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Path Integral Approach for Quantum Motion on Spaces of Non-constant Curvature According to Koenigs: Three Dimensions

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Abstract

In this contribution a path integral approach for the quantum motion on three-dimensional spaces according to Koenigs, for short"Koenigs-Spaces", is discussed. Their construction is simple: One takes a Hamiltonian from three-dimensional flat space and divides it by a three-dimensional superintegrable potential. Such superintegrable potentials will be the isotropic singular oscillator, the Holt-potential, the Coulomb potential, or two centrifugal potentials, respectively. In all cases a non-trivial space of non-constant curvature is generated. In order to obtain a proper quantum theory a curvature term has to be incorporated into the quantum Hamiltonian. For possible bound-state solutions we find equations up to twelfth order in the energy E.

1 Introduction

In this contribution I discuss the quantum motion on three-dimensional spaces of non-constant curvature according to Koenigs [24], which I will call for short "Koenigs-spaces". The construction of such a space is simple. One takes a three-dimensional flat Hamiltonian, $\mathcal{H}$, including some potential V, and divides $\mathcal{H}$ by a function $f(x, y, z)$ ($(x, y, z) \in \mathbb{R}^3$) such that f takes on the form of a metric:

$$\mathcal{H}_{\text{Koenigs}} = \frac{\mathcal{H}}{f(x, y, z)} \quad . \tag{1.1}$$

Such a construction leads to a very rich structure, and attempts to classify such systems are e.g. due to Kalnins et al. [18, 19] and Daskaloyannis and Ypsilantis [2].

Simpler examples of such spaces are the two- and three-dimensional Darboux spaces, where one chooses the function f in such a way that it depends only on one variable [10, 20], respectively their three-dimensional analogue [11]. Another choice consists whether one chooses for f some arbitrary potential (or some superintegrable potential) and taking into account that the Poisson bracket structure of the observables makes up a reasonable simple algebra [2, 6, 20].

In previous publications we have analyzed the quantum motion on Darboux spaces by means of the path integral [10, 14] and on two-dimensional Koenigs-spaces [12]. The path integral approach [5, 16, 22, 26] served as a powerful tool to calculate the propagator, respectively the Green function of the quantum motion in such spaces. In the present contribution I apply the path integral technique to five kinds of Koenigs-spaces, where a specific three-dimensional superintegrable potential [13] for the function f is chosen. They are the three-dimensional

Table 1: Coordinates in three-dimensional Euclidean space

Coordinate System	Coordinates
I. Cartesian	$x = x'$, $y = y'$, $z = z'$
II. Circular Polar	$x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$, $z = z'$
III. Circular Elliptic	$x = d\cosh\mu\cos\nu$, $y = d\sinh\mu\sin\nu$,$z = z'$
IV. Circular Parabolic	$x = \frac{1}{2}(\eta^2 - \xi^2)$, $y = \xi\eta$, $z = z'$
V. Sphero-Conical	$x = r\mathrm{sn}(\alpha, k)\mathrm{dn}(\beta, k')$, $y = r\mathrm{cn}(\alpha, k)\mathrm{cn}(\beta, k')$ $z = r\mathrm{dn}(\alpha, k)\mathrm{sn}(\beta, k')$
VI. Spherical	$x = r\sin\vartheta\cos\varphi$, $y = r\sin\vartheta\sin\varphi$, $z = r\cos\vartheta$
VII. Parabolic	$x = \xi\eta\cos\varphi$, $y = \xi\eta\sin\varphi$, $z = \frac{1}{2}(\eta^2 - \xi^2)$
VIII. Prolate Spheroidal	$x = d\sinh\mu\sin\nu\cos\varphi$, $y = d\sinh\mu\sin\nu\sin\varphi$ $z = d\cosh\mu\cos\nu$
IX. Oblate Spheroidal	$x = d\cosh\mu\sin\nu\sin\varphi$, $y = d\cosh\mu\sin\nu\sin\varphi$ $z = d\sinh\mu\cos\nu$
X. Ellipsoidal	$x = k^2\sqrt{a^2 - c^2}\,\mathrm{sn}\alpha\mathrm{sn}\beta\mathrm{sn}\gamma$ $y = -(k^2/k')\sqrt{a^2 - c^2}\,\mathrm{cn}\alpha\mathrm{cn}\beta\mathrm{cn}\gamma$ $z = (\mathrm{i}/k')\sqrt{a^2 - c^2}\,\mathrm{dn}\alpha\mathrm{dn}\beta\mathrm{dn}\gamma$
XI. Paraboloidal	$x = 2d\cosh\alpha\cos\beta\sinh\gamma$, $y = 2d\sinh\alpha\sin\beta\cosh\gamma$ $z = d(\cosh^2\alpha + \cos^2\beta - \cosh^2\gamma)$

Table 2: The three-dimensional maximally super-integrable potentials

Potential $V(x,y,z)$, $\mathbf{x}=(x,y,z)\in\mathbb{R}^3$	Coordinate System
$V_1=\dfrac{M}{2}\omega^2\mathbf{x}^2+\dfrac{\hbar^2}{2m}\left(\dfrac{k_1^2-\frac{1}{4}}{x^2}+\dfrac{k_2^2-\frac{1}{4}}{y^2}+\dfrac{k_3^2-\frac{1}{4}}{z^2}\right)$	Cartesian Spherical Circular Polar Circular Elliptic Conical Oblate Spheroidal Prolate Spheroidal Ellipsoidal
$V_2=\dfrac{M}{2}\omega^2(x^2+y^2+4z^2)+\dfrac{\hbar^2}{2m}\left(\dfrac{k_1^2-\frac{1}{4}}{x^2}+\dfrac{k_2^2-\frac{1}{4}}{y^2}\right)$	Cartesian Parabolic Circular Polar Circular Elliptic
$V_3=-\dfrac{\alpha}{\sqrt{x^2+y^2+z^2}}+\dfrac{\hbar^2}{2m}\left(\dfrac{k_1^2-\frac{1}{4}}{x^2}+\dfrac{k_2^2-\frac{1}{4}}{y^2}\right)$	Conical Spherical Parabolic Prolate Spheroidal II
$V_4=\dfrac{\hbar^2}{2m}\left(\dfrac{k_1^2x}{y^2\sqrt{x^2+y^2}}+\dfrac{k_2^2-\frac{1}{4}}{y^2}+\dfrac{k_3^2-\frac{1}{4}}{z^2}\right)$	Spherical Circular Elliptic II Circular Parabolic Circular Polar
$V_5=\dfrac{\hbar^2}{2m}\left(\dfrac{k_1^2x}{y^2\sqrt{x^2+y^2}}+\dfrac{k_2^2-\frac{1}{4}}{y^2}\right)-k_3z$	Circular Polar Circular Elliptic II Circular Parabolic Parabolic

isotropic singular oscillator (Section II), the Holt-potential (section III), the three-dimensional Coulomb-potential (Section IV), and two centrifugal potentials (Section V and VI). The last Section is devoted to a summary and a discussion of the achieved results.

In Table 1 I have displayed the 11 coordinate systems in $\mathbb{R}^3$. In a previous article [13] we have discussed in much detail the minimally and maximally superintegrable systems in $\mathbb{R}^3$. There are five maximally superintegrable and seven minimally superintegrable system. The maximally superintegrable potentials have the property that these systems have five functionally independent integrals of motion (classical mechanics), respectively five observables (quantum mechanics). The minimally superintegrable instead have only four functionally independent integrals of motion, respectively four observables. In [13] we have called these superintegrable systems "Smorodinsky-Winternitz potentials". In Table 2 I have indicated the coordinate systems in which the five maximally superintegrable systems in $\mathbb{R}^3$ are separable. The cases where an explicit path integration is possible are underlined.

2 Koenigs-Space $K_{\rm I}$ with Isotropic Singular Oscillator

We start with the first example, where we take for the metric terms

$$\begin{aligned} \mathrm{d}s^2 &= f_I(x,y,z)(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2) \ , && (2.1)\\ f_I(x,y,z) &= \alpha(x^2+y^2+z^2)+\frac{\beta_x}{x^2}+\frac{\beta_y}{y^2}+\frac{\beta_z}{z^2}+\delta \ , && (2.2)\end{aligned}$$

and $\alpha,\beta_x,\beta_y,\beta_z,\delta$ are constants. The classical Hamiltonian and Lagrangian in $\mathbb{R}^3$ with the isotropic singular oscillator as the superintegrable potential have the form:

$$\begin{aligned} \mathcal{L} &= \frac{m}{2}\Big((\dot x^2+\dot y^2+\dot z^2)-\omega^2(x^2+y^2+z^2)\Big)-\frac{\hbar^2}{2m}\bigg(\frac{k_x^2+\frac12}{x^2}+\frac{k_y^2+\frac12}{y^2}+\frac{k_z^2+\frac12}{z^2}\bigg) \ , && (2.3)\\ \mathcal{H} &= \frac{p_x^2+p_y^2+p_z^2}{2m}+\frac{m}{2}\omega^2(x^2+y^2+z^2)+\frac{\hbar^2}{2m}\bigg(\frac{k_x^2+\frac12}{x^2}+\frac{k_y^2+\frac12}{y^2}+\frac{k_z^2+\frac12}{z^2}\bigg) \ . && (2.4)\end{aligned}$$

(The specific choice of the constant $+\frac12$ has practical reasons which will become clear in the sequel.) Counting constants, there are nine independent constants: $\alpha,\beta_{x,y,z},\delta$, and $\omega,k_{x,y,z}$. A tenth constant can be added by adding a further constant $\tilde\delta$ into the potential of the Hamiltonian. It will be omitted in the following. The first Koenigs-space $K_{\rm I}$ is constructed by considering

$$\mathcal{H}_{K_{\rm I}}=\frac{\mathcal{H}}{f_I(x,y,z)} \ , \qquad (2.5)$$

hence for the Lagrangian (with potential)

$$\begin{aligned} \mathcal{L}_{K_{\rm I}} &= \frac{m}{2}f_I(x,y,z)(\dot x^2+\dot y^2+\dot z^2)\\ &\quad -\frac{1}{f_I(x,y,z)}\bigg[\frac{m}{2}\omega^2(x^2+y^2+z^2)+\frac{\hbar^2}{2m}\bigg(\frac{k_x^2+\frac12}{x^2}+\frac{k_y^2+\frac12}{y^2}\bigg)+\frac{k_z^2+\frac12}{z^2}\bigg] \ . && (2.6)\end{aligned}$$

Setting the potential in square-brackets equal to zero yields the Lagrangian for the free motion in $K_{\rm I}$. With this information we can set up the path integral in $K_{\rm I}$ including a potential. Because the space is three-dimensional, the quantum potential $\propto\hbar^2$ does not have the simple form as in the two-dimensional case [12]. This is due to the fact that two-dimensional spaces are conformally flat, and has the consequence that in the path integral the additional quantum potential $\propto\hbar^2$ can be set to zero by choosing an appropriate lattice. This lattice corresponds to the product form path integral, i.e. we have for diagonal metric $g_{ab}=f_a^2\delta_{ab}$ the quantum potential

$$\Delta V=\frac{\hbar^2(D-2)}{8m}\sum_a\frac{(D-4)f_{a,a}^2+2f_af_{a,aa}}{f_a^4} \ . \qquad (2.7)$$

Table 3: Some special cases for the space $K_{\rm I}$

Metric	Space	ΔV
$f_I(x,y,z)$	Koenigs space $K_{\rm I}$	$\Delta V_1+\dfrac{3\hbar^2}{8m}\bigg(\dfrac{1}{h_x^2}+\dfrac{1}{h_y^2}+\dfrac{1}{h_z^2}\bigg)$
$\dfrac{bu^2-a}{u^2}$	Three-dimensional Darboux Space $D_{\rm II}$	$\Delta V_1+\dfrac{3\hbar^2}{8m(bu^2-a)}$
$\dfrac{1}{u^2}$	Three-dimensional Hyperboloid	$\dfrac{3\hbar^2}{8m}$
1	$\mathbb{R}^3$	0

Obviously, $\Delta V=0$ for $D=2$. For our purposes we rewrite the metric term in the following way:

$$f_I(x,y,z)=\frac{\alpha x^2(x^2+y^2+z^2)+\beta_x+\dfrac{x^2\beta_y}{y^2}+\dfrac{x^2\beta_z}{z^2}+\delta x^2}{x^2}\equiv\frac{h_x^2}{x^2}\ , \tag{2.8}$$

and similarly in terms of h_y and h_z. This gives for the $x=x_1$-part of ΔV

$$\Delta V_x = \Delta V_{1,x}+\Delta V_{2,x} \tag{2.9}$$

$$\Delta V_{1,x} = \frac{\hbar^2}{8m}\frac{2x^2h_xh_{x,xx}-2xh_xh_{x,x}-x^2h_xh_{x,x}^2}{h_x^4}\ ,\qquad \Delta V_{2,x}=\frac{3\hbar^2}{8mh_x^2}\ . \tag{2.10}$$

Repeating the procedure for the $y=x_2$- and $z=x_3$-coordinate we get

$$\Delta V = \Delta V_1+\Delta V_2=\sum_{i=1}^3\Delta V_{1,x_i}+\Delta V_2 \tag{2.11}$$

$$\Delta V_2 = \frac{3\hbar^2}{8m}\bigg(\frac{1}{h_x^2}+\frac{1}{h_y^2}+\frac{1}{h_z^2}\bigg)=\frac{3\hbar^2}{8mf_I}\bigg(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\bigg)\ . \tag{2.12}$$

Note that if we choose for $h\equiv 1$ that there is only one summand in the last equation with $\Delta V_2=\Delta V=3\hbar^2/8m$ (this is the case for the three-dimensional hyperboloid).

In Table 3 I have displayed some special cases of $K_{\rm I}$. From [10] we know that the free motion in the three-dimensional Darboux space $D_{\rm II}$ is separable in all eleven coordinate systems listed in Table 1.

We now repeat our reasoning from [11]: The part ΔV_1 disturbs a proper quantum treatment of the three-dimensional Koenigs space, and we *set up our quantum theory with an effective Lagrangian*

$$\mathcal{L}_{K_{\rm I}}^{\rm eff}=\mathcal{L}_{K_{\rm I}}+\Delta V_1\ . \tag{2.13}$$

Actually, our effective Lagrangian corresponds to the subtraction of a curvature term in $\mathcal{H}$ [21]. The canonical momentum operators are constructed by

$$p_{x_i}=\frac{\hbar}{\rm i}\bigg(\frac{\partial}{\partial x_i}+\frac{\Gamma_i}{2}\bigg)\ ,\qquad \Gamma_i=\frac{\partial}{\partial x_i}\ln\sqrt{g}\ , \tag{2.14}$$

with $x_1 = x, x_2 = y, , x_3 = z$ and $g = \det(g_{ab})$, (g_{ab}) the metric tensor. The Hamiltonian then has the form

$$\begin{aligned}
\mathcal{H}^{\text{eff}}_{K_\text{I}} &= -\frac{\hbar^2}{2m}\Delta_{LB} + \frac{1}{f_I(x,y,z)}\bigg[\frac{m}{2}\omega^2(x^2+y^2+z^2) \\
&\qquad + \frac{\hbar^2}{2m}\bigg(\frac{k_x^2+\frac{1}{2}}{x^2} + \frac{k_y^2+\frac{1}{2}}{y^2} + \frac{k_z^2+\frac{1}{2}}{z^2}\bigg)\bigg] - \Delta V_1 \qquad (2.15) \\
&= \frac{1}{2m}\frac{1}{\sqrt{f_I}}(p_x^2+p_y^2+p_z^2)\frac{1}{\sqrt{f_I}} + \frac{1}{f_I}\bigg[\frac{m}{2}\omega^2(x^2+y^2+z^2) \\
&\qquad + \frac{\hbar^2}{2m}\bigg(\frac{k_x^2+\frac{1}{2}}{x^2} + \frac{k_y^2+\frac{1}{2}}{y^2} + \frac{k_z^2+\frac{1}{2}}{z^2}\bigg)\bigg] + \Delta V_2\ . \qquad (2.16)
\end{aligned}$$

For the path integral in the product lattice definition we obtain by means of a space-time transformation [16, 22] (ΔV_2 inserted)

$$\begin{aligned}
K^{(K_\text{I})}(x'',x',y'',y',z',z'';T) &= \int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\int\limits_{y(t')=y'}^{y(t'')=y''}\mathcal{D}y(t)\int\limits_{z(t')=z'}^{z(t'')=z''}\mathcal{D}z(t) f_I(x,y,z) \\
\times\exp\Bigg(\frac{\text{i}}{\hbar}\int_{t'}^{t''}\bigg\{&\frac{m}{2}f_I(x,y,z)(\dot{x}^2+\dot{y}^2+\dot{z}^2) \\
-\frac{1}{f_I(x,y,z)}\bigg[\frac{m}{2}\omega^2(x^2+y^2+z^2) &+ \frac{\hbar^2}{2m}\bigg(\frac{k_x^2-\frac{1}{4}}{x^2} + \frac{k_y^2-\frac{1}{4}}{y^2} + \frac{k_z^2-\frac{1}{4}}{z^2}\bigg)\bigg]\bigg\}\text{d}t\Bigg) \qquad (2.17) \\
G^{(K_\text{I})}(x'',x',y'',y',z',z'';E) &= \frac{\text{i}}{\hbar}(f_I'f_I'')^{-\frac{1}{4}}\int_0^\infty \text{d}s'' K^{(K_\text{I})}(x'',x',y'',y',z',z'';s'')\text{e}^{\text{i}\delta\cdot Es''/\hbar}, \qquad (2.18)
\end{aligned}$$

(note the change of constant to $-\frac{1}{4}$) with the time-transformed path integral $K^{(K_\text{I})}(s'')$ given by ($\widetilde{\omega}^2 = \omega^2 - 2\alpha E/m$)

$$\begin{aligned}
K^{(K_\text{I})}(x'',x',y'',y',z',z'';s'') &= \int\limits_{x(0)=x'}^{x(s'')=x''}\mathcal{D}x(s)\int\limits_{y(0)=y'}^{y(s'')=y''}\mathcal{D}y(s)\int\limits_{z(0)=z'}^{z(s'')=z''}\mathcal{D}z(s) \\
\times\exp\Bigg\{\frac{\text{i}}{\hbar}\int_0^{s''}&\bigg[\frac{m}{2}\Big((\dot{x}^2+\dot{y}^2+\dot{z}^2) - \widetilde{\omega}^2(x^2+y^2+z^2)\Big) \\
-\frac{\hbar^2}{2m}\bigg(\frac{k_x^2-2m\beta_xE/\hbar^2-\frac{1}{4}}{x^2} &+ \frac{k_y^2-2m\beta_yE/\hbar^2-\frac{1}{4}}{y^2} + \frac{k_z^2-2m\beta_zE/\hbar^2-\frac{1}{4}}{z^2}\bigg)\bigg]\text{d}s''\Bigg\}\ . \qquad (2.19)
\end{aligned}$$

The path integrals in the variables x, y, z are path integrals for the radial harmonic oscillator, however with energy-dependent coefficients. We also see that the only effect of the constant δ consists of an additional phase in s''-integral which has consequences for the energy spectrum.

2.1 Koenigs-Space K_I with Isotropic Singular Oscillator in Polar Coordinates

We switch in the usual way to three-dimensional polar coordinates (r, ϑ, φ), and abbreviate $\tilde{k}_x^2 = k_x^2 - 2m\beta_xE/\hbar^2$, $\tilde{k}_y^2 = k_y^2 - 2m\beta_yE/\hbar^2$ and $\tilde{k}_z^2 = k_y^2 - 2m\beta_zE/\hbar^2$, respectively. In the variables ϑ, φ we obtain path integrals for the Pöschl–Potential, and in the variable r a radial path integral. The successive path integrations therefore yield

$$K^{(K_I)}(r'', r', \vartheta'', \vartheta'', \varphi'', \varphi'; s'') = \sum_{n_\varphi} \Phi_{n_\varphi}^{(\tilde{k}_y, \tilde{k}_x)}(\varphi'') \Phi_{n_\varphi}^{(\tilde{k}_y, \tilde{k}_x)}(\varphi') \sum_{n_\theta} \Phi_{n_\theta}^{(\tilde{k}_z, \lambda_1)}(\theta'') \Phi_{n_\theta}^{(\tilde{k}_z, \lambda_1)}(\theta')$$
$$\times \frac{m\widetilde{\omega}\sqrt{r'r''}}{\mathrm{i}\hbar \sin\widetilde{\omega}s''} \exp\left[-\frac{m\widetilde{\omega}}{2\mathrm{i}\hbar}(r'^2 + r''^2)\cot\widetilde{\omega}s''\right] I_{\lambda_2}\left(\frac{m\widetilde{\omega}r'r''}{\mathrm{i}\hbar\sin\widetilde{\omega}s''}\right) \ . \tag{2.20}$$

Here $\lambda_1 = 2n_\varphi + \tilde{k}_x + \tilde{k}_y + 1$, $\lambda_2 = 2n_\vartheta + \tilde{k}_z + \lambda_1 + 1$, and the $\Phi_{n_\varphi}^{(\tilde{k}_y, \tilde{k}_x)}(\varphi)$ are the wave-functions for the Pöschl-Teller potential, which are given by [1, 3, 7, 23]

$$V^{(PT)}(x) = \frac{\hbar^2}{2m}\left(\frac{\alpha^2 - \frac{1}{4}}{\sin^2 x} + \frac{\beta^2 - \frac{1}{4}}{\cos^2 x}\right) \tag{2.21}$$

$$\Phi_n^{(\alpha,\beta)}(x) = \left[2(\alpha+\beta+2l+1)\frac{l!\Gamma(\alpha+\beta+l+1)}{\Gamma(\alpha+l+1)\Gamma(\beta+l+1)}\right]^{1/2} \times (\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2} P_n^{(\alpha,\beta)}(\cos 2x) \ . \tag{2.22}$$

The $P_n^{(\alpha,\beta)}(z)$ are Gegenbauer polynomials [8] and $I_\lambda(z)$ is the modified Bessel function [8]. Performing the s''-integration we obtain the Green function $G^{(K_I)}(E)$ [8, 16]:

$$G^{(K_I)}(r'', r', \vartheta'', \vartheta'', \varphi'', \varphi'; E)$$
$$= (f_I' f_I'')^{-\frac{1}{4}} \sum_{n_\varphi} \Phi_{n_\varphi}^{(\tilde{k}_y, \tilde{k}_x)}(\varphi'') \Phi_{n_\varphi}^{(\tilde{k}_y, \tilde{k}_x)}(\varphi') \sum_{n_\theta} \Phi_{n_\theta}^{(\tilde{k}_z, \lambda_1)}(\theta'') \Phi_{n_\theta}^{(\tilde{k}_z, \lambda_1)}(\theta')$$
$$\times \frac{\Gamma[\frac{1}{2}(1+\lambda_2 - \delta\cdot E/\hbar\widetilde{\omega})]}{\hbar\widetilde{\omega}\sqrt{r'r''}\,\Gamma(1+\lambda)} W_{\delta\cdot E/2\widetilde{\omega}, \lambda_2/2}\left(\frac{m\widetilde{\omega}}{\hbar}r_>^2\right) M_{\delta\cdot E/2\widetilde{\omega}, \lambda_2/2}\left(\frac{m\widetilde{\omega}}{\hbar}r_<^2\right) \ . \tag{2.23}$$

$M_{\mu,\nu}(z)$ and $W_{\mu,\nu}(z)$ are Whittaker-functions [8], and $r_<, r_>$ is the smaller/larger of r', r''. The poles of the Γ-function give the energy-levels of the bound states:

$$\tfrac{1}{2}(1+\lambda_2 - \delta\cdot E/\hbar\widetilde{\omega}) = -n_r \ , \tag{2.24}$$

which is equivalent to ($N = n_r + n_\vartheta + n_\varphi = 0, 1, 2, \ldots$):

$$\begin{aligned} \delta\cdot E &= \hbar\widetilde{\omega}(2N + \tilde{k}_x + \tilde{k}_y + \tilde{k}_z + 3) \qquad (2.25) \\ &= \hbar\sqrt{\omega^2 - \frac{2\alpha}{m}E}\left(2N + \sqrt{k_x^2 - \frac{2m\beta_x}{\hbar^2}E} + \sqrt{k_y^2 - \frac{2m\beta_y}{\hbar^2}E} + \sqrt{k_z^2 - \frac{2m\beta_z}{\hbar^2}E} + 3\right) \ . \end{aligned}$$

In general, this quantization condition is an equation of twelfth order in E. Such an equation cannot be solved generally, however, we cam study some special cases:

1. The case $k_1 = k_2 = K_3 = \omega = 0$:

$$E_N = -\frac{2\alpha\hbar^2}{m}\frac{(2N+3)^2}{(\delta + 2\sqrt{\alpha\beta})^2} \ . \tag{2.26}$$

   For $\alpha < 0$ this gives an infinite well-defined bound state spectrum. For $\alpha > 0$ the spectrum is negative infinite. Usually this means that a particle will fall into the center and the wave-functions are *not* well defined. However, let us recall that the spectrum on the SU(1,1) hyperboloid gives a positive continuous spectrum *and* a negative infinite discrete spectrum. Hence, unphysical for real particles such a spectrum can be given a physical meaning nevertheless: One has to re-interprete the motion on the hyperboloid (space with curvature) by dimensional reduction to a potential problem in flat space: In the case of the SU(1,1) hyperboloid the modified Pöschl–Teller potential emerges and the negative infinite spectrum is gets a cut yielding only *finite number* of well-defined bound-states [1].

2. The case $k_1 = k_2 = k_3 = \alpha = 0$:
$$E_N = -\frac{\beta^\omega 2}{2\delta^2}\left(1 \mp \sqrt{1 - \frac{2\delta\hbar}{m\beta\omega}(2N+3)}\right)^2 \quad . \tag{2.27}$$
This gives for $\beta \neq 0$ semi-bound states with positive real part (2.28).

3. The case $k_1 = k_2 = k_3 = \alpha = \beta = 0, \delta > 0$:
$$E_N = \frac{\hbar\omega}{\delta}(2N+3) \quad . \tag{2.28}$$

4. The case $\beta_1 = \beta_2 = \beta_3 = \alpha = 0, \delta > 0$:
$$E_N = \frac{\hbar\omega}{\delta}(2N + k_1 + k_2 + k_3 + 3) \quad , \tag{2.29}$$

and we recover the flat space limit.

If we know the bound state energy E_N, we can determine the wave-functions according to
$$\Psi_N^{(K_{\rm I})}(r,\theta,\varphi) = N_N f_I^{-1/4} \Phi_{n_\varphi}^{(\tilde k_y,\tilde k_x)}(\varphi)\Phi_{n_\theta}^{(\tilde k_z,\lambda_1)}(\theta)\Phi_{n_r}^{(RHO,\lambda)}(r) \quad , \tag{2.30}$$
with the normalization constant N_N determined by evaluating the residuum in the Green function (2.23), and the $\Phi_N^{(RHO,\lambda)}(r)$ are the wave-functions of the radial harmonic oscillator [16]:
$$\Psi_n^{(RHO,\lambda)}(r) = \sqrt{\frac{2m}{\hbar}\frac{n!}{\Gamma(n+\lambda+1)}}\, r\left(\frac{m\omega}{\hbar}r\right)^{\lambda/2} \exp\left(-\frac{m\omega}{2\hbar}r^2\right) L_n^{(\lambda)}\left(\frac{m\omega}{\hbar}r^2\right) \quad . \tag{2.31}$$
We can recover the flat space limit with $\alpha = \beta_{x_i} = 0$ with the correct spectrum $E_N = \hbar\omega(N + k_x + k_y + k_z + 3)/\delta$.

2.2 Koenigs-Space $K_{\rm I}$ with Isotropic Singular Oscillator in Cartesian Coordinates

Instead of switching to polar coordinates we keep the Cartesian system and obtain
$$\begin{aligned}
&K^{(K_{\rm I})}(x'',x',y'',y',z',z'';s'')\\
&= \prod_{i=1}^{3} \frac{m\tilde\omega\sqrt{x_i'x_i''}}{{\rm i}\hbar\sin\tilde\omega s''}\exp\left[-\frac{m\tilde\omega}{2{\rm i}\hbar}(x_i'^2 + x_i''^2)\cot\tilde\omega s''\right] I_{\tilde k_{x_i}}\left(\frac{m\tilde\omega x_i' x_i''}{{\rm i}\hbar\sin\tilde\omega s''}\right)\\
&= \sum_{n_x}\Phi_{n_x}^{(RHO,\tilde k_x)}(x')\Phi_{n_x}^{(RHO,\tilde k_x)}(x'')\sum_{n_y}\Phi_{n_y}^{(RHO,\tilde k_y)}(y')\Phi_{n_y}^{(RHO,\tilde k_y)}(y'')\\
&\qquad\times\sum_{n_z}\Phi_{n_z}^{(RHO,\tilde k_z)}(z')\Phi_{n_z}^{(RHO,\tilde k_z)}(z'')\exp\left\{\frac{\rm i}{\hbar}\left[\hbar\tilde\omega\Big(2N + \tilde k_x + \tilde k_y + \tilde k_z + 3\Big)\right]s''\right\} .
\end{aligned}\tag{2.32}$$

Performing the s''-integration yields for the energy-spectrum the same result as before. The wave-functions are given by

$$\Psi_N^{(K_{\rm I})}(x,y,z) = N_N f_I^{-1/4}\Phi_{n_x}^{(RHO,\tilde k_x)}(x)\Phi_{n_x}^{(RHO,\tilde k_y)}(y)\Phi_{n_z}^{(RHO,\tilde k_z)}(z) \ , \tag{2.33}$$

with the normalization constant N_N determined by evaluating the residuum in the Green function for the energy-levels determined by (2.25). Note that all coefficients $\tilde k_x, \tilde k_y, \tilde k_z$ are energy-dependent.

As it is well-known [13], the singular isotropic is separable also in circular polar, circular elliptic, conical, oblate and prolate spheroidal and ellipsoidal coordinates, from which only the circular polar coordinate system (ϱ,φ,z) allows an explicit solution which is very easily obtained: The principal difference just consists of replacing the product of the two radial oscillator wave-functions in x and y by a product of a Pöschl–Teller wave-function in φ and radial oscillator wave-function in ϱ [13]. The energy-spectrum, of course, remains the same and is again determined by (2.25). We omit further details because this case does not give anything new.

2.3 Koenigs-Space $K_{\rm I}$ with Zero Constants

We now consider the Koenigs space $K_{\rm I}$ with constants set to zero, denoted by $K_{\rm I}^{(0)}$. This gives for the corresponding space-time transformed path integral (2.19)

$$\begin{aligned}
K^{(K_{\rm I}^{(0)})}(x'',x',y'',y',z',z'';s'') &= \int\limits_{x(0)=x'}^{x(s'')=x''}\mathcal{D}x(s)\int\limits_{y(0)=y'}^{y(s'')=y''}\mathcal{D}y(s)\int\limits_{z(0)=z'}^{z(s'')=z''}\mathcal{D}z(s)\\
\times\exp\Bigg\{\frac{\rm i}{\hbar}\int_0^{s''}&\bigg[\frac{m}{2}(\dot x^2+\dot y^2+\dot z^2)+\alpha E(x^2+y^2+z^2)\\
+\frac{\hbar^2}{2m}&\bigg(\frac{2m\beta_x E/\hbar^2+3/4}{x^2}+\frac{2m\beta_y E/\hbar^2+3/4}{y^2}+\frac{2m\beta_z E/\hbar^2+3/4}{z^2}\bigg)\bigg]{\rm d}s''\Bigg\} \ .
\end{aligned} \tag{2.34}$$

Obviously, this path integral can be separated in all the coordinate systems in which the singular isotropic oscillator is separable.

Let us investigate the case for the Cartesian coordinates. We consider the quantization condition (2.25). We have to set $\omega = k_x = k_y = k_z = 0\alpha \neq 0$, which yields:

$$\delta\cdot E_N = \hbar\sqrt{-\frac{2\alpha}{m}E_N}\left(2N+3+\tilde\beta\sqrt{-\frac{2m}{\hbar^2}E_N}\right) \ , \tag{2.35}$$

with $\tilde\beta = \sqrt{\beta_x}+\sqrt{\beta_y}+\sqrt{\beta_z}$. This quantization condition is a quadratic equation in the energy E and has the solution

$$E_N = -\frac{8\alpha^2\tilde\beta^2}{m(\delta^2+4\tilde\beta^2}(2N+3)^2\left(1\pm\sqrt{1-\frac{\hbar^2(\delta^2+4\tilde\beta^2)}{4\alpha\tilde\beta^2}}\right)^2 \ . \tag{2.36}$$

As an easy special case we consider $\tilde\beta = 0$, then

$$E_N = \frac{2\alpha\hbar^2}{m\delta^2}(2N+3)^2 \ . \tag{2.37}$$

Therefore we obtain an infinite discrete spectrum for $\delta \neq 0$. The spectrum can either be positive ($\alpha > 0$), or negative ($\alpha < 0$). Such infinite negative energy spectra are well-known for spaces with indefinite metric, for instance for the SU(1,1)-manifold [1, 7, 23]. Such spectra can be used by dimensional reduction for potential problems in flat space yielding finite negative discrete spectra. For the case of the infinite positive spectrum we are done with the corresponding wave-functions:

$$\Psi_N^{(K_{\rm I}^{(0)})}(n_x,n_y,n_z) = N_N f_I^{-1/4} \Phi_{n_x}^{(RHO,\tilde k_x)}(x)\Phi_{n_x}^{(RHO,\tilde k_y)}(y)\Phi_{n_z}^{(RHO,\tilde k_z)}(z) \ , \tag{2.38}$$

with the normalization constant N_N determined by evaluating the residuum in the Green function for the energy-levels (2.35). Note that all coefficients $\tilde k_x, \tilde k_y, \tilde k_z$ are energy-dependent. We omit the path integral representations in the other coordinate systems.

3 Koenigs-Space $K_{\rm II}$ with Holt-Potential

Next we consider for the metric terms

$$\begin{aligned} {\rm d}s^2 &= f_{II}(x,y,z)({\rm d}x^2+{\rm d}y^2+{\rm d}z^2) \ , &(3.1)\\ f_{II}(x,y,z) &= \alpha(x^2+y^2+4z^2)+\frac{\beta_x}{x^2}+\frac{\beta_y}{x^y}+\delta \ , &(3.2)\end{aligned}$$

and $\alpha, \beta_x, \beta_y, \delta$ are constants. The classical Hamiltonian and Lagrangian in $\mathbb{R}^3$ with the Holt-potential as the superintegrable potential have the form:

$$\begin{aligned} \mathcal{L} &= \frac{m}{2}\Big((\dot x^2+\dot y^2+\dot z^2)-\omega^2(x^2+y^2+4z^2)\Big)-\frac{\hbar^2}{2m}\bigg(\frac{k_x^2+\frac12}{x^2}+\frac{k_y^2+\frac12}{y^2}\bigg) \ , &(3.3)\\ \mathcal{H} &= \frac{p_x^2+p_y^2+p_z^2}{2m}+\frac{m}{2}\omega^2(x^2+y^2+4z^2)+\frac{\hbar^2}{2m}\bigg(\frac{k_x^2+\frac12}{x^2}+\frac{k_y^2+\frac12}{y^2}\bigg) \ . &(3.4)\end{aligned}$$

Counting constants, there are seven independent constants: $\alpha, \beta_x, \beta_y, \delta$, and ω, k_x, k_y. An eighth constant can be added by adding a further constant $\tilde\delta$ into the potential of the Hamiltonian, which is omitted. The second Koenigs-space $K_{\rm II}$ with potential is now constructed by considering

$$\mathcal{H}_{K_{\rm II}} = \frac{\mathcal{H}}{f_{II}(x,y,z)} \ . \tag{3.5}$$

From the discussion in the Section II it is obvious how to construct the path integral on $K_{\rm II}$. Again, we introduce the functions h similar as in (2.8), but, there is now a new feature. From the construction of ΔV_2 according to (2.8) we obtain also a z-dependent term $\propto 3\hbar^2/8mh_z^2$. For the three-dimensional Holt-potential there is, however, no such term $\propto 1/z^2$. In fact, the same situation occurs also for the Coulomb potential, see the next Section. Such a term $\propto 3\hbar^2/8mh_z^2$ would not spoil the separability in the Cartesian coordinate system and the two circular systems (polar and parabolic), but it spoils the separability in parabolic coordinates.

Similarly, as in the previous Section we make the choice of symmetry preservation and add the critical term $\propto 3\hbar^2/8mh_z^2$ into the Lagrangian such that it is canceled after quantization, i.e.

$$\mathcal{L}_{K_{\rm II}}^{\rm eff} = \mathcal{L}_{K_{\rm II}} + \Delta V_1 + \Delta V_2(z) \ . \tag{3.6}$$

Table 4: Some special cases for the space $K_{\rm II}$

Metric	Space	ΔV
$f_{II}(x,y,z)$	Koenigs space $K_{\rm II}$	$\Delta V_1+\dfrac{3\hbar^2}{8m}\biggl(\dfrac{1}{h_x^2}+\dfrac{1}{h_y^2}+\dfrac{1}{h_z^2}\biggr)$
$\dfrac{bu^2-a}{u^2}$	Three-dimensional Darboux Space $D_{\rm II}$	$\Delta V_1+\dfrac{3\hbar^2}{8m(bu^2-a)}$
$\dfrac{1}{u^2}$	Three-dimensional Hyperboloid	$\dfrac{3\hbar^2}{8m}$
1	$\mathbb{R}^3$	0

In Table 4 I have listed some special cases of the Koenigs space $K_{\rm II}$. It is in fact the same, up to scaling, as for $K_{\rm I}$.

We proceed straightforward to the time-transformed path integral $K^{(K_{\rm II})}(s'')$ which has the form

$$K^{(K_{\rm II})}(x'',x',y'',y',z',z';s'')=\int\limits_{x(0)=x'}^{x(s'')=x''}\mathcal{D}x(s)\int\limits_{y(0)=y'}^{y(s'')=y''}\mathcal{D}y(s)\int\limits_{z(0)=z'}^{z(s'')=z''}\mathcal{D}z(s)$$

$$\times\exp\Biggl\{\frac{\rm i}{\hbar}\int_0^{s''}\Biggl[\frac{m}{2}\Bigl((\dot x^2+\dot y^2+\dot z^2)-\widetilde\omega^2(x^2+y^2+4z^2)\Bigr)-\frac{\hbar^2}{2m}\biggl(\frac{\tilde k_x^2-\frac14}{x^2}+\frac{\tilde k_y^2-\frac14}{y^2}\biggr)\Biggr]{\rm d}s''\Biggr\}\ . \tag{3.7}$$

Again, $\widetilde\omega^2=\omega^2-2\alpha E/m$, $\tilde k_{x_{1,2}}^2=k_{x_{1,2}}^2-2m\beta_{x_{1,2}}E/\hbar^2$. We have in the variables x,y a singular oscillator with frequency $\widetilde\omega$, and in the variable z an oscillator with frequency $2\widetilde\omega$.

Form [13] we know that the Holt-potential is separable in four coordinate systems: Cartesian, parabolic circular polar and circular elliptic coordinates, respectively. Only in Cartesian and circular polar coordinates a closed solution is possible. In Cartesian coordinates we take the respective solution as expanded into the wave-functions and get

$$\begin{aligned}&K^{(K_{\rm II})}(x'',x',y'',y',z',z';s'')\\&=\sum_{n_x}\Psi_{n_x}^{(RHO,\tilde k_x)}(x'')\Psi_{n_x}^{(RHO,\tilde k_x)*}(x')\sum_{n_y}\Psi_{n_y}^{(RHO,\tilde k_y)}(x'')\Psi_{n_y}^{(RHO,\tilde k_y)*}(y')\sum_{n_z}\Psi_{n_z}^{(HO)}(z'')\Psi_{n_z}^{(HO)*}(z')\\&\qquad\times{\rm e}^{-{\rm i}s''\widetilde\omega(2n_x+2n_y+n_z+5/2)+\tilde k_x+\tilde k_y)}\ .\end{aligned} \tag{3.8}$$

Here, the $\Psi_{n_z}^{(HO)}(z)$ denote the wave-functions of the harmonic oscillator with its Hermite polynomials. Performing the s''-integration similarly as in (2.18) we get the quantization condition $(N=2(n_x+n_y)+n_z)$

$$\delta\cdot E_N=\hbar\Bigl(\omega^2-\frac{2\alpha}{m}E_N\Bigr)^{1/2}\left(N+\sqrt{k_x^2-\frac{2m\beta_x}{\hbar^2}E_N}+\sqrt{k_y^2-\frac{2m\beta_y}{\hbar^2}E_N}+\tfrac52\right)\ . \tag{3.9}$$

In general, this is an equation of eighth order in E_N. The solution in terms of the wave-functions then has the form

$$\Psi_N^{(K_{\rm II})}(x,y,z)=N_N f_{II}^{-1/4}\Psi_{n_x}^{(RHO,\tilde k_x)}(x)\Psi_{n_y}^{(RHO,\tilde k_y)}(y)\Psi_{n_z}^{(HO)}(z)\ , \tag{3.10}$$

and the normalization constant N_N is determined by the residuum of the corresponding Green function at the energy E_N from (3.9). The correct flat space limit with $\alpha=\beta_x=\beta_y=0$ is easily recovered with spectrum $E_N=\hbar\omega(2N+\frac{5}{2}+k_x+k_y)$, and similarly other special cases as in the previous Section.

The case of the circular polar coordinate system is very easily obtained. The principal difference just consists of replacing the product of the two radial oscillator wave-functions in x and y by a product of a Pöschl–Teller wave-function in φ and radial oscillator wave-function in ϱ [13]. The energy-spectrum, of course, remains the same and is again determined by (3.9).

Similarly as in $K_{\rm I}$, we can also consider the case of all constants set to zero in the space $K_{\rm II}$, denoted by $K_{\rm II}^{(0)}$. The calculations are very similar to the previous section, yielding the quantization condition ($\tilde\beta=\sqrt{\beta_x}+\sqrt{\beta_y}$)

$$E_N(\delta^2+4\tilde\beta^2)-\frac{2\alpha\hbar^2}{m}(N+\tfrac{5}{2})^2=-\frac{4\alpha\tilde\beta}{m}(N+\tfrac{5}{2})\sqrt{-2mE_N}\ . \tag{3.11}$$

As an easy special case we consider $\tilde\beta=0$, then

$$E_N=\frac{2\alpha\hbar^2}{m\delta^2}(N+\tfrac{5}{2})^2\ , \tag{3.12}$$

which yields for $\delta\neq 0$ either a positive discrete spectrum ($\alpha>0$) or negative discrete spectrum ($\alpha<0$), these case have been already discussed in the previous section.

4 Koenigs-Space $K_{\rm III}$ with Coulomb-Potential

In the next example we consider a metric which corresponds to the three-dimensional Coulomb potential ($r^2=x^2+y^2+z^2$)

$$\begin{aligned} {\rm d}s^2 &= f_{III}(x,y,z)({\rm d}x^2+{\rm d}y^2+{\rm d}z^2)\ , && (4.1)\\ f_{III}(x,y,z) &= -\frac{\alpha_1}{\sqrt{x^2+y^2+z^2}}+\frac{\beta}{x^2}+\frac{\gamma}{y^2}+\delta && (4.2)\end{aligned}$$

and $\alpha_1,\beta,\gamma,\delta$ are constants. The classical Hamiltonian and Lagrangian in $\mathbb{R}^3$ with the Coulomb potential as the superintegrable potential have the form:

$$\begin{aligned}\mathcal{L} &= \frac{m}{2}(\dot x^2+\dot y^2+\dot z^2)+\frac{\alpha_2}{r}-\frac{\hbar^2}{2mr^2\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right) && (4.3)\\ \mathcal{H} &= \frac{p_x^2+p_y^2+p_z^2}{2m}-\frac{\alpha_2}{r}+\frac{\hbar^2}{2mr^2\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)\ . && (4.4)\end{aligned}$$

Counting constants, there are seven independent constants: $\alpha_1,\beta,\gamma,\delta$, and α_2,k_1,k_2. An eight constants can be added by adding a further constant $\tilde\delta$ into the potential of the Hamiltonian, which is again omitted. The third Koenigs-space $K_{\rm III}$ is constructed by considering

$$\mathcal{H}_{K_{\rm III}}=\frac{\mathcal{H}}{f_{III}(x,y,z)}\ . \tag{4.5}$$

Table 5: Some special cases for the space $K_{\rm III}$

Metric	Space	ΔV
$f_{III}(x,y,z)$	Koenigs space $K_{\rm III}$	$\Delta V_1+\dfrac{3\hbar^2}{8m}\left(\dfrac{1}{h_x^2}+\dfrac{1}{h_y^2}+\dfrac{1}{h_z^2}\right)$
$\frac{\alpha_1}{r}$	Special Koenigs space $K_{\rm III}^{\alpha_1}$	$\Delta V_1+\dfrac{3\hbar^2}{8m}\left(\dfrac{1}{h_x^2}+\dfrac{1}{h_y^2}+\dfrac{1}{h_z^2}\right)$
$\dfrac{bu^2-a}{u^2}$	Three-dimensional Darboux Space $D_{\rm II}$	$\Delta V_1+\dfrac{3\hbar^2}{8m(bu^2-a)}$
$\dfrac{1}{u^2}$	Three-dimensional Hyperboloid	$\dfrac{3\hbar^2}{8m}$
1	$\mathbb{R}^3$	0

In Table 5 I have displayed some special cases for $K_{\rm III}$. Again, the special cases are very similar as in the two previous cases, except for only $\alpha_1\neq 0$, and $\alpha_1,\delta\neq 0$.

We proceed to the time-transformed path integral $K^{(K_{\rm III})}(s'')$ which has the form

$$K^{(K_{\rm III})}(r'',r',\vartheta'',\vartheta',\varphi'',\varphi';s'')=\int\limits_{r(0)=r'}^{r(s'')=r''}\mathcal{D}r(s)\int\limits_{\vartheta(0)=\vartheta'}^{\vartheta(s'')=\vartheta''}\mathcal{D}\vartheta(s)\int\limits_{\varphi(0)=\varphi'}^{\varphi(s'')=\varphi''}\mathcal{D}\varphi(s)r^2\sin\vartheta$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\Big(\dot r^2+r^2(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)\Big)+\frac{\tilde\alpha}{r}-\frac{\hbar^2}{2mr^2\sin^2\vartheta}\left(\frac{\tilde k_1^2-\frac14}{\cos^2\varphi}+\frac{\tilde k_2^2-\frac14}{\sin^2\varphi}\right)\right]ds''\right\}.\tag{4.6}$$

Here, $\tilde k_1^2=k_1^2-2m\beta E/\hbar^2$, $\tilde k_2^2=k_2^2-2m\gamma E/\hbar^2$, $\tilde\alpha=\alpha_2-\alpha_1 E$. This path integral for the Coulomb potential has been discussed extensively in literature and the solution in terms of the Green function has been obtained by many authors, e.g.[4, 16, 9, 17, 27]. We obtain for the Green function in polar coordinates ($\lambda_1=2n_\varphi\pm\tilde k_1\pm\tilde k_2+1$, $\lambda_2=l+\lambda_1+\frac12$, $\kappa=\tilde\alpha\sqrt{-m/2\delta\cdot E}/\hbar$)

$$G^{(K_{\rm III})}(r'',r',\vartheta'',\vartheta',\varphi'',\varphi';E)=(f'_{III}f''_{III})^{-\frac14}\sum_{n_\varphi=0}^\infty\Phi_n^{(\pm\tilde k_2,\pm\tilde k_1)}(\varphi'')\Phi_n^{(\pm\tilde k_2,\pm\tilde k_1)}(\varphi')$$
$$\times\sum_{l=0}^\infty(l+\lambda_1+\tfrac12)\frac{\Gamma(l+\lambda_1+1)}{l!}P_{\lambda_1+l}^{-\lambda_1}(\cos\vartheta'')P_{\lambda_1+l}^{-\lambda_1}(\cos\vartheta')$$
$$\times\frac{1}{r'r''}\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\Gamma(\frac12+\lambda_2-\kappa)}{\Gamma(2\lambda_2+1)}W_{\kappa,\lambda_2}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right)M_{\kappa,\lambda_2}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right)\ .\tag{4.7}$$

Bound states are determined by the poles of the Green function, respectively by the poles of the Γ-function, i.e.

$$\frac12+\lambda_2-\kappa=-n_r\ ,\tag{4.8}$$

or more explicitly

$$2+2n_\varphi+l+n_r+\sqrt{k_1^2-\frac{2m\beta E_N}{\hbar^2}}+\sqrt{k_2^2-\frac{2m\gamma E_N}{\hbar^2}}-\frac{\alpha_2-\alpha_1E_N}{\hbar}\sqrt{-\frac{m}{2\delta\cdot E_N}}=0\ . \tag{4.9}$$

This is again an equation of twelfth order in the energy E.

We consider some special cases of (4.9):

1. For $\beta=\gamma=\alpha_1=0$ we obtain the usual Coulomb potential energy spectrum ($N=2+2n_\phi+l+n_r$):

$$E_N=-\frac{m\alpha_2^2}{2\delta\hbar^2(N+k_1+k_2)^2}\ . \tag{4.10}$$

2. For $k_1=k_2=\alpha_2=0$ we obtain the special space $K_{\mathrm{III}}^{\alpha_1}$ ($\tilde\beta=\sqrt{\beta}+\sqrt{\gamma}$):

$$E_N=-\frac{\hbar^2N^2}{2m(\tilde\beta+\frac12\alpha_1/\sqrt{\delta})^2}\ . \tag{4.11}$$

3. For $k_1=k_2=0$ we obtain the special space $K_{\mathrm{III}}^{\alpha_1}$ with an additional Coulomb potential ($\tilde N^2=N^2+2m\alpha_2\hat\beta/(\sqrt{\delta}\hbar^2)$, $\hat\beta=\tilde\beta+\alpha_1/(2\sqrt{\delta})$):

$$E_N=-\frac{\hbar^2\tilde N^2}{4m\hat\beta^2}\left(1\mp\sqrt{1-\frac{4m^2\alpha_2^2\hat\beta^2}{\delta\hbar^4\tilde N^4}}\right)\ . \tag{4.12}$$

Note that for the upper-sign in the square-root term we get well-defined bound states for $N\to\infty$:

$$E_N=-\frac{m\alpha_2^2}{2\delta\hbar^2\tilde N^2}\ , \tag{4.13}$$

i.e. a Coulomb spectrum. However, note the complicated involvement of the various constants, in particular the shift $N^2\to\tilde N^2$.

In either case the wave-functions are given by

$$\begin{aligned}\Psi_{n_r,l,n_phi}(r,\theta,\phi)&=f_{III}^{-1/4}\Phi_n^{(\pm\tilde k_2,\pm\tilde k_1)}(\varphi)\sqrt{(l+\lambda_1+\tfrac12)\frac{\Gamma(l+\lambda_1+1)}{l!}}P_{\lambda_1+l}^{-\lambda_1}(\cos\vartheta'')\\ &\times N_N\frac{2}{(n+\lambda_1+\frac12)^2}\left[\frac{2l!}{a^3(l+\lambda_2+\frac12)\Gamma(l+2\lambda_2+1)}\right]^{1/2}\left(\frac{2r}{a(l+\lambda_2+\frac12)}\right)^{\lambda_2}\\ &\times\exp\left(-\frac{r}{a(l+\lambda_2+\frac12)}\right)L_l^{(2\lambda_2)}\left(\frac{2r}{a(l+\lambda_2+\frac12)}\right)\ ,\end{aligned} \tag{4.14}$$

provided the spectrum is bounded from below and the additional normalization constant N_N is determined by the poles of the Green function (4.7) at the energy-levels determined by (4.9).

As it is well-known, the Coulomb potential is also separable in conical, parabolic and and prolate spheroidal coordinates [13]. In conical and prolate spheroidal coordinates no closed solutions in terms of well-known higher transcendental functions can be found. In parabolic coordinates, we have the same dependence in the variable φ as for polar coordinates, and in the variables ξ and η we get for the discrete spectrum a product of Laguerre polynomials and exponentials,

Table 6: Some special cases for the space K_{IV}

Metric	Space	ΔV
$f_{IV}(x,y,z)$	Koenigs space K_{IV}	$\Delta V_1+\dfrac{3\hbar^2}{8m}\left(\dfrac{1}{h_x^2}+\dfrac{1}{h_y^2}+\dfrac{1}{h_z^2}\right)$
$\dfrac{bu^2-a}{u^2}$	Three-dimensional Darboux Space D_{II}	$\Delta V_1+\dfrac{3\hbar^2}{8m(bu^2-a)}$
$\dfrac{1}{u^2}$	Three-dimensional Hyperboloid	$\dfrac{3\hbar^2}{8m}$
1	$\mathbb{R}^3$	0

actually wave-functions very similar as in the polar variable r. The discrete spectrum, of course, remains the same. In [13] this has been discussed in great detail and will not be repeated here.

The continuous spectrum is usually given in terms of M-Whittaker functions. In the present case, this is quite an involved problem due to the complicated structure of the indices. Both indices κ and λ_2 are complex valued. This, in general leads to an energy spectrum $E_p > c$ with some constant $c > 0$. For instance, in the case of the three-dimensional hyperboloid the constant is given by $c = \hbar^2/2m$, whereas in the case of the Coulomb potential in flat space $c = 0$.

5 Koenigs-Space K_{IV} with Centrifugal Potential I

In the next example we consider a metric which corresponds to the three-dimensional centrifugal potential

$$\begin{aligned} \mathrm{d}s^2 &= f_{IV}(x,y,z)(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2)\ , && (5.1)\\ f_{IV}(x,y) &= \frac{\hbar^2}{2m}\left(\frac{\alpha x}{y^2\sqrt{x^2+y^2}}+\frac{\beta}{y^2}+\frac{\gamma}{z^2}\right)+\delta && (5.2) \end{aligned}$$

and $\alpha_1,\beta,\gamma,\delta$ are constants. The classical Hamiltonian and Lagrangian in $\mathbb{R}^3$ with this potential in spherical coordinates have the form

$$\begin{aligned} \mathcal{L} &= \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)-\frac{\hbar^2}{2mr^2}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2+k_2^2-\frac14}{4\sin^2\frac{\varphi}{2}}+\frac{k_2^2-k_1^2-\frac14}{4\cos^2\frac{\varphi}{2}}\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\right)\ , && (5.3)\\ \mathcal{H} &= \frac{p_x^2+p_y^2+p_z^2}{2m}+\frac{\hbar^2}{2mr^2}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2+k_2^2-\frac14}{4\sin^2\frac{\varphi}{2}}+\frac{k_2^2-k_1^2-\frac14}{4\cos^2\frac{\varphi}{2}}\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\right)\ . && (5.4) \end{aligned}$$

Counting constants, there are seven independent constants: $\alpha,\beta,\gamma,\delta$, and k_1,k_2,k_3. An eight constants can be added by adding a further constant $\tilde{\delta}$ into the potential of the Hamiltonian, which is omitted.

In Table 6 I have displayed some special cases for the space K_{IV}. Again, the special cases are very similar as in the previous cases.

The fourth Koenigs-space $K_{\rm IV}$ is constructed by considering

$$\mathcal{H}_{K_{\rm IV}}=\frac{\mathcal{H}}{f_{IV}(x,y,z)}\quad . \tag{5.5}$$

We write down the path integral formulation for $K^{(K_{\rm IV})}(s'')$

$$K^{(K_{\rm IV})}(r'',r',\vartheta'',\vartheta',\varphi'',\varphi';s'')=\int\limits_{r(0)=r'}^{r(s'')=r''}\mathcal{D}r(s)\int\limits_{\vartheta(0)=\vartheta'}^{\vartheta(s'')=\vartheta''}\mathcal{D}\vartheta(s)\int\limits_{\varphi(0)=\varphi'}^{\varphi(s'')=\varphi''}\mathcal{D}\varphi(s)r^2\sin\vartheta$$
$$\times\exp\Bigg\{\frac{\rm i}{\hbar}\int_0^{s''}\Bigg[\frac{m}{2}\Big(\dot r^2+r^2(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)\Big)$$
$$-\frac{\hbar^2}{2mr^2}\Bigg(\frac{1}{\sin^2\vartheta}\bigg(\frac{\tilde k_1^2+\tilde k_2^2-\frac14}{4\sin^2\frac{\varphi}{2}}+\frac{\tilde k_2^2-\tilde k_1^2-\frac14}{4\cos^2\frac{\varphi}{2}}\bigg)+\frac{\tilde k_3^2-\frac14}{\cos^2\vartheta}\Bigg)\Bigg]{\rm d}s''\Bigg\}\,. \tag{5.6}$$

Here, $\tilde k_1^2=k_2^2+k_1^2-2m(\beta+\alpha)E/\hbar^2$, $\tilde k_2^2=k_2^2-k_1^2+2m(\beta-\alpha)E/\hbar^2$, $\tilde k_3^2=k_3^2-2m\gamma E/\hbar$. We obtain for the path integral $K^{(K_{\rm IV})}(s'')$ ($\lambda_1=n+(\tilde k_1+\tilde k_2+1)/2$, $\lambda_2=2m+\lambda_1\pm\tilde k_3+1$)

$$K^{(K_{\rm IV})}(r'',r',\vartheta'',\vartheta',\varphi'',\varphi';s'')$$
$$=\frac{(r'r''\sin\theta'\sin\theta'')^{-1/2}}{2}\sum_{n=0}^\infty\Psi_n^{(\tilde k_1,\tilde k_2)}\bigg(\frac{\varphi'}{2}\bigg)\Psi_n^{(\tilde k_1,\tilde k_2)}\bigg(\frac{\varphi''}{2}\bigg)\sum_{m=0}^\infty\Phi_m^{(\lambda_1,\pm\tilde k_3)}(\vartheta')\Phi_m^{(\lambda_1,\pm\tilde k_3)}(\vartheta'')$$
$$\times\frac{m}{{\rm i}\hbar s''}\exp\bigg[\frac{{\rm i}m}{2\hbar s''}(r'^2+r''^2)\bigg]I_{\lambda_2}\bigg(\frac{mr'r''}{{\rm i}\hbar s''}\bigg)\quad . \tag{5.7}$$

Therefore the Green function $G^{(K_{\rm IV})}(E)$ has the form ($\tilde E=E-\delta$)

$$G^{(K_{\rm IV})}(r'',r',\vartheta'',\vartheta',\varphi'',\varphi';E)=(f'_{IV}f''_{IV})^{-\frac14}\frac{(r'r''\sin\theta'\sin\theta'')^{-1/2}}{2}$$
$$\times\sum_{n=0}^\infty\Psi_n^{(\tilde k_1,\tilde k_2)}\bigg(\frac{\varphi'}{2}\bigg)\Psi_n^{(\tilde k_1,\tilde k_2)}\bigg(\frac{\varphi''}{2}\bigg)\sum_{m=0}^\infty\Phi_m^{(\lambda_1,\pm\tilde k_3)}(\vartheta')\Phi_m^{(\lambda_1,\pm\tilde k_3)}(\vartheta'')$$
$$\times\frac{2m}{\hbar^2}I_{\lambda_2}\bigg(\sqrt{-2m\tilde E}\,\frac{r_<}{\hbar}\bigg)K_{\lambda_2}\bigg(\sqrt{-2m\tilde E}\,\frac{r_>}{\hbar}\bigg)\quad . \tag{5.8}$$

The analysis of this Green function is complicated due to the complicated index which has imaginary parts for $2m\gamma E/\hbar^2>k_3^2$ etc. This, in general leads to modified K-Bessel-functions as wave-functions (c.f. Liouville quantum mechanics [16]) with a continuous energy spectrum $E_p>c$ with some constant $c>0$. For instance, in the case of the three-dimensional hyperboloid the constant is given by $c=\hbar^2/2m$. We do not discuss these issues any further.

6 Koenigs-Space $K_{\rm V}$ with Centrifugal Potential II

In the last example we consider a metric which corresponds to the three-dimensional linear-centrifugal potential

$$\begin{aligned}{\rm d}s^2&=f_V(x,y,z)({\rm d}x^2+{\rm d}y^2+{\rm d}z^2)\ , &(6.1)\\ f_V(x,y)&=\frac{\hbar^2}{2m}\bigg(\frac{\alpha x}{y^2\sqrt{x^2+y^2}}+\frac{\beta}{y^2}\bigg)+\gamma z+\delta &(6.2)\end{aligned}$$

Table 7: Some special cases for the space K_{V}

Metric	Space	ΔV
$f_V(x,y,z)$	Koenigs space K_{V}	$\Delta V_1+\dfrac{3\hbar^2}{8m}\bigg(\dfrac{1}{h_x^2}+\dfrac{1}{h_y^2}+\dfrac{1}{h_z^2}\bigg)$
γu	Three-dimensional Darboux space D_{I}	0
$\dfrac{bu^2-a}{u^2}$	Three-dimensional Darboux Space D_{II}	$\Delta V_1+\dfrac{3\hbar^2}{8m(bu^2-a)}$
$\dfrac{1}{u^2}$	Three-dimensional Hyperboloid	$\dfrac{3\hbar^2}{8m}$
1	$\mathrm{I\!R}^3$	0

and $\alpha_1,\beta,\gamma,\delta$ are constants. The classical Hamiltonian and Lagrangian in $\mathrm{I\!R}^3$ with this potential have the form

$$\begin{aligned}
\mathcal{L} &= \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)-\frac{\hbar^2}{2m}\Bigg(\frac{k_1^2x}{y^2\sqrt{x^2+y^2}}+\frac{k_2^2-\frac14}{y^2}\Bigg)-k_3z\ , &(6.3)\\
\mathcal{H} &= \frac{p_x^2+p_y^2+p_z^2}{2m}+\frac{\hbar^2}{2m}\Bigg(\frac{k_1^2x}{y^2\sqrt{x^2+y^2}}+\frac{k_2^2-\frac14}{y^2}\Bigg)+k_3z\ . &(6.4)
\end{aligned}$$

Counting constants, there are seven independent constants: $\alpha,\beta,\gamma,\delta$, and k_1,k_2,k_3. An eight constants can be added by adding a further constant $\tilde\delta$ into the potential of the Hamiltonian, which is omitted. The fifth Koenigs-space K_{V} is constructed by considering

$$\mathcal{H}_{K_{\mathrm{V}}}=\frac{\mathcal{H}}{f_V(x,y,z)}\ . \tag{6.5}$$

In Table 7 I have displayed some special cases for K_{V}. Again, the special cases are very similar as in the previous cases. However, the new special case which appears is the three dimensional Darboux space D_{I} from [11]. From [11] we know that the free motion separates in seven coordinate systems, i.e. in Cartesian, the three circular systems, the parabolic, the paraboloidal system, and a rotated Cartesian system.

We write down the path integral formulation for $K^{(K_{\mathrm{V}})}(s'')$ in circular polar coordinates

$$\begin{aligned}
K^{(K_{\mathrm{V}})}(\varrho'',\varrho',\varphi'',\varphi',z'',z';s'') &= \int\limits_{\varrho(0)=\varrho'}^{\varrho(s'')=\varrho''}\mathcal{D}\varrho(s)\varrho\int\limits_{\varphi(0)=\varphi'}^{\varphi(s'')=\varphi''}\mathcal{D}\varphi(s)\int\limits_{z(0)=z'}^{z(s'')=z''}\mathcal{D}z(s)\\
\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\Bigg[\frac{m}{2}\Big(\dot\varrho^2+\varrho^2\dot\varphi^2+\dot z^2\Big)&-\frac{\hbar^2}{2m\varrho^2}\Bigg(\frac{\tilde k_1^2-\frac14}{4\cos^2\frac{\varphi}{2}}+\frac{\tilde k_2^2-\frac14}{4\sin^2\frac{\varphi}{2}}-\frac14\Bigg)-\tilde k_3 z\Bigg]\mathrm{d}s''\Bigg\}\ . \qquad (6.6)
\end{aligned}$$

Here, $\tilde k_1^2=k_2^2+k_1^2-2m(\beta+\alpha)E/\hbar^2$, $\tilde k_2^2=k_2^2-k_1^2+2m(\beta-\alpha)E/\hbar^2$, $\tilde k_3=k_3-2m\gamma E/\hbar$. This path integral has the solution ($\lambda_1=n+\frac12(\tilde k_1+\tilde k_2+1)$):

$$K^{(K_{\rm V})}(\varrho'',\varrho',\varphi'',\varphi',z'',z';s'')$$
$$= \left(\frac{m}{2\pi i\hbar s''}\right)^{1/2} \exp\left[\frac{i}{\hbar}\left(\frac{m}{2s''}(z''-z')^2 - \frac{k_3 T}{2}(z'+z'') - \frac{k_3^2 s''^3}{24m}\right)\right]$$
$$\times \frac{m}{2i\hbar s''} \sum_{n=0}^{\infty} \Phi_n^{(\tilde{k}_1,\tilde{k}_2)}\left(\frac{\phi'}{2}\right) \Phi_n^{(\tilde{k}_1,\tilde{k}_2)}\left(\frac{\phi''}{2}\right) \exp\left[-\frac{m}{2i\hbar s''}(\rho'^2+\rho''^2)\right] I_{\lambda_1}\left(\frac{m\rho'\rho''}{i\hbar s''}\right). \qquad (6.7)$$

The analysis of this Green function is again very complicated due to the complicated index which has imaginary parts for $2m\gamma E/\hbar^2 > k_3^2$ etc. yielding a continuous spectrum. As in the previous section, these issues will not be discussed further.

7 Summary and Discussion

In this contribution I have discussed a path integral approach for spaces of non-constant curvature according to Koenigs, which I have for short called "Koenigs-spaces" $K_{\rm I}$–$K_{\rm V}$, respectively. I have found a very rich structure of the spectral properties of the quantum motion on Koenigs-spaces. In the general case with potential, in three spaces the quantization condition is determined by an equation up to twelfth order in the energy E. Such an equation cannot be solved explicitly, however special cases can be studied. We found also constraints on the parameters for the well-definedness on the wave-functions. For the remaining two spaces no quantization condition was formulated, because it is known that in the corresponding cases of superintegrable potentials in $\mathbb{R}^3$ only a continuous spectrum exists.

Let us note a further feature of these spaces. It is obvious that our solutions remain on a formal level. Neither have we specified an embedding space, nor have we specified boundary conditions on our spaces. Let us consider the space $K_{\rm V}$: We set $\alpha=\beta=\delta=0$ and $\gamma=1$. In this case we obtain a metric which corresponds to the three-dimensional Darboux space $D_{\rm I}$ (modulo change of variables), as discussed in [11]. In $D_{\rm I}$ boundary conditions and the signature of the ambient space is very important, because choosing a positive or a negative signature of the ambient space changes the boundary conditions, and hence the quantization conditions [14, 20]. Including an appropriate potential, bound states defined by a transcendental equation can be found.

Furthermore, we can recover the three-dimensional Darboux space $D_{\rm II}$ [11, 20] by setting in our examples in the potential function f all constant to zero except those corresponding to the $1/x^2$-singularity and the constant $\delta=1$. However, we did not discuss these cases in detail.

In our approach we have chosen examples of superintegrable potentials in three-dimensional space, i.e. the isotropic singular oscillator, the Holt potential, the Coulomb potential, and two centrifugal potentials, respectively. I did not consider the minimally superintegrable potentials. There are eight of them [13], however they contain always an unspecified function F depending on the radial variable r, say, leading to some unspecified spectrum.

I have omitted the discussion of the continuous spectrum. This is mostly due to lack of the specification of the ambient space. For instance, in the Darboux space $D_{\rm II}$ we know that the continuous spectrum has the form of $E_p \propto (\hbar^2/2m)p^2 + \text{constant}$. The wave-functions are proportional to K-Bessel functions [10]. However, in Darboux space $D_{\rm I}$ there is no such constant, and the wave-functions have a different form. Furthermore, $D_{\rm II}$ contains as special cases the three-dimensional Euclidean plane and the Hyperbolic space, respectively.

However, a more detailed study of these special case would require some additional input from a physics point of view: Can a space of non-constant curvature (Koenigs or Darboux space) model actually curved space-time? And how such a global or local model can give rise to observable physical effects? These issues are beyond the scope of this article and will not be discussed here any further.

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