DESY 07-109

DESY 07-109
arXiv:0708.0583

Branes in the GL(1|1) WZNW-Model

Thomas Creutzig[(1)], Thomas Quella[(2)] and Volker Schomerus[(1)]

[(1)] DESY Theory Group, DESY Hamburg
Notkestrasse 85, D-22603 Hamburg, Germany
e-mail: thomas.creutzig@desy.de, volker.schomerus@desy.de

[(2)] KdV Institute for Mathematics, University of Amsterdam,
Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands
e-mail: tquella@science.uva.nl

August 2007

Abstract

We initiate a systematic study of boundary conditions in conformal field theories with target space supersymmetry. The WZNW model on GL(1|1) is used as a prototypical example for which we find the complete set of maximally symmetric branes. This includes a unique brane of maximal super-dimension 2|2, a 2-parameter family of branes with super-dimension 0|2 and an infinite set of fully localized branes possessing a single modulus. Members of the latter family can only exist along certain lines on the bosonic base, much like fractional branes at orbifold singularities. Our results establish that all essential algebraic features of Cardy-type boundary theories carry over to the non-rational logarithmic WZNW model on GL(1|1).

1 Introduction

Field theories with target space supersymmetry have received considerable attention lately, because of their interesting applications in both condensed matter theory and in string theory. This applies in particular to 2-dimensional conformal field theories with space-time (internal) supersymmetry. They describe critical behavior in many systems with disorder [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and they provide building blocks for string theory in AdS backgrounds [11, 12, 13, 14].[1]

Conformal field theories with target space supersymmetry have some properties that, for a long time, were considered rather exotic. In fact, the correlators of such theories very often possess logarithmic singularities on the world-sheet. In condensed matter theory, these had been seen in various examples starting from [15]. But it was only recently [16] that the appearance of logarithms in correlation functions was understood as a rather generic consequence of internal supersymmetry in CPT invariant local quantum field theory.

In many respects, logarithmic conformal field theories behave rather differently from the well studied unitary rational models (see, e.g., [17, 18] and references therein). It has proven particularly difficult to construct examples of local logarithmic conformal field theories. Until recently, the only example that was fully understood was that of a triplet model [19]. The problems may be traced back to the non-diagonalizability of the generator $D = L_0 + \bar{L}_0$ of scale transformations which is one of the characteristic features of any logarithmic conformal field theory. Since locality implies that the generator $R = L_0 - \bar{L}_0$ of rotations must be diagonalizable with integer valued spectrum, the left and right moving sector in a logarithmic conformal field theory must conspire in an intricate way to ensure locality.

Against all these odds, recent work on WZNW models on type I supergroups [16, 20, 21, 22] is now supplying us with a large number of local logarithmic conformal field theories. This remarkable progress is closely linked to the existence of an action principle for these logarithmic models. The latter furnishes valuable geometric insights in addition to efficient computational tools. These provide an explicit solution of the WZNW model for the supergroups GL(1|1) [16], SU(2|1) [21] and PSL(2|2) [20] along with powerful

[1] We should stress that in latter context, gauge fixing the Green-Schwarz superstring leads to non-relativistic theories which may have very different properties from what we are about to describe.

results and predictions for generic supergroups of type I [22].

It is natural and important to extend these developments beyond the bulk theory and to include world-sheets with boundaries. Systems with boundaries are highly relevant for applications (see e.g. [23, 24, 25] for an incomplete review of applications and many further references), often more so than theories on closed surfaces. Moreover, boundary conformal field theory also displays rather rich mathematical structures (see e.g. [26, 27, 28, 29, 30] or [31, 32, 33, 34] for various directions and further references), in particular related to modular properties, fusion etc. All this is very poorly understood for general logarithmic conformal field theories, see however [35, 36, 37, 38, 39, 40] and especially [41] for recent progress in specific models. WZNW models on supergroups present themselves as an ideal playground to extend many of the beautiful results of unitary rational conformal field theory to logarithmic models. Even the simplest models are mathematically rich and physically relevant.

The aim of this work is to initiate a systematic study of boundary conditions in WZNW models on supergroups based on the example of GL(1|1).[2] Let us list the main results of this paper in more detail. Recall that maximally symmetric boundary conditions in conformal field theories carry two labels. The first one refers to the choice of a gluing condition between left and right moving chiral fields. The second label parametrizes different boundary conditions associated with the same gluing condition. In uncompactified free field theory, for example, the two labels correspond to the dimension of the brane and its transverse position. The relation between these labels and the branes' geometry becomes more intricate when the world-sheet theory is interacting.

In the second section we shall describe the possible ways in which we can glue left and right movers in the GL(1|1) WZNW model. We shall see that there are essentially two choices, corresponding to what we shall call *untwisted* and *twisted* branes. Most of this work is then devoted to the *untwisted branes*. We shall discuss in section 3 that all untwisted branes satisfy Dirichlet boundary conditions for the two bosonic coordinates. Hence, they describe objects that are fully localized in the bosonic base of the supergroup GL(1|1). The position of these branes is parametrized by a pair (z_0, y_0) of real numbers. For generic choices of y_0, the untwisted branes extend along the two fermionic directions of GL(1|1) and there exists a non-vanishing B-field. But on the lines $y_0 = 2\pi s$, for any

[2] Spectra of supersymmetric coset models with open boundary conditions were also studied previously, in particular in [42, 43].

integer s, there exists an additional set of branes which are localized in the fermionic directions as well as the bosonic ones, i.e. they are truly point-like.

After a detailed study of the branes' geometry we shall provide exact boundary states for generic and non-generic untwisted branes on GL(1|1) in section 4. There, we shall also discuss what happens when a generic brane is moved onto one of the lines $y_0 = 2\pi s$: It turns out to split into a pair of non-generic branes with a transverse separation that is proportional to the level of the WZNW model. Section 5 contains a detailed discussion of the relation between our findings for boundary conditions in a local logarithmic conformal field theory and the usual Cardy case of unitary rational models [44]. We shall see that in both cases branes are parametrized by irreducible representations of the current algebra. Furthermore, the spectra between any two branes can be determined by fusion. Similar results for the $p = 2$ triplet model have been obtained in [41]. In the case of GL(1|1) WZNW model we will establish that most of the boundary spectra are not fully reducible. This applies in particular to the spectrum of boundary operators on a single generic brane. Section 6 is devoted to a brief study of twisted branes on GL(1|1). We shall find that these satisfy Neumann boundary conditions in the bosonic coordinates.

2 Gluing Conditions for $\widehat{\mathrm{gl}}(1|1)$ Symmetric Branes

Branes on supergroups come in different families or types. They are characterized by the way in which left and right moving chiral fields are glued along the boundary (see e.g. [45]). Mathematically, the various possible gluing conditions correspond to automorphisms of the chiral symmetry. If two gluing automorphisms differ by an inner automorphism, the associated branes are related to each other by simple translation on the target space.

The chiral symmetry of the GL(1|1) WZNW model is a $\widehat{\mathrm{gl}}(1|1)$ current superalgebra. Its metric preserving automorphisms will be classified in the first subsection up to the possible composition with an inner automophism. In addition to the trivial automorphism we shall find one non-trivial outer automorphism Ω. Some general facts about the associated gluing conditions for supercurrents and their geometrical interpretation are collected in the second subsection.

2.1 Automorphisms of the $\widehat{\mathrm{gl}}(1|1)$ current superalgebra

In this subsection, we determine the relevant gluing automorphisms Ω for branes in the GL(1|1) WZNW model. An automorphism of the $\widehat{\mathrm{gl}}(1|1)$ current superalgebra is admissible as a gluing automorphism if it acts trivially on the Virasoro Sugawara field T. When restricted to the zero mode algebra, any such automorphisms Ω gives rise to an automorphism ω of the underlying finite dimensional Lie superalgebra gl(1|1). If Ω leaves T invariant, the corresponding automorphism ω acts trivially on the associated quadratic Casimir element C of gl(1|1). Our first goal is therefore to classify all automorphisms ω of gl(1|1) with the additional property that $\omega(C) = C$.

The Lie superalgebra gl(1|1) is generated by two bosonic elements E, N and two fermionic elements $\Psi^{\pm}$, subject to the relations

$$[N, \Psi^{\pm}] \;=\; \pm\Psi^{\pm} \quad , \quad \{\Psi^{-}, \Psi^{+}\} \;=\; E \quad . \tag{2.1}$$

In addition, the element E is central, i.e. it commutes with all other elements of gl(1|1). The relevant quadratic Casimir element C of gl(1|1) is given by

$$C \;=\; (2N-1)E + 2\Psi^{-}\Psi^{+} + \frac{1}{k}E^2 \quad . \tag{2.2}$$

Since E is central, one has the freedom of adding a quadratic polynomial in E. The choice we have made here is the one that corresponds to the Virasoro Sugawara field of the $\widehat{\mathrm{gl}}(1|1)$ current superalgebra at level k that has been used in [16]. In this context the subleading term in k should be thought as a quantum renormalization. Adding additional contributions in E^2 does not change the qualitative features of the model.

A straightforward calculation shows, that the Casimir preserving automorphisms of gl(1|1) come in two families,

$$\omega_\alpha^{(0)}(E) = \quad E \quad , \quad \omega_\alpha^{(0)}(N) = \quad N \quad , \quad \omega_\alpha^{(0)}(\Psi^{\pm}) = \quad e^{\pm i\alpha}\Psi^{\pm} \tag{2.3}$$

$$\omega_\alpha^{(1)}(E) = -E \quad , \quad \omega_\alpha^{(1)}(N) = -N \quad , \quad \omega_\alpha^{(1)}(\Psi^{\pm}) = \pm e^{\pm i\alpha}\Psi^{\mp} \; . \tag{2.4}$$

With E being central, the only non-trivial bosonic inner automorphisms Ad_α are provided by conjugation with $\exp(i\alpha N)$. Looking back onto the eqs. (2.3), we observe that $\omega_\alpha^{(0)} = \mathrm{Ad}_\alpha$, i.e. the automorphisms $\omega_\alpha^{(0)}$ are all inner. Furthermore, any two members of the second family $\omega_\alpha^{(1)}$ are related by conjugation with some $\exp(i\alpha N)$. Hence, it suffices to consider one representative $\omega = \omega^{(1)}_{\alpha=0}$. We conclude that, up to composition with inner automorphisms, there exist two admissible automorphisms of gl(1|1), namely the trivial automorphism $\omega^{(0)} = \mathrm{id}$ and the non-trivial $\omega = \omega^{(1)}_0$. Note that the latter squares to an inner automorphism.

Let us now show that both automorphisms lift to admissible automorphisms of the current superalgebra $\widehat{\mathrm{gl}}(1|1)$. This current algebra is generated by the modes of the chiral fields $E(z), N(z)$ and $\Psi^{\pm}(z)$ with relations,

$$[E_n, N_m] \;=\; -km\delta_{n+m} \;,\quad [N_n, \Psi^{\pm}_m] \;=\; \pm\Psi^{\pm}_{n+m} \;,\quad \{\Psi^{-}_n, \Psi^{+}_m\} \;=\; E_{n+m} + km\delta_{n+m} \;. \tag{2.5}$$

All other (anti-)commutators vanish and the number k is known as the level of $\widehat{\mathrm{gl}}(1|1)$. The action of $\omega^{(0)} = \mathrm{id}$ on gl(1|1) lifts to the trivial automorphism $\Omega^{(0)} = \mathrm{id}$ on $\widehat{\mathrm{gl}}(1|1)$. In case of $\omega^{(1)}$, its properties guarantee that

$$\Omega(E_n) \;=\; -E_n \quad,\quad \Omega(N_n) \;=\; -N_n \quad,\quad \Omega(\Psi^{\pm}_n) \;=\; \pm\Psi^{\mp}_n$$

is consistent with the level dependent terms in eqs. (2.5). Furthermore, the modes of the stress energy tensor take the form [15]

$$\begin{aligned} L_n \;=\; & \frac{1}{2k}\left(2N_nE_0 - E_n + 2\Psi^{-}_n\Psi^{+}_0 + \frac{1}{k}E_nE_0\right) \\ & + \frac{1}{k}\sum_{m>0}\left(E_{n-m}N_m + N_{n-m}E_m + \Psi^{-}_{n-m}\Psi^{+}_m - \Psi^{+}_{n-m}\Psi^{-}_m + \frac{1}{k}E_{n-m}E_m\right) \end{aligned}$$

It is easy to check that the L_n are indeed invariant under the action of Ω. Consequently we have found two classes of automorphisms of $\widehat{\mathrm{gl}}(1|1)$ that are admissible as gluing automorphisms.

2.2 Types of boundary conditions

Let us consider a WZNW model on the upper half of the complex plane. Boundary conditions along the boundary at $z = \bar{z}$ preserve conformal invariance of the model if and only if the two chiral components of the stress energy tensor T agree all along the boundary, i.e.

$$T(z) \;=\; \overline{T}(\bar{z}) \quad \text{for } z = \bar{z} \;. \tag{2.6}$$

In any WZNW model, the stress energy tensor T is constructed out of the chiral currents. A boundary condition is said to be maximally symmetric if left and right moving currents can be identified along the boundary, up to the action of an automorphism Ω,

$$J^a(z) \;=\; \Omega\big(\bar{J}^a(\bar{z})\big) \quad \text{for } z=\bar{z}\ . \tag{2.7}$$

where $J^a = E, N, \Psi^{\pm}$ when we deal with the GL(1|1) model. For Ω we can insert any of the automorphisms we have discussed in the previous subsection.

It will be convenient to rewrite the gluing conditions (2.7) in terms of those fields that appear in the action of the GL(1|1) WZNW model. In principle, there exist various choices that come with different parametrizations of the supergroup GL(1|1). One possible set of coordinate fields is introduced through

$$g \;=\; e^{ic_-\Psi^-}\, e^{iXE+iYN}\, e^{ic_+\Psi^+}\quad . \tag{2.8}$$

The fields X and Y are bosonic while $c_\pm$ are fermionic. Let us also recall that the (anti-)holomorphic currents of the WZNW model are given by

$$J(z) \;=\; -k\partial g g^{-1} \quad \text{and} \quad \bar{J}(\bar{z}) \;=\; k g^{-1}\bar{\partial} g\quad .$$

Inserting our specific choice of the paramerization (2.8), the currents take the following form

$$\bar{J} \;=\; kie^{iY}\bar{\partial}c_-\Psi^- + k\big(i\bar{\partial}X - (\bar{\partial}c_-)c_+e^{iY}\big)E + ki\bar{\partial}YN + k(i\bar{\partial}c_+ - c_+\bar{\partial}Y)\Psi^+ \tag{2.9}$$

and

$$J \;=\; -k(i\partial c_- - c_-\partial Y)\Psi^- - k\big(i\partial X - c_-(\partial c_+)e^{iY}\big)E - ki\partial YN - kie^{iY}\partial c_+\Psi^+. \tag{2.10}$$

The various components of these Lie superalgebra valued (anti-)holomorphic currents can be projected out with the help of the super-trace

$$\mathrm{str}(NE) \;=\; \mathrm{str}(\Psi^+\Psi^-) \;=\; -1\ . \tag{2.11}$$

We conclude that $E(z) = \mathrm{str}(J(z)E) = ki\partial Y$ and similar expressions hold for the other three holomorphic currents and their anti-holomorphic counterparts.

Let us briefly recall how to extract the branes' geometry from the gluing conditions. Locally, the action of a WZNW model on any (super-)group looks as follows

$$S(X) \;\sim\; \int_\Sigma d^2z(g_{\mu\nu} + B_{\mu\nu})\partial X^\mu \bar{\partial} X^\nu. \tag{2.12}$$

with a (graded) antisymmetric 2-form potential B of the WZ 3-form $H = dB$ and a (graded) symmetric metric g. Vanishing of the boundary contributions to the variation leaves us with two choices: We can either impose Dirichlet boundary conditions $\partial_p X^\mu = 0$ or require that

$$g_{\mu\nu}\,\partial_n X^\mu(z,\bar{z}) \;=\; iB_{\mu\nu}\,\partial_p X^\mu(z,\bar{z}) \;\;\text{for } z=\bar{z}\;. \tag{2.13}$$

In general, some combination of these two possibilities occurs. The gluing conditions (2.7) for our currents (2.9) and (2.10) can always be brought into standard form by splitting the derivatives ∂ and $\bar{\partial}$ into ∂_p and ∂_n. Following the reasoning that was first proposed in [46] for bosonic WZNW models (see also [47] for a different approach), one may show that maximally symmetric branes on super-groups are localized along ω twisted super-conjugacy classes

$$C^\omega(b) \;=\; \left\{\omega(g)bg^{-1} \;\middle|\; g \text{ in } G\right\} \tag{2.14}$$

where b can be any element of the bosonic subgroup and ω is now regarded as an automorphism of the supergroup rather than its Lie superalgebra. For the GL(1|1) WZNW model, a more detailed derivation of this statement along with an explicit description of the resulting brane geometries will be given below.

3 Untwisted Branes: Geometry and Particle limit

This section is devoted to the geometry of branes associated with the trivial gluing automorphism. We shall show that such branes are localized at a point (x_0, y_0) on the bosonic base of GL(1|1). For generic choices y_0, they stretch out along the fermionic directions, i.e. the fermionic fields obey Neumann type boundary conditions. When $y_0 = 2\pi s$, $s \in \mathbb{Z}$, on the other hand, the corresponding branes are point-like. These geometric insights from the first part of the section are then used in the second part to study branes in the particle limit in which the level k is sent to infinity. Most importantly, we shall provide minisuperspace analogues of the boundary states for both generic and non-generic untwisted branes, see eqs. (3.25) and (3.27), respectively.

3.1 Geometric interpretation of untwisted branes

In the previous section we have made a number of general statements concerning the geometry of maximally symmetric branes on (super-)group target spaces. Here, we want to step back a bit and work out the precise form of the boundary conditions for coordinate fields. We shall continue to use the specific parametrization (2.8) of GL(1|1). Insertion of our explicit formulas (2.9) and (2.10) for left and right moving currents into the gluing condition (2.7) with $\Omega = \mathbb{I}$ gives

$$\begin{aligned} \partial_p Y \ &= \ 0 \quad , \quad \partial_p Z \ = \ 0 \quad , \quad \text{for } z \ = \ \bar{z} \ , \\ \text{where} \quad & Z = X + ic_- c_+ (e^{-iY} - 1)^{-1} \end{aligned} \tag{3.1}$$

and ∂_p denotes the derivative along the boundary. In other words, both bosonic fields Y and Z satisfy Dirichlet boundary conditions. Untwisted branes in the GL(1|1) WZNW model are therefore parameterized by the constant values (y_0, z_0) the two bosonic fields Y, Z assume along the boundary. For the two basic fermionic fields we obtain similarly

$$\begin{aligned} \pm\, 2 \sin^2(Y/2) \partial_n d_\pm \ &= \ \sin(Y)\, \partial_p d_\pm \quad , \quad \text{for } z \ = \ \bar{z} \ , \\ \text{where} \quad d_\pm \ &= \ c_\pm e^{iY/2} \sin^{-1}(Y/2)/2i \ . \end{aligned} \tag{3.2}$$

Thereby, the fermionic directions are seen to satisfy Neumann boundary conditions with a constant B-field whose strength depends on the position of the brane along the bosonic base. We shall provide explicit formulas below. For the moment let us point out that the condition (3.2) degenerates whenever the value y_0 of the bosonic field Y on the boundary approaches an integer multiple of 2π. In fact, when $y_0 = 2\pi s, s \in \mathbb{Z}$ we obtain Dirichlet boundary conditions in all directions, bosonic and fermionic ones,

$$\partial_p Y \ = \ \partial_p Z = \partial_p d_\pm \ = \ 0 \ \text{ for } z \ = \ \bar{z}. \tag{3.3}$$

In the following, we shall refer to the branes with parameters $(z_0, y_0 \neq 2\pi s)$ as *generic (untwisted) branes.* These branes are localized at the point (z_0, y_0) of the bosonic base and they stretch out along the fermionic directions. A localization at points $(z_0, 2\pi s), s \in \mathbb{Z}$, implies Dirichlet boundary conditions for the fermionic fields. We shall refer to the corresponding branes as *non-generic (untwisted) branes.*

We have seen in the description of our gluing conditions that it was advantageous to introduce fields Z and $d_\pm$ instead of X and $c_\pm$. They correspond to a new choice of coordinates on the supergroup GL(1|1)

$$g \ = \ e^{ic_-\Psi^-} e^{ixE+iyN} e^{ic_+\Psi^+} \ = \ e^{id_-\Psi^-} e^{-id_+\Psi^+} e^{izE+iyN} e^{id_+\Psi^+} e^{-id_-\Psi^-} \tag{3.4}$$

that is particularly adapted to the description of untwisted branes. In fact, we recall from our general discussion that untwisted branes are localized along conjugacy classes.

It is therefore natural to introduce a parametrization in which supergroup elements g are obtained by conjugating bosonic elements $g_0 = \exp(iz_0 E + iy_0 N)$ with exponentials of fermionic generators. From equation (3.4) it is also easy to read off that conjugacy classes containing a bosonic group element g_0 contain two fermionic directions as long as $y_0 \neq 2\pi s$. In case $y_0 = 2\pi s$, conjugation of g_0 with the fermionic factors is a trivial operation and hence the conjugacy classes consist of points only.

It is instructive to work out the form of the background metric and B-field in our new coordinates. To this end, let us recall that

$$ds^2 \; = \; \mathrm{str}\big((g^{-1}dg)^2\big) \; = \; 2dxdy - 2e^{iy}d\eta_- d\eta_+ \quad . \tag{3.5}$$

Here, the super-coordinates $x, y, \eta_\pm$ correspond to our coordinate fields $X, Y, c_\pm$. Similarly, the Wess-Zumino 3-form on the supergroup GL(1|1) is given by

$$H \; = \; \frac{2}{3}\,\mathrm{str}(g^{-1}dg)^{\wedge 3} \; = \; 2ie^{iy}d\eta_- \wedge d\eta_+ \wedge dy \quad . \tag{3.6}$$

After the appropriate change of coordinates from $(x, y, \eta_\pm)$ to $(z, y, \zeta_\pm)$, the metric reads

$$ds^2 \; = \; 2dzdy + 8\sin^2(y/2)d\zeta_- d\zeta_+ \tag{3.7}$$

and the H field becomes

$$H \; = \; 4i\big(\cos(y) - 1\big)d\zeta_- \wedge d\zeta_+ \wedge dy \quad . \tag{3.8}$$

It is easy to check that $H = dB$ possesses a 2-form potential B given by

$$B \; = \; 4i\sin(y)\, d\zeta_- \wedge d\zeta_+ \; + 2i\zeta_+ d\zeta_- \wedge dy - 2i\zeta_- d\zeta_+ \wedge dy \quad . \tag{3.9}$$

Upon pull back to the untwisted branes we can set $dy = 0$ and the B-field becomes

$$\pi^*_{\mathrm{brane}}\, B \; = \; 4i\sin(y)\, d\zeta_- \wedge d\zeta_+ \quad . \tag{3.10}$$

This expression together with our formula (3.7) for the metric allow to recast the boundary conditions (3.2) for the fermionic fields in theories with generic untwisted boundary conditions in the familiar form (2.13).

3.2 Boundary states in the minisuperspace theory

As in the analysis of the bulk GL(1|1) model [16] it is very instructive to study the properties of untwisted branes in the so-called particle or minisuperspace limit. Thereby we obtain predictions for several field theory quantities in the limit where the level k tends to infinity. Our first aim is to present formulas for the minisuperspace analogue of Ishibashi states. Using our insights from the previous subsection we shall then propose candidate boundary states for the particle limit and expand them in terms of Ishibashi states.

Let us begin by recalling a few basic facts about the supergroup GL(1|1) or rather the space of functions Ł2 it determines, see [16]. The latter is spanned by the elements

$$e_0(e,n) \;=\; e^{iex+iny} \;, \quad e_\pm(e,n) \;=\; \eta_\pm e_0(e,n) \quad e_2(e,n) \;=\; \eta_-\eta_+ e_0(e,n) \;. \tag{3.11}$$

where the coordinates are the same as in the previous subsection. Right and left invariant vector fields take the following form

$$R_E \;=\; i\partial_x \;, \quad R_N \;=\; i\partial_y + \eta_-\partial_- \;, \quad R_+ \;=\; -e^{-iy}\partial_+ - i\eta_-\partial_x \;, \quad R_- \;=\; -\partial_- \;, \tag{3.12}$$

and

$$L_E \;=\; -i\partial_x \;, \quad L_N \;=\; -i\partial_y - \eta_+\partial_+ \;, \quad L_- \;=\; e^{-iy}\partial_- - i\eta_+\partial_x \;, \quad L_+ \;=\; \partial_+ \;, \tag{3.13}$$

These vector fields generate two (anti-)commuting copies of the underlying Lie superalgebra gl(1|1). For the reader's convenience we also wish to reproduce the invariant Haar measure on GL(1|1),

$$d\mu \;=\; e^{-iy} dx dy d\eta_+ d\eta_- \;. \tag{3.14}$$

The decomposition of Ł2 with respect to both left and right regular action was analyzed in [16]. Here, we are most interested in properties of the adjoint action $\mathrm{ad}_X = R_X + L_X$ since it is this combination of the symmetry generators that is preserved by the untwisted D-branes.

Our first aim is to construct a canonical basis in the space of (co-)invariants. By definition, a (co-)invariant $|\psi\rangle\rangle$ ($\langle\langle\psi|$) is a state (linear functional) satisfying

$$\mathrm{ad}_X|\psi\rangle\rangle \;=\; (R_X + L_X)|\psi\rangle\rangle \;=\; 0 \quad , \quad \langle\langle\psi|\,\mathrm{ad}_X \;=\; \langle\langle\psi|(R_X + L_X) \;=\; 0 \;. \tag{3.15}$$

These two linear conditions resemble the so-called Ishibashi conditions in boundary conformal field theory. In the minisuperspace theory, it is easy to describe the space of solutions. One may check by a short computation that a generic invariant takes the form

$$|e,n\rangle\!\rangle_0 \;=\; \frac{1}{2\pi\sqrt{e}}\big(e_0(e,n) - e_0(e,n-1) + e e_2(e,n)\big) \quad . \tag{3.16}$$

The pre-factor $1/2\pi\sqrt{e}$ is determined by a normalization condition to be spelled out below. We note that the function $|e,n\rangle\!\rangle_0$ is obtained by taking the super-trace of supergroup elements in the typical representation $\langle e,n\rangle$.[3] To each of the invariants $|e,n\rangle\!\rangle_0$ we can assign a co-invariant ${}_0\langle\!\langle e,n| : \mathrm{L}2 \to \mathbb{C}$ through

$${}_0\langle\!\langle e,n| \;=\; \int d\mu\, \frac{1}{2\pi\sqrt{e}}\big(e_0(-e,-n+1) - e_0(-e,-n) - e e_2(-e,-n+1)\big) \ . \tag{3.17}$$

Our normalization of both $|e,n\rangle\!\rangle_0$ and the dual invariant ${}_0\langle\!\langle e,n|$ ensures that

$${}_0\langle\!\langle e,n|(-1)^F u_1^{\frac{1}{2}(L_E-R_E)} u_2^{\frac{1}{2}(L_N-R_N)}|e',n'\rangle\!\rangle_0 \;=\; \delta(n'-n)\,\delta(e'-e)\,\chi_{\langle e,n\rangle}(u_1,u_2)$$

where $\chi_{\langle e,n\rangle}(u_1,u_2) = u_1^e\left(u_2^{n-1} - u_2^n\right)$ is the super-character of the typical representation $\langle e,n\rangle$ of gl(1|1). If we re-scale the invariants $|e,n\rangle\!\rangle_0$ and then send e to zero we obtain another family of invariants,

$$|0,n\rangle\!\rangle_0 \;:=\; \lim_{e\to 0}\sqrt{e}\,|e,n\rangle\!\rangle_0 \;=\; e_0(0,n) - e_0(0,n-1) \quad . \tag{3.18}$$

Similarly, we define the dual ${}_0\langle\!\langle 0,n|$ as a limit of ${}_0\langle\!\langle -e,-n+1|\sqrt{e}$. By construction, the states $|0,n\rangle\!\rangle_0$ and the associated linear forms possess vanishing overlap with each other and with the states $|e,n\rangle\!\rangle_0$,

$${}_0\langle\!\langle 0,n|u_1^{\frac{1}{2}(L_E-R_E)} u_2^{\frac{1}{2}(L_N-R_N)}|e',n'\rangle\!\rangle_0 \;=\; 0 \tag{3.19}$$

for all e', including $e'=0$. This does certainly not imply that ${}_0\langle\!\langle 0,n|$ acts trivially on the space of functions.

It is easy to see that the functions $|0,n\rangle\!\rangle_0$ do not yet span the space of invariants. What we are missing is a family of additional states $|n\rangle\!\rangle_0$ which is given by

$$|n\rangle\!\rangle_0 \;=\; \frac{1}{2\pi}\,e_0(0,n) \quad \text{for} \quad n \;\in\; [0,1[\ .$$

[3] Our conventions for the representation theory of gl(1|1) are the same as in [48]. In particular, $\langle e,n\rangle$ denotes a 2-dimensional graded representation of gl(1|1). Let us agree to consider the state with smaller N-eigenvalue as even (bosonic). The same representation with opposite grading shall receive an additional prime, i.e. it is denoted by $\langle e,n\rangle'$.

The corresponding dual co-invariants are given by the prescription

$$ {}_0\langle\!\langle n| \;=\; \frac{1}{2\pi}\int d\mu \sum_{m\in\mathbb{Z}} e_2(0,-n+m+1) \quad . \tag{3.20} $$

Our normalization ensures that

$$ {}_0\langle\!\langle n|(-1)^F u_1^{\frac{1}{2}(L_E-R_E)} u_2^{\frac{1}{2}(L_N-R_N)}|n'\rangle\!\rangle_0 \;=\; \delta(0)\,\delta(n'-n)\,\chi_{\langle n\rangle}(u_1,u_2) \tag{3.21} $$

where $\chi_{\langle n\rangle}(u_1,u_2) = u_2^n$. The divergent factor $\delta(0)$ stems from the infinite volume of our target space and it could absorbed into the normalization of the Ishibashi state. Let us observe that the co-invariants ${}_0\langle\!\langle n|$ may be obtained by a limiting procedure from ${}_0\langle\!\langle e,n|$,

$$ {}_0\langle\!\langle n| \;=\; -\lim_{e\to 0}\frac{1}{\sqrt{e}}\sum_m {}_0\langle\!\langle e,n+m| \quad . \tag{3.22} $$

A similar construction can be performed with the Ishibashi states $|e,n\rangle\!\rangle_0$ to give the formal invariants $\sum_m e_2(0,n+m)$. They are formally dual to co-invariants given by $\int d\mu e_0(0,-n+1)$. In our discussion, and in particular when we wrote eq. (3.20), we have implicitly equipped Ł2 with a topology that excludes to consider $\sum_m e_2(0,n+m)$ as a true function. While the dual functional $\int d\mu e_0(0,-n+1)$ does not suffer from any such problem, it so happens not to appear in the construction of boundary states. This is why we do not bother giving it a proper name.

It is our aim now to determine the coupling of bulk modes to branes in the minisuperspace limit. In the particle limit, the bulk 1-point functions are linear functionals $f\mapsto\langle f\rangle$ on the space Ł2 of functions such that $\langle \mathrm{ad}_X f\rangle = 0$, i.e. they are co-invariants. The first family of co-invariants we shall describe corresponds to branes in generic positions (z_0,y_0). Since these are localized at a point (z_0,y_0) on the bosonic base and delocalized along the fermionic directions, their density is given by

$$ \begin{aligned} \rho_{(z_0,y_0)} =&\; -2i\sin(y_0/2)\,\delta(y-y_0)\,\delta(z-z_0) \\ =&\; -2i\sin(y_0/2)\,\delta(y-y_0)\,\delta\big(x-i\eta_-\eta_+(1-e^{-iy})^{-1}-z_0\big) \quad . \end{aligned} \tag{3.23} $$

The constant prefactor $-2i\sin(y_0/2)$ was chosen simply to match the normalization of our boundary states below. Obviously, the density $\rho_{(z_0,y_0)}$ is invariant under the adjoint action. It gives rise to a family of co-invariants through the prescription

$$ f\mapsto\langle f\rangle_\rho \;:=\; \int d\mu\,\rho(x,y,\eta_\pm)\,f(x,y,\eta_\pm) \quad . \tag{3.24} $$

Geometrically, the integral computes the strength of the coupling of a bulk mode f to a brane with mass density ρ. It is not difficult to check that our functional $\langle\cdot\rangle_{(z_0,y_0)}$ admits an expansion in terms of dual Ishibashi states as follows,

$$\begin{aligned}\langle\ \cdot\ \rangle_{(z_0,y_0)} \equiv\ {}_0\langle z_0, y_0| \ &= \int dedn\,\sqrt{e}\,e^{i(n-1/2)y_0+iz_0e}\ {}_0\langle\langle e,n| \\ &= \int_{e\neq 0} dedn\,\sqrt{e}\,e^{i(n-1/2)y_0+iz_0e}\ {}_0\langle\langle e,n| + \int dn\,e^{i(n-1/2)y_0}\ {}_0\langle\langle 0,n| \quad .\end{aligned} \tag{3.25}$$

In the second line of this formula we have separated typical and atypical contributions to the boundary state. Considering that the state ${}_0\langle\langle 0,n|$ is obtained through the limiting procedure ${}_0\langle\langle 0,n| = \lim_{e\to 0}\sqrt{e}\ {}_0\langle\langle e,n|$, the second term is the natural continuation of the first. In this sense, we may drop the condition $e\neq 0$ in the first integration and combine typical and atypical terms into the single integral appearing in the first line. We observe that all $\langle\cdot\rangle_{(z_0,y_0)}$ vanish on functions $e_0(e,n)$ with $e=0$.

Let us now turn to the non-generic branes. These are localized also in the fermionic directions. Hence, their density takes the form

$$\rho^s_{z_0} \ = \ (-1)^s\,\delta(y-2\pi s)\,\delta(x-z_0)\,\delta(\eta_+)\,\delta(\eta_-) \tag{3.26}$$

where s is an integer. When this density is inserted into the general prescription (3.24), we obtain another family of co-invariants. Its expansion in terms of Ishibashi states reads

$$\begin{aligned}\langle\ \cdot\ \rangle^s_{z_0} = \ {}_0\langle z_0; s| \ &= \int dedn\,\frac{1}{\sqrt{e}}\,e^{2\pi i(n-1/2)s+iez_0}\ {}_0\langle\langle e,n| \\ &= \int_{e\neq 0} dedn\,\frac{1}{\sqrt{e}}\,e^{2\pi i(n-1/2)s+iez_0}\ {}_0\langle\langle e,n| - \int_0^1 dn\,e^{2\pi i(n-1/2)s}\ {}_0\langle\langle n| \quad .\end{aligned} \tag{3.27}$$

Once more, the second line displays typical and atypical contributions to the boundary state separately. In passing from the first to the second line, we exploited $s\in\mathbb{Z}$ along with our observation (3.22).

The two families $\langle\cdot\rangle_{(z_0,y_0)}$ with $y_0\neq 2\pi s$ and $\langle\cdot\rangle^s_{z_0}$ are not entirely independent. In fact, we note that boundary states from the generic family may be 're-expanded' in terms of members from the non-generic family when the paremeter y_0 tends to $2\pi s$. The precise relation is

$$\lim_{y_0\to 2\pi s}\langle f\rangle_{(z_0,y_0)} \ = \ \frac{1}{i}\frac{\partial}{\partial z_0}\langle f\rangle^s_{z_0} \tag{3.28}$$

for all elements $f \in Ł2$. We shall find that both families of co-invariants can be lifted to the full field theory. An analogue of relation (3.28) also holds in the field theory. It tells us that, for special values of the parameters, branes from the generic family decompose into a superposition of two branes from the non-generic family. Their distance is finite for finite level but tends to zero as k is sent to infinity.

4 Untwisted Boundary States and Their Spectra

We are now prepared to spell out the boundary states and boundary spectra for maximally symmetric branes with trivial gluing conditions. As we have argued in the previous section, they come in two different families. After a few comments on the relevant Ishibashi states, we construct the boundary states for branes in generic positions in the second subsection. Branes in non-generic position are constructed in the third part of this section.

4.1 Characters and Ishibashi states

In this subsection we shall provide a list of untwisted Isibashi states from which the boundary states of the GL(1|1) WZNW model will be built in consecutive subsections. By definition, an untwisted Ishibashi state is a solution of the following set of linear relations

$$\big(X_n + \bar{X}_{-n}\big)\,|\Psi\rangle\!\rangle \;=\; 0 \quad \text{ for } \quad X = E, N, \Psi^{\pm} \; . \tag{4.1}$$

These relation lift our invariance conditions (3.15) from the particle model to the full field theory. It is obvious that solutions must be in one-to-one correspondence to invariants in the minisuperspace theory.

To begin with, there exists a 2-parameter family of typical Ishibashi states $|e, n\rangle\!\rangle$ with $e \neq mk$ and $n \in \mathbb{R}$. They can be uniquely characterized by their relative overlaps

$$\langle\!\langle e, n|(-1)^{F^c} q^{L^c_0 - \frac{c}{24}} u^{N^c_0} |e', n'\rangle\!\rangle \;=\; \delta(n' - n)\delta(e' - e)\; \chi_{\langle e,n\rangle}(u, q) \tag{4.2}$$

where $L^c_0 = (L_0 + \bar{L}_0)/2, N^c_0 = (N_0 - \bar{N}_0)/2$ and $\hat{\chi}_{\langle e,n\rangle}$ denotes the unspecialized super-characters for typical representations. It takes the form

$$\hat{\chi}_{\langle e,n\rangle}(u, q) \;=\; u^{n-1} q^{\frac{e}{2k}(2n-1+e/k)+1/8}\, \theta\Big(\mu - \frac{1}{2}(\tau + 1), \tau\Big)/\eta(\tau)^3$$

where μ is related to u by $u = \exp(2\pi i\mu)$ and similarly for $q = \exp(2\pi i\tau)$, as usual. In comparison to the minisuperspace theory we have set $u_1 = 1$ and $u_2 = u$. Since E_0

and $\bar{E}_0$ are central the dependence on u_1 can be re-introduced simply by multiplying the character functions with u_1^e. When e is a multiple of the level, $\hat{\chi}_{\langle e,n\rangle}$ are the characters of reducible representations which contain two atypical irreducible building blocks. As in the particle theory, we shall also define $|mk,n\rangle\rangle$ and $\langle\langle mk,n|$ by a limiting procedure,

$$|mk,n\rangle\rangle \;=\; \lim_{e\to mk} \sin^{1/2}(\pi e/k)|e,n\rangle\rangle \quad , \quad \langle\langle mk,n| \;=\; \lim_{e\to mk} \sin^{1/2}(\pi e/k)\langle\langle e,n| \quad . \qquad (4.3)$$

The Ishibashi states $|0,n\rangle\rangle$ possess vanishing overlap among each other and with the typical Ishibashi states.

In addition, we introduce a family of atypical Ishibashi states $|n\rangle\rangle^{(m)}$ and ${}^{(m)}\langle\langle n|$ for $n \in [0,1[, m \in \mathbb{Z}$. These correspond to the states $|n\rangle\rangle_0$ and ${}_0\langle\langle n|$ that appeared in our discussion of the particle limit. Once more, we may characterize the Ishibashi states by their overlaps

$${}^{(m)}\langle\langle n|(-1)^{F^c} q^{L_0^c-\frac{c}{24}} u^{N_0^c}|n'\rangle\rangle^{(m)} \;=\; \delta(n'-n)\delta(m-m')\ \hat{\chi}^{(m)}_{\langle n\rangle}(u,q) \quad . \qquad (4.4)$$

Here, $\hat{\chi}^{(m)}_{\langle n\rangle}$ denotes the unspecialized super-character of the atypical representation $\langle n\rangle^{(m)}$, see Appendix A.3 for details, i.e.

$$\hat{\chi}^{(m)}_{\langle n\rangle}(u,q) \;=\; \frac{u^n}{1-zq^m}\,\frac{q^{\frac{m}{2}(m+2n+1)+1/8}\theta\Big(\mu-\frac{1}{2}(\tau+1),\tau\Big)}{\eta(\tau)^3} \quad . \qquad (4.5)$$

It is important to stress that most atypical states are obtained in eqs. (4.3) as limits of typical Ishibashi states.

To summarize, we have constructed a family of Ishibashi states $|e,n\rangle\rangle, e,n \in \mathbb{R}$, one for each Kac module of the affine current algebra $\widehat{\mathrm{gl}}(1|1)$. In addition, there is one 'small' family of Ishibashi states $|n\rangle\rangle^{(m)}$ with $m \in \mathbb{Z}$ and $n \in [0,1[$. This second set of states is in one-to-one correspondence with the set of atypical blocks of $\widehat{\mathrm{gl}}(1|1)$.[4]

4.2 The generic boundary state

In this section, we propose the boundary state corresponding to a generic brane localized at (z_0,y_0) with $y_0 \neq 2\pi s$ and perform a non-trivial Cardy consistency check [44]. Therefore, we need to know the modular properties of the characters. They are easily computed with the help of [49] and we list them in appendix A.4.

[4] Two atypical irreducibles π and π' are said to be part of the same block if there exists a sequence of irreps $\pi_0 = \pi, \pi_1, \ldots, \pi_{N-1}, \pi_N = \pi'$ such that any pair π_i, π_{i+1} of consecutive irreps in the sequence appears in the composition series of some indecomposable. The two $\widehat{\mathrm{gl}}(1|1)$ representations $\langle n\rangle^{(m)}$ and $\langle n\rangle^{(m)}$ are part of the same block whenever $m = m'$ and $n - n' \in \mathbb{Z}$.

Proposition 4.1. (Generic boundary state) *The boundary state of branes associated with generic position parameters* z_0*,* y_0 *is*

$$|z_0, y_0\rangle \;=\; \sqrt{\frac{2i}{k}} \int dedn\;\; \exp\bigl(i(n-1/2)y_0 + iez_0\bigr)\;\; \sin^{1/2}(\pi e/k)\;\; |e,n\rangle\!\rangle\;. \qquad (4.6)$$

We shall argue below that these boundary states give rise to elementary branes if and only if the parameter $y_0 \notin 2\pi\mathbb{Z}$*.*

Before we show that our Ansatz for the generic boundary states produces the expected boundary spectrum, let us make a few comments. To begin with, it is instructive to compare the coefficients of the Ishibashi states in $|z_0, y_0\rangle$ with the minisuperspace result eq. (3.25). If we send k to infinity, the factor $\sin^{1/2}(\pi e/k)$ is proportional to the factor $\sqrt{e}$ that appears in the 1-point coupling of bulk modes in the minisuperspace theory. The replacement $\sqrt{e} \to \sin^{1/2}(\pi e/k)$ is necessary to ensure that the field theory couplings are invariant under spectral flow. Let us also stress that the integration in formula (4.6) extends over all e, including $e = mk$. Using our Ishibashi states $|mk, n\rangle\!\rangle$ from eq. (4.3), we may rewrite the generic boundary states as

$$\begin{aligned} |z_0, y_0\rangle \;=\; & \sqrt{\frac{2i}{k}} \int_{e\neq mk} dedn\;\; \exp\bigl(i(n-1/2)y_0 + iez_0\bigr)\;\; \sin^{1/2}(\pi e/k)\;\; |e,n\rangle\!\rangle \\ & + \sqrt{\frac{2i}{k}} \sum_m \int dn \exp\bigl(i(n-1/2)y_0 + imkz_0\bigr)\;\; |mk,n\rangle\!\rangle \quad . \end{aligned}$$

The second line displays explicitly how closed string states in atypical representations couple to generic branes.

In order to check the consistency of our proposal for the boundary states with worldsheet duality, we compute the spectrum between a pair of generic branes,

$$\begin{aligned} \langle z_0, y_0|(-1)^{F^c} \tilde{q}^{L_0^c} \tilde{z}^{N_0^c} |z_0', y_0'\rangle \;&=\; \tfrac{2i}{k} \textstyle\int de' dn' e^{i(n'-\frac{1}{2})(y_0'-y_0)+ie'(z_0'-z_0)} \sin(\pi e'/k) \hat{\chi}_{\langle e',n'\rangle}(\tilde{\mu},\tilde{\tau}) \\ &=\; \hat{\chi}_{\langle e,n\rangle}(\mu,\tau) \;-\; \hat{\chi}_{\langle e,n+1\rangle}(\mu,\tau) \qquad (4.7) \end{aligned}$$

where the momenta e, n are related to the coordinates of the branes according to

$$e \;=\; \frac{k(y_0'-y_0)}{2\pi} \quad , \quad n \;=\; \frac{k(z_0'-z_0)}{2\pi} - \frac{y_0'-y_0}{2\pi}\;.$$

To begin with, the result is a combination of characters with integer coefficients. Hence, it can be consistently interpreted as the partition function for open strings that stretch in between the two branes. If we put both branes into the same position (z_0, y_0), then the result specializes to

$$\langle z_0, y_0|(-1)^{F^c}\tilde{q}^{L_0^c}\tilde{u}^{N_0^c}|z_0, y_0\rangle \;=\; \hat{\chi}_{\langle 0,0\rangle}(\mu,\tau) \;-\hat{\chi}_{\langle 0,1\rangle}(\mu,\tau) \;=\; \hat{\chi}_{\mathcal{P}_0}(\mu,\tau). \tag{4.8}$$

In the last step we have observed that the super-characters of the representation spaces over the two atypical Kac modules $\langle 0,0\rangle$ and $\langle 0,1\rangle'$ combine into the character of the representation that is generated from the projective cover $\mathcal{P}_0$. This outcome was expected: it signals that the state space of open strings on a generic branes contains no bosonic zero modes and two fermionic ones. The latter give rise to the four ground states of the projective cover. This is in agreement with the fact that generic branes stretch out along the fermionic directions.

There is one important subtlety in our interpretation of the result (4.8) that we do not want to gloss over. While the character of the projective cover $\hat{\mathcal{P}}_0$ is the same as that of the two affine Kac modules, the corresponding representations are not. The characters are blind against the nilpotent parts in L_0 and hence they cannot distinguish between an indecomposable and its composition series. But for the conformal field theory, the difference is important. In particular, the generator L_0 is diagonalizable on all Kac modules, atypical or not, but it has a nilpotent contribution in the $\widehat{\mathrm{gl}}(1|1)$-module over $\mathcal{P}_0$. Hence, if the boundary spectrum does transform in $\hat{\mathcal{P}}_0$, then some boundary correlators are guaranteed to display logarithmic singularities when two boundary coordinates come close to each other. The information we obtained from the boundary states using world-sheet duality alone is not sufficient to make any rigorous statements on the existence of such logarithms. But in the minisuperspace limit $k \to 0$ we have clearly identified the projective cover $\mathcal{P}_0$ as the relevant structure. Since L_0 is not diagonalizable in that limit, it cannot be so for finite level k.

4.3 Non generic point-like branes

Let us now turn to the boundary states of non-generic untwisted branes in the GL(1|1) WZNW model. From our discussion of the geometry we expect them to be parametrized by a single real modulus z_0 and to possess a spectrum without any degeneracy of ground states. These expectations will be met. Let us begin by spelling out the formula for the non-generic boundary states.

Proposition 4.2. (Non-generic boundary states) *The boundary states of elementary branes associated with non-generic position parameters z_0 and $y_0 = 2\pi s, s \in \mathbb{Z}$, are given by*

$$|z_0; s\rangle \;=\; \frac{1}{\sqrt{2ki}} \int dedn\ \exp\bigl(2\pi i(n-1/2)s + iez_0\bigr)\ \sin^{-1/2}(\pi e/k)\ |e,n\rangle\!\rangle\ . \tag{4.9}$$

If we send the level k to infinity in the boundary states $|z_0; s\rangle$, then the coefficient of the Ishibashi state $|e, s\rangle\!\rangle$ gets replaced by $1/\sqrt{e}$ and thereby it reproduces the coupling (3.27) of bulk modes in the minisuperspace theory. Once more, the replacement $1/\sqrt{e} \mapsto \sin^{-1/2}(\pi e/k)$ is necessary to ensure spectral flow symmetry of the field theoretic couplings.

Just like their cousins $|z_0; s\rangle_0$ in minisuperspace (see eq. (3.27)), the boundary states $|z_0; s\rangle$ couple to atypical Ishibashi states, though this is again somewhat hidden in our notations. We can make this coupling more explicit by rewriting $|z_0; s\rangle$ in the form,

$$\begin{aligned} |z_0; s\rangle = \frac{1}{\sqrt{2ki}} \int_{e\neq mk} & dedn\ \exp\bigl(2\pi i(n-1/2)s + iez_0\bigr)\ \sin^{-1/2}(\pi e/k)\ |e,n\rangle\!\rangle \\ & - \frac{1}{\sqrt{2ki}} \sum_m \int_0^1 dn\ \exp\bigl(2\pi i(n-1/2)s + imkz_0\bigr)\ |n\rangle\!\rangle^{(m)}\ . \end{aligned} \tag{4.10}$$

Note that the non-generic boundary states only involve to the special family $|n\rangle\!\rangle^{(m)}$ of atypical Ishibashi states. In case of generic boundary states, we had found non-vanishing couplings to the regular atypical Ishibashi states $|mk, n\rangle\!\rangle$.

Let us verify that the proposed boundary states produce a consistent open string spectrum. In order to do so, we investigate the overlap between two non-generic boundary states $|z_0; s\rangle$ and $|z_0'; s'\rangle$,

$$\begin{aligned} \langle z_0; s|(-1)^{F^c} \tilde{q}^{L_0^c} \tilde{z}^{N_0^c} |z_0'; s'\rangle \;&=\; \int \frac{de'dn'}{2ki} \frac{e^{2\pi i(n'-1/2)(s'-s)+ie'(z_0'-z_0)}}{\sin(\pi e'/k)}\ \hat{\chi}_{\langle e',n'\rangle}(\tilde{\mu}, \tilde{\tau}) \\ &=\; \hat{\chi}^{(m)}_{\langle n\rangle}(\mu, \tau) \end{aligned} \tag{4.11}$$

where the labels n and m in the character are related to the branes' parameters through

$$n \;=\; \frac{k(z_0'-z_0)}{2\pi} + s - s' \quad , \quad m = s' - s\ . \tag{4.12}$$

$\hat{\chi}^{(m)}_{\langle n\rangle}$ are characters of atypical irreducible representation of $\widehat{\mathrm{gl}}(1|1)$. For $m = 0$ the corresponding representations are generated from the 1-dimensional irreducible atypical representations $\langle n\rangle$ of the finite-dimensional Lie superalgebra gl(1|1) by application of current algebra modes. The representations with $m \neq 0$ are obtained from those with $m = 0$ by spectral flow (see Appendix A).

We also want to look at the spectrum of boundary operators that can be inserted on a boundary if we impose non-generic boundary conditions with parameters z_0 and s. Specializing eq. (4.11) to the case with $z_0' = z_0$ and $s' = s$ we find

$$\langle z_0; s|(-1)^{F^c}\tilde{q}^{L_0^c}\tilde{u}^{N_0^c}|z_0; s\rangle \;=\; \hat{\chi}^{(0)}_{\langle 0\rangle}(\mu, \tau) \quad .$$

Hence, the spectrum consists of states that are generated from a single invariant ground state $|0\rangle$ by application of current algebra modes with negative mode indices. In particular, the zero modes of the fermions act trivially on ground states. This is in agreement with our geometric insights according to which non-generic branes are localized in all directions, including the two fermionic ones.

We may now ask what happens if we send the parameter y_0 of the generic brane to $y_0 = 2\pi s$. From our formulas for boundary states we deduce that

$$|z_0, 2\pi s\rangle \;=\; \int \frac{dedn}{\sqrt{2ki}}\, \frac{e^{ie(z_0+\frac{\pi}{k})} - e^{ie(z_0-\frac{\pi}{k})}}{\sin^{1/2}(\pi e/k)}\, e^{2\pi i(n-1/2)s}\,|e, n\rangle\!\rangle \;=\; |z_0 + \pi/k; s\rangle - |z_0 - \pi/k; s\rangle \; .$$

In other words, when a generic brane is moved onto one of the special lines $y_0 = 2\pi s$, it decomposes into a brane-anti-brane pair. Its constituents sit in positions $z_0 \pm \pi/k$ and possess the same discrete parameter s. This relation between non-generic branes and generic branes in non-generic positions is a field theoretic analogue of the equation (3.28) we discovered in the minisuperspace theory.

5 Comparison with Cardy's Theory

Let us recall a few rather basis facts concerning branes in rational unitary conformal field theory. For simplicity we shall restrict to cases with a charge conjugate modular invariant and a trivial gluing automorphism Ω (the so-called 'Cardy case'). This will allow a comparison with the results of the previous subsections. In the Cardy case, elementary boundary conditions turn out to be in one-to-one correspondence with the irreducible representations of the chiral algebra [44]. Let us label these by J, with $J = 0$ being reserved for the vacuum representation. The boundary condition with label $J = 0$ has a rather simple spectrum containing only the vacuum representation $\mathcal{H}_0$. More generally, if we impose the boundary condition $J = 0$ on one side of the strip and any other elementary boundary condition on the other, the spectrum consists of a single irreducible $\mathcal{H}_J$. Finally, the spectrum between two boundary conditions with label J_1 and J_2 is determined by the fusion of J_1 and J_2. We shall now discuss that all these statements carry over to untwisted branes in the GL(1|1) WZNW model. The fusion procedure, however, can provide spectra containing indecomposables that are not irreducible.

5.1 Brane parameters and representations

We proposed that the GL(1|1) WZNW model possesses two families of elementary branes. The first one is referred to as the generic family and its members are parametrized by (z_0, y_0) with $y_0 \neq 2\pi s, s \in \mathbb{Z}$. Boundary states for the generic branes were provided in subsection 4.2. These are also defined for integer $y_0/2\pi$ but we have argued that the corresponding branes are not elementary. They rather correspond to superpositions of branes from the second family. This second family consists of branes with only one continuous modulus z_0 and a discrete parameter s. Their boundary states can be found in subsection 4.3.

There is one distinguished brane in this second family with $z_0 = 0$ and $s = 0$. We propose that it plays the role of the $J = 0$ brane in rational conformal field theory. In order to confirm this idea, we compute the spectrum of open strings stretching between $z_0 = 0, s = 0$ and any of the other elementary branes. If the second brane is non-generic with parameters z_0, s, the relative spectrum reads

$$\langle 0; 0|(-1)^{F^c} \tilde{q}^{L_0^c} \tilde{u}^{N_0^c} |z_0; s\rangle \ = \ \hat{\chi}^{(m)}_{\langle n\rangle}(\mu, \tau) \tag{5.1}$$

where the parameter n on the character is

$$n \ = \ n(z_0; s) \ = \ \frac{kz_0}{2\pi} - s \quad , \quad m \ = \ m(z_0; s) \ = \ s \ . \tag{5.2}$$

Indeed, we see that the open string spectrum corresponds to a single irreducible atypical module of $\widehat{\mathrm{gl}}(1|1)$, in agreement with the expectations from rational conformal field theory.

Let us now consider the case in which the second brane is located in a generic position (z_0, y_0). From the boundary state we find

$$\langle 0; 0|(-1)^{F^c} \tilde{q}^{L_0^c} \tilde{u}^{N_0^c} |z_0, y_0\rangle \ = \ \hat{\chi}_{\langle e,n\rangle}(\mu, \tau) \ , \tag{5.3}$$

where the parameters of the character on the right hand side are

$$e \ = \ e(z_0, y_0) \ = \ \frac{ky_0}{2\pi} \quad , \quad n \ = \ n(z_0, y_0) \ = \ \frac{kz_0}{2\pi} - \frac{y_0}{2\pi} + \frac{1}{2} \ . \tag{5.4}$$

As long as $y_0/2\pi$ is not an integer, e is not a multiple of the level and therefore, $\hat{\chi}_{\langle e,n\rangle}$ is the character of a single irreducible representation of $\widehat{\mathrm{gl}}(1|1)$.

At this point we have found that all our elementary branes are labelled by irreducible representations of $\widehat{\mathrm{gl}}(1|1)$. In case of the elementary generic branes, the relation between the position moduli $(z_0, y_0), y_0 \neq 2\pi m$, and representation labels $\langle e, n\rangle, e \neq mk$, is provided by eq. (5.4). All typical irreducible representations of $\widehat{\mathrm{gl}}(1|1)$ appear in this correspondence. For the non-generic branes the relation between their parameters $(z_0; s)$ and the representation labels of an atypical irreducible can be found in eq. (5.2). Once more, all atypical irreducibles appear in this correspondence. Hence, branes in the GL(1|1) WZNW model are in one-to-one correspondence with irreducible representations of the current superalgebra $\widehat{\mathrm{gl}}(1|1)$, just as in rational conformal field theory.

5.2 Brane spectra and fusion

Let us now analyze whether we can find the spectrum between a pair of elementary branes through fusion of the corresponding current algebra representations. For the convenience of the reader we have listed the relevant fusion rules for irreducible representations of the current superalgebra $\widehat{\mathrm{gl}}(1|1)$ in Appendix A.5.

The spectrum between two typical branes with parameters (z_0, y_0) and (z_0', y_0') has been computed in eq. (4.7). We can convert the brane parameters into representation labels with the help of eq. (5.4) and then exploit the known fusion product of the corresponding representations. In case $y_0' - y_0 \neq 2\pi\mathbb{Z}$ we find

$$\left\langle \frac{ky_0}{2\pi}, \frac{kz_0}{2\pi} - \frac{y_0}{2\pi} + \frac{1}{2} \right\rangle^* \otimes_F \left\langle \frac{ky_0'}{2\pi}, \frac{kz_0'}{2\pi} - \frac{y_0'}{2\pi} + \frac{1}{2} \right\rangle \tag{5.5}$$

$$\cong \left\langle \frac{k(y_0'-y_0)}{2\pi}, \frac{k(z_0'-z_0)}{2\pi} - \frac{y_0'-y_0}{2\pi} + 1 \right\rangle \oplus \left\langle \frac{k(y_0'-y_0)}{2\pi}, \frac{k(z_0'-z_0)}{2\pi} - \frac{y_0'-y_0}{2\pi} \right\rangle'$$

Here, $\otimes_F$ denotes the fusion product and we used the rule $\langle e, n\rangle^* = \langle -e, -n+1\rangle'$ for the conjugation of representations. Then we inserted the known fusion rules while keeping track of whether the representation is fermionic or bosonic. The result agrees nicely with the true spectrum we computed earlier.

When the difference $(y_0' - y_0)/2\pi = m$ is an integer, the fusion of the two representations on the left hand side of (5.5) results in a single indecomposable. It is the image of the affine representation over the projective cover $\hat{\mathcal{P}}_{(k(z_0'-z_0)-(y_0'-y_0))/2\pi}$ under m units of spectral flow, i.e.

$$\left\langle \frac{ky_0}{2\pi}, \frac{kz_0}{2\pi} - \frac{y_0}{2\pi} + \frac{1}{2} \right\rangle^* \otimes_F \left\langle \frac{ky_0'}{2\pi}, \frac{kz_0'}{2\pi} - \frac{y_0'}{2\pi} + \frac{1}{2} \right\rangle \; = \; \left(\mathcal{P}^{(m)}_{(k(z_0'-z_0)-(y_0'-y_0))/2\pi} \right)' \tag{5.6}$$

where $m = (y_0' - y_0)/2\pi$. Our minisuperspace theory along with the boundary states confirm this result in the case $y_0 = y_0'$ and $z_0 = z_0'$ (see our discussion at the end of section 4.2). For other choices of the parameters, we only see that the fusion rules provide a representation with the correct character. Whether the true state space is given by a single indecomposable or by a sum of Kac modules or even irreducibles cannot be resolved rigorously with the methods we have at our disposal. Nevertheless, it seems very likely that the projective cover is what appears since this is the only result which is also consistent with spectral flow symmetry.

The fusion between atypical irreducibles is rather simple. It leads to a prediction for the spectrum between two non-generic branes that should be checked against our earlier result (4.11),

$$\left(\left\langle \frac{kz_0}{2\pi} - s \right\rangle^{(s)} \right)^* \otimes_F \left\langle \frac{kz_0'}{2\pi} - s' \right\rangle^{(s')} \; \cong \; \left\langle \frac{k(z_0' - z_0)}{2\pi} + s - s' \right\rangle^{(s'-s)} \; .$$

Once more, the findings from world-sheet duality are consistent with the fusion prescription. There is one final check to be performed. It concerns the spectrum between a non-generic brane with parameters $(z_0; s)$ and a generic one with moduli (z_0, y_0). From the fusion we find

$$\left(\left\langle \frac{kz_0}{2\pi} - s \right\rangle^{(s)} \right)^* \otimes_F \left\langle \frac{ky_0'}{2\pi}, \frac{kz_0'}{2\pi} - \frac{y_0'}{2\pi} + \frac{1}{2} \right\rangle \; = \; \left\langle -sk + \frac{ky_0'}{2\pi}, \frac{k(z_0' - z_0)}{2\pi} - \frac{y_0'}{2\pi} + s + \frac{1}{2} \right\rangle \; . \tag{5.7}$$

It may not come as a big surprise that this fusion rule correctly predicts the spectrum between a generic and a non-generic brane. In fact, from our formulas for boundary states and modular transformation we find

$$\begin{aligned} \left\langle z_0; s \right| (-1)^{F^c} \tilde{q}^{L_0^c} \tilde{u}^{N_0^c} \left| z_0', y_0' \right\rangle \; &= \; \hat{\chi}_{\langle e,n \rangle}(\mu, \tau) \\ \text{where} \quad e \; = \; -ks + \frac{ky_0'}{2\pi} \quad , \quad n \; &= \; \frac{k(z_0' - z_0)}{2\pi} - \frac{y_0'}{2\pi} + s + \frac{1}{2} \; . \end{aligned} \tag{5.8}$$

In conclusion we found that the spectra between any pair of elementary branes may be determined by the fusion of the corresponding irreducible representations. It is important to stress that the fusion product of irreducible representations can produce representations that are not fully reducible.

6 Twisted Brane: Geometry and Boundary State

This final section contains a brief discussion of twisted branes. By definition, twisted branes in the gl(1|1) model preserve one copy of the affine Lie superalgebra $\widehat{\mathrm{gl}}(1|1)$. The construction of the relevant generators differs from the case of untwisted branes by the action of an outer (gluing) automorphism Ω on anti-holomorphic bulk currents. We shall find that there is a single twisted brane boundary condition corresponding to a brane which extends in both bosonic and fermionic directions. As for untwisted branes, we shall first extract the brane's geometry from the gluing conditions. Thereafter, we study the unique Ishibashi and boundary state in the particle limit. Finally, the minisuperspace results are lifted to the full field theory.

In the case of the automorphism Ω, we can easily bring the associated gluing conditions (2.7) for super-currents into the form

$$\begin{aligned} \partial_n Y = \ & 0 \qquad , \quad \partial_n \bar{\xi} = \ i e^{-iY} \partial_p \xi \\ \partial_n X - \ 2ie^{iY} \xi \partial_n \bar{\xi} = \ & 0 \qquad , \quad \partial_n \xi = \ -ie^{iY} \partial_p \bar{\xi} \ , \end{aligned} \tag{6.1}$$

for all $z = \bar{z}$. Here, we have redefined the fermionic fields $\xi = \frac{e^{iY}}{2}(c_+ + c_-)$ and $\bar{\xi} = \frac{1}{2}(c_- - c_+)$. The bosonic fields, on the other hand, remain unaltered. This parametrization is motivated by a new choice of coordinates on the supergroup GL(1|1)

$$\begin{aligned} g \ = \ e^{ic_-\Psi^-} e^{ixE+iyN} e^{ic_+\Psi^+} \ &= \ e^{i\bar{\xi}\Psi^-} e^{-i\xi\Psi^+} e^{izE+iyN} e^{-i\xi\Psi^-} e^{-i\bar{\xi}\Psi^+} \\ &= \ \Omega(e^{i\bar{\xi}\Psi^+} e^{i\xi\Psi^-}) e^{izE+iyN} e^{-i\xi\Psi^-} e^{-i\bar{\xi}\Psi^+} \end{aligned} \tag{6.2}$$

which is obtained by twisted conjugation of bosonic elements with fermionic ones.

We can re-express the metric and H-field in terms of the new coordinates $x, y, \xi\, \bar{\xi}$,

$$\begin{aligned} ds^2 = \ & 2dxdy + 4d\xi d\bar{\xi} - 4i\xi dy d\bar{\xi}, \\ H = \ & 2ie^{-iy} d\xi \wedge d\xi \wedge dy - 2ie^{iy} d\bar{\xi} \wedge d\bar{\xi} \wedge dy) \ . \end{aligned}$$

Using our expression for the metric we infer the following formula for the B-field from our gluing conditions (6.1),

$$B \ = \ -2e^{-iy} d\xi \wedge d\xi - 2e^{iy} d\bar{\xi} \wedge d\bar{\xi} \ .$$

It is straightforward to verify that that $dB = H$. We conclude that twisted branes are stretched out into all directions of our supergroup.

Consequently, the space of functions on a twisted D-brane is given by Ł2. Since twisted branes admit an action of GL(1|1) the space of functions carries an action of the Lie superalgebra gl(1|1), namely the twisted adjoint action $\mathrm{ad}_X^\Omega = R_X + L_X^\Omega$ where

$$L_E^\Omega \;=\; i\partial_x \;, \quad L_N^\Omega \;=\; i\partial_y + \eta_+\partial_+ \;, \quad L_-^\Omega \;=\; -\partial_+ \;, \quad L_+^\Omega \;=\; e^{-iy}\partial_- - i\eta_+\partial_x \quad .$$

The generators R_X are given by the same formulas as above. Analyzing the representation content of Ł2 we then find three different kinds of representations. These include the typicals $\langle -2k, -2l+1\rangle$ which are generated by $e_0(k,l) = \exp(ikx+ily), \xi e_0(k,l-1)$. We recall that in our conventions for $\langle e,n\rangle$ the state with smaller N eigenvalue is taken to be bosonic. Furthermore, there exist typicals $\langle -2k,-2l+2\rangle'$ generated by $\bar\xi e_0(k,l), e_0(k,l-1)+2k\xi\bar\xi e_0(k,l-1)$. In this case, the state with lower N eigenvalue is fermionic, hence the prime $'$. Finally, representations with vanishing eigenvalue of E decompose into projective covers of atypicals. In summary, under the twisted adjoint action, the space of functions decomposes as

$$Ł2^{\text{twisted}} \;\cong\; \int_{e\neq 0} dedn \Big[\,\langle e,n\rangle \oplus \langle e,n\rangle'\Big] \;\oplus\; \int dn\,\mathcal{P}_n \quad .$$

We see that fermionic and bosonic states with any given eigenvalue of E and N come in pairs. Therefore, the supertrace of $u_1^{L_E^\Omega - R_E} u_2^{L_N^\Omega - R_N}$ vanishes identically.

Concerning the construction of minisuperspace Ishibashi states $|\psi\rangle\!\rangle_0^\Omega$ satisfying the twisted invariance condition

$$\big(R_X + L_X^\Omega\big)\,|\psi\rangle\!\rangle_0^\Omega \;=\; 0 \tag{6.3}$$

we observe that the space of functions on GL(1|1) contains a single element invariant under the twisted adjoint action, namely the constant function

$$|0\rangle\!\rangle_0^\Omega \;=\; e_0(0,0) \quad .$$

Its dual is given by

$${}_0^\Omega\langle\!\langle 0| \;=\; \int d\mu\, e_0(0,0) = \int d\mu \quad .$$

The linear functional ${}_0^\Omega\langle\!\langle 0|$ is indeed the unique twisted co-invariant on GL(1|1). We note that $|0\rangle\!\rangle_0^\Omega$ and ${}_0^\Omega\langle\!\langle 0|$ possess vanishing overlap, i.e.

$${}_0^\Omega\langle\!\langle 0|(-1)^F u_1^{L_E^\Omega - R_E} u_2^{L_N^\Omega - R_N} |0\rangle\!\rangle_0^\Omega \;=\; 0$$

simply because the relevant integrand contains no fermionic zero modes.

Having the semi-classical Ishibashi state at our disposal, we can turn to the boundary state. Our geometric interpretation of twisted branes suggests that their semi-classical density is given by $\rho(x, y, \xi, \bar{\xi}) = 1$, corresponding to a brane that fills the entire target space. We see that

$$\langle \, \cdot \, \rangle^{\Omega} \; = \; {}_0\langle \Omega | \; = \; \int d\mu \; = \; {}^{\Omega}_{0}\langle\!\langle 0 | \quad .$$

All this lifts straightforwardly to the full field theory. We obtain unique Ishibashi states $|0\rangle\!\rangle^{\Omega}$ and ${}^{\Omega}\langle\!\langle 0|$ which we can identify with the boundary states,

$$|\Omega\rangle \; = \; |0\rangle\!\rangle^{\Omega} \quad , \quad \langle \Omega | \; = \; {}^{\Omega}\langle\!\langle 0|$$

just as in the case of Neumann boundary conditions for a free uncompactified boson. The interaction between two such branes is encoded in the overlap

$$\langle \Omega | (-1)^{F^c} \tilde{q}^{L_0^c} u^{N_0^c} | \Omega \rangle \; = \; 0 \quad ,$$

where $N_0^c = \big(\Omega(N_0) - \bar{N}_0\big)/2$. Through the modular bootstrap, vanishing of this overlap implies that the boundary partition function vanishes as well. In our minisuperspace approximation we did observe already that contributions from bosonic and fermionic states to the partition function cancel each other. The same holds true for the full field theory since creation operators also come in pairs. Hence, our results are consistent with the world-sheet duality.

Admittedly, the simplest version of the modular bootstrap does not constrain the form of our boundary states very significantly. But there exists more stringent tests, such as bootstrap relations involving the overlap between twisted and untwisted D-branes [50, 51]. We have no doubt that these can be worked out to confirm our proposal for the twisted boundary state.

7 Conclusions

In this work we have studied maximally symmetric branes in the WZNW model on the simplest supergroup GL(1|1). Following previous reasoning for bosonic models [46] we have shown that such branes are localized along (twisted) super-conjugacy classes, an insight that generalizes straightforwardly to other supergroup target spaces. As in the case of the $p = 2$ triplet theory [41], untwisted branes turn out to be in one-to-one correspondence with irreducible representations of the current algebra. This correspondence relies on the existence of an 'identity' brane whose spectrum consists of the irreducible vacuum representation only. The spectrum between the identity and any other elementary brane is built from a single irreducible of $\widehat{\mathrm{gl}}(1|1)$ and any such irreducible appears in this way. Moreover, one can compute the spectrum between any two elementary branes by fusion of affine representations. What we have just listed are characteristic features of Cardy's theory for rational non-logarithmic conformal field theories. Our work proves that they extend at least to one of the simplest logarithmic field theory and it seems very likely that they hold more generally in all WZNW models on (type I) supergroups, see also [41] for related findings in the $p = 2$ triplet theory.

In spite of these parallels to bosonic WZNW models, branes on supergroups possess a much richer spectrum of possible geometries. Whereas Dirichlet branes on a purely bosonic torus, for example, are all related by translation, we discovered the existence of atypical lines on the bosonic base of the GL(1|1) WZNW model. The distance between any two such neighboring parallel lines is controlled by the level k. When a typical untwisted brane is moved onto one of these lines, it splits into two atypical ones. Individual atypical branes possess a single modulus that describes their dislocation along the atypical lines. In order for them to leave an atypical line they must combine with a second atypical brane. Processes of this kind model the formation of long multiplets from shorts. Hence, on more general group manifolds, more than just two atypical branes may be required to form a generic brane. Let us stress, however, that the notions of long (typical) and short (atypical) multiplets which are relevant for such processes derive directly from the representation theory of the affine Lie superalgebra. Thereby, all spectral flow symmetries are built into our description. We also wish to point out the obvious similarities with so-called fractional branes at orbifold singularities, see e.g. the discussions in section 4.3 of [52].

Another interesting and new feature of branes on GL(1|1) is the occurrence of boundary spectra that cannot be decomposed into a direct sum of irreducibles. In particular we have shown that the spectrum of boundary operators on a single generic brane is described by the projective cover of the vacuum module. For more general group manifolds, we expect the corresponding projective cover to be present as well, though along with additional stuff. The generator L_0 of dilatations is not diagonalizable on projective covers, see e.g. [16]. According to the usual reasoning, this implies the existence of logarithmic singularities in boundary correlation functions on branes in generic positions. As we have remarked before, the modular bootstrap alone did not allow for such a strong conclusion as it is blind to all nilpotent contributions within L_0. But in addition to the standard conformal field theory analysis, our investigation of the GL(1|1) WZNW model also draws from the existence of the geometric regime at large level k. The presence of projective covers is easily understood in the minisuperspace theory and it must persist when field theoretic corrections are taken into account.

There are a few obvious extensions of the above analysis that seem to merit closer investigation. These include the computation of boundary correlation functions for twisted and untwisted branes in the GL(1|1) model. We expect that correlators with a small number of bulk and/or boundary insertions may be computed using free field techniques, as in the case of bulk models [16, 20]. It would also be interesting to study the various brane geometries that can come up on other supergroup manifolds. We plan to report on both issues in the near future.

Note added: While we were in the final stages of preparing this manuscript, a related paper [53] appeared which discusses branes in triplet models with $p \geq 2$. The results of Gaberdiel and Runkel show that branes in triplet models share many features with the outcome of our analysis. In particular, for trivial gluing automorphism, branes in both models are labelled by irreducible representations of the chiral algebra. Also the labels for relevant Ishibashi states follow the same pattern: We have found one 'generic' Ishibashi state for each Kac module and an exceptional family with members being associated to atypical blocks. When the same rules are applied to the triplet models, we obtain a set of Ishibashi states that seems closely related to those used in [53]. Furthermore, Gaberdiel and Runkel also find that the partition function for any pair of boundary conditions may be determined by fusion of representations. The existence of a geometric regime for the GL(1|1) WZNW model allows us to go one step further. It gives us full control over the structure of the state space and thereby also over the nilpotent contributions to L_0 which are not visible in partition functions. Fusion of $\widehat{\mathrm{gl}}(1|1)$ representations was shown to correctly reproduce the state spaces of boundary theories in the GL(1|1) WZNW model. Let us stress, however, that the triplet and the GL(1|1) WZNW model are close cousins (see e.g. the discussion in [22]). It would therefore be somewhat premature to claim that all these structures will be present in more general logarithmic conformal field theories.

Acknowledgements

We are grateful to Matthias Gaberdiel, Gerhard Götz, Ingo Runkel, Hubert Saleur and Aliosha Semikhatov for useful discussions and comments. This research was supported in part by the EU Research Training Network grants "ForcesUniverse" (contract number MRTN-CT-2004-005104), "Superstring Theory" (contract number MRTN-CT-2004-512194) and by the PPARC rolling grant PP/C507145/1. Part of this work has been performed while Thomas Quella was working at King's College London, funded by a PPARC postdoctoral fellowship under reference PPA/P/S/2002/00370.

A The Representation Theory of $\widehat{\mathrm{gl}}(1|1)$

A.1 Spectral flow automorphisms

A useful tool for the investigation of the current algebra $\widehat{gl}(1|1)$ and its representations are spectral flow automorphisms. The first one, γ_m, leaves the modes N_n invariant and acts on the remaining ones as

$$\gamma_m(E_n) \; = \; E_n + km\delta_{n0} \; , \quad \gamma_m(\Psi_n^\pm) \; = \; \Psi_{n\pm m}^\pm \quad . \tag{A.1}$$

The previous transformation also induces a modification of the energy momentum tensor which is determined by

$$\gamma_m(L_n) \; = \; L_n + mN_n \quad . \tag{A.2}$$

Since the rank of GL(1|1) is two, there is a second one parameter family of spectral flow automorphisms $\tilde{\gamma}_\zeta$ which is parametrized by a continuous number ζ. It is rather trivial in the sense that its action does not act on the mode numbers,

$$\tilde{\gamma}_\zeta(N_n) \; = \; N_n + k\,\zeta\,\delta_{n0} \;\; \text{and} \;\; \tilde{\gamma}_\zeta(L_n) \; = \; L_n + \zeta\,E_n \quad . \tag{A.3}$$

All other modes of the currents are left invariant.

The two spectral flow symmetries above induce a map on the set of representations of $\widehat{\mathrm{gl}}(1|1)$. Given any representation ρ we obtain two new ones by defining $\rho_m = \rho \circ \gamma_m$ and $\tilde{\rho}_\zeta = \rho \circ \tilde{\gamma}_\zeta$. The latter is not very exciting but the former will play a crucial role below. Let us thus state in passing that the super-characters of these representations are related by

$$\chi_{\rho_m}(\mu, \tau) \; = \; \chi_\rho(\mu + m\tau, \tau) \quad . \tag{A.4}$$

This formula gives severe restrictions on the nature of the representations ρ_m.

A.2 Some formulas concerning Theta functions

Let us recall some facts about the theta function in one variable, the reference is Mumford's first book [49]. $\theta(\mu,\tau)$ is the unique holomorphic function on $\mathbb{C}\times\mathbb{H}$, such that

$$\begin{aligned}
\theta(\mu+1,\tau) &= \theta(\mu,\tau),\\
\theta(\mu+\tau,\tau) &= e^{-\pi i\tau}e^{-2\pi i\mu}\theta(\mu,\tau),\\
\theta(\mu+\frac{1}{2},\tau+1) &= \theta(\mu,\tau),\\
\theta(\mu/\tau,-1/\tau) &= \sqrt{-i\tau}e^{\pi i\mu^2/\tau}\theta(\mu,\tau)\\
\lim_{\mathrm{Im}(\tau)\to\infty}\theta(\mu,\tau) &= 1 \ .
\end{aligned} \tag{A.5}$$

The theta functions has a simple expansion as an infinite product,

$$\theta(\mu,\tau) \ = \ \prod_{m=0}^{\infty}\big(1-q^m\big)\prod_{n=0}^{\infty}\big(1+u^{-1}q^{n+1/2}\big)\big(1+uq^{n+1/2}\big) \quad , \tag{A.6}$$

where $q=e^{2\pi i\tau}$ and $u=e^{2\pi i\mu}$. The $\widehat{\mathrm{gl}}(1|1)$ characters in the RR sector we shall present in the next section have a simple expression in terms of the variant

$$\theta\Big(\mu-\frac{1}{2}(\tau+1),\tau\Big) \ = \ (1-u)\prod_{n=1}^{\infty}\big(1-q^n\big)\big(1-uq^n\big)\big(1-u^{-1}q^n\big) \quad . \tag{A.7}$$

Its behavior under modular S transformations which send the arguments of the theta function to $\tilde{\tau}=-1/\tau$ and $\tilde{\mu}=\mu/\tau$ can be deduced from the properties above. One simply finds

$$\theta\Big(\tilde{\mu}-\frac{1}{2}(\tilde{\tau}+1),\tilde{\tau}\Big) \ = \ i\sqrt{-i\tilde{\tau}}\,e^{\pi i\tilde{\mu}^2/\tilde{\tau}}\,u^{1/2}\tilde{u}^{-1/2}\,q^{-1/8}\tilde{q}^{1/8}\,\theta\Big(\mu-\frac{1}{2}(\tau+1),\tau\Big) \quad . \tag{A.8}$$

A.3 Representations and their characters

In this appendix we review the representations of the current superalgebra $\widehat{\mathrm{gl}}(1|1)$ that are relevant for our discussion in the main text. We shall slightly deviate from the presentation in [16] in putting even more emphasis on the role of the spectral flow automorphism (A.1). The latter is the only constituent which leads to a substantial difference between the representation theory of the finite dimensional subalgebra gl(1|1) and that of its affinization $\widehat{\mathrm{gl}}(1|1)$.

All irreducible representations of $\widehat{\mathrm{gl}}(1|1)$ are quotients of Kac modules. Just as for $\mathrm{gl}(1|1)$, we distinguish between Kac modules $\langle e,n\rangle$ and anti Kac modules $\overline{\langle e,n\rangle}$. These symbols have been chosen since the ground states transform in the corresponding representations of the horizontal subalgebra $\mathrm{gl}(1|1)$.[5] For $e \notin k\mathbb{Z}$ both types of representations will be called typical, otherwise atypical. Typical representations are irreducible and one has the equivalence $\langle e,n\rangle \cong \overline{\langle e,n\rangle}$. The super-character of (anti) Kac modules can easily be found to be

$$\hat{\chi}_{\langle e,n\rangle}(\mu,\tau) \;=\; \hat{\chi}_{\overline{\langle e,n\rangle}}(\mu,\tau) \;=\; u^{n-1}q^{\frac{e}{2k}(2n-1+e/k)+1/8}\theta\Big(\mu-\frac{1}{2}(\tau+1),\tau\Big)/\eta(\tau)^3 \quad . \tag{A.9}$$

When writing down this expression we assumed the ground state with quantum numbers $(E_0,N_0)=(e,n)$ to be fermionic. The spectral flow γ_m transforms the characters of Kac modules according to

$$\gamma_m : \quad \chi_{\langle e,n\rangle}(\mu,\tau) \;\mapsto\; (-1)^m\chi_{\langle e+mk,n-m\rangle}(\mu,\tau) \quad . \tag{A.10}$$

This equation should be interpreted as defining a map between representations. We recognize that $\langle e,n\rangle$ is transformed into $\langle e+mk,n-m\rangle$ under γ_m and that the parity of the module is changed if m is odd. A change of parity occurs if the interpretation of what are bosonic and what are fermionic states is altered compared to the standard choice.

The equivalence between Kac modules and anti Kac modules is destroyed for $e \in k\mathbb{Z}$. For these values the representations $\langle mk,n\rangle$ and $\overline{\langle mk,n\rangle}$ degenerate and exhibit a single singular vector which can be found on energy level $|m|$, see [16] for details.[6] This statement is particularly clear for $m=0$ when the singular vector is a ground state. In view of eq. (A.10) the attentive reader will have anticipated that the residual cases $e=mk$ simply arise by applying the spectral flow automorphism γ_m.

The structure of the Kac modules may be inferred from their composition series. According to our previous statements the Kac module $\langle mk,n\rangle$ contains precisely one irreducible submodule denoted by $\langle n-1\rangle^{(m)}$. The quotient of $\langle mk,n\rangle$ by the submodule $\langle n-1\rangle^{(m)}$ turns out to be the irreducible representation $\big(\langle n\rangle^{(m)}\big)'$. Hence, one can describe

[5] We would like to stress that the representations $\langle mk,n\rangle$ and $\overline{\langle mk,n\rangle}$ are inequivalent for $m \in \mathbb{Z}$ even though their ground states transform identically as long as $m \neq 0$. The reason becomes clear below.

[6] In order to avoid confusion we would like to emphasize that the construction in [16] gives rise to Kac modules for $m<0$ and anti Kac modules for $m>0$. The remaining modules cannot be obtained through Verma modules of the sort considered there.

the representation using the composition series

$$\langle mk, n\rangle : \quad \big(\langle n\rangle^{(m)}\big)' \;\longrightarrow\; \langle n-1\rangle^{(m)} \quad . \tag{A.11}$$

Again, all this can be understood best for $m = 0$ where the statement reduces to well-known facts about Kac modules of the finite dimensional subalgebra gl(1|1). This remark especially implies that the atypical irreducible representations $\langle n\rangle^{(0)}$ are built over the one-dimensional gl(1|1)-module $\langle n\rangle$. They are transformed into the remaining representations $\langle n\rangle^{(m)}$ under the spectral flow automorphism γ_m. For $m \neq 0$, the ground states of $\langle n\rangle^{(m)}$ can easily be seen to form the gl(1|1)-module $\langle mk, n-m\rangle$. The information contained in the composition series (A.11) may be used to calculate the super-characters of the atypical irreducible representations $\langle n\rangle^{(m)}$. Following the ideas of [54] one simply finds

$$\begin{aligned}\hat{\chi}^{(m)}_{\langle n\rangle}(\mu,\tau) &= \sum_{l=0}^{\infty} \hat{\chi}_{\langle mk, n+l+1\rangle}(\mu,\tau) \\ &= \frac{u^n}{1-uq^m}\,\frac{q^{\frac{m}{2}(2n+m+1)+1/8}\theta\Big(\mu - \frac{1}{2}(\tau+1),\tau\Big)}{\eta(\tau)^3} \quad .\end{aligned} \tag{A.12}$$

Analogous results hold for anti Kac modules.

Finally we need to discuss the projective covers of irreducible representations. The typical representations $\langle e, n\rangle$ with $e \notin k\mathbb{Z}$ are projective themselves. But the atypical representations $\langle n\rangle^{(m)}$ have more complicated projective covers whose composition series reads

$$\mathcal{P}^{(m)}_n : \quad \big(\langle n\rangle^{(m)}\big)' \;\longrightarrow\; \langle n+1\rangle^{(m)} \oplus \langle n-1\rangle^{(m)} \;\longrightarrow\; \big(\langle n\rangle^{(m)}\big)' \quad . \tag{A.13}$$

An alternative description of the projective covers is in terms of their Kac composition series $\mathcal{P}^{(m)}_n : \langle mk, n\rangle \to \langle mk, n+1\rangle'$. Consequently, the characters of projective covers are given by

$$\hat{\chi}_{\mathcal{P}^{(m)}_n}(\mu,\tau) \;=\; \hat{\chi}_{\langle mk,n\rangle}(\mu,\tau) - \hat{\chi}_{\langle mk,n+1\rangle}(\mu,\tau) \ . \tag{A.14}$$

These statements can once again be checked explicitly for $m = 0$ and then generalized to arbitrary values of m by means of the spectral flow transformation. For future convenience we shall silently omit the superscript $^{(m)}$ in the case that $m = 0$.

A.4 Some modular transformations

In this section we list the modular transformations of all the affine characters appearing in the previous section. Since all these representations may be expressed in terms of Kac modules it is sufficient to know the transformation

$$\hat{\chi}_{\langle e',n'\rangle}(\mu,\tau) \;=\; -\frac{1}{k}\int dedn\;\exp\frac{2\pi i}{k}\Big[e'(n-1/2)+e(n'-1/2)+e'e/k\Big]\,\hat{\chi}_{\langle e,n\rangle}(\tilde{\mu},\tilde{\tau}) \;\;. \tag{A.15}$$

to derive the remaining ones. Using the series representation (A.12) one, e.g., obtains the following behavior for characters of atypical representations,

$$\hat{\chi}^{(m)}_{\langle n'\rangle}(\mu,\tau) \;=\; \frac{1}{2ki}\int dedn\;\frac{\exp 2\pi i\big[e/k(n'+m)+m(n-1/2)\big]}{\sin(\pi e/k)}\,\hat{\chi}_{\langle e,n\rangle}(\tilde{\mu},\tilde{\tau}) \;\;. \tag{A.16}$$

Similarly, using the Kac composition series for projective covers we deduce

$$\begin{aligned}\hat{\chi}_{\mathcal{P}^{(m)}_{n'}}(\mu,\tau) \;&=\; \hat{\chi}_{\langle mk,n'\rangle}(\mu,\tau)-\hat{\chi}_{\langle mk,n'+1\rangle}(\mu,\tau) \\ &=\; \frac{2i(-1)^m}{k}\int dedn\;\exp 2\pi i\Big[e/k(n'+mk)+mn\Big]\,\sin(\pi e/k)\,\hat{\chi}_{\langle e,n\rangle}(\tilde{\mu},\tilde{\tau}) \;\;.\end{aligned} \tag{A.17}$$

The alternating signs in these formulas arise since the spectral flow changes the parity of representations for odd values of m.

A.5 Fusion rules of the $\widehat{\mathrm{gl}}(1|1)$ current algebra

Up to the need to incorporate the spectral flow automorphism and the additional atypical representations induced from it, the fusion rules of $\widehat{\mathrm{gl}}(1|1)$ agree precisely with the tensor product decomposition of gl(1|1)-modules, see e.g. [48]. Given any two integers, $m_1, m_2 \in \mathbb{Z}$, we thus find

$$\begin{aligned}\langle e_1,n_1\rangle\otimes\langle e_2,n_2\rangle \;&\cong\; \begin{cases}\langle e_1+e_2,n_1+n_2\rangle'\oplus\langle e_1+e_2,n_1+n_2-1\rangle & ,\; e_1{+}e_2\notin k\mathbb{Z}\\ \mathcal{P}^{(m)}_{n_1+n_2-1} & ,\; e_1{+}e_2=mk\end{cases}\\ \langle n_1\rangle^{(m_1)}\otimes\langle n_2\rangle^{(m_2)} \;&\cong\; \langle n_1+n_2\rangle^{(m_1+m_2)}\\ \langle n_1\rangle^{(m_1)}\otimes\langle e_2,n_2\rangle \;&\cong\; \langle m_1k+e_2,n_1+n_2\rangle \;\;.\end{aligned} \tag{A.18}$$

The prime $'$ in the first line indicates that the representation has the opposite parity compared to our standard choice.

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