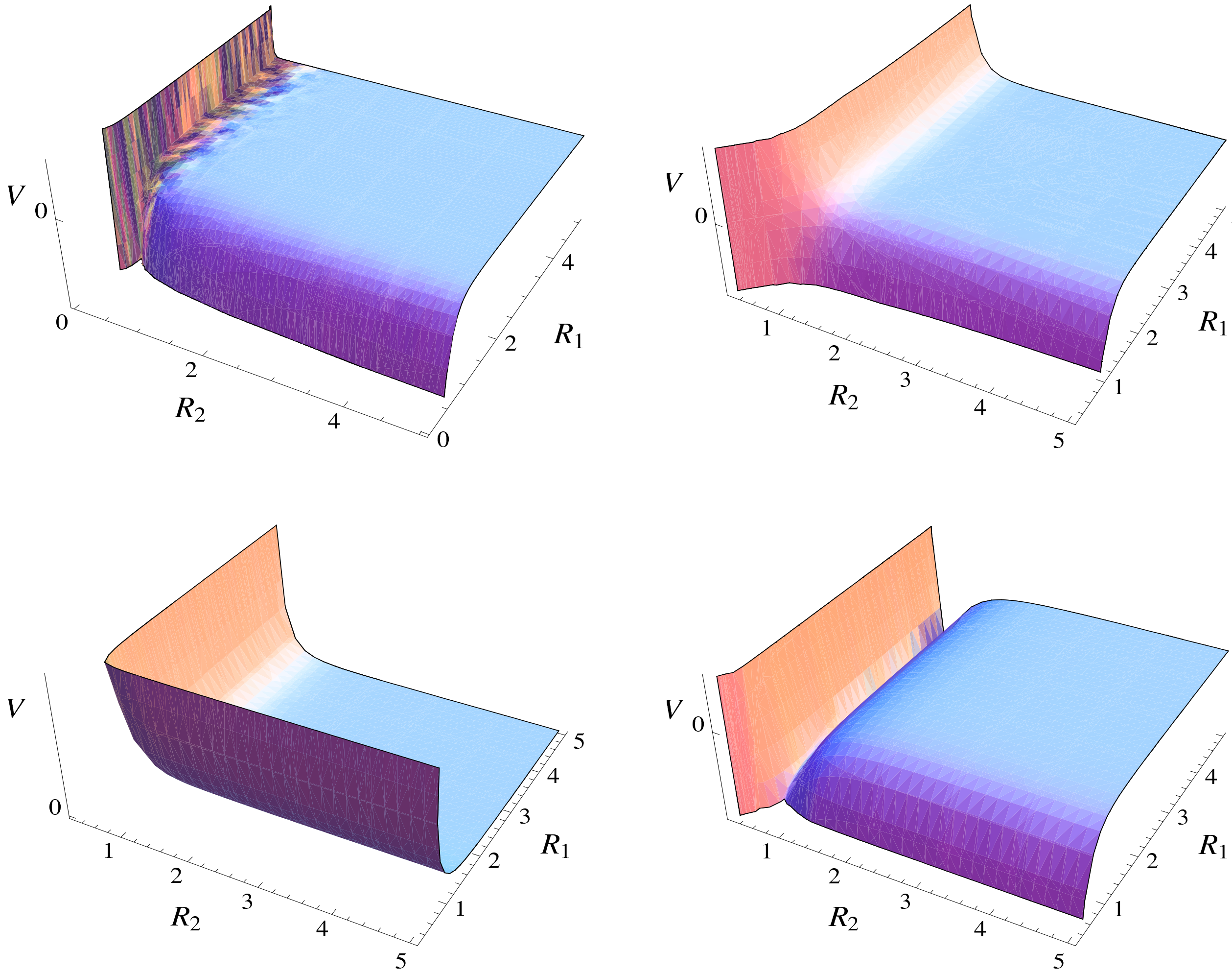


Figure 2: The different contributions to the Casimir energy from the bulk vector multiplet and the hypermultiplets of the Higgs sector (see text). From top left to bottom right we have the contributions from the vector multiplet, the **10**-plets $H_{1,2}$, the **10**-plets $H_{3,4}$, and the **16**-plets $\Phi, \Phi^c$.

which, together with (29) and $m_H^2 = -\lambda\mu^2/(\Lambda^2 V)$, yields

$$V_H = -\frac{1}{720\pi}\frac{\lambda\mu^2}{\Lambda^2 V}\left(-5\frac{R_2}{R_1^3} + \frac{5}{2}\frac{R_1}{R_2^3} + \dots\right) . \tag{36}$$

For the **10**-plets $H_{3,4}$ the choice of parities is different, leading to color triplets as zero modes. The corresponding Casimir energy is given by

$$\begin{aligned} V'_H &= 12\left(V^{0,0}_{m_H} - V^{0,0}\right) + 12\left(V^{0,1/2}_{m_H} - V^{0,1/2}\right) \\ &\quad +8\left(V^{1/2,0}_{m_H} - V^{1/2,0}\right) + 8\left(V^{1/2,1/2}_{m_H} - V^{1/2,1/2}\right) . \end{aligned} \tag{37}$$

Here we have neglected the supersymmetric brane masses (cf. [24]) which cancel in the behavior at small $R_{1,2}$,

$$V'_H = -\frac{1}{720\pi}\frac{\lambda'\mu^2}{\Lambda^2 V}\left(10\frac{R_2}{R_1^3} + \frac{5}{2}\frac{R_1}{R_2^3} + \dots\right) . \tag{38}$$

In the same way one obtains for the **16**-plets,

$$\begin{aligned} V_\Phi &= 2\left(V^{0,0}_{M'} - V^{0,0}_{M}\right) + 16\left(V^{0,1/2}_{M'} - V^{0,1/2}_{M}\right) + 24\left(V^{1/2,1/2}_{M'} - V^{1/2,1/2}_{M}\right) \\ &\quad +8\left(V^{1/2,0}_{m_\Phi} - V^{1/2,0}\right) + 14\left(V^{0,0}_{m_\Phi} - V^{0,0}\right) , \end{aligned} \tag{39}$$

with $M' = \sqrt{M^2 + m_\Phi^2}$, which yields for small radii

$$V_\Phi = -\frac{1}{720\pi}\frac{\lambda''\mu^2}{\Lambda^2 V}\left(4\frac{R_2}{R_1^3} - 11\frac{R_1}{R_2^3} + \dots\right) . \tag{40}$$

The four contributions to the Casimir energy, $V_g$, $V_H$, $V'_H$ and $V_\Phi$ are displayed in Figure 2. Note that features at larger radii, like the profile in the $R_2$-direction for $V_\Phi$, can be lost in the simplified expression where we keep only the leading term in $\mu^2$. The behavior at small radii however is unchanged and obvious from the analytic expressions given above. Note that only $V'_H$ is repulsive in all directions at small radii.

To leading order in $1/\Lambda$, the Casimir energy is determined by the contribution from hypermultiplets since the gaugino mass is stronger volume suppressed than the scalar masses. Depending on signs and magnitude of $\lambda$, $\lambda'$ and $\lambda''$, the resulting behavior at small radii can be attractive or repulsive. As an example, we shall assume in the following $\lambda' < 0, |\lambda'| \gg |\lambda|, |\lambda''|$ which yields a repulsive behavior at small radii.

# 5 Stabilization of the Compact Dimensions

In the previous section we have calculated quantum corrections to the effective potential at small radii and we have seen that, depending on the supersymmetry breaking parameters, the behavior can be attractive or repulsive. In the latter case a bulk cosmological term can lead to stabilization of the compact dimensions [18]. As we shall show in this section, stabilization can also follow from the interplay of the Higgs mechanism in 6D, which generates bulk mass terms, and supersymmetry breaking on the brane.

Consider the mass $M$ generated by spontaneous symmetry breaking as discussed in Section 2 (cf. (6)),

$$M^2 \simeq g_6^2 \langle \Phi^c \rangle^2 = g_4^2 V \langle \Phi^c \rangle^2 \ , \tag{41}$$

where $g_6$ has dimension length and $g_4 = g_6/\sqrt{V}$ is dimensionless. For simplicity, we shall assume that $M$ is small compared to the Kaluza-Klein masses and approximately constant.

In orbifold compactifications of the heterotic string expectation values $\langle \Phi \rangle$ can be induced by localized Fayet-Iliopoulos terms. Vanishing of the D-terms then implies

$$V \langle \Phi^c \rangle^2 = C \Lambda^2 \ , \tag{42}$$

where $C \ll 1$ is a loop factor and $\Lambda$ is the string scale or, more generally, the UV cutoff of the model. For instance, in the 6D model of [16] one finds for the localized anomalous $U(1)$'s, $C\Lambda^2 \sim g M_{\mathrm{P}}^2/(384\pi^2)$.

Supersymmetry breaking by a brane field $S$, with $\mu = F_S/\Lambda$ , leads to a 'classical' vacuum energy density,

$$\begin{aligned} V^{(0)} &= -\lambda'' \int d^2 y \int d^4\theta \frac{1}{\Lambda^4} \delta^2(y) \langle S^\dagger S (\Phi^\dagger \Phi + \Phi^{c\dagger}\Phi^c) \rangle \simeq -\lambda'' \ \frac{\mu^2}{\Lambda^2} \ \langle \Phi^c \rangle^2 \\ &= -\lambda'' \ \frac{\mu^2 C}{V} \ , \end{aligned} \tag{43}$$

with $V = (2\pi)^2 R_1 R_2$. For $\lambda'' > 0$, $V^{(0)}$ is attractive at large radii. Note that this supersymmetry breaking mass term does not lead to a negative mass squared for $\Phi$ and $\Phi^c$ since these fields are assumed to be stabilized by much larger supersymmetry preserving masses at the minimum. We assume that no tachyonic mass terms are generated for fields whose expectation values are not fixed by the D–term potential.

The classical energy density $V^{(0)}$ together with the Casimir energy $V^{(1)} = V'_H$ yields the total energy density,

$$\begin{aligned} V_{\mathrm{tot}}(R_1, R_2) &= V^{(0)}(R_1, R_2) + V^{(1)}(R_1, R_2) \\ &= -\frac{1}{288\pi^3} \frac{\mu^2 \lambda'}{\Lambda^2} \left( \frac{1}{R_1^4} + \frac{1}{4R_2^4} \right) - \frac{\lambda''}{4\pi^2} \frac{\mu^2 C}{R_1 R_2} \ . \end{aligned} \tag{44}$$

The effective potential is attractive at large radii and, for $\lambda' < 0$, i.e. $m^2_{H_{3,4}} > 0$, repulsive at small radii. One easily verifies that the effective potential $V_{\mathrm{tot}}$ has a stable minimum at

$$R_1^{\mathrm{min}} = \sqrt{2} R_2^{\mathrm{min}} \ , \quad R_2^{\mathrm{min}} = \frac{2^{1/4}}{12\sqrt{\pi}} \sqrt{\frac{-\lambda'}{\lambda''}} \frac{1}{M} \ . \tag{45}$$

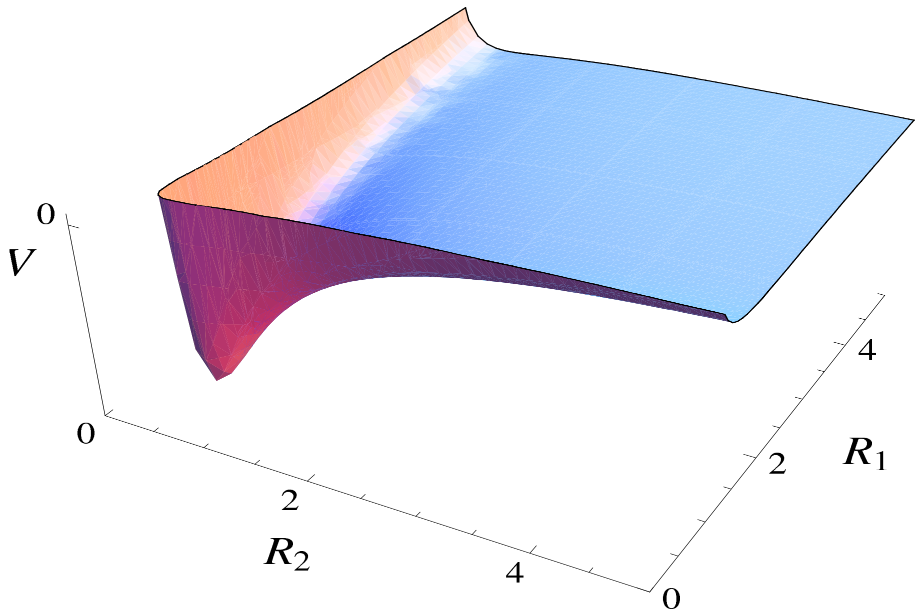


Figure 3: Casimir energy of the **10**-plets $H_3$ and $H_4$ together with the classical energy density from the supersymmetry breaking brane.

Here $M$ is the mass given by Eq. (41) at the minimum, and we have assumed $g_4(V_{\min}) \simeq 1/\sqrt{2}$, as it is the case for standard model gauge interactions. As Figure 3 illustrates, the total energy density $V_{\mathrm{tot}}$ is very flat for large radii.

In orbifold compactifications of the heterotic string one typically has $M \sim M_{\mathrm{GUT}}$. It is very remarkable that the interplay of gauge and supersymmetry breaking has lead to a stabilization at $R^{\min} \sim 1/M_{\mathrm{GUT}}$, independent of the scale $\mu$ of supersymmetry breaking. The reason is that both, the classical vacuum energy density as well as the one-loop Casimir energy are proportional to $\mu^2$ which therefore does not affect the position of the minimum. Another interesting implication of the potential is that for $\mu \ll M_{\mathrm{GUT}}$,

$$\Delta V_{\mathrm{tot}}(R^{\min}) = V_{\mathrm{tot}}(\infty) - V_{\mathrm{tot}}(R^{\min}) \;\sim\; \mu^2 M_{\mathrm{GUT}}^2 \;\ll\; M_{\mathrm{GUT}}^4 \,. \tag{46}$$

Note that the energy density $V_{\mathrm{tot}}$ is negative at the minimum. It has to be tuned to zero by means of a brane cosmological constant. In a full supergravity treatment of stabilization also the interactions of the supersymmetry breaking brane field with the radion fields have to be taken into account.

The fact that the energy density difference $V_{\mathrm{tot}}(\infty) - V_{\mathrm{tot}}(R^{\min})$ is much smaller than $M_{\mathrm{GUT}}^4$ has important cosmological consequences. In the thermal phase of the early universe, the volume of the compact dimensions and, correspondingly, the value of 4D coupling constants begins to change already at temperatures $T \sim \sqrt{\mu M_{\mathrm{GUT}}} \ll M_{\mathrm{GUT}}$ (cf. [31]).

# 6 Conclusions

We have calculated the one-loop Casimir energy for bulk fields on the orbifold $T^2/\mathbb{Z}_2^3$. As expected, depending on the boundary conditions, the behavior at small radii can be attractive or repulsive. For the considered supersymmetric model, the Casimir energy is proportional to the scale of supersymmetry breaking. The relative strength of the couplings of the different bulk fields to the supersymmetry breaking brane field then determines whether the behavior of the total energy density is repulsive or attractive at small radii.

Quantum corrections also modify the behavior at large radii. In orbifold compactifications with $U(1)$ gauge factors, generically Fayet-Iliopoulos terms are generated locally at the orbifold fixed points. This leads to a breaking of these $U(1)$ gauge symmetries by the Higgs mechanism. Since the symmetry breaking is induced by local terms, the generated masses scale like $M \sim 1/\sqrt{V}$ with the volume of the compact dimensions.

The coupling of the bulk Higgs field to the supersymmetry breaking brane field gives rise to a classical contribution to the total energy density which scales like $1/V$ with the volume. Depending on the sign of the coupling, the behavior of the energy density at large radii can be attractive or repulsive. An attractive behavior at large radii, together with a repulsive behavior due to the Casimir energy at small radii, can stabilize the compact dimensions. Since the supersymmetry breaking scale factorizes, the vacuum size of the compact dimensions is determined by the remaining mass scale, the mass $M$ generated by the Higgs mechanism, $R^{\text{min}} \sim 1/M \sim 1/M_{\text{GUT}}$. At the minimum the energy density $V_{\text{tot}}$ is negative and has to be tuned to zero by adding a brane cosmological term.

The characteristic feature of the described stabilization mechanism is a potential well much smaller than the GUT scale, $\Delta V_{\text{tot}}(R^{\text{min}}) \sim \mu^2 M_{\text{GUT}}^2 \ll M_{\text{GUT}}^4$. Clearly, this has important cosmological consequences, both for the thermal phase of the early universe as well as a possible earlier inflationary phase.

# Acknowledgments

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# A Mode Expansion on $T^2/\mathbb{Z}_2^3$

The orbifold $T^2/\mathbb{Z}_2^3$ has four fixed points which we denote by $y_{\mathrm{O}} = (0,0)$, $y_{\mathrm{PS}} = (\pi R_1/2, 0)$, $y_{\mathrm{GG}} = (0, \pi R_2/2)$ and $y_{\mathrm{fl}} = (\pi R_1/2, \pi R_2/2)$ (cf. [32]). The possible boundary conditions of functions on this orbifold are characterized by three parities, $(a, b = +, -)$,

$$\begin{aligned}
\phi_{\pm ab}(y_{\mathrm{O}} - y) &= \pm\phi_{\pm ab}(y_{\mathrm{O}} + y) \,, \\
\phi_{a\pm b}(y_{\mathrm{PS}} - y) &= \pm\phi_{a\pm b}(y_{\mathrm{PS}} + y) \,, \\
\phi_{ab\pm}(y_{\mathrm{GG}} - y) &= \pm\phi_{ab\pm}(y_{\mathrm{GG}} + y) \,.
\end{aligned} \tag{47}$$

It is straightforward to define an orthonormal basis on the torus. The mode expansion of functions with the boundary conditions (47) then reads explicitly,

$$\phi_{+++}(x,y) = \frac{1}{\sqrt{2\pi^2 R_1 R_2 2^{\delta_{n,0}\delta_{m,0}}}} \left[\delta_{0,m}\sum_{n=0}^{\infty} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\right] \phi_{+++}^{(2m,2n)}(x) \times \cos\left(\frac{2my_1}{R_1} + \frac{2ny_2}{R_2}\right) , \tag{48a}$$

$$\phi_{++-}(x,y) = \frac{1}{\sqrt{2\pi^2 R_1 R_2}} \left[\delta_{0,m}\sum_{n=0}^{\infty} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\right] \phi_{++-}^{(2m,2n+1)}(x) \times \cos\left(\frac{2my_1}{R_1} + \frac{(2n+1)y_2}{R_2}\right) , \tag{48b}$$

$$\phi_{+-+}(x,y) = \frac{1}{\sqrt{2\pi^2 R_1 R_2}} \left[\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\right] \phi_{+-+}^{(2m+1,2n)}(x) \times \cos\left(\frac{(2m+1)y_1}{R_1} + \frac{(2n)y_2}{R_2}\right) , \tag{48c}$$

$$\phi_{+--}(x,y) = \frac{1}{\sqrt{2\pi^2 R_1 R_2}} \left[\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\right] \phi_{+--}^{(2m+1,2n+1)}(x) \times \cos\left(\frac{(2m+1)y_1}{R_1} + \frac{(2n+1)y_2}{R_2}\right) , \tag{48d}$$

$$\phi_{-++}(x,y) = \frac{1}{\sqrt{2\pi^2 R_1 R_2}} \left[\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\right] \phi_{-++}^{(2m+1,2n+1)}(x) \times \sin\left(\frac{(2m+1)y_1}{R_1} + \frac{(2n+1)y_2}{R_2}\right) , \tag{48e}$$

$$\phi_{-+-}(x,y) = \frac{1}{\sqrt{2\pi^2 R_1 R_2}} \left[\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\right] \phi_{-+-}^{(2m+1,2n)}(x) \times \sin\left(\frac{(2m+1)y_1}{R_1} + \frac{2ny_2}{R_2}\right) , \tag{48f}$$

$$\phi_{--+}(x,y) = \frac{1}{\sqrt{2\pi^2 R_1 R_2}} \left[ \delta_{0,m} \sum_{n=0}^{\infty} + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \right] \phi_{++-}^{(2m,2n+1)}(x) \times \sin\left( \frac{2my_1}{R_1} + \frac{(2n+1)y_2}{R_2} \right) , \tag{48g}$$

$$\phi_{---}(x,y) = \frac{1}{\sqrt{2\pi^2 R_1 R_2}} \left[ \delta_{0,m} \sum_{n=0}^{\infty} + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \right] \phi_{---}^{(2m,2n)}(x) \times \sin\left( \frac{2my_1}{R_1} + \frac{(2n)y_2}{R_2} \right) . \tag{48h}$$

# B Evaluation of Casimir Sums

Our evaluation of the Casimir double sums requires two single sums which we shall now consider. The first sum reads

$$\widetilde{F}(s;a,c) \equiv \sum_{m=0}^{\infty} \frac{1}{\left[(m+a)^2 + c^2\right]^s} . \tag{49}$$

This is a series of the generalized Epstein-Hurwitz zeta type. The result can be found in [30] and is given by

$$\begin{aligned} \widetilde{F}(s;a,c) =& \frac{c^{-2s}}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m+s)}{m!} c^{-2m} \zeta_H(-2m,a) + \sqrt{\pi} \frac{\Gamma(s-\frac{1}{2})}{2\Gamma(s)} c^{1-2s} \\ &+ \frac{2\pi^s}{\Gamma(s)} c^{1/2-s} \sum_{p=1}^{\infty} p^{s-1/2} \cos(2\pi p a) K_{s-1/2}(2\pi p c) , \end{aligned} \tag{50}$$

where $\zeta_H(s,a)$ is the Hurwitz zeta-function. Note that this is not a convergent series but an asymptotic one. In the following it will be important that $\zeta_H(-2m,0) = \zeta_H(-2n,1/2) = 0$ for $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. In our case, the first sum in $\widetilde{F}(s;a,c)$ thus reduces to a single term. For $a = 1/2$ the sum vanishes, and for $a = 0$ only the first term contributes; with $\zeta_H(0,0) = 1/2$ one obtains $c^{-2s}/2$.

The second, related sum is given by

$$F(s;a,c) \equiv \sum_{m=-\infty}^{\infty} \frac{1}{\left[(m+a)^2 + c^2\right]^s} . \tag{51}$$

Using the two identities ($m \in \mathbb{N}$)

$$\zeta_H(-2m,a) = -\zeta_H(-2m,1-a) , \tag{52}$$

$$F(s;a,c) = \widetilde{F}(s;a,c) + \widetilde{F}(s;1-a,c) , \tag{53}$$

one easily obtains, in agreement with [18],

$$F(s;a,c) = \frac{\sqrt{\pi}}{\Gamma(s)}|c|^{1-2s}\left[\Gamma\left(s-\tfrac{1}{2}\right) + 4\sum_{p=1}^{\infty}\cos(2\pi pa)(\pi\, p\,|c|)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi\, p\,|c|)\right] \ . \quad (54)$$

These two sums provide the basis for our evaluation of the Casimir sums.

## B.1 Casimir Sum (I) on $\mathbf{T^2}/\mathbb{Z}_2^3$

We first consider the summation

$$\left[\sum\right]_{m,n} = \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} \ . \quad (55)$$

In this case the Casimir energy (cf. (21)) is obtained from

$$\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\left[e^2(m+\alpha)^2 + (n+\beta)^2 + \kappa^2\right]^{-s} \ , \quad (56)$$

where we have shifted $s \to s+2$ and defined $\kappa^2 = \frac{R_2^2}{4}M^2$. Using the expression for $F(s;a,c)$, we can perform the sum over $n$,

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\left[e^2(m+\alpha)^2 + (n+\beta)^2 + \kappa^2\right]^{-s} \\
&= \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_{m=0}^{\infty}(e^2(m+\alpha)^2+\kappa^2)^{1/2-s} \\
&\quad + \frac{4\sqrt{\pi}}{\Gamma(s)}\sum_{p=1}^{+\infty}\cos(2\pi p\beta)\sum_{m=0}^{\infty}(\pi\, p)^{s-\frac{1}{2}}\left(\sqrt{e^2(m+\alpha)^2+\kappa^2}\right)^{\frac{1}{2}-s} \\
&\qquad\qquad K_{s-\frac{1}{2}}(2\pi\, p\,\sqrt{e^2(m+\alpha)^2+\kappa^2}) \\
&\equiv \ f_1(s) + f_2(s) \ . 
\end{aligned} \quad (57)$$

Let us consider $f_1(s)$ first. The sum over $m$ can be performed with the help of $\widetilde{F}(s;a,c)$,

$$\begin{aligned}
f_1(s) =& \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_{m=0}^{\infty}(e^2(m+\alpha)^2+\kappa^2)^{1/2-s} \\
=& \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\kappa^{1-2s}\zeta_H(0,\alpha) + \frac{\pi}{2(s-1)}\frac{\kappa^{2-2s}}{e} \\
&+ \frac{2\pi^s}{\Gamma(s)}e^{-s}\kappa^{1-s}\sum_{p=1}^{\infty}p^{s-1}\cos(2\pi p\alpha)K_{s-1}(2\pi p\left(\tfrac{\kappa}{e}\right)) \ . 
\end{aligned} \quad (58)$$

Recalling the shift in $s$, we can now write $\zeta(s)$ (21) as

$$
\begin{aligned}
\zeta(s) =& \frac{1}{32\pi^2}\left(\frac{4}{R_2^2}\right)^{-s}\frac{\mu_r^{2s+4}}{s(s+1)}\Bigg\{\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\kappa^{1-2s}\zeta_H(0,\alpha)+\frac{\pi}{2(s-1)}\frac{\kappa^{2-2s}}{e} \\
&+\frac{2\pi^s}{\Gamma(s)}e^{-s}\kappa^{1-s}\sum_{p=1}^{\infty}p^{s-1}\cos(2\pi p\alpha)K_{s-1}(2\pi p\left(\tfrac{\kappa}{e}\right)) \\
&+\frac{4\sqrt{\pi}}{\Gamma(s)}\sum_{p=1}^{+\infty}\cos(2\pi p\beta)\sum_{m=0}^{\infty}(\pi\, p)^{s-\frac{1}{2}}\left(\sqrt{e^2(m+\alpha)^2+\kappa^2}\right)^{\frac{1}{2}-s} \\
&\qquad\qquad K_{s-\frac{1}{2}}(2\pi\, p\,\sqrt{e^2(m+\alpha)^2+\kappa^2})\Bigg\}\ .
\end{aligned}
\tag{59}
$$

Now we have to differentiate with respect to $s$ and set $s=-2$. Since $\Gamma(-2)=\infty$, the derivative has only to act on $\Gamma(s)$ if the corresponding term is inversely proportional to $\Gamma(s)$. Performing the differentiation, using

$$
\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{\Gamma(s)}\bigg|_{s=-2}=-\frac{\Gamma'(s)}{\Gamma(s)^2}\bigg|_{s=-2}=2\ , \tag{60}
$$

and $K_a(z)=K_{-a}(z)$, and substituting again $e=R_2/R_1$ and $\kappa^2=R_2M/2$, we finally obtain for the Casimir energy,

$$
\begin{aligned}
V_{\mathrm{M}}^{\alpha,\beta(I)} =& \frac{M^5R_2}{120\pi}\zeta_H(0,\alpha)+\frac{M^6R_1R_2}{768\pi}\left(\frac{11}{12}-\log\left(\frac{M}{\mu_r}\right)\right) \\
&-\frac{1}{8\pi^4}\frac{M^3R_2}{R_1^2}\sum_{p=1}^{\infty}\frac{\cos(2\pi p\alpha)}{p^3}K_3(\pi pMR_1) \\
&-\frac{2}{\pi^4}\frac{1}{R_2^4}\sum_{p=1}^{\infty}\frac{\cos(2\pi p\beta)}{p^{5/2}}\sum_{m=0}^{\infty}\left(\frac{R_2}{R_1}\sqrt{(m+\alpha)^2+\frac{M^2R_1^2}{4}}\right)^{\frac{5}{2}} \\
&\qquad K_{5/2}\left(2\pi\, p\,\tfrac{R_2}{R_1}\sqrt{(m+\alpha)^2+M^2R_1^2/4}\right)\ .
\end{aligned}
\tag{61}
$$

The second term corresponds to a finite part of the 6D cosmological constant. The dependence on the regularization scale $\mu_r$ shows that an infinite contribution has been subtracted.

## B.2 Casimir Sum (II) on $\mathbf{T^2}/\mathbb{Z}_2^3$

The second relevant summation is

$$
\Big[\sum\Big]_{m,n}=\left[\delta_{m,0}\sum_{n=0}^{\infty}+\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\right]\ . \tag{62}
$$

For the corresponding boundary conditions one has $\alpha = 0$. The Casimir sum can then be written as

$$\begin{aligned}
&\left[\delta_{m,0}\sum_{n=0}^{\infty}+\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\right]\left[e^2m^2+(n+\beta)^2+\kappa^2\right]^{-s} \\
=&\left[\delta_{m,0}\sum_{n=0}^{\infty}+\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}-\delta_{m,0}\sum_{n=-\infty}^{\infty}\right]\left[e^2m^2+(n+\beta)^2+\kappa^2\right]^{-s} , \qquad (63)
\end{aligned}$$

where we again shifted $s \to s+2$ and set $\frac{R_2^2}{4}M^2 = \kappa^2$. The double sum is the sum (I) which we have already evaluated. Using

$$\sum_{n=-\infty}^{-1}\left[(n+\beta)^2+\kappa^2\right]^{-s} = \sum_{n=0}^{\infty}\left[(n+1-\beta)^2+\kappa^2\right]^{-s} . \qquad (64)$$

one easily finds for the remaining piece[2]

$$\begin{aligned}
f_3(s) &= -\sum_{n=0}^{\infty}\left[(n+1-\beta)^2+\kappa^2\right]^{-s} \\
&= -\kappa^{-2s}\zeta_H(0,1-\beta) - \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{2\Gamma(s)}\kappa^{1-2s} \\
&- \frac{2\pi^s}{\Gamma(s)}\kappa^{1/2-s}\sum_{p=1}^{\infty}p^{s-1/2}\cos(2\pi p(1-\beta))K_{s-1/2}(2\pi p\kappa) . \qquad (65)
\end{aligned}$$

Differentiating the corresponding contribution to $\zeta(s)$, setting $s=-2$, and substituting $\kappa = R_2M/2$ yields the Casimir energy,

$$\begin{aligned}
V_M^{0,\beta(II)} &= V_M^{0,\beta(I)} \\
&+\frac{M^4}{64\pi^2}\left(\frac{3}{2}-2\log\left(\frac{M}{\mu_r}\right)\right)\zeta_H(0,1-\beta) - \frac{1}{240\pi}M^5R_2 \\
&-\frac{1}{\pi^4}\frac{1}{R_2^4}\sum_{p=1}^{\infty}\frac{\cos(2\pi p(1-\beta))}{p^{5/2}}\left(\frac{MR_2}{2}\right)^{5/2}K_{5/2}\left(\pi pMR_2\right) . \qquad (66)
\end{aligned}$$

The first of the additional terms does not depend on the radii. It represents a finite contribution to the brane cosmological term. The dependence on the regularization scale $\mu_r$ again shows that a divergent contribution has been subtracted.

[2] Note that $\zeta_H(0,1) = -1/2$, and $\zeta_H(-2m,1) = 0$ for $m \in \mathbb{N}$.

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