Preprint typeset in JHEP style - HYPER VERSION

DESY 08-$$$
YITP-SB-08-01

Rigid supersymmetry with boundaries

Dmitry V. Belyaev

Deutsches Elektronen-Synchrotron, DESY-Theory
Notkestrasse 85, 22603 Hamburg, Germany
E-mail: dmitry.belyaev@desy.de

Peter van Nieuwenhuizen

C. N. Yang Institute for Theoretical Physics, SUNY at Stony Brook
Stony Brook, NY 11794-3840, USA
E-mail: vannieu@max2.physics.sunysb.edu

ABSTRACT: We construct rigidly supersymmetric bulk-plus-boundary actions, both in x-space and in superspace. For each standard supersymmetric bulk action a minimal supersymmetric bulk-plus-boundary action follows from an extended F- or D-term formula. Additional separately supersymmetric boundary actions can be systematically constructed using co-dimension one multiplets (boundary superfields). We also discuss the orbit of boundary conditions which follow from the Euler-Lagrange variational principle.

Contents

1. Introduction

Since its beginning, research in supersymmetry (susy) has mainly been concerned with constructing invariant actions, and deducing the consequences of their field equations. However, the field equations are only half of the information one needs for a mathematically well-posed problem; the other half are the boundary conditions (BC) one must impose on the fields. In susy (and supergravity) one usually assumes that fields fall off sufficiently fast at (spacelike and timelike) infinity and that boundary terms which arise from partial integration may be omitted. However, if there is a boundary, this assumption is unwarranted, and one must face the issue of BC. In this article we present a thorough study of BC in models of rigid susy with a timelike boundary.

We distinguish between two kinds of BC: those which are needed to keep the action invariant under rigid susy, and those which arise from the Euler-Lagrange (EL) field equations. The first set is off-shell, the second set is on-shell. Our main philosophy is to construct bulk-plus-boundary actions which are susy by themselves (under half of bulk susy), so no BC are needed to cancel boundary terms in the susy variation of the action. (This approach was first advocated in [1, 2].) We call such models "susy without BC." We develop an extension of the usual tensor calculus which gives the boundary action which one must add to the bulk action to obtain "susy without BC." Once this boundary action has been constructed, one can study the EL variation of the bulk-plus-boundary action. In the bulk it gives standard field equations, but boundary terms arise which can only be canceled by imposing BC on some of the fields. It follows that the BC one obtains in this way are, to begin with, BC on on-shell fields. However, once a set of such BC has been obtained, one can also require that they hold for off-shell fields. For example, in a path integral approach where fields are, of course, off-shell, we might still impose such BC on these off-shell fields. We shall first study the various possibilities in the examples considered below, and come back to more definite statements in the conclusions.

As always, one has the option of using the x-space (component) approach, or the superspace approach. In an earlier article [3] we analyzed a particular supergravity model ($N = 1$ supergravity in 2+1 dimensions), and since the superspace approach for supergravity is rather complicated, we cast that article entirely in x-space. However, the superspace approach of rigid susy is much simpler, and thus we shall first derive our new results in x-space, but then recast these results into superspace.

Our program of constructing invariant actions consists of two parts. First we obtain actions with "susy without BC" by adding suitable actions on the boundary; these boundary actions are not susy by themselves but merely complete the bulk actions, and we shall have to find an appropriate superspace description for them. Next, for some models it will turn out that we need to construct another action on the boundary which is susy by itself; this action can be described by x-space or superspace methods in one dimension less (boundary superfields).

Before introducing our extension of the tensor calculus, it may be helpful to point out some possible pitfalls. First, the boundary terms one obtains from partially integrating terms in the susy variation of the action are in general different from those in the EL variation of the actions. Thus even if "susy without BC" holds, one will in general nead EL BC. Second, BC on spacelike surfaces (initial conditions) have physically a very different meaning from BC on timelike surfaces (genuine BC, at all times). We consider only the latter, and choose as boundary the hypersurface at $x^3 = 0$. However, from a space-time point of view, one can treat these two sets of BC on equal footing; technically this is achieved by introducing projection operators $P_\pm = \frac{1}{2}(1 \pm n^\mu \gamma_\mu)$ where n^μ is the normal to the boundary, and decomposing the susy parameters into eigenspinors $\epsilon_\pm$ of this projection operator. This procedure was used in [4], but note that in that article a very different philosophy was used: no "susy without BC" was implemented, and the consistency of the complete set of susy BC and EL BC was studied (the "orbit of BC"). Since (half of) bulk susy is unbroken in our case, and auxiliary fields are present, the study of the orbit of BC can be written as BC on boundary superfields. Finally, it is of course true that in varying actions on the boundary one may again need to partially integrate, thus obtaining boundary terms on the boundary. We assume that all total derivatives on the boundary vanish. This is not necessary, but it simplifies the analysis.

Let us now introduce our extension of the usual tensor calculus which takes boundaries into account. As an example, consider the usual F-term formula for an invariant action in the bulk. Decomposing the integration measure $d^{m+1}x$ into a measure $d^m x$ on the boundary and dx^3 away from the boundary, one has

$$S = \int_{\mathcal{M}} dx^3 d^m x F \tag{1.1}$$

Since F varies into a total derivative, $\delta F = \bar{\epsilon}\gamma^\mu \partial_\mu \psi$, the variation of S is equal to a boundary term $\delta S = -\int d^m x (\bar{\epsilon}\gamma^3\psi)$. We shall introduce a susy parameter ϵ_+ satisfying $\bar{\epsilon}_+\gamma^3 = -\bar{\epsilon}_+$. Then $\delta S = \int d^m x\, \bar{\epsilon}_+\psi$, and since $\delta A = \bar{\epsilon}\psi$, we find a suitable action $S_{\text{boundary}} = \int d^m x\, A$ on the boundary, whose ϵ_+ variation cancels the variation of the bulk action. So, the usual F-term formula is extended to the following "$F+A$" formula for a bulk-plus-boundary action[1]

$$S = \int_{\mathcal{M}} dx^3 d^m x\, F - \int_{\partial\mathcal{M}} d^m x\, A \tag{1.2}$$

We will find that this extended F-term formula works both in 3 and 4 dimensions. In what follows, we will indicate other ways in which this formula can be derived and will apply it to various models of rigid susy in 3 and 4 dimensions.

The presence of boundary terms may modify EL BC. Consider as an example the action for the spinning string, which is the same as the 2D Wess-Zumino (WZ) action,

$$S_{\text{WZ}} = \int d\sigma dt \mathcal{L}, \quad \mathcal{L} = -\partial_\mu X \partial^\mu X - \overline{\psi}\gamma^\mu \partial_\mu \psi + F^2 \tag{1.3}$$

[1]In [3] we derived the analog of this "$F+A$" formula in supergravity. There, instead of *components* F and A, one needs to use the corresponding *densities*.

The susy variation $\delta\mathcal{L} = \partial_\mu(-\bar{\epsilon}\gamma^\mu\gamma^\nu\psi\partial_\nu X + \bar{\epsilon}\gamma^\mu\psi F)$ leads to a boundary term at $\sigma = 0$ which can be canceled (when $\epsilon = \epsilon_+$) by adding the following boundary action at $\sigma = 0$,

$$S_b = -\int dt \Big[XF + X\partial_\sigma X \Big] \tag{1.4}$$

(The first term is the term denoted by "A" in (1.2) while the second term is produced if one rewrites the action $X\partial_\mu\partial^\mu X$ as obtained from the tensor calculus as $-\partial_\mu X\partial^\mu X + \partial_\sigma(X\partial_\sigma X)$ and uses $\int d\sigma\partial_\sigma(X\partial_\sigma X) = -X\partial_\sigma X$.) From the EL variation of S_{WZ}, if one requires that all coefficients of varied fields vanish, one obtains a set of EL BC which is too strong,

$$X = F - \partial_\sigma X = \psi_+ = \psi_- = 0 \tag{1.5}$$

As explained below (see section 4.1) for the 3D WZ model, which is very similar to the 2D WZ model, one can add a separately susy action on the boundary,

$$S_{\rm b}(extra) = \int dt \Big(XF + X\partial_\sigma X - \frac{1}{2}\bar{\psi}\psi \Big) \tag{1.6}$$

The total string action now becomes

$$S = S_{\rm WZ} - \frac{1}{2}\int dt \bar{\psi}\psi \tag{1.7}$$

and one finds now the same EL BC for X as before, $\delta X\partial_\sigma X = 0$, while for ψ one finds $\bar{\psi}_+\delta\psi_- = 0$. These are the usual Dirichlet or Neumann conditions for X and the Neveu-Schwarz or Ramond conditions for ψ.[2] They are needed to make the EL variation of S vanish on-shell, but they are not needed to make the ϵ_+ susy variation of S vanish (off-shell).

The EL variations on the boundary are of the form "$p\delta q$," and one might expect that one might choose either $p = 0$ or $q = \text{const}$ as BC for each field (which would give 2^N sets of BC where N is the number of q's). However, this is incorrect: consistency of the EL BC with susy [6, 7, 5, 4] leaves only two families of BC [8, 4]. These families become shorter when auxiliary fields are properly incorporated. Then, as we show in section 4.1, each family corresponds to a BC on a boundary superfield [1, 9]. This nice result is of course due to "susy without BC."

We remark that our "susy without BC" approach, and the "$F + A$" formula in particular, can be applied to a variety of physically interesting models, including those involving strings and branes, solitons and instantons. Some of the applications were discussed in [3].

[2]For the 2D case, our conventions give $x^3 = \sigma$ and $\gamma^3 = \gamma^\sigma$. Taking a particular representation of gamma matrices (which we avoid in this paper) one can rewrite our two-component spinors $\psi_\pm$ in terms of one-component spinors $\psi^\pm$ and recover the usual form of the NS and R conditions, $\psi^+ = \pm\psi^-$ (see e.g. [5]).

2. Extended tensor calculus

In this section, we present extensions of the standard F- and D-term formulae for the 3D and 4D cases.[3] In both dimensions, we use Cartesian coordinates x^μ to describe the bulk $\mathcal{M}$ and assume that the boundary $\partial\mathcal{M}$ is at $x^3 = 0$ and is parametrized by x^m. In $\mathcal{M}$, $x^3 > 0$. In the presence of the boundary, half of susy is (spontaneously) broken. We choose to preserve the half parametrized by $\epsilon_+ = P_+\epsilon$, where $P_\pm = \frac{1}{2}(1 \pm \gamma^3)$. Then $\gamma^3\epsilon_+ = \epsilon_+$ but $\overline{\epsilon}_+\gamma^3 = -\overline{\epsilon}_+$.

2.1 3D extended F-term formula

Consider the 3D $N = 1$ scalar multiplet $\Phi_3 = (A, \psi, F)$,

$$\delta A = \overline{\epsilon}\psi, \quad \delta\psi = \gamma^\mu\epsilon\partial_\mu A + F\epsilon, \quad \delta F = \overline{\epsilon}\gamma^\mu\partial_\mu\psi \tag{2.1}$$

The standard F-term formula gives a bulk action $\int_\mathcal{M} d^3xF$ that is not susy in the presence of the boundary. Its susy variation gives rise to a boundary term $-\int_{\partial\mathcal{M}} d^2x(\overline{\epsilon}\gamma^3\psi)$. Our extended F-term formula,

$$S = \int_\mathcal{M} d^3xF - \int_{\partial\mathcal{M}} d^2xA \tag{2.2}$$

gives a bulk-plus-boundary action that is invariant under ϵ_+ susy. Indeed, $\delta F = \partial_\mu(\overline{\epsilon}\gamma^\mu\psi)$ yields a boundary term $-\overline{\epsilon}_+\gamma^3\psi = \overline{\epsilon}_+\psi$ under the ϵ_+ susy. Clearly, the corresponding variation of A on the boundary, $\delta A = \overline{\epsilon}_+\psi$, cancels the contribution from the bulk.

Another 3D $N = 1$ multiplet, which includes a 3D vector v_μ, is the spinor multiplet $\Psi_3 = (\chi, M, v_\mu, \lambda)$ with the following susy transformation rules,

$$\begin{aligned} \delta\chi &= M\epsilon + \gamma^\mu\epsilon v_\mu, \quad \delta M = -\frac{1}{2}\overline{\epsilon}\lambda + \overline{\epsilon}\gamma^\mu\partial_\mu\chi \\ \delta v_\mu &= -\frac{1}{2}\overline{\epsilon}\gamma_\mu\lambda + \overline{\epsilon}\partial_\mu\chi, \quad \delta\lambda = 2\gamma^{\mu\nu}\epsilon\partial_\mu v_\nu \end{aligned} \tag{2.3}$$

The highest component, λ, transforms into a total derivative but, as λ is a fermion, we cannot use a "λ-term formula" for constructing susy actions.

2.2 4D extended F- and D-term formulae

For the 4D $N = 1$ scalar (chiral) multiplet $\Phi_4 = (A, B, \psi, F, G)$,

$$\begin{aligned} &\delta A = \overline{\epsilon}\psi, \quad \delta B = -i\overline{\epsilon}\gamma_5\psi \\ &\delta\psi = \gamma^\mu\partial_\mu(A - i\gamma_5 B)\epsilon + (F + i\gamma_5 G)\epsilon \\ &\delta F = \overline{\epsilon}\gamma^\mu\partial_\mu\psi, \quad \delta G = i\overline{\epsilon}\gamma_5\gamma^\mu\partial_\mu\psi \end{aligned} \tag{2.4}$$

the extended F-term formula is

$$S = \int_\mathcal{M} d^4xF - \int_{\partial\mathcal{M}} d^3xA \tag{2.5}$$

[3] Our spinors are Majorana spinors, so $\overline{\psi} \equiv \psi^\dagger i\gamma^0$ is equal to $\overline{\psi} = \psi^T C$ where $C\gamma^\mu C^{-1} = -(\gamma^\mu)^T$ and $C^T = -C$. Furthermore, $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + \gamma^{\mu\nu}$ with $\eta^{\mu\nu} = (-1, +1, \ldots, +1)$ and in $d = 4$ we use γ_5 with $\gamma_5^2 = 1$.

Alternatively, one can use the extended G-term formula,

$$S = \int_{\mathcal{M}} d^4x G - \int_{\partial\mathcal{M}} d^3x B \tag{2.6}$$

In both cases, we find a bulk-plus-boundary action that is invariant under ϵ_+ susy.

For the 4D $N=1$ vector multiplet $V_4 = (C, \chi, H, K, v_\mu, \lambda, D)$,

$$\begin{aligned}
&\delta C = i\bar{\epsilon}\gamma_5\chi, \quad \delta\chi = (i\gamma_5 H - K - \gamma^\mu v_\mu + i\gamma^\mu\gamma_5\partial_\mu C)\epsilon \\
&\delta H = i\bar{\epsilon}\gamma_5\gamma^\mu\partial_\mu\chi + i\bar{\epsilon}\gamma_5\lambda, \quad \delta K = -\bar{\epsilon}\gamma^\mu\partial_\mu\chi - \bar{\epsilon}\lambda \\
&\delta v_\mu = -\bar{\epsilon}\partial_\mu\chi - \bar{\epsilon}\gamma_\mu\lambda, \quad \delta\lambda = \gamma^{\mu\nu}\epsilon\partial_\mu v_\nu + i\gamma_5 D\epsilon, \quad \delta D = i\bar{\epsilon}\gamma_5\gamma^\mu\partial_\mu\lambda
\end{aligned} \tag{2.7}$$

the highest component, D, transforms into a total derivative and the standard D-term formula gives a bulk action $S = \int_{\mathcal{M}} d^4x D$. This action is not susy in the presence of the boundary. Our extended D-term formula is

$$S = \int_{\mathcal{M}} d^4x D + \int_{\partial\mathcal{M}} d^3x (H - \partial_3 C) \tag{2.8}$$

and it gives bulk-plus-boundary actions that are invariant under ϵ_+ susy. (Here and hereafter we assume that total tangential ∂_m derivatives integrated over the boundary vanish.)

The extended D-term formula can be derived from the extended F-term formula. Indeed, given a vector multiplet V_4, we can construct the following scalar multiplet,

$$\Phi_4[V_4] = (-H,\ K,\ -i\gamma_5(\lambda + \gamma^\mu\partial_\mu\chi),\ D + \partial^\mu\partial_\mu C,\ -\partial^\mu v_\mu) \tag{2.9}$$

Applying (2.5) to this multiplet, we recover (2.8). Clearly, the F-term formula covers all cases, and is also simpler.

3. Applications

The extended F- and D-term formulae of the previous section can be applied to a variety of composite multiplets. This allows straightforward construction of susy bulk-plus-boundary actions that are minimal extensions of known (bulk) actions. In this section, we will consider several examples of this procedure. Generically, terms linear in the (bulk) auxiliary fields appear in the boundary actions. We will find that in some, but not all, cases these terms can be eliminated by adding separately susy boundary actions. It will follow that, generically, "susy without BC" requires the presence of auxiliary fields.

3.1 3D Wess-Zumino model

Given a 3D scalar multiplet $\Phi_3(A) = (A, \psi, F)$, we can construct a "kinetic" scalar multiplet whose lowest component is F,

$$T(\Phi_3) \equiv \Phi_3(F) = (F,\ \gamma^\mu \partial_\mu \psi,\ \partial_\mu \partial^\mu A) \tag{3.1}$$

The product of $\Phi_3(A)$ and $\Phi_3(F)$ gives another 3D scalar multiplet,

$$\Phi_3(AF) = (AF,\ F\psi + A\gamma^\mu \partial_\mu \psi,\ F^2 + A\partial_\mu \partial^\mu A - \overline{\psi}\gamma^\mu \partial_\mu \psi) \tag{3.2}$$

Applying our extended F-term formula (2.2) to this multiplet, we find the following bulk-plus-boundary action,

$$S = \int_{\mathcal{M}} d^3x (F^2 - \partial_\mu A \partial^\mu A - \overline{\psi}\gamma^\mu \partial_\mu \psi) - \int_{\partial\mathcal{M}} d^2x (AF + A\partial_3 A) \tag{3.3}$$

where we partially integrated to arrive at the standard form for the bulk action. This is the 3D Wess-Zumino (WZ) model supplemented by a particular boundary term. The bulk-plus-boundary action is, by construction, invariant under ϵ_+ susy, as can be explicitly verified.

We observe that the bulk auxiliary field F appears linearly on the boundary. Therefore, eliminating F via its field equation would require imposing a boundary condition $A = 0$. The action without auxiliary fields would then necessarily be "susy with BC." To be able to eliminate F while preserving "susy without BC," we will now look for a separately susy boundary action that cancels the term linear in F.

Separately susy boundary actions can be constructed systematically using co-dimension one multiplets. To this extent, we split the 3D $N = 1$ multiplet Φ_3 into two 2D $N = (1, 0)$ multiplets under the ϵ_+ susy. (The third coordinate, x^3, will appear in the 2D multiplets as a parameter.) Defining $\psi_\pm = P_\pm \psi$, we find

$$\begin{aligned} &\delta A = \overline{\epsilon}_+ \psi_-, \quad \delta\psi_- = \gamma^m \epsilon_+ \partial_m A \\ &\delta\psi_+ = (F + \partial_3 A)\epsilon_+, \quad \delta(F + \partial_3 A) = \overline{\epsilon}_+ \gamma^m \partial_m \psi_+ \end{aligned} \tag{3.4}$$

so that we find a scalar and a spinor 2D $N = (1,0)$ multiplet,[4]

$$\Phi_2 = (A,\ \psi_-), \quad \Psi_2 = (\psi_+,\ F + \partial_3 A) \tag{3.5}$$

Their product is another spinor multiplet,

$$\Phi_2 \times \Psi_2 = (A\psi_+,\ A(F + \partial_3 A) - \overline{\psi}_+\psi_-) \tag{3.6}$$

whose highest component transforms into a total ∂_m derivative. Therefore, the following action

$$\int_{\partial\mathcal{M}} d^2x(AF + A\partial_3 A - \overline{\psi}_+\psi_-) \tag{3.7}$$

is invariant under ϵ_+ susy. Adding it to (3.3), the first two terms cancel, and we obtain

$$S = \int_{\mathcal{M}} d^3x(F^2 - \partial_\mu A\partial^\mu A - \overline{\psi}\gamma^\mu\partial_\mu\psi) - \int_{\partial\mathcal{M}} d^2x\frac{1}{2}\overline{\psi}\psi \tag{3.8}$$

where we used $\overline{\psi}\psi = 2\overline{\psi}_+\psi_-$. Setting $F = 0$ in the action and susy transformations, we arrive at the 3D WZ model without auxiliary fields that is still "susy without BC."

3.2 4D Wess-Zumino model

The 4D WZ model will turn out to be more subtle. We start again with the scalar multiplet $\Phi_4 = (A, B, \psi, F, G)$ and construct the kinetic multiplet,

$$T(\Phi_4) = (F,\ -G,\ \gamma^\mu\partial_\mu\psi,\ \Box A,\ -\Box B) \tag{3.9}$$

where $\Box = \partial_\mu\partial^\mu$. Their product gives the following scalar multiplet

$$\begin{aligned}\Phi_4 \times T(\Phi_4) = \ &(AF + BG, \quad -AG + BF, \quad (A + i\gamma_5 B)\gamma^\mu\partial_\mu\psi + (F - i\gamma_5 G)\psi, \\ &A\Box A + B\Box B + F^2 + G^2 - \overline{\psi}\gamma^\mu\partial_\mu\psi, \quad -A\Box B + B\Box A + i\overline{\psi}\gamma_5\gamma^\mu\partial_\mu\psi)\end{aligned} \tag{3.10}$$

Applying our extended F-term formula (2.5) to this multiplet, we find, after some partial integration, the following action

$$\begin{aligned}S = &\int_{\mathcal{M}} d^4x\Big[-\partial_\mu A\partial^\mu A - \partial_\mu B\partial^\mu B - \overline{\psi}\gamma^\mu\partial_\mu\psi + F^2 + G^2\Big] \\ &- \int_{\partial\mathcal{M}} d^3x\Big[A(F + \partial_3 A) + B(G + \partial_3 B)\Big]\end{aligned} \tag{3.11}$$

This action is invariant under ϵ_+ susy by construction. As in the 3D case, we find terms linear in the bulk auxiliary fields, F and G, in the boundary action. However, unlike the 3D

[4] The highest component of Ψ_2, $F + \partial_3 A$, transforms into a total ∂_m derivative under ϵ_+ susy. Integrating this component over the 3D manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ gives our "$F + A$" formula (2.2). The four-dimensional extended F- and D-term formulae can also be derived in such a way using co-dimension one multiplets.

case, we will find that one cannot eliminate both terms by adding a separately susy boundary action.

To construct separately susy boundary actions, we split the 4D $N=1$ scalar multiplet Φ_4 into two 3D $N=1$ scalar multiplets under ϵ_+ susy. (The fourth coordinate, x^3, will appear in the 3D multiplets as a parameter.) Defining $\psi_\pm = P_\pm\psi$, we find

$$\begin{aligned}
\delta A &= \overline{\epsilon}_+\psi_-, \quad \delta\psi_- = \gamma^m\epsilon_+\partial_m A + i\gamma_5(G+\partial_3 B)\epsilon_+, \quad \delta(G+\partial_3 B) = i\overline{\epsilon}_+\gamma_5\gamma^m\partial_m\psi_- \\
\delta B &= -i\overline{\epsilon}_+\gamma_5\psi_+, \quad \delta\psi_+ = i\gamma_5\gamma^m\epsilon_+\partial_m B + (F+\partial_3 A)\epsilon_+, \quad \delta(F+\partial_3 A) = \overline{\epsilon}_+\gamma^m\partial_m\psi_+
\end{aligned}$$

so that the two 3D multiplets contained in Φ_4 are[5]

$$\Phi_A = (A,\ \psi_-,\ G+\partial_3 B), \quad \Phi_B = (B,\ -i\gamma_5\psi_+,\ -F-\partial_3 A) \tag{3.12}$$

Their product yields another 3D scalar multiplet

$$\Phi_A\times\Phi_B = \Big(AB,\quad -i\gamma_5 A\psi_+ + B\psi_-,\quad -A(F+\partial_3 A) + B(G+\partial_3 B) + \overline{\psi}_+\psi_-\Big) \tag{3.13}$$

The highest component of this multiplet can be used to construct the following separately susy boundary actions,

$$\alpha\int_{\partial\mathcal{M}} d^3x\Big[-A(F+\partial_3 A) + B(G+\partial_3 B) + \frac{1}{2}\overline{\psi}\psi\Big] \tag{3.14}$$

where we used $\overline{\psi}\psi = 2\overline{\psi}_+\psi_-$. We observe that adding this action to (3.11) with $\alpha=-1$ or $\alpha=+1$, we can cancel either the term linear in F or the term linear in G, but not both. Therefore, eliminating auxiliary fields in the 4D WZ model cannot be done while maintaining the "susy without BC" property (as a boundary condition $A=0$ or $B=0$ arises in the process). Turning this around, we see that generically auxiliary fields are required for "susy without BC."

3.3 3D Maxwell model

For the 3D $N=1$ spinor multiplet $\Psi_3 = (\chi, M, v_\mu, \lambda)$, in (2.3), the kinetic multiplet $T(\Psi_3)$ is a spinor multiplet whose lowest component is λ,

$$T(\Psi_3) \equiv \Psi_3(\lambda) = (\lambda,\ \ 0,\ \ -\epsilon_{\mu\nu\rho}F^{\nu\rho},\ \ 2\gamma^\mu\partial_\mu\lambda) \tag{3.15}$$

where $F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$ and $\gamma^{\mu\nu\rho} = -\epsilon^{\mu\nu\rho}$ (then $-\gamma^\mu\epsilon_{\mu\nu\rho} = \gamma_{\nu\rho}$). We will take $\epsilon^{013} = +1$ so that

$$\gamma^0\gamma^3 = \gamma^1 = \gamma_1, \quad \gamma^1\gamma^3 = \gamma^0 = -\gamma_0 \tag{3.16}$$

The product of $T(\Psi_3)$ with itself gives the following 3D $N=1$ scalar multiplet,

$$T(\Psi_3)\times T(\Psi_3) = (\overline{\lambda}\lambda,\ \ -2\gamma^{\mu\nu}\lambda F_{\mu\nu},\ \ 4F_{\mu\nu}F^{\mu\nu} + 2\overline{\lambda}\gamma^\mu\partial_\mu\lambda) \tag{3.17}$$

[5]We keep here the 4D gamma matrices to describe 3D multiplets. This description avoids explicit decomposition of the gamma matrices at the price of an unusual definition of susy transformations.

Applying our "$F+A$" formula (2.2) to this multiplet, we obtain

$$S=\int_{\mathcal{M}} d^3x\Big[4F_{\mu\nu}F^{\mu\nu}+2\overline{\lambda}\gamma^\mu\partial_\mu\lambda\Big]-\int_{\partial\mathcal{M}} d^2x(\overline{\lambda}\lambda) \tag{3.18}$$

This action is invariant under ϵ_+ susy and contains the usual susy Maxwell action in the bulk (up to an overall normalization constant).[6] As there is no auxiliary field in this action, there is no particular reason to add a separately susy boundary action.

3.4 4D Maxwell model

The Maxwell action for the 4D $N=1$ vector multiplet $V_4=(C,\chi,H,K,v_\mu,\lambda,D)$ can be written using the F-term formula applied to the following composite scalar multiplet,

$$\begin{aligned}\Phi_4 &= (A_c,\ B_c,\ \psi_c,\ F_c,\ G_c) \\ &= (\overline{\lambda}\lambda,\ \overline{\lambda}i\gamma^5\lambda,\ -\gamma^{\mu\nu}F_{\mu\nu}\lambda+2i\gamma^5\lambda D,\ F_{\mu\nu}F^{\mu\nu}+2\overline{\lambda}\gamma^\mu\partial_\mu\lambda-2D^2,\ \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma})\end{aligned} \tag{3.19}$$

Our extended F-term formula (2.5) applied to this multiplet gives (up to our factor $-1/2$)

$$S=\int_{\mathcal{M}} d^4x\Big[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}-\overline{\lambda}\gamma^\mu\partial_\mu\lambda+D^2\Big]+\int_{\partial\mathcal{M}} d^3x\frac{1}{2}\overline{\lambda}\lambda \tag{3.20}$$

This bulk-plus-boundary action is "susy without BC" by construction. The auxiliary field D appears only in the bulk and we can eliminate it by its field equation (set $D=0$) while preserving the "susy without BC" property. Adding a separately susy boundary action in this case is, therefore, not required.

It is instructive, however, to discuss an alternative derivation of the same action. As in the 3D case, we can first construct the kinetic multiplet $T(V_4)$, a composite 4D vector multiplet whose lowest component is D,

$$T(V_4)=(D,\ \gamma^\mu\partial_\mu\lambda,\ 0,\ 0,\ -\partial^\mu F_{\mu\nu},\ -\partial_\mu\partial^\mu\lambda,\ -\partial_\mu\partial^\mu D) \tag{3.21}$$

Unlike the 3D case, however, the 4D Maxwell action arises not from $T(V_4)\times T(V_4)$, but from $V_4\times T(V_4)$. The latter is a composite 4D vector multiplet whose lowest component is $\widetilde{C}=CD$. Among other components of $\widetilde{V}_4=V_4\times T(V_4)$ we find, in particular,

$$\begin{aligned}\widetilde{H} &= HD-\frac{1}{2}\overline{\chi}\gamma^\mu\partial_\mu\lambda \\ \widetilde{D} &= D^2-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}-\overline{\lambda}\gamma^\mu\partial_\mu\lambda+\partial_\mu\Big[-C\partial^\mu D+\frac{1}{2}\overline{\chi}\gamma^\mu\gamma^\nu\partial_\nu\lambda+F^{\mu\nu}v_\nu\Big]\end{aligned} \tag{3.22}$$

Our extended D-term formula (2.8) applied to $\widetilde{V}_4$ gives

$$\begin{aligned}\widetilde{S} = &\int_{\mathcal{M}} d^4x\Big[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}-\overline{\lambda}\gamma^\mu\partial_\mu\lambda+D^2\Big] \\ &-\int_{\partial\mathcal{M}} d^3x\Big[v^mF_{3m}-D(H-\partial_3C)-\overline{\chi}_+\gamma^\mu\partial_\mu\lambda\Big]\end{aligned} \tag{3.23}$$

[6] One would have to define $\lambda^{\rm new}=\lambda/2$ to get canonical kinetic terms for v_μ and $\lambda^{\rm new}$ at the same time. Our choice of λ in 3D followed from a natural parametrization of the corresponding superfield Γ_α, see (5.1). In 4D, our λ is canonically defined.

This bulk-plus-boundary action is "susy without BC" by construction. We see, however, that the boundary action contains now a term linear in the auxiliary field D. Again, we would like to find a separately susy boundary action which, upon adding it to $\widetilde{S}$, would cancel this term. A systematic search for such an action would require decomposing V_4 into co-dimension one (3D) multiplets and then using tensor calculus to construct a 3D scalar multiplet whose F component contains the $D(H-\partial_3 C)$ combination. Instead of following this tedious procedure, we simply deduce the answer by noting that since both S and $\widetilde{S}$, in (3.20) and (3.23), are "susy without BC," so is their difference,

$$S - \widetilde{S} = \int_{\partial\mathcal{M}} d^3x \Big[\frac{1}{2}\overline{\lambda}\lambda + v^m F_{3m} - D(H - \partial_3 C) - \overline{\chi}_+\gamma^\mu\partial_\mu\lambda\Big] \tag{3.24}$$

One can verify that this boundary action is, indeed, invariant under ϵ_+ susy.

3.5 3D Chern-Simons model

Returning to the 3D case and taking now the product of Ψ_3 with $T(\Psi_3)$, we find another 3D $N = 1$ scalar multiplet

$$\Psi_3 \times T(\Psi_3) = \Big(\overline{\chi}\lambda, \quad (M - \gamma^\mu v_\mu)\lambda - \gamma^{\mu\nu}F_{\mu\nu}\chi, \quad \partial_\mu(\overline{\chi}\gamma^\mu\lambda) + 2\epsilon^{\mu\nu\rho}v_\mu F_{\nu\rho} + \overline{\lambda}\lambda\Big) \tag{3.25}$$

The 3D extended F-term formula (2.2) now gives

$$S = \int_{\mathcal{M}} d^3x \Big[\partial_\mu(\overline{\chi}\gamma^\mu\lambda) + 2\epsilon^{\mu\nu\rho}v_\mu F_{\nu\rho} + \overline{\lambda}\lambda\Big] - \int_{\partial\mathcal{M}} d^2x(\overline{\chi}\lambda) \tag{3.26}$$

Using $\overline{\chi}\lambda = \overline{\chi}_+\lambda_- + \overline{\chi}_-\lambda_+$ and $\overline{\chi}\gamma^3\lambda = -\overline{\chi}_+\lambda_- + \overline{\chi}_-\lambda_+$, the action simplifies to

$$S = \int_{\mathcal{M}} d^3x \Big[2\epsilon^{\mu\nu\rho}v_\mu F_{\nu\rho} + \overline{\lambda}\lambda\Big] - \int_{\partial\mathcal{M}} d^2x(2\overline{\chi}_-\lambda_+) \tag{3.27}$$

By construction, this action is invariant under ϵ_+ susy. Its bulk Lagrangian $2\epsilon^{\mu\nu\rho}v_\mu F_{\nu\rho} + \overline{\lambda}\lambda$ is the usual 3D susy Chern-Simons Lagrangian.[7] In this model (unlike the Maxwell case) λ is nonpropagating. We observe that it appears quadratically in the bulk and linearly on the boundary. On the other hand, χ and M are the fields that would be set to zero in the 3D Wess-Zumino (WZ) gauge.[8] As χ appears in the boundary action, we conclude that, generically, "susy without BC" requires such fields to be present. In other words, it may not be possible to impose the WZ gauge and to still have a bulk-plus-boundary action that is "susy without BC."

[7] Combining the Maxwell and Chern-Simons actions would require introducing a dimensionful (mass) parameter. In fact, the Chern-Simons action gives rise naturally to a gauge-invariant (up to a boundary variation) mass term for the 3D vector field v_μ.

[8] The usual gauge transformation, $\delta_g v_\mu = \partial_\mu A$, is extended in superspace to $\delta\Psi_3 = D\Phi_3$, see (5.8), where $\Phi_3 = (A, \psi, F)$ is now a multiplet of parameters. This gives $\delta\chi = \psi$, $\delta M = F$, $\delta v_\mu = \partial_\mu A$ and $\delta\lambda = 0$. If this transformation is a symmetry of an action, one can impose the WZ gauge: set $\chi = M = 0$. However, (3.27) is not invariant under such transformation (though its variation is only a boundary term).

It is instructive to consider two cases when the boundary action in (3.27) vanishes. The first case is when the WZ gauge $\chi = M = 0$ is imposed. The bulk action then varies into[9]

$$\delta S = \int_{\partial\mathcal{M}} d^2x \Big[-2(\bar{\epsilon}_+\gamma^m\lambda_+)v_m \Big] \tag{3.28}$$

The second case is when one eliminates the auxiliary field λ by its field equation (that is, by setting $\lambda = 0$ in (2.3) and (3.27)). The bulk action now varies into

$$\delta S = \int_{\partial\mathcal{M}} d^2x \Big[-2(\bar{\epsilon}_+\gamma^{mn}\chi_-)F_{mn} \Big] \tag{3.29}$$

We see that in both cases one needs to impose some boundary conditions to make the susy variation vanish, that is only "susy with BC" is possible when some of the "auxiliary" fields are absent.

Returning to the action (3.27), we now ask if it is possible to find a separately susy boundary action that allows to remove the term linear in the auxiliary field λ. Once again, to construct such an action we split the 3D $N = 1$ spinor multiplet Ψ_3 into 2D $N = (1,0)$ multiplets under ϵ_+ susy. First, we find that

$$\begin{aligned}
&\delta\chi_+ = \epsilon_+(M + v_3), \quad \delta(M + v_3) = \bar{\epsilon}_+\gamma^m\partial_m\chi_+ \\
&\delta v_3 = \bar{\epsilon}_+(\frac{1}{2}\lambda_- + \partial_3\chi_-), \quad \delta(\frac{1}{2}\lambda_- + \partial_3\chi_-) = \gamma^m\epsilon_+\partial_m v_3 \\
&\delta\chi_- = \gamma^m\epsilon_+ v_m, \quad \delta v_m = -\frac{1}{2}\bar{\epsilon}_+\gamma_m\lambda_+ + \bar{\epsilon}_+\partial_m\chi_-, \quad \delta\lambda_+ = \gamma^{mn}F_{mn}\epsilon_+
\end{aligned} \tag{3.30}$$

which gives a spinor multiplet $(\chi_+,\ M + v_3)$, a scalar multiplet $(v_3,\ \frac{1}{2}\lambda_- + \partial_3\chi_-)$ and a multiplet $(\chi_-,\ v_m,\ \frac{1}{2}\lambda_+)$. The latter is, in fact, further reducible under ϵ_+ susy. To see this, we first note that (3.16) implies

$$\gamma^0\epsilon_+ = \gamma^1\epsilon_+, \quad \gamma^0\chi_- = -\gamma^1\chi_- \tag{3.31}$$

Defining $v_\pm = v_0 \pm v_1$ and $\partial_\pm = \partial_0 \pm \partial_1$, and using identities like

$$\gamma^m\epsilon_+\partial_m = \gamma^1\epsilon_+\partial_+, \quad \gamma^m\chi_- v_m = -\gamma^1\chi_- v_- \tag{3.32}$$

we find after a little algebra another spinor and scalar multiplet

$$\begin{aligned}
&\delta(\gamma^1\chi_-) = \epsilon_+ v_+, \quad \delta v_+ = \bar{\epsilon}_+\gamma^m\partial_m(\gamma^1\chi_-) \\
&\delta v_- = \bar{\epsilon}_+[\gamma^1\lambda_+ + \partial_-\chi_-], \quad \delta[\gamma^1\lambda_+ + \partial_-\chi_-] = \gamma^m\epsilon_+\partial_m v_-
\end{aligned} \tag{3.33}$$

We conclude that the 3D $N = 1$ spinor multiplet $\Psi_3 = (\chi, M, v_\mu, \lambda)$ splits into the following four 2D $N = (1,0)$ multiplets,

$$\begin{aligned}
&\Psi_2 = (\chi_+,\ M + v_3), \quad \Phi_2 = (v_3,\ \frac{1}{2}\lambda_- + \partial_3\chi_-) \\
&\Psi_2' = (\gamma^1\chi_-,\ v_+), \quad \Phi_2' = (v_-,\ \gamma^1\lambda_+ + \partial_-\chi_-)
\end{aligned} \tag{3.34}$$

[9]As is well-known, in the WZ gauge one must add a compensating gauge transformation to the susy transformation. To keep $\delta\chi = \delta M = 0$, we need a gauge transformation with $(A_c, \psi_c, F_c) = (0, -\gamma^\mu\epsilon v_\mu, \frac{1}{2}\bar{\epsilon}\lambda)$. The resulting susy transformations, as follows from (2.3), are $\delta v_\mu = -\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda$ and $\delta\lambda = \gamma^{\mu\nu}F_{\mu\nu}\epsilon$.

Multiplying Ψ_2' with Φ_2', we obtain a composite 2D $N=(1,0)$ spinor multiplet,

$$\Psi_2' \times \Phi_2' = (\gamma^1 \chi_- v_-, \;\; v_+ v_- + \overline{\chi}_- \lambda_+ + \overline{\chi}_- \gamma^1 \partial_- \chi_-) \tag{3.35}$$

Its highest component transforms into a total ∂_m derivative under ϵ_+ susy so that

$$2 \int_{\partial \mathcal{M}} d^2 x \left[v_+ v_- + \overline{\chi}_- \lambda_+ + \overline{\chi}_- \gamma^1 \partial_- \chi_- \right] \tag{3.36}$$

is invariant under ϵ_+ susy. Adding this boundary action to (3.27), we obtain[10]

$$S = \int_{\mathcal{M}} d^3 x \left[2\epsilon^{\mu\nu\rho} v_\mu F_{\nu\rho} + \overline{\lambda} \lambda \right] + 2 \int_{\partial \mathcal{M}} d^2 x \left[v_+ v_- + \overline{\chi}_- \gamma^1 \partial_- \chi_- \right] \tag{3.37}$$

This bulk-plus-boundary action is invariant under ϵ_+ susy. The term linear in the auxiliary field λ no longer appears in the boundary action so that we can eliminate it (set $\lambda = 0$) while preserving "susy without BC."

We observe that the elimination of the term linear in λ from the boundary action has turned the hitherto pure-gauge bulk fermionic field χ_- into a dynamical boundary field. The distinctive feature of the Chern-Simons model that is responsible for this effect is that it is gauge invariant only up to a boundary term. In this sense it is very similar to supergravity theories where the (super)diffeomorphism invariance of the bulk action also holds only up to a boundary term [3]. Therefore, it is expected that some of the usual pure gauge degrees of freedom (usually removed by imposing the WZ gauge) will become important for bulk-plus-boundary supergravity theories.

[10]The boundary action in (3.37) can also be written as $-2 \int_{\partial \mathcal{M}} d^2 x (v_m v^m + \overline{\chi}_- \gamma^m \partial_m \chi_-)$.

4. Euler-Lagrange variation and boundary conditions

Our extended F- and D-term formulae give bulk-plus-boundary actions that are "susy without BC." Nevertheless, BC do arise if one requires the Euler-Lagrange (EL) variation to vanish. The BC one finds in this way have to be consistent with susy: the susy variation of a given BC may generate a new BC which has to be added to the total set of BC, and the susy variation of this new BC may generate yet another BC, etc. The total set of BC forms a (finite or infinite) "susy orbit" of BC [6, 7, 4]. In this section, using the 3D and 4D Wess-Zumino models as examples, we show that one needs to consider only *finite* susy orbits when auxiliary fields are present. In the next section, we will show that such orbits arise naturally as BC on superfields once one passes to the formulation in terms of co-dimension one (boundary) superfields [1, 9].

4.1 3D Wess-Zumino model

First, we simplify our equations by writing bulk-plus-boundary actions as bulk Lagrangians with appropriate total ∂_3 derivatives. The 3D Wess-Zumino bulk-plus-boundary action (3.3) is then written as the following Lagrangian,

$$\mathcal{L} = \mathcal{L}_B + \partial_3 \mathcal{L}_b, \quad \mathcal{L}_B = F^2 - \partial_\mu A \partial^\mu A - \overline{\psi}\gamma^\mu \partial_\mu \psi, \quad \mathcal{L}_b = AF + A\partial_3 A \tag{4.1}$$

The EL variation of $\mathcal{L}_B$ gives

$$\delta\mathcal{L}_B = (EOM) - \partial_3[2\delta A \partial_3 A + \overline{\psi}\gamma^3 \delta\psi] \tag{4.2}$$

where $(EOM) = 2(F\delta F + \delta A \partial_\mu \partial^\mu A - \delta\overline{\psi}\gamma^\mu \partial_\mu \psi)$ and we dropped an (insignificant) total ∂_m derivative. For the EL variation of the total Lagrangian $\mathcal{L}$ we then find

$$\delta\mathcal{L} = (EOM) + \partial_3\Big[A\delta F + (F - \partial_3 A)\delta A + A\delta(\partial_3 A) + \overline{\psi}_+ \delta\psi_- - \overline{\psi}_- \delta\psi_+\Big] \tag{4.3}$$

Requiring this to vanish for arbitrary variations of the fields on the boundary gives the following set of BC,

$$A = F - \partial_3 A = \psi_+ = \psi_- = 0 \tag{4.4}$$

which is obviously too strong. $(A, \partial_3 A)$ and (ψ_-, ψ_+) can be thought of as (q, p) pairs of canonically conjugated variables with respect to the x^3 direction. Therefore, acceptable BC would be conditions on p *or* q, but not on both of them at the same time.

For the modified action (3.8), we have $\mathcal{L}_b = \overline{\psi}_+ \psi_-$ so that

$$\delta\mathcal{L} = (EOM) + \partial_3\Big[-2(\partial_3 A)\delta A + 2\overline{\psi}_+ \delta\psi_-\Big] \tag{4.5}$$

We see that the boundary piece of the EL variation is in the "$p\delta q$" form, so that Neumann (N) BC "$p = 0$" follow from requiring $\delta\mathcal{L}$ to vanish for arbitrary δq on the boundary, or one can set "$q = \text{const}$" as Dirichlet (D) BC. In the case at hand,[11]

$$N: (\partial_3 A, \psi_+) = 0, \quad D: (A, \psi_-) = \text{const} \tag{4.6}$$

[11] When "const" stands for a multiplet (or a superfield), it is understood that only the lowest component is a non-zero constant, whereas higher components have to be zero by susy.

The Dirichlet BC form a closed susy orbit, see (3.4), but the Neuman BC, $(\partial_3 A, \psi_+) = 0$, do not form a closed orbit. Indeed, (3.4) indicates that $(F + \partial_3 A, \psi_+) = 0$ would be closed under ϵ_+ susy, whereas (2.1) says that omitting F would lead to an infinite orbit of conditions involving restrictions of bulk equations of motion to the boundary (as was observed in [4]).

However, the same action (3.8) can be shown to give rise to Neumann BC which do form a closed susy orbit. This is achieved by a field redefinition in accordance with the structure of the co-dimension one multiplets (3.5).[12] Defining $F' = F + \partial_3 A$, we find that

$$\mathcal{L}_B = F^2 - (\partial_3 A)^2 + \dots = (F')^2 - 2F'\partial_3 A + \dots \tag{4.7}$$

Using F' as an independent bulk field gives, instead of (4.5),

$$\delta\mathcal{L} = (EOM) + \partial_3\Big[-2F'\delta A + 2\overline{\psi}_+ \delta\psi_- \Big] \tag{4.8}$$

and instead of (4.6), we find the following susy orbits of BC,

$$N : \Psi_2 = 0, \quad D : \Phi_2 = \text{const} \tag{4.9}$$

where Φ_2 and Ψ_2 are defined in (3.5). We will see later that these BC follow naturally in the superspace formulation with co-dimension one superfields. It then will also become obvious that if, instead of adding (3.7) to (3.3), as we did to obtain (3.8), we *subtract* it, the resulting bulk-plus-boundary action would have flipped sets of BC,

$$N : \Phi_2 = 0, \quad D : \Psi_2 = \text{const} \tag{4.10}$$

4.2 4D Wess-Zumino model

The analysis of the EL variation and associated BC for the 4D Wess-Zumino model is very similar to the 3D case. We find that the sum of (3.11) and (3.14), with $\alpha = \pm 1$, gives actions whose EL variations are in the "$p\delta q$" form provided we use $F' = F + \partial_3 A$ and $G' = G + \partial_3 B$ as independent bulk fields. The corresponding BC are

$$\begin{aligned} \alpha = +1 \quad &\Rightarrow \quad N : \Phi_A = 0, \quad D : \Phi_B = \text{const} \\ \alpha = -1 \quad &\Rightarrow \quad N : \Phi_B = 0, \quad D : \Phi_A = \text{const} \end{aligned} \tag{4.11}$$

where Φ_A and Φ_B are defined in (3.12).

[12] The necessity of such field redefinitions was discussed in [1] for a particular 5D susy model.

5. Superspace approach

In this section we will demonstrate how the results derived so far in the susy tensor calculus approach follow from superspace. In particular, we will explain how co-dimension one superfields can be obtained by projection with superspace covariant derivatives. We will discuss only the 3D case (with 2D boundaries).[13]

5.1 Superfields and superspace covariant derivatives

A superfield glues components of a susy multiplet into a single object (a field over superspace). Using the same letter for a multiplet and the corresponding superfield, the 3D $N = 1$ superfields are[14]

$$\begin{aligned} \Phi &= (A, \psi_\alpha, F) = A + \overline{\theta}\psi + \theta^2 F \\ \Gamma_\alpha &= (\chi_\alpha, M, v_\mu, \lambda_\alpha) = \chi_\alpha + \theta_\alpha M + (\gamma^\mu\theta)_\alpha v_\mu + \theta^2 \Big[\lambda_\alpha - (\gamma^\mu\partial_\mu\chi)_\alpha\Big] \end{aligned} \tag{5.1}$$

where θ_α is an anticommuting parameter (a two-component 3D Majorana spinor) and

$$\theta^2 = \frac{1}{2}\overline{\theta}\theta = \frac{1}{2}\theta^T C\theta = \frac{1}{2}\theta_\alpha C^{\alpha\beta}\theta_\beta = \frac{1}{2}\theta^\alpha\theta_\alpha \tag{5.2}$$

Here we introduced spinor indices α that so far have been hidden in our notation. Keeping these indices explicit is often convenient in superspace calculations. In our conventions,

$$\begin{gathered} \overline{\theta}\psi = \theta^\alpha\psi_\alpha, \quad (\gamma^\mu\theta)_\alpha = (\gamma^\mu)_\alpha{}^\beta\theta_\beta, \quad \theta^\alpha = \theta_\beta C^{\beta\alpha}, \quad \theta_\alpha = \theta^\beta C_{\beta\alpha}, \quad \theta_\alpha\theta_\beta = -C_{\alpha\beta}\theta^2 \\ C_{\alpha\beta}C^{\beta\gamma} = \delta_\alpha{}^\gamma, \quad C_{\alpha\beta} = -C_{\beta\alpha}, \quad \gamma^\mu_{\alpha\beta} \equiv (\gamma^\mu)_\alpha{}^\gamma C_{\gamma\beta} = \gamma^\mu_{\beta\alpha} \end{gathered} \tag{5.3}$$

Susy transformations of superfields are generated by differential operators Q_α,

$$\delta\Phi = \overline{\epsilon}Q\Phi, \quad \delta\Gamma_\alpha = \overline{\epsilon}Q\Gamma_\alpha; \quad Q_\alpha = \partial_\alpha - (\gamma^\mu\theta)_\alpha\partial_\mu, \quad \partial_\alpha = \frac{\partial}{\partial\theta^\alpha} \tag{5.4}$$

On the component level, this gives the transformations (2.1) and (2.3). In our conventions, $\partial_\alpha\theta^\beta = \delta_\alpha{}^\beta$ and $\partial_\alpha\theta_\beta = C_{\alpha\beta}$, so that introducing

$$D_\alpha = \partial_\alpha + (\gamma^\mu\theta)_\alpha\partial_\mu \tag{5.5}$$

we obtain the following algebra

$$\{Q_\alpha, Q_\beta\} = 2\gamma^\mu_{\alpha\beta}\partial_\mu, \quad \{Q_\alpha, D_\beta\} = 0, \quad \{D_\alpha, D_\beta\} = -2\gamma^\mu_{\alpha\beta}\partial_\mu \tag{5.6}$$

[13] The 4D case (with 3D boundaries) can be discussed along similar lines but is more involved. If one chooses the original approach to superspace due to Salam and Strathdee [10], then one can keep the 4D gamma matrices in a general representation and use them to describe co-dimension one (3D) superfields. A more conventional approach [11, 12] uses two-component spinors, which assumes a particular representation of the 4D gamma matrices from the start. One way to define co-dimension one superfields in this approach was described in [13]. Their definition by projection with covariant derivatives can also be established. This was essentially done by Siegel in [14], only there the dependence of fields on extra coordinates was suppressed.

[14] Our 3D superspace conventions are close to those in [12].

The second property, $\{Q, D\} = 0$, implies that D_α are superspace covariant derivatives,

$$\delta(D_{\alpha_1} \dots D_{\alpha_n} \Phi) = \bar{\epsilon} Q(D_{\alpha_1} \dots D_{\alpha_n} \Phi) \tag{5.7}$$

For example, $D_\alpha \Phi$ is a spinor multiplet like Γ_α. This is used to define superfield gauge transformations as

$$\delta_g \Gamma_\alpha = (\delta\chi_\alpha,\ \delta M,\ \delta v_\mu,\ \delta\lambda_\alpha) = D_\alpha \Phi = (\psi_\alpha,\ F,\ \partial_\mu A,\ 0) \tag{5.8}$$

so that v_μ transforms like a gauge field and λ_α is gauge-invariant. When such a superfield transformation is a symmetry of the action, one can impose a Wess-Zumino gauge: $\chi_\alpha = M = 0$.

As the indices α are two-dimensional, $[D_\alpha, D_\beta]$ is proportional to $C_{\alpha\beta}$ and we find

$$D_\alpha D_\beta = -\gamma^\mu_{\alpha\beta} \partial_\mu - C_{\alpha\beta} D^2, \quad D^2 = \frac{1}{2} D^\alpha D_\alpha \tag{5.9}$$

As the complete antisymmetrization of three two-dimensional indices gives zero, we find the following identity

$$D_\alpha D_\beta D_\gamma = \frac{1}{2} D_\alpha \{D_\beta, D_\gamma\} - \frac{1}{2} D_\beta \{D_\alpha, D_\gamma\} + \frac{1}{2} D_\gamma \{D_\alpha, D_\beta\} \tag{5.10}$$

It then follows that an arbitrary product of D_α can be written as a linear combination of 1, D_α and D^2 with ∂_μ-dependent coefficients. For example,

$$D^\alpha D_\beta D_\alpha = 0, \quad D^2 D_\alpha = -D_\alpha D^2 = (\gamma^\mu D)_\alpha \partial_\mu, \quad D^2 D^2 = \partial_\mu \partial^\mu \tag{5.11}$$

In turn, this implies that all the independent components of a 3D $N = 1$ superfield S can be defined in terms of lowest components of S, $D_\alpha S$ and $D^2 S$. For example,

$$\begin{aligned} \Phi &= (A, \psi_\alpha, F) = (\Phi,\ \ D_\alpha \Phi,\ \ -D^2 \Phi)_| \\ \Gamma_\alpha &= (\chi_\alpha, M, v_\mu, \lambda_\alpha) = \Big(\Gamma_\alpha,\ \ -\frac{1}{2}\overline{D}\Gamma,\ \ -\frac{1}{2}\overline{D}\gamma^\mu\Gamma,\ \ -D^2\Gamma_\alpha + (\gamma^\mu \partial_\mu \Gamma)_\alpha\Big)_| \end{aligned} \tag{5.12}$$

where the bar "|" indicates setting $\theta = 0$. The fact that λ_α is a gauge-invariant component field corresponds to the fact that

$$w_\alpha = -D^\beta D_\alpha \Gamma_\beta = -D^2 \Gamma_\alpha + (\gamma^\mu \partial_\mu \Gamma)_\alpha \tag{5.13}$$

is a gauge-invariant superfield. We find (compare with Γ_α in (5.1)),

$$w_\alpha = (\lambda_\alpha,\ 0,\ -\epsilon_{\mu\nu\rho} F^{\nu\rho},\ 2(\gamma^\mu \partial_\mu \lambda)_\alpha)\ = \lambda_\alpha + (\gamma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta^2 (\gamma^\mu \partial_\mu \lambda)_\alpha \tag{5.14}$$

Note that $-D^2\Phi$ and w_α correspond to the kinetic multiplets (3.1) and (3.15), respectively.

5.2 Co-dimension one superfields

We now proceed to decompose the 3D $N = 1$ superfields Φ and Γ_α into 2D $N = (1,0)$ superfields transforming in the standard way under ϵ_+ susy. First, we write

$$\overline{\epsilon}Q = \overline{\epsilon}_+ Q_- + \overline{\epsilon}_- Q_+, \quad \epsilon_\pm = P_\pm \epsilon, \quad Q_\pm \equiv P_\pm Q \tag{5.15}$$

where $P_\pm = \frac{1}{2}(1 \pm \gamma^3)$. From (5.4), using $\mu = (m, 3)$, we obtain

$$\begin{aligned} Q_- &= Q'_- + \theta_- \partial_3, \quad Q'_{-\alpha} \equiv \partial_{-\alpha} - (\gamma^m \theta_+)_\alpha \partial_m, \quad \partial_{-\alpha} \equiv \frac{\partial}{\partial \theta_+^\alpha} \\ Q_+ &= Q'_+ - \theta_+ \partial_3, \quad Q'_{+\alpha} \equiv \partial_{+\alpha} - (\gamma^m \theta_-)_\alpha \partial_m, \quad \partial_{+\alpha} \equiv \frac{\partial}{\partial \theta_-^\alpha} \end{aligned} \tag{5.16}$$

By definition, Q_- is the generator of ϵ_+ susy transformations on 3D $N = 1$ superfields,

$$\delta_+ \Phi = (\overline{\epsilon}_+ Q_-)\Phi, \quad \delta_+ \Gamma_\alpha = (\overline{\epsilon}_+ Q_-)\Gamma_\alpha \tag{5.17}$$

On the other hand, Q'_- has the standard form for the generator of ϵ_+ susy transformations on 2D $N = (1,0)$ superfields. The two operators are related as follows (as was also observed and used in [15]),

$$Q'_- = Q_- - \theta_- \partial_3 = e^{+\overline{\theta}_+ \theta_- \partial_3} Q_- e^{-\overline{\theta}_+ \theta_- \partial_3} \tag{5.18}$$

Therefore, writing

$$\boxed{\Phi = e^{-\overline{\theta}_+ \theta_- \partial_3} \left[\widehat{A} + \overline{\theta}_- \widehat{\psi}_+ \right]} \tag{5.19}$$

we find

$$\delta_+ \Phi = (\overline{\epsilon}_+ Q_-)\Phi = e^{-\overline{\theta}_+ \theta_- \partial_3} \left\{ \overline{\epsilon}_+ Q'_- \left[\widehat{A} + \overline{\theta}_- \widehat{\psi}_+ \right] \right\} = e^{-\overline{\theta}_+ \theta_- \partial_3} \left[(\delta_+ \widehat{A}) + \overline{\theta}_- (\delta_+ \widehat{\psi}_+) \right] \tag{5.20}$$

so that the θ_+-dependent objects $\widehat{A}$ and $\widehat{\psi}_+$ defined by (5.19) are, indeed, 2D $N = (1,0)$ superfields. The θ_+ expansions of these superfields follow from (5.19),

$$\widehat{A} + \overline{\theta}_- \widehat{\psi}_+ = e^{+\overline{\theta}_+ \theta_- \partial_3} \Phi = (1 + \overline{\theta}_+ \theta_- \partial_3)(A + \overline{\theta}_+ \psi_- + \overline{\theta}_- \psi_+ + \overline{\theta}_+ \theta_- F) \tag{5.21}$$

which gives[15]

$$\boxed{\widehat{A} = A + \overline{\theta}_+ \psi_-, \quad \widehat{\psi}_+ = \psi_+ + \theta_+ (F + \partial_3 A)} \tag{5.22}$$

[15]We denote the co-dimension one superfields by the same letter as the corresponding lowest component, but with a hat on it. These lowest components can be obtained by setting $\theta_+ = 0$ in the 3D superfield, e.g. $\Phi(\theta_+ = 0) = A + \overline{\theta}_- \psi_+$.

The superfield transformations $\delta_+\widehat{A} = (\overline{\epsilon}_+ Q'_-)\widehat{A}$ and $\delta_+\widehat{\psi}_+ = (\overline{\epsilon}_+ Q'_-)\widehat{\psi}_+$ give rise to the component susy transformations (3.4).

The co-dimension one superfields can also be defined by projection with superspace covariant derivatives. To this extent, we decompose D_α into $D_{\pm\alpha} = (P_\pm D)_\alpha$,

$$\begin{aligned} D_- &= D'_- - \theta_-\partial_3, \quad D'_{-\alpha} \equiv \partial_{-\alpha} + (\gamma^m\theta_+)_\alpha\partial_m \\ D_+ &= D'_+ + \theta_+\partial_3, \quad D'_{+\alpha} \equiv \partial_{+\alpha} + (\gamma^m\theta_-)_\alpha\partial_m \end{aligned} \tag{5.23}$$

and observe that

$$D'_+ = D_+ - \theta_+\partial_3 = e^{+\overline{\theta}_+\theta_-\partial_3} D_+ e^{-\overline{\theta}_+\theta_-\partial_3} \tag{5.24}$$

Acting with $D_{+\alpha}$ on Φ and setting $\theta_- = 0$ then gives

$$D_+\Phi_{|\theta_-=0} = D'_+\Big[\widehat{A} + \overline{\theta}_-\widehat{\psi}_+\Big]_{|\theta_-=0} = \widehat{\psi}_+ \tag{5.25}$$

where we used that

$$\partial_{+\alpha}\theta_-^\beta = (P_+)_\alpha{}^\gamma\partial_\gamma\theta^\delta(P_+)_\delta{}^\beta = (P_+)_\alpha{}^\gamma(P_+)_\gamma{}^\beta = (P_+)_\alpha{}^\beta \tag{5.26}$$

As a result, the co-dimension one decomposition of Φ by projection is given by

$$\boxed{\widehat{A} = \Phi_{|\theta_-=0}, \quad \widehat{\psi}_+ = D_+\Phi_{|\theta_-=0}} \tag{5.27}$$

The decomposition of the 3D $N = 1$ spinor multiplet Γ_α is quite similar. We find,

$$\begin{aligned} \Gamma_+ &= e^{-\overline{\theta}_+\theta_-\partial_3}\Big[\widehat{\chi}_+ - \gamma^1\theta_-\widehat{v}_-\Big] \\ \Gamma_- &= e^{-\overline{\theta}_+\theta_-\partial_3}\Big[\widehat{\chi}_- + \theta_-(\widehat{M} - \widehat{v}_3)\Big] \end{aligned} \tag{5.28}$$

where

$$\begin{aligned} \widehat{\chi}_+ &= \chi_+ + \theta_+(M + v_3), \quad \widehat{v}_- = v_- + \overline{\theta}_+\gamma^1[\lambda_+ + \gamma^1\partial_-\chi_-] \\ \widehat{\chi}_- &= \chi_- + \gamma^1\theta_+ v_+, \quad (\widehat{M} - \widehat{v}_3) = (M - v_3) - \overline{\theta}_+[\lambda_- - \gamma^1\partial_+\chi_+ + 2\partial_3\chi_-] \end{aligned} \tag{5.29}$$

Observing that $-\overline{D}'_-\widehat{\chi}_+ = M + v_3 + \overline{\theta}_+\gamma^1\partial_+\chi_+$, we further find

$$\widehat{M} = M + \overline{\theta}_+\Big[-\frac{1}{2}\lambda_- + \gamma^1\partial_+\chi_+ - \partial_3\chi_-\Big], \quad \widehat{v}_3 = v_3 + \overline{\theta}_+\Big[\frac{1}{2}\lambda_- + \partial_3\chi_-\Big] \tag{5.30}$$

The multiplets $\widehat{\chi}_+$, $\widehat{\chi}_-$, $\widehat{v}_-$ and $\widehat{v}_3$ match those in (3.34). These multiplets can also be defined by projection. For the following, we only note that

$$\Gamma_{-|\theta_-=0} = \widehat{\chi}_-, \quad (D_{+\alpha}\Gamma_{+\beta})_{|\theta_-=0} = (P_+\gamma^1)_{\alpha\beta}\widehat{v}_- \tag{5.31}$$

5.3 Co-dimension one decomposition of 3D Lagrangians

In 3D, an $N=1$ susy Lagrangian is usually defined as the F-term of a scalar superfield,

$$\mathcal{L} = F = [\Phi]_F = \int d^2\theta \Phi = -D^2\Phi_| \tag{5.32}$$

Such a Lagrangian transforms into a total $\partial_\mu = (\partial_m, \partial_3)$ derivative and is not susy in the presence of a boundary. Using the following identity (that will be proven shortly),

$$D^2 = \overline{D}_- D_+ + \partial_3 \tag{5.33}$$

we find that the following modified Lagrangian,

$$\mathcal{L}' = F + \partial_3 A = [\Phi]_F + \partial_3(\Phi_|) = -\overline{D}_- D_+ \Phi_| = -\overline{D}'_- \widehat{\psi}_{+|\theta_+=0} = [\widehat{\psi}_+]_f \tag{5.34}$$

is written as the f-term of a 2D $N=(1,0)$ spinor superfield $\widehat{\psi}_+ = \psi_+ + \theta_+ f$. Therefore, under ϵ_+ susy, it transforms into a total ∂_m derivative and is susy in the presence of a boundary at $x^3 = \text{const}$. This way we recover our "$F+A$" formula (2.2) and also obtain a way to rewrite the resulting modified Lagrangian in terms of co-dimension one superfields.

To prove (5.33), we first project (5.9) with $P_\pm$ to find that

$$\begin{aligned} D_{-\alpha} D_{+\beta} &= (P_-)_\alpha{}^\gamma (P_+)_\beta{}^\delta D_\gamma D_\delta \\ &= -(P_-\gamma^\mu P_-)_{\alpha\beta}\partial_\mu - (P_- P_-)_{\alpha\beta} D^2 = (P_-)_{\alpha\beta}(\partial_3 - D^2) \end{aligned} \tag{5.35}$$

where we used $(P_-)_{\alpha\beta} = -(P_+)_{\beta\alpha}$ as follows from $(P_\pm)_{\alpha\beta} = \frac{1}{2}(C_{\alpha\beta} \pm \gamma^3_{\alpha\beta})$. Contraction with $C^{\alpha\beta}$ gives

$$C^{\alpha\beta} D_{-\alpha} D_{+\beta} = \overline{D}_- D_+ = -(P_-)_\alpha{}^\alpha(\partial_3 - D^2) = -(\partial_3 - D^2) \tag{5.36}$$

which proves (5.33). Altogether, (5.9) decomposes as

$$\begin{aligned} D_{-\alpha} D_{-\beta} &= -(\gamma^m P_+)_{\alpha\beta}\partial_m, \quad & D_{-\alpha} D_{+\beta} &= (P_-)_{\alpha\beta}(\partial_3 - D^2) \\ D_{+\alpha} D_{+\beta} &= -(\gamma^m P_-)_{\alpha\beta}\partial_m, \quad & D_{+\beta} D_{-\alpha} &= (P_-)_{\alpha\beta}(\partial_3 + D^2) \end{aligned} \tag{5.37}$$

from which we find that

$$\{D_{\pm\alpha}, D_{\pm\beta}\} = -2(P_\pm\gamma^m)_{\alpha\beta}\partial_m, \quad \{D_{-\alpha}, D_{+\beta}\} = 2(P_-)_{\alpha\beta}\partial_3 \tag{5.38}$$

Now we are ready to apply the formalism to specific examples.

5.4 3D Wess-Zumino model

We start with the 3D Lagrangian,

$$\mathcal{L} = D^2(\Phi D^2 \Phi)_| = F^2 - \overline{\psi}\gamma^\mu \partial_\mu \psi + A\partial_\mu \partial^\mu A \tag{5.39}$$

The modified Lagrangian (5.34) is given by

$$\mathcal{L}' = \mathcal{L} + \partial_3(AF) = D_-^\alpha D_{+\alpha}(\Phi D^2 \Phi)_| \tag{5.40}$$

and corresponds to the bulk-plus-boundary action (3.3). To write this Lagrangian in terms of co-dimension one superfields, we have to move the $D_{+\alpha}$ past all D_- and then set $\theta_- = 0$. Using

$$D^2 = \overline{D}_- D_+ + \partial_3, \quad D_{+\alpha} D^2 = -(\gamma^m D_-)_\alpha \partial_m - D_{+\alpha}\partial_3 \tag{5.41}$$

(the second identity follows from (5.11) by projection), we find

$$\mathcal{L}' = D_-^\alpha \Big[(D_{+\alpha}\Phi)\partial_3 \Phi - \Phi \partial_3 (D_{+\alpha}\Phi) + (D_{+\alpha}\Phi)(\overline{D}_- D_+ \Phi) - \Phi(\gamma^m D_-)_\alpha \partial_m \Phi \Big]_| \tag{5.42}$$

Setting $\theta_- = 0$ gives

$$\mathcal{L}' = D_-'^\alpha \Big[\widehat{\psi}_{+\alpha}\partial_3 \widehat{A} - \widehat{A}\partial_3 \widehat{\psi}_{+\alpha} + \widehat{\psi}_{+\alpha}(\overline{D}'_- \widehat{\psi}_+) - \widehat{A}(\gamma^m D'_-)_\alpha \partial_m \widehat{A} \Big]_{|\theta_+ = 0} \tag{5.43}$$

This Lagrangian is written in terms of 2D $N = (1,0)$ superfields and is manifestly ϵ_+ susy (it varies into a total ∂_m derivative) in the presence of a boundary at $x^3 = \text{const}$. The EL variation, on the other hand, gives

$$\delta\mathcal{L}' = (EOM) + \partial_3 \Big\{ D_-'^\alpha \Big[\widehat{\psi}_{+\alpha}\delta\widehat{A} - \widehat{A}\delta\widehat{\psi}_{+\alpha} \Big]_{|\theta_+ = 0} \Big\} \tag{5.44}$$

We observe that $\widehat{A}$ and $\widehat{\psi}_+$ are conjugated superfields, with respect to the "time derivative" ∂_3, but the boundary variation is not in the "$p\delta q$" form. It is however easy to see which separately susy boundary Lagrangians can be added to bring the boundary piece of the EL variation to the "$p\delta q$" form. Defining

$$\mathcal{L}'_\pm = \mathcal{L}' \pm \partial_3 \Delta, \quad \Delta = D_-'^\alpha [\widehat{\psi}_{+\alpha}\widehat{A}]_{|\theta_+ = 0} = -A(F + \partial_3 A) + \overline{\psi}_- \psi_+ \tag{5.45}$$

we find that the boundary piece of the EL variation and the corresponding Neumann (N) and Dirichlet (D) boundary conditions are

$$\begin{array}{lllllll} \delta\mathcal{L}'_+ & \Rightarrow & 2\widehat{\psi}_+ \delta\widehat{A} & \Rightarrow & N: \widehat{\psi}_+ = 0, & D: & \widehat{A} = \text{const} \\ \delta\mathcal{L}'_- & \Rightarrow & -2\widehat{A}\delta\widehat{\psi}_+ & \Rightarrow & N: \quad \widehat{A} = 0, & D: & \widehat{\psi}_+ = \text{const} \end{array} \tag{5.46}$$

The boundary Lagrangian Δ corresponds to the one in (3.7).

Instead of (5.39), one could start with an alternative 3D Lagrangian,

$$\mathcal{L}_2 = -D^2\Big(\frac{1}{2}D^\alpha\Phi D_\alpha\Phi\Big)_| = F^2 - \overline{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu A\partial^\mu A \tag{5.47}$$

that differs from (5.39) by a total ∂_μ derivative. The modified Lagrangian (5.34) is now

$$\mathcal{L}_2' = \mathcal{L}_2 + \partial_3\Big(\frac{1}{2}\overline{\psi}\psi\Big) = -\overline{D}^\alpha_- D_{+\alpha}\Big(\overline{D}_-\Phi D_+\Phi\Big)_| \tag{5.48}$$

which in terms of co-dimension one superfields becomes

$$\mathcal{L}_2' = D'^\alpha_-\Big[2\widehat{\psi}_{+\alpha}\partial_3\widehat{A} + (D'^\beta_-\widehat{\psi}_{+\alpha})\widehat{\psi}_{+\beta} + \partial_m\widehat{A}(\gamma^m D'_-)_\alpha\widehat{A}\Big]_{|\theta_+=0} \tag{5.49}$$

This way we get directly a Lagrangian whose boundary piece of the EL variation is in the "$p\delta q$" form. One can check that $\mathcal{L}_2'$ differs from $\mathcal{L}_+'$ by an (insignificant) total ∂_m derivative.

Adding a superpotential would not change the form of the superfield boundary conditions. To see this, let us consider

$$\mathcal{L}_3 = -D^2\Big[W(\Phi)\Big]_| = -\frac{1}{2}W''(A)\overline{\psi}\psi + W'(A)F \tag{5.50}$$

The modified Lagrangian (5.34) is[16]

$$\mathcal{L}_3' = \mathcal{L}_3 + \partial_3\Big[W(A)\Big] = -\overline{D}_- D_+\Big[W(\Phi)\Big]_| = D'^\alpha_-\Big[-W(\widehat{A})\widehat{\psi}_{+\alpha}\Big]_{|\theta_+=0} \tag{5.51}$$

Obviously, adding this to $\mathcal{L}_\pm'$ would not change the BC (5.46). However, on the component level, one could look for the form of BC with the auxiliary field F eliminated. Then the superpotential W would explicitly appear in the BC as in that case $2F = -W'(A)$.

5.5 3D Chern-Simons model

The superfield 3D Lagrangian for the Chern-Simons model is

$$\mathcal{L} = -D^2(\overline{w}\Gamma)_| = 2\epsilon^{\mu\nu\rho}v_\mu F_{\nu\rho} + \overline{\lambda}\lambda + \partial_\mu(\overline{\chi}\gamma^\mu\lambda) \tag{5.52}$$

The modified Lagrangian (5.34) is

$$\mathcal{L}' = \mathcal{L} + \partial_3(\overline{\lambda}\chi) = -\overline{D}^\alpha_-\Big[D_{+\alpha}\overline{w}\Gamma\Big]_| \tag{5.53}$$

Using (5.13) and (5.41), we find that

$$\begin{aligned} D_{+\alpha}(\overline{w}\Gamma) &= D_{+\alpha}(-\overline{\Gamma}D^2\Gamma + \overline{\Gamma}\gamma^\mu\partial_\mu\Gamma) \\ &= -(D_{+\alpha}\overline{\Gamma})D^2\Gamma + \overline{\Gamma}(D_{+\alpha}D^2\Gamma) + (D_{+\alpha}\overline{\Gamma})\gamma^\mu\partial_\mu\Gamma - \overline{\Gamma}\gamma^\mu\partial_\mu(D_{+\alpha}\Gamma) \\ &= -(D_{+\alpha}\overline{\Gamma})\partial_3\Gamma - \overline{\Gamma}\partial_3(D_{+\alpha}\Gamma) + (D_{+\alpha}\overline{\Gamma})\gamma^3\partial_3\Gamma - \overline{\Gamma}\gamma^3\partial_3(D_{+\alpha}\Gamma) + (\text{no } \partial_3) \\ &= -2(D_{+\alpha}\overline{\Gamma}_+)\partial_3\Gamma_- - 2\overline{\Gamma}_-\partial_3(D_{+\alpha}\Gamma_+) + (\text{no } \partial_3) \end{aligned} \tag{5.54}$$

[16]The fact that the bulk superpotential $W(A)$ is a natural boundary Lagrangian was observed in [16]. In [5] this was also derived using superspace methods, but the general philosophy of that work was to use BC for susy. Here we emphasize that the co-dimension one superspace methods give rise to bulk-plus-boundary actions that are "susy without BC."

where we dropped terms not involving ∂_3. As a result,

$$\mathcal{L}' = 2\overline{D}_-^\alpha \Big[(D_{+\alpha}\overline{\Gamma}_+)\partial_3\Gamma_- + \overline{\Gamma}_-\partial_3(D_{+\alpha}\Gamma_+) + (\text{no } \partial_3)\Big]_| \tag{5.55}$$

Setting $\theta_- = 0$ and using (5.31), we arrive at

$$\mathcal{L}' = 2D'^\alpha_- \Big[\widehat{v}_-\partial_3(\gamma^1\widehat{\chi}_-)_\alpha - (\gamma^1\widehat{\chi}_-)_\alpha\partial_3\widehat{v}_- + (\text{no } \partial_3)\Big]_{|\theta_+=0} \tag{5.56}$$

This shows that $\widehat{v}_-$ and $\widehat{\chi}_-$ are the conjugated co-dimension one superfields for the Chern-Simons model. Again, we can define two Lagrangians for which the boundary piece of the EL variation is in the "$p\delta q$" form,

$$\mathcal{L}'_\pm = \mathcal{L}' \pm 2\partial_3\Delta, \quad \Delta = D'^\alpha_-\Big[\widehat{v}_-(\gamma^1\widehat{\chi}_-)_\alpha\Big]_{|\theta_+=0} = v_+v_- + \overline{\chi}_-\lambda_+ + \overline{\chi}_-\gamma^1\partial_-\chi_- \tag{5.57}$$

The boundary piece of the EL variation and the superfield Neumann and Dirichlet BC for these Lagrangians are as follows,

$$\begin{array}{lllll} \delta\mathcal{L}'_+ & \Rightarrow & 4\widehat{v}_-\delta(\gamma^1\widehat{\chi}_-) & \Rightarrow & N: \widehat{v}_- = 0, \quad D: \widehat{\chi}_- = \text{const} \\ \delta\mathcal{L}'_- & \Rightarrow & -4(\gamma^1\widehat{\chi}_-)\delta\widehat{v}_- & \Rightarrow & N: \widehat{\chi}_- = 0, \quad D: \widehat{v}_- = \text{const} \end{array} \tag{5.58}$$

The boundary Lagrangian Δ corresponds to the one in (3.36).

Note that deriving these BC in the component formulation is tricky as one has to choose appropriate independent bulk fields (namely, $\lambda'_- = \lambda_- - \gamma^1\partial_+\chi_+ + 2\partial_3\chi_-$) as dictated by the way fields appear in the co-dimension one superfields.

5.6 3D Maxwell model

The superfield 3D Lagrangian for the Maxwell model is

$$\mathcal{L} = -D^2(\overline{w}w)_| = 4F_{\mu\nu}F^{\mu\nu} + 2\overline{\lambda}\gamma^\mu\partial_\mu\lambda \tag{5.59}$$

The modified Lagrangian (5.34) is

$$\mathcal{L}' = \mathcal{L} + \partial_3(\overline{\lambda}\lambda) = -D^\alpha_- D_{+\alpha}(2\overline{w}_+w_-)_| \tag{5.60}$$

where the projections $w_\pm$, as follows from (5.13), are

$$\begin{aligned} w_+ &= \gamma^m\partial_m\Gamma_- - \overline{D}_-D_+\Gamma_+ \\ w_- &= \gamma^m\partial_m\Gamma_+ - \overline{D}_-D_+\Gamma_- - 2\partial_3\Gamma_- \end{aligned} \tag{5.61}$$

To find conjugated co-dimension one superfields in this model, we perform the co-dimension one decomposition of the EL variation $\delta\mathcal{L}'$ and look for terms with ∂_3 acting on variations of superfields. Using $D_{+\alpha}\overline{D}_-D_+ = -2\partial_3 D_{+\alpha} + (\text{no } \partial_3)$, we find that

$$D_{+\alpha}(\overline{w}_+\delta w_- + \overline{w}_-\delta w_+) = -2(D_{+\alpha}\overline{w}_+)\partial_3\delta\Gamma_- - 2\overline{w}_-\partial_3(D_{+\alpha}\Gamma_+) + (\text{no } \partial_3\delta\Gamma) \tag{5.62}$$

Therefore, the EL variation of $\mathcal{L}'$ reads

$$\delta\mathcal{L}' = (EOM) + 4\partial_3\Big\{D_-^\alpha\Big[\overline{w}_-\delta(D_{+\alpha}\Gamma_+) + (D_{+\alpha}\overline{w}_+)\delta\Gamma_-\Big]_|\Big\} \tag{5.63}$$

This shows that, unlike the Wess-Zumino and Chern-Simons models, here we have *two pairs* of conjugated co-dimension one superfields and the EL variation is already in the "$p\delta q$" form.

To write this more explicitly, we need an analog of (5.31) for w_α. First, we find that

$$\begin{aligned} w_+ &= \lambda_+ + \theta_+F_{+-} + 2\gamma^1\theta_-F_{-3} + \overline{\theta}_+\theta_-(-\gamma^1\partial_-\lambda_- + \partial_3\lambda_+) \\ w_- &= \lambda_- - \theta_-F_{+-} + 2\gamma^1\theta_+F_{+3} + \overline{\theta}_+\theta_-(\gamma^1\partial_+\lambda_+ - \partial_3\lambda_-) \end{aligned} \tag{5.64}$$

where $F_{+-} = \partial_+v_- - \partial_-v_+$, $F_{+3} = \partial_+v_3 - \partial_3v_+$, $F_{-3} = \partial_-v_3 - \partial_3v_-$ (or, equivalently, $F_{+-} = -2F_{01}$, $F_{+3} = F_{03} + F_{13}$, $F_{-3} = F_{03} - F_{13}$) with $v_\pm = v_0 \pm v_1$ and $\partial_\pm = \partial_0 \pm \partial_1$. This leads to the following decomposition,

$$\begin{aligned} w_+ &= e^{-\overline{\theta}_+\theta_-\partial_3}\Big[\widehat{\lambda}_+ + 2\gamma^1\theta_-\widehat{F}_{-3}\Big] \\ w_- &= e^{-\overline{\theta}_+\theta_-\partial_3}\Big[\widehat{\lambda}_- - \theta_-\widehat{F}_{+-}\Big] \end{aligned} \tag{5.65}$$

where

$$\begin{aligned} &\widehat{\lambda}_+ = \lambda_+ + \theta_+F_{+-}, \quad \widehat{F}_{-3} = F_{-3} + \frac{1}{2}\overline{\theta}_+(\partial_-\lambda_- - 2\gamma^1\partial_3\lambda_+) \\ &\widehat{\lambda}_- = \lambda_- + 2\gamma^1\theta_+F_{+3}, \quad \widehat{F}_{+-} = F_{+-} + \overline{\theta}_+\gamma^1\partial_+\lambda_+ \end{aligned} \tag{5.66}$$

These superfields can also be defined by projection. We only need two of the projections,

$$w_{-|\theta_-=0} = \widehat{\lambda}_-, \quad (D_{+\alpha}w_{+\beta})_{|\theta_-=0} = -2(P_+\gamma^1)_{\alpha\beta}\widehat{F}_{-3} \tag{5.67}$$

Together with (5.31), this allows us to rewrite (5.63) as

$$\delta\mathcal{L}' = (EOM) + 4\partial_3\Big\{-D_-'^\alpha\Big[(\gamma^1\widehat{\lambda}_-)_\alpha\delta\widehat{v}_- + 2\widehat{F}_{-3}(\gamma^1\delta\widehat{\chi}_-)\Big]_{|\theta_+=0}\Big\} \tag{5.68}$$

This clearly shows $(\widehat{\lambda}_-, \widehat{v}_-)$ and $(\widehat{F}_{-3}, \widehat{\chi}_-)$ as the two pairs of conjugated co-dimension one superfields. (In components, we have

$$\begin{aligned} \delta\mathcal{L}' = (EOM) + 4\partial_3\Big\{&2F_{+3}\delta v_- + 2F_{-3}\delta v_+ \\ &+\overline{\lambda}_-\delta(\lambda_+ + \gamma^1\partial_-\chi_-) + \delta\overline{\chi}_-(\gamma^1\partial_-\lambda_- - 2\partial_3\lambda_+)\Big\} \end{aligned} \tag{5.69}$$

Proving this on the component level is rather tricky, as one has to define $\lambda'_+ = \lambda_+ + \gamma^1\partial_-\chi_-$ and $\lambda'_- = \lambda_- - \gamma^1\partial_+\chi_+ + 2\partial_3\chi_-$ and consider them as independent bulk fields.)

In the Maxwell model, we can define *four* Lagrangians with different sets of BC. Namely,

$$\mathcal{L}'_1 = \mathcal{L}', \quad \mathcal{L}'_2 = \mathcal{L}' + 4\partial_3\Delta_1, \quad \mathcal{L}'_3 = \mathcal{L}' + 4\partial_3\Delta_2, \quad \mathcal{L}'_4 = \mathcal{L}' + 4\partial_3(\Delta_1 + \Delta_2) \tag{5.70}$$

with

$$\Delta_1 = D'^{\alpha}_{-}\Big[(\gamma^1\widehat{\lambda}_-)_\alpha\widehat{v}_-\Big]_{|\theta_+=0}, \quad \Delta_2 = D'^{\alpha}_{-}\Big[2\widehat{F}_{-3}(\gamma^1\widehat{\chi}_-)_\alpha\Big]_{|\theta_+=0} \tag{5.71}$$

The Neumann BC in the four cases are, respectively,

$$(\widehat{\lambda}_-, \widehat{F}_{-3}) = 0, \quad (\widehat{v}_-, \widehat{F}_{-3}) = 0, \quad (\widehat{\lambda}_-, \widehat{\chi}_-) = 0, \quad (\widehat{v}_-, \widehat{\chi}_-) = 0 \tag{5.72}$$

Each of these four sets of BC is closed under ϵ_+ susy. The first set is also gauge-invariant.

6. Conclusions

In this article we have made a systematic study of boundary conditions (BC) in rigidly supersymmetric (susy) models. We first analyzed the models in x-space, and were able to construct susy bulk-plus-boundary actions which were susy by themselves, without the need for BC. We called such actions "susy without BC." To achieve this, we had to add boundary actions which completed the bulk actions, but which themselves were not susy. In some cases we ended up with models which contained boundary terms which were linear in auxiliary fields. Since elimination of auxiliary fields in such models gave too strong BC, we added separately susy actions on the boundary which canceled the terms linear in auxiliary fields.

In the tensor calculus approach, the key to the construction of susy bulk-plus-boundary actions was our extended F-term formula (or "$F + A$" formula): (2.2) in 3D and (2.5) in 4D. In 4D, we found also an extended D-term formula (2.8). For constructing separately susy boundary actions, we needed in addition to decompose bulk susy multiplets into a set of co-dimension one multiplets out of which, using standard tensor calculus methods, we could construct susy boundary actions.

To construct the susy bulk-plus-boundary actions in superspace (which we discussed explicitly only for the 3D case), we used the decomposition in (5.33),

$$D^2 = \overline{D}_- D_+ + \partial_3 \tag{6.1}$$

where $D = D_\alpha$ are the usual superspace covariant derivatives (with the spin index α), and $D_\pm = P_\pm D$ with $P_\pm = \frac{1}{2}(1 \pm \gamma^3)$. The modified Lagrangian $\mathcal{L}' = (-D^2 + \partial_3)\Phi_|$ for a composite superfield Φ consisted of the usual bulk term F from $-D^2\Phi_|$, and the boundary term A from $\partial_3\Phi_|$ which is to be added on the boundary. So, starting from the Lagrangian $\mathcal{L}' = -\overline{D}_- D_+\Phi_| = -D^{\alpha}_{-} D_{+\alpha}\Phi_|$, the nonsupersymmetric boundary term "A" which completes the bulk action "F" is included from the start.

The operators $D_+ = D_{+\alpha}$ were used to decompose a bulk superfield which depends on θ_+ and θ_- into a set of co-dimension one superfields which depend only on θ_+. While the components of a superfield are defined by acting on it with D_α and setting $\theta_+ = \theta_- = 0$, see (5.12), we defined the co-dimension one superfields by acting on the parent superfield with $D_{+\alpha}$ and setting $\theta_{-\alpha} = 0$, see (5.27). This approach led naturally to the foliation of bulk superfields into co-dimension one (boundary) superfields which is similar to the decomposition of $N = 2$ superfields into $N = 1$ superfields. Using these co-dimension one superfields we could construct separately susy boundary actions using the usual superspace methods. The susy covariant derivatives D'_- of the lower-dimensional superspace (which depend on θ_+ and ∂_m, but not on θ_- and ∂_3) were obtained by setting $\theta_- = 0$ in D_-.

We conclude that the component approach and the superspace approach remain equivalent in the presence of boundaries.

An issue we want now to confront concerns the BC for Euler-Largange (EL) variations. In various cases we were able to add separately susy boundary actions such that the EL variation of the action was of the form "$p\delta q$" on the boundary. We thus imposed either $p = 0$ or $q =$ const on the boundary as BC for on-shell fields, in other words as the BC which make the field equations to a mathematically well-posed problem. Should one also use these BC for off-shell fields, for example in path integrals? We do not believe so as it is natural to preserve "susy without BC." If one does impose BC off-shell, the boundary action can be simplified (and in our examples it would vanish), but the resulting bulk-plus-boundary action would not be "susy without BC." If the boundary terms in the EL variation of the action are not of the form "$p\delta q$" (but in the cases we studied they could always be cast into this form by adding a suitable separately susy action on the boundary), we believe that any set of BC, which makes this boundary term vanish *on-shell*, is allowed. Taking any set of (on-shell) BC requires, of course, to study their consistency and to construct the orbit of BC. This orbit is particularly simple in our "susy without BC" formulation: then the orbit is just a boundary superfield (provided we keep enough auxiliary fields).

The results of the present article for rigidly susy models in x-space and superspace, and those of [3] for locally susy models in x-space, have settled some of the questions we had about susy models with boundaries. We are now interested in tackling the Horava-Witten model in 11D [17, 18], and various (susy) AdS/CFT and Randall-Sundrum models in dimensions greater than four. In these cases full sets of auxiliary fields are not known (or do not exist), and thus no complete superspace formulations are available. Therefore, many of our constructions are not directly applicable. However, in our articles we also studied the issue of eliminating auxiliary fields while preserving "susy without BC," and in many cases it was indeed possible to do so. Therefore, we expect that some of the higher dimensional models can be made "susy without BC."

Acknowledgments. We thank the C. N. Yang Institute for Theoretical Physics at SUNY Stony Brook and Deutsches Electronen-Synchrotron DESY in Hamburg for hospitality extended to us during visits related to this project. The research of D.V.B. was supported in part by the German Science Foundation (DFG). The research of P.v.N. was supported by the NSF grant no. PHY-0354776.

References

[1] D. V. Belyaev, JHEP **0601**, 046 (2006).

[2] D. V. Belyaev, JHEP **0601**, 047 (2006).

[3] D. V. Belyaev and P. van Nieuwenhuizen, arXiv:0711.2272 [hep-th], to appear in JHEP.

[4] P. van Nieuwenhuizen and D. V. Vassilevich, Class. Quant. Grav. **22**, 5029 (2005).

[5] U. Lindstrom, M. Rocek and P. van Nieuwenhuizen, Nucl. Phys. B **662**, 147 (2003).

[6] P. Di Vecchia, B. Durhuus, P. Olesen and J. L. Petersen, Nucl. Phys. B **207**, 77 (1982); Nucl. Phys. B **217**, 395 (1983).

[7] Y. Igarashi, Phys. Rev. D **30**, 1812 (1984).

[8] S. W. Hawking, Phys. Lett. B **126**, 175 (1983).

[9] D. V. Belyaev, JHEP **0608**, 032 (2006) [arXiv:hep-th/0605282].

[10] A. Salam and J. A. Strathdee, Nucl. Phys. B **76**, 477 (1974); Phys. Rev. D **11**, 1521 (1975).

[11] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, 2nd Edition, Princeton University Press, 1992.

[12] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, Front. Phys. **58**, 1 (1983) [arXiv:hep-th/0108200].

[13] Y. Sakamura, Nucl. Phys. B **656**, 132 (2003).

[14] W. Siegel, Nucl. Phys. B **156**, 135 (1979).

[15] N. Sakai and Y. Tanii, Prog. Theor. Phys. **83**, 968 (1990).

[16] N. P. Warner, Nucl. Phys. B **450**, 663 (1995).

[17] P. Horava and E. Witten, Nucl. Phys. B **460**, 506 (1996); Nucl. Phys. B **475**, 94 (1996).

[18] I. G. Moss, Phys. Lett. B **577**, 71 (2003); Nucl. Phys. B **729**, 179 (2005).