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Combinatorial Quantisation of $GL(1|1)$ Chern-Simons Theory I: The Torus

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Abstract: Chern-Simons Theories with gauge super-groups appear naturally in string theory and they possess interesting applications in mathematics, e.g. for the construction of knot and link invariants. This paper is the first in a series where we propose a new quantisation scheme for such super-group Chern-Simons theories on 3-manifolds of the form $\Sigma \times \mathbb{R}$. It is based on a simplicial decomposition of an n-punctured Riemann surface $\Sigma = \Sigma_{g,n}$ of genus g and allows to construct observables of the quantum theory for any g and n from basic building blocks, most importantly the so-called monodromy algebra. In this paper we restrict to the torus case, i.e. we assume that $\Sigma = \mathbb{T}^2$, and to the gauge super-group $G = GL(1|1)$. We construct the corresponding space of quantum states for the integer level k Chern-Simons theory along with an explicit representation of the modular group $SL(2,\mathbb{Z})$ on these states. The latter is shown to be equivalent to the Lyubachenko-Majid action on the centre of a restricted version of the quantised universal enveloping algebra of the Lie super-algebra $gl(1|1)$ at the primitive k-th root of unity.

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1 Introduction

Chern-Simons theories and their quantisation are an important research topic with many links in particular to mathematics, such as the theory of 3-manifold invariants and knot theory. Their role in this context was first developed in the seminal paper [1] and then further explored through much subsequent work. Chern-Simons theories also play an important role in physics. They provide key examples of topological field theories and thereby are relevant for topological phases of matter and in particular for quantum Hall fluids, see e.g. [2–5] and many references therein.

Most of the past research and applications have been developed for Chern-Simons theories in which the gauge group is an ordinary (Lie) group. The generalisation to gauge supergroups, that is also the subject of this work, has received limited attention in the past, see e.g. [6–10]. There exist various motivations, both from physics and from mathematics, to consider Chern-Simons theories in which the gauge connection takes values in a Lie superalgebra. In particular, these models appear in the context of brane constructions. As observed in [11], Chern-Simons theories can emerge by topological twisting from the intersection of N D3 and NS5 branes in 10-dimensional type IIB superstring theory. Three of the four extended directions of the D3 branes are assumed to extend along the NS5 branes while the forth direction runs along one of the transverse coordinates x. The NS5 branes split this transverse direction x into two disconnected parts and if we split our stack of $N = n + m$ D3 branes into n that extend to the left and m extending to the right of the NS5 branes, then the topologically twisted effective theory on the 3-dimensional intersection was shown to be Chern-Simons theory with gauge supergroup $U(n|m)$ [9, 11]. The level k of the Chern-Simons theory is determined by the complexified string coupling, see [10] for details and references. The brane construction we sketched here is closely related to the realisation of Chern-Simons theory through a Kapustin-Witten topological twist [12] of 4-dimensional $N = 4$ supersymmetric Yang-Mills theory. The latter arises as the low energy effective field theory on a stack of D3 branes. Related constructions of Chern-Simons theories with gauge supergroup were also explored in [13].

On the more mathematical side, Chern-Simons theory possesses the relation with invariants of knots/links and 3-manifolds. If the gauge group is $G = SU(2)$, for example, expectation values of Wilson lines in the fundamental representation give rise to the famous Jones polynomial. For other groups and representations one obtains other classes of polynomials that have also been explored extensively. Knot invariants for Lie supergroups have not been explored as much, see however [6, 7] and more recent developments in [14, 15]. If the Wilson line operators are evaluated in maximally atypical representations of the gauge supergroup, the expectation values of Wilson lines turn out to be identical to the ones for a cohomologically reduced bosonic theory, see e.g. [16]. In the case the representations are not maximally atypical, on the other hand, one expects some new invariants. Such representations possess zero super-dimension which causes sever problems when one attempts to extend the usual constructions for bosonic (or purely even) gauge groups. It is one of the motivations of our program to develop a systematic route towards such generalizations that work for arbitrary supergroups and representations.

We should also mention here that a way to overcome the problem of vanishing dimensions was already proposed in [17] where one uses so-called re-normalized or modified dimensions that have nice topological properties generalising those of Reshetikhin–Turaev type. This more categorical approach has launched an avalanche of results [15, 18–24] in a direction related to our (though not quite directly). We however do not follow this rather abstract route and instead use a combinatorial approach based on graph algebras that is inspired by lattice gauge theory.

Another important aspect of Chern-Simons theories is their intimate relation with 2-dimensional Wess–Zumino–Novikov–Witten conformal field theories. According to common folklore, the state space of Chern-Simons theory on 3-manifold M of the form $M = \Sigma \otimes \mathbb{R}$ with an n-punctured Riemann surface $\Sigma = \Sigma_{g,n}$ of genus g coincides with the space of conformal n-point blocks of the WZNW theory. For gauge supergroups, however, the relation has not been explored well enough. While WZNW models for gauge supergroups have been constructed systematically [25], at least on surfaces of genus $g = 0$, the state spaces of associated Chern-Simons theories on $M = \Sigma \otimes \mathbb{R}$ were only constructed for a few supergroups and surfaces Σ, see in particular [10] for an extensive discussion of the $GL(1|1)$ Chern-Simons theory for a surface Σ of genus $g = 1$. At least in this special case it was shown that the state spaces coincide with the spaces of conformal blocks, just as expected. Through the approach we develop below we recover the same state space as in [10], but in a way that makes the generalisation to arbitrary supergroups and surfaces Σ of any genus rather straightforward. To lay the foundations for such an extension is indeed one of the main goals of this work.

In order to do so, we extend the combinatorial approach to the Hamiltonian quantisation of Chern-Simons theory that was first developed in a series of papers [26–29], and then consequently axiomatized in [30]. It applies to cases in which the underlying 3-manifold $M = \Sigma \times \mathbb{R}$ splits into a spacial 2-dimensional Riemann surface Σ and a time direction $\mathbb{R}$. The classical phase space of this theory is provided by the space of all gauge fields on Σ modulo gauge transformations. The idea of the combinatorial quantisation is to replace the continuous space Σ through a lattice (simplicial decomposition). While for most gauge theories such a lattice discretisation is only an approximation, for Chern-Simons theories it is exact due to the topological nature, at least as long as the lattice properly encodes the topology of the underlying surface Σ. In lattice gauge theory, the group valued holonomies of the gauge fields along the links of the lattice describe field configurations and gauge transformations act on these holonomies at the vertices. In the classical theory the space of such field configurations comes equipped with a Poisson bracket that respects the gauge transformations. The combinatorial quantisation developed in [26, 27] is achieved by replacing the algebra of functions on the link through a q-deformed algebra with a deformation parameter q that is determined by the level k of the Chern-Simons theory. It can be shown that the algebra of gauge invariant observables depends only on the underlying surface, not on the lattice discretisation. Therefore it is possible to work with one canonical lattice, one for each surface Σ. For an n-punctured Riemann surface of genus g, this canonical lattice has $2g + n$ links and a single vertex. The quantum "graph" algebra corresponding to such a lattice is made out of elementary blocks – monodromy algebras for each closed link – where the algebraic relations between different cycles elements are encoded by the quantum R-matrix.

These graph algebras had a reincarnation recently within the context of factorisation homology, a notion that was originally introduced by Beilinson and Drinfeld [31] as an abstraction from chiral conformal field and then extended to a topological setting in [32–34]. In [35] these general concepts were made explicit for 2-dimensional surfaces and the resulting algebras were found to agree with those that were introduced in [26, 27].

The algebra of observables of the Chern-Simons theory carries an action of the mapping class (or Teichmüller) group of the underlying surface Σ, i.e. of the group of orientation preserving homeomorphisms $Homeo^+(\Sigma)$ of the surface Σ divided by its identity component $Homeo_0^+(\Sigma)$. The latter consists of homeomorphisms that can be smoothly deformed to the identity. This group is generated by so-called Dehn twists. These are special homeomorphisms that are associated to non-contractible cycles γ of Σ. They amount to cutting Σ along γ, rotating one of the resulting boundary circles by 2π and then gluing the circles back together. In the special case of a torus, i.e. a Riemann surface $\Sigma_{1,0}$ this mapping class group is given by the modular group $SL(2,\mathbb{Z})$. As usual in quantum physics this action of the mapping class group on observables lifts to a projective action on the space of states. For Chern-Simons theories with bosonic gauge groups G, the latter was worked out in [29, 36] and it was shown to agree with the Reshetikhin-Turaev representation of the mapping class group [37–39]. Let us note that this representation is intimately related to knot and 3-manifold invariants [40]. The relation is based on the representation of 3-manifolds through Heegaard splitting into two handlebodies of genus g. In particular, we can take a closed 3-sphere S^3 and remove a handlebody H_3 from it. By definition, the boundary ∂H_3 of the 3-manifold H_3 is a Riemann surface $\Sigma = \partial H_3$. Gluing this handlebody H_3 back into its complement $S^3 \setminus H_3$ with a non-trivial element from the mapping class group of the surface Σ one obtains some 3-manifold M. The resulting relation between 3-manifolds and elements of the mapping class group may be employed to build 3-manifold invariants from representations of the mapping class group [41]. There exists another widely known representation of 3-manifolds through Dehn surgery on a (framed) knot or link complement which assigns 3-manifolds to framed links. When combined with the previous construction one also obtains a map from elements of the mapping class group to framed links, see [40] for an explicit construction. Hence, representations of the mapping class group are intimately related with invariants of 3-manifolds and of links. This may explain our focus on the mapping class group and its representations.

The main goal of this paper is to discuss the quantisation of Chern-Simons theory for one of simplest Lie supergroups, namely the supergroup $GL(1|1)$. While this will allow us to be extremely explicit, the supergroup $GL(1|1)$ is sufficiently non-trivial to provide a prototypical example, at least for Chern-Simons theories with gauge supergroup of type I. As we mentioned above, we expect an intimate relation between Chern-Simons theory and 2-dimensional WZNW models. Supergroup versions of the latter were studied extensively, see [42] for a review. The first complete solution of the $GL(1|1)$ model was worked out in [43]. This solution was then generalized in several steps to type I supergroups [25, 44]. In the end it turned out that all the crucial elements of the theory were already visible in the $GL(1|1)$ example, see also [42].

In this work we describe a first step of a longer programme which aims at the construction of Chern-Simons theory at both integer and non-integer levels and for arbitrary gauge supergroups on a manifold $M = \Sigma \otimes \mathbb{R}$ with a Riemann surface Σ of any genus and any number of punctures. Compared to our goal, the actual constructions and results we shall describe below may seem rather modest at first. In fact, we shall focus on the Lie supergroup $GL(1|1)$, a surface Σ of genus $g = 1$ and (odd) integer level k. Overcoming all our restrictive assumptions is actually less of an issue than it may naively appear. As we mentioned before, we do not expect the extension to other supergroups to create any new problems. Furthermore, the combinatorial quantisation we explore here is ideally suited to address surfaces of higher genus. The restriction to integer level may actually seem the most problematic since representations of the modular group $SL(2,\mathbb{Z})$ associated with $U_q\, gl(1|1)$ for generic q are not known so far. Nevertheless, we will construct such representations within our approach in a forthcoming paper.

In the next section we extend the combinatorial quantisation developed in [26, 27, 45] for semisimple (modular) Hopf algebras to not necessarily semisimple super Hopf algebras of finite dimension, at least for the torus $\Sigma = \mathbb{T}^2$. In this case, the associated mapping class group coincides with the modular group $SL(2,\mathbb{Z})$. We describe two different actions of the latter. The first one is the action on observables and states of Chern-Simons theory on $M = \mathbb{T}^2 \times \mathbb{R}$. Our construction follows the general procedure in [29] and generalizes the latter to finite-dimensional ribbon and factorisable (i.e. with non-degenerate monodromy) super Hopf algebras. The second action of the modular group we shall describe is the action on the centre of such super Hopf algebras introduced by Lyubachenko and Majid [46]. We believe that the two actions are (projectively) equivalent in cases where they are both well defined, see our explicit conjecture in Section 2.8, but we check this claim only for the case of $GL(1|1)$.

In Section 3, we illustrate the general construction of quantisation at a simple (bosonic) example, namely where the lattice gauge group is played by the group algebra of a finite cyclic group. Section 4 contains our main example which is relevant for Chern-Simons theory with the gauge supergroup $GL(1|1)$ at integer odd level k. There we introduce the restricted version of the deformed universal enveloping algebra $U_q\, gl(1|1)$, denoted by $\overline{U}_q\, gl(1|1)$, where the deformation parameter q satisfies $q^p = 1$ for p odd integer. This is a finite-dimensional ribbon factorisable Hopf algebra. The connection to the level is simple: $p = k$ (it would be however shifted by the dual Coxeter number for other supergroups). In this case, we then describe the combinatorial approach to the Hamiltonian quantisation of Chern-Simons theory on the manifold $M = \mathbb{T}^2 \times \mathbb{R}$. Special attention is paid to the action of the modular group on observables and states. We construct this action in all detail and verify that we obtain a representation of the modular group indeed, and in Section 4.9 we finally compare this representation with the one obtained in [10] based on the brane construction discussed above. Next in Section 4.10, we discuss the (projective) action of the modular group on the centre of $\overline{U}_q\, gl(1|1)$ that is defined following Lyubachenko-Majid and show that the latter is projectively equivalent to our representation on states of the Chern-Simons theory. In the concluding section we outline how the results of this work can be possibly extended to other Lie superalgebras, for surfaces of higher genus and beyond the cases of q a root of unity.

We should also note that at the final stage of writing this paper we became aware of very recent results [47] on a related subject that proves our Conjecture from Section 2.8 in the purely even case.

2 Torus observables for super Hopf algebras

In this first section we provide background material and outline the main constructions and results. These will be illustrated through explicit examples in later sections. Our discussion starts with a short review of ribbon super Hopf algebras. Then we turn to the construction of monodromies and handle algebras within the framework of combinatorial quantization that was developed for bosonic gauge groups in [26–28, 48]. We extend these algebras to allow for gauge supergroups where the underlying super Hopf algebra comes from (restricted) deformed universal enveloping of the corresponding Lie superalgebra. The associated spaces of Chern-Simons states and an action of the modular group on these states are discussed in Sections 2.5 and 2.6, following and extending the semisimple cases [29] when necessary. We conjecture that the resulting representation of the modular group is (projectively) equivalent to the Lyubashenko-Majid action on the center of the underlying super Hopf algebra. For convenience of the reader we review the latter in Section 2.7. The conjectured equivalence between representations is not proven in general, but in our two key examples to be discussed in Sections 3 and 4.

2.1 Conventions on super Hopf algebras

In this part we recall some basics about $\mathbb{Z}_2$-graded ribbon Hopf algebras over $\mathbb{C}$. We begin with $\mathbb{Z}_2$-graded algebras, and then recall useful identities in the theory of integrals, and define $\mathbb{Z}_2$-graded ribbon Hopf algebras.

A $\mathbb{Z}_2$-graded algebra $\mathcal{G}$ over the field of complex numbers $\mathbb{C}$ is a complex $\mathbb{Z}_2$-graded vector space equipped with a multiplication map $m\colon\ \mathcal{G}\otimes\mathcal{G}\to\mathcal{G}$ and a unit map $\eta\colon\ \mathbb{C}\to\mathcal{G}$ which are $\mathbb{C}$-linear and respect the grading. In other words, $\mathcal{G}$ decomposes into two subspaces $\mathcal{G}=\mathcal{G}_0\oplus\mathcal{G}_1$, called even and odd respectively, on which the multiplication acts as follows

$$m\colon\ \mathcal{G}_i\otimes\mathcal{G}_j\to\mathcal{G}_{i+j},$$

where the index is taken modulo 2. The grade of the element $a\in\mathcal{G}_i$ is defined as $|a|=i$, and we call it *even* if $|a|=0$ and *odd* otherwise. The multiplication and the unit satisfy the standard algebra axioms, including associativity. Following physics conventions we will also refer to this structure as a *superalgebra*.

Having two super algebras $\mathcal{G}$ and $\mathcal{H}$, we define $\mathbb{Z}_2$-graded algebra structure on the tensor product $\mathcal{G}\otimes\mathcal{H}$ by

$$(a_1\otimes b_1)\cdot(a_2\otimes b_2)=(-1)^{|b_1||a_2|}a_1a_2\otimes b_1b_2,\qquad a_1,a_2\in\mathcal{G},\ b_1,b_2\in\mathcal{H}\ . \tag{2.1}$$

To define a $\mathbb{Z}_2$-graded Hopf algebra, we also require the co-product $\Delta\colon\ \mathcal{G}\to\mathcal{G}\otimes\mathcal{G}$ and the co-unit $\epsilon\colon\ \mathcal{G}\to\mathbb{C}$ maps to exist. Both of them should be $\mathbb{Z}_2$-graded algebra homomorphisms, where $\mathbb{C}$ is purely even, and they are assumed to satisfy the following co-associativity and co-unit axioms:

$$(\Delta\otimes id)\circ\Delta=\Delta\circ(id\otimes\Delta), \tag{2.2}$$

$$(\epsilon\otimes id)\circ\Delta=id=(id\otimes\epsilon)\circ\Delta. \tag{2.3}$$

Let us also introduce the opposite co-product

$$\Delta^{\mathrm{op}}:=\tau\circ\Delta, \tag{2.4}$$

where we used the flip map of super vector spaces $\tau\colon\ \mathcal{G}\otimes\mathcal{G}\to\mathcal{G}\otimes\mathcal{G}$ defined on homogeneous elements as

$$\tau(a\otimes b)=(-1)^{|a||b|}b\otimes a\ . \tag{2.5}$$

Further, we require a grade preserving map $S\colon\ \mathcal{G} \to \mathcal{G}$, called an antipode, which is an algebra anti-homomorphism, that is

$$S(ab) = (-1)^{|a||b|} S(b)S(a), \qquad a, b \in \mathcal{G}, \tag{2.6}$$

and a co-algebra anti-homomorphism, that is

$$\Delta \circ S = (S \otimes S) \circ \Delta^{\mathrm{op}}. \tag{2.7}$$

In addition, it also satisfies

$$m \circ (id \otimes S) \circ \Delta = m \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon. \tag{2.8}$$

Finally, a $\mathbb{Z}_2$-graded algebra $\mathcal{G}$ equipped with a co-product Δ, a co-unit ϵ and an antipode S is called *$\mathbb{Z}_2$-graded* or *super Hopf algebra*, and will be denoted by the same symbol $\mathcal{G}$ as the underlying algebra in the following.

A super Hopf algebra is *quasi-triangular* if its tensor square admits an invertible element called *universal R-matrix* $R \in (\mathcal{G} \otimes \mathcal{G})_0$ satisfying the following relations

$$\begin{aligned} &R \cdot \Delta(x) = \Delta^{\mathrm{op}}(x) \cdot R, \qquad x \in \mathcal{G}, \\ &(id \otimes \Delta)(R) = R_{13} \cdot R_{12}, \\ &(\Delta \otimes id)(R) = R_{13} \cdot R_{23}, \end{aligned} \tag{2.9}$$

where we set

$$\begin{aligned} &R_{12} = R \otimes 1, \quad R_{23} = 1 \otimes R, \\ &R_{13} = (id \otimes \tau)(R \otimes 1), \end{aligned} \tag{2.10}$$

with τ as defined in equation (2.5).

One can define a monodromy matrix $M \in \mathcal{G} \otimes \mathcal{G}$ using the universal R-matrix

$$M = R' \cdot R, \tag{2.11}$$

where

$$R' = R_{21} = \tau(R)$$

and we keep using '$\cdot$' notation to emphasise that the product is as in equation (2.1). We will call a monodromy matrix M *non-degenerate* if it can be expanded

$$M = \sum_{i=1}^{\dim(\mathcal{G})} f_i \otimes g_i \tag{2.12}$$

with $\{f_i\}$ and $\{g_i\}$ being two bases in $\mathcal{G}$. If such an expansion of M exists, we call $\mathcal{G}$ *factorisable.*

A $\mathbb{Z}_2$-graded Hopf algebra $\mathcal{G}$ is called *ribbon* if it admits a so-called ribbon element $\boldsymbol{v} \in \mathcal{G}$, which is an even central element satisfying

$$M \cdot \Delta(\boldsymbol{v}) = \boldsymbol{v} \otimes \boldsymbol{v}, \qquad S(\boldsymbol{v}) = \boldsymbol{v}. \tag{2.13}$$

We note that in the case of semisimple ribbon and factorisable $\mathcal{G}$, the name "modular Hopf algebra" is also used because representations of $\mathcal{G}$ provide then a modular category.

In a ribbon super Hopf algebra, we have the identities

$$\boldsymbol{v}^2 = \boldsymbol{u}S(\boldsymbol{u}), \qquad \epsilon(\boldsymbol{v}) = 1, \tag{2.14}$$

where we used the so-called Drinfeld element

$$\boldsymbol{u} = m \circ (S \otimes id)(R') \ . \tag{2.15}$$

One can find an explicit expression for the ribbon element from the properties of a right integral in the manner described below.

2.2 Integrals and co-integrals

We now review standard facts from the theory of integrals [49] for a Hopf algebra. We will use the same theory in our super algebra setting, as super Hopf algebras are normal Hopf algebras too.

A linear form $\mu \in \mathcal{G}^*$ will be called a *right integral of* $\mathcal{G}$ if it satisfies

$$(\mu \otimes id) \circ \Delta(x) = \mu(x)\mathbf{1}, \tag{2.16}$$

for all $x \in \mathcal{G}$. Similarly, one can define a left integral, with μ hitting the second tensor factor instead. If an integral exists it is unique up to a scalar. Moreover, it is known that a finite-dimensional Hopf algebra always allows such integrals [50]. However in general a right integral does not have to coincide with a left one. Such a deviation of a right integral from being a left one is measured by a group-like element $\boldsymbol{a}$ called *co-modulus*: [1]

$$(id \otimes \mu) \circ \Delta(x) = \mu(x)\boldsymbol{a}. \tag{2.17}$$

A right co-integral of $\mathcal{G}$ is an element $\boldsymbol{c} \in \mathcal{G}$ such that

$$\boldsymbol{c}x = \epsilon(x)\boldsymbol{c} \ , \qquad x \in \mathcal{G} \ . \tag{2.18}$$

We note that this notion is actually dual to the notion of the integral: under a canonical identification between $\mathcal{G}$ and $\mathcal{G}^{**}$, the element $\boldsymbol{c}$ defines a right integral of $\mathcal{G}^*$. We can likewise define the left co-integral using instead the left multiplication by x. Non-trivial right and left co-integrals are unique up to scalar [50]. A Hopf algebra is called *unimodular* if its right co-integral is also left.

In the case when we have a universal R-matrix and corresponding M-matrix is non-degenerate, the integral can be normalised by

$$(\mu \otimes \mu)(M) = 1. \tag{2.19}$$

This will be the case for our examples below. From now on, we will consider only finite-dimensional quasi-triangular Hopf algebras with a non-degenerate monodromy matrix.

[1] It is called *the distinguished group-like* element in [49].

If the co-modulus can be expressed as a square of a group-like element in $\mathcal{G}$, i.e.

$$\boldsymbol{a} = \boldsymbol{g}^2, \tag{2.20}$$

then such an element $\boldsymbol{g}$ satisfies $S^2(x) = \boldsymbol{g}x\boldsymbol{g}^{-1}$, for $x \in \mathcal{G}$, and it is called a *balancing element*.

The balancing element is important for two reasons. First, it provides us the ribbon element

$$\boldsymbol{v} = \boldsymbol{g}^{-1}\boldsymbol{u}, \tag{2.21}$$

where we recall the Drinfeld element (2.15). This is a concrete formula for the ribbon element that will be used in the following sections.

Secondly, the balancing element provides us with a notion of *quantum trace* over a representation $\pi\colon \mathcal{G} \to \mathrm{End}(V)$,

$$\mathrm{str}_q\big(\pi(x)\big) := \mathrm{str}\big(\pi(\boldsymbol{g}^{-1}x)\big)\ , \qquad x \in \mathcal{G}\ , \tag{2.22}$$

where $\mathrm{str}(-) = \mathrm{tr}(\omega(-))$ is the supertrace with ω the parity map. The quantum trace over a representation π of $\mathcal{G}$ can be used to produce a central element of $\mathcal{G}$:

$$z_\pi := \big((\mathrm{str}_q \circ \pi) \otimes id\big)(M)\ \in\ Z(\mathcal{G}), \tag{2.23}$$

see e.g. [51] for non-graded case. Though in general not all central elements of $\mathcal{G}$ can be produced this way: the map $\mathrm{Rep}\,\mathcal{G} \to Z(\mathcal{G})$ defined in (2.23) is surjective if only if the algebra $\mathcal{G}$ is semisimple.

2.3 Reconstruction of $\mathcal{G}$

Let $\mathcal{G}$ be a finite-dimensional factorisable (super) Hopf algebra. We recall that the monodromy matrix M from (2.11) is an element in $\mathcal{G} \otimes \mathcal{G}$ which can be expanded as in (2.12) where the two sets of elements f_i and g_i both provide a basis of $\mathcal{G}$. The algebra $\mathcal{G}$ can be reconstructed from such non-degenerate M. Indeed, the linear map

$$\mathcal{G}^* \to \mathcal{G}, \qquad f \mapsto (f \otimes id)(M) \tag{2.24}$$

is an isomorphism if and only if M is non-degenerate, e.g. one can run here over f being the dual elements f_i^* to the basis f_i to recover all basis elements $g_i \in \mathcal{G}$.

The reconstruction of $\mathcal{G}$ from M can be processed also on the algebraic level. We first recall that M satisfies an "exchange" relation:

$$R_{21} \cdot M_{13} \cdot R_{12} \cdot M_{23} = M_{23} \cdot R_{21} \cdot M_{13} \cdot R_{12}, \tag{2.25}$$

which follows straightforwardly from the relations (2.9), and here we used conventions for M_{ij} as in (2.10). We then think of the above relation as a set of $\dim(\mathcal{G})^2$ (anti-)commutation relations for the elements in the third tensor factor, each relation corresponding to a basis element of $\mathcal{G}\otimes\mathcal{G}$. In terms of the basis expansions (2.12) this means the following: Let us introduce $R = \sum_{i,j} a_i \otimes b_j$ which serve us as "structure constants" matrix. Then the above equation can be written in components as follows

$$\begin{aligned}
&\sum_{i,j,k,l,m,n=1}^{\dim(\mathcal{G})} (-1)^{|a_i|(|b_j|+|f_k|+|a_l|)+|g_k|(|a_l|+|b_m|+|f_n|)} b_j f_k a_l \otimes a_i b_m f_n \otimes g_k g_n = \\
&= \sum_{i,j,k,l,m,n=1}^{\dim(\mathcal{G})} (-1)^{|a_i|(|b_j|+|f_k|+|a_l|)+|f_n||b_m|+|f_k|(|a_l|+|b_m|)} b_j f_k a_l \otimes f_n a_i b_m \otimes g_n g_k,
\end{aligned} \tag{2.26}$$

where we simplified the minus signs by taking into account that the monodromy matrix is an even element in $\mathcal{G} \otimes \mathcal{G}$, i.e. $|f_i| = |g_i|$. It is clear that using the (anti-)commutation relations of $\mathcal{G}$ one can arrange the elements on the second tensor factor of the right hand side of the above equation in such a way that they agree with those on the second tensor factor on the left, and by equating the corresponding terms we thus obtain defining relations for the third tensor factor in terms of the basis elements g_k. In fact, using equation (2.25) one can reconstruct the relations of the initial algebra $\mathcal{G}$ without knowing the commutation relations of the elements on the third tensor factor. We will pursue a similar treatment for an algebra which we will define as the handle algebra $\mathcal{T}$.

2.4 Handle algebra $\mathcal{T}$ and its Fock representation

In this section, we describe how to define a so-called handle algebra for a given ribbon super Hopf algebra. We will see that certain elements of the handle algebra give a realisation of the $SL(2,\mathbb{Z})$ group, i.e. the mapping class group of the torus, acting on its Fock-type representation.

Let $\mathcal{G}$ be a finite-dimensional factorisable (super) Hopf algebra. One can define an algebra using the notion of *universal element* [45], which belongs to a tensor product of the Hopf algebra $\mathcal{G}$ and of the algebra being defined, subject to a set of equations. We have already encountered in Section 2.3 an example of a universal element given by the monodromy matrix M of the Hopf algebra $\mathcal{G}$, which one can regard as an element in $\mathcal{G} \otimes \mathcal{G}$.

The handle algebra $\mathcal{T}$ is defined using a pair of universal elements $A, B \in \mathcal{G} \otimes \mathcal{T}$ subject to the exchange relations

$$R_{21} \cdot A_{13} \cdot R_{12} \cdot A_{23} = A_{23} \cdot R_{21} \cdot A_{13} \cdot R_{12}, \tag{2.27}$$

$$R_{21} \cdot B_{13} \cdot R_{12} \cdot B_{23} = B_{23} \cdot R_{21} \cdot B_{13} \cdot R_{12}, \tag{2.28}$$

$$R_{12}^{-1} \cdot A_{13} \cdot R_{12} \cdot B_{23} = B_{23} \cdot R_{21} \cdot A_{13} \cdot R_{12}, \tag{2.29}$$

which are equations in the vector space $\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{T}$. In a fixed basis in $\mathcal{T}$, these equations can be written explicitly in the same manner as (2.26). In other words, the handle algebra $\mathcal{T}$ is generated algebraically by the images of A and B under the “evaluation” map

$$f \otimes id : \ \mathcal{G} \otimes \mathcal{T} \to \mathcal{T} \ , \qquad \text{for } f \in \mathcal{G}^* \ . \tag{2.30}$$

In particular, running over all $f \in \mathcal{G}^*$ and applying this map to A we recover a subalgebra in $\mathcal{T}$ isomorphic to $\mathcal{G}$, as this single element A satisfies the same relation (2.25) as the monodromy matrix M does. We will thus call $A = M(a)$ the universal element or monodromy corresponding to the a-cycle of the torus, and the corresponding algebra *loop* or *monodromy algebra*. Similarly, we call $B = M(b)$ the monodromy of the b-cycle. Therefore, the universal elements A and B generate two subalgebras $\mathcal{T}^{(a)}$ and $\mathcal{T}^{(b)}$ of $\mathcal{T}$, which are both isomorphic to $\mathcal{G}$, and the units of these subalgebras coincide with the unit of $\mathcal{T}$. However, the elements from $\mathcal{T}^{(a)}$ do not commute with those from $\mathcal{T}^{(b)}$ – the third exchange relation (2.29) provides non-trivial commutation relations between elements of such subalgebras. Explicit examples of this construction will be provided in the next two sections for both semisimple and non-semisimple cases.

The handle algebra $\mathcal{T}$ has a representation (this is motivated by [29, Thm. 21])

$$D\colon\ \mathcal{T} \to \mathrm{End}_{\mathbb{C}}\,\mathfrak{R}$$

on a finite-dimensional vector space $\mathfrak{R}$ that has the form of a Fock module which is constructed in two steps:

1. Introduce a vacuum state $|0\rangle \in \mathfrak{R}$ which is by definition left invariant with respect to the universal element B associated to the b-cycle

$$\big\{(id \otimes D)B\big\}(1 \otimes |0\rangle) = 1 \otimes |0\rangle. \tag{2.31}$$

This is an equation on elements of $\mathcal{G} \otimes \mathfrak{R}$. The only solution of this equation is

$$g^{(b)}|0\rangle = \epsilon(g)|0\rangle, \tag{2.32}$$

where $g \in \mathcal{G}$ and $g^{(b)}$ is the corresponding element in $\mathcal{T}^{(b)}$ under the isomorphism $\mathcal{G} \cong \mathcal{T}^{(b)}$. We also note that practically we can rewrite the above equation (2.31) "component-wise" on representations π of $\mathcal{G}$ as

$$\big\{(\pi \otimes D)B\big\}(v \otimes |0\rangle) = v \otimes |0\rangle,$$

for all representations π and for all vectors v in the representation space of π. This gives us a system of equations for the action of elements from the second tensor factor, i.e. the algebra $\mathcal{T}^{(b)}$, on the vacuum, c.f. [29, Eq. (7.3)][2].

2. The Fock module $\mathfrak{R}$ is then defined as a free module over $\mathcal{T}^{(a)}$ generated from the vacuum $|0\rangle$. In other words, the rest of the vectors $|f\rangle \in \mathfrak{R}$ that belong to the representation space is obtained by applying the elements from the subalgebra $\mathcal{T}^{(a)}$ on the vacuum without imposing extra relations, and these are

$$|f\rangle = \big\{(f \otimes D)A\big\}|0\rangle, \qquad \text{for } f \in \mathcal{G}^*\ . \tag{2.33}$$

As the solution for A will be provided by the non-degenerate monodromy matrix $M = M(a)$, in this case the Fock module is isomorphic to $\mathcal{G}$ as a vector space. We thus see by construction that $\mathfrak{R}$ is a representation of $\mathcal{T}^{(a)}$, it is actually isomorphic to the regular representation of $\mathcal{G} \cong \mathcal{T}^{(a)}$. We thus only need to show that the action of $\mathcal{T}^{(b)}$ is well-defined on such vectors – this follows from the third exchange relation (2.29) in $\mathcal{T}$ that provides commutation relations

[2] We note that we however use a different normalisation, which facilitates a comparison of two $SL(2,\mathbb{Z})$ actions discussed below.

between elements in $\mathcal{T}^{(a)}$ and $\mathcal{T}^{(b)}$. These are obtained from equations analogous to (2.26) and they are of the form:[3]

$$\sum_{k,n=1}^{\dim(\mathcal{G})} f_{k,n}\, g_k^{(a)} \cdot g_n^{(b)} = \sum_{k,n=1}^{\dim(\mathcal{G})} \tilde{f}_{k,n}\, g_n^{(b)} \cdot g_k^{(a)} \ , \tag{2.34}$$

where $f_{k,n}$ and $\tilde{f}_{k,n}$ are complex numbers (possibly zero), and $g_k^{(a)}$ and $g_n^{(b)}$ are elements in the basis expansions (2.12) corresponding to $M(a)$ and $M(b)$ respectively. Under the action of $\mathcal{T}^{(b)}$ on the free $\mathcal{T}^{(a)}$-module $\mathfrak{R}$, we can always use (2.34) to pass elements of $\mathcal{T}^{(b)}$ through those of $\mathcal{T}^{(a)}$ until they reach the vacuum where we already fixed the action via (2.32).

We recall that in the case when $\mathcal{G}$ is semisimple, it is known that the handle algebra $\mathcal{T}$ can be also constructed as the Heisenberg double of the Hopf algebra $\mathcal{G}$. We did not investigate an analogue of this in the non-semisimple case but we believe that such an isomorphism to a Heisenberg double also holds. Furthermore, due to this relation to the Heisenberg double, the handle algebra in the semisimple case admits a unique irreducible representation given by the Fock module defined above. Again, in the non-semisimple case it is an open problem of classification of representations of $\mathcal{T}$ but for the analysis of $SL(2,\mathbb{Z})$ action below we will need the Fock module only.

2.5 Gauge invariant subalgebra and its representation

In analogy with the construction in the semisimple case [45], our next step in defining the $SL(2,\mathbb{Z})$ action is to introduce the so-called gauge invariant subalgebra $\mathcal{A}$ in the handle algebra $\mathcal{T}$. To this end let us first define a so-called "adjoint" action of the Hopf algebra $\mathcal{G}$ on $\mathcal{T}$. We recall that $\mathcal{T}$ is generated by elements from its $a-$ and $b-$ cycle subalgebras $\mathcal{T}^{(i)} \subset \mathcal{T}$, for $i = a, b$, as was discussed in Section 2.4. Moreover, we have an algebra isomorphism:

$$\kappa^{(i)}\colon\ \mathcal{G} \to \mathcal{T}^{(i)} \ , \quad M \mapsto M(i) \tag{2.35}$$

written in terms of the universal elements. We can now define the $\mathcal{G}$-action of "adjoint" type on these subalgebras:

$$x(f) = \sum_{(x)} (-1)^{|f||x''|} \kappa^{(i)}(x') \cdot f \cdot \kappa^{(i)}\big(S(x'')\big) \ , \qquad x \in \mathcal{G} \ , \ f \in \mathcal{T}^{(i)} \ , \tag{2.36}$$

where we used the standard Sweedler notation for co-product components

$$\Delta(x) = \sum_{(x)} x' \otimes x''. \tag{2.37}$$

In particular, using the Hopf algebra axioms we have $x(1) = \epsilon(x)1$.

The action (2.36) makes $\mathcal{T}$ a module algebra over the Hopf algebra $\mathcal{G}$, i.e. the action is compatible with the multiplication in $\mathcal{T}$ in the sense that

$$x(f \cdot g) = \sum_{(x)} (-1)^{|f||x''|} x'(f) \cdot x''(g) \ , \tag{2.38}$$

[3] Formally, we have $\dim(\mathcal{G})^2$ number of such relations, though not necessarily all of them are algebraically independent.

where $x \in \mathcal{G}$ and $f, g \in \mathcal{T}$. Therefore, one can construct a smash product algebra $\bar{\mathcal{T}} = \mathcal{T} \rtimes \mathcal{G}$ by defining the multiplication

$$(f \otimes x) \cdot (g \otimes y) = \sum_{(x)} (-1)^{|g||x''|} f \cdot x'(g) \otimes x'' \cdot y, \tag{2.39}$$

where $f, g \in \mathcal{T}$ and $x, y \in \mathcal{G}$. We will denote the element $f \otimes x$ as $f \cdot \iota(x)$, where

$$\iota\colon\ \mathcal{G} \to \bar{\mathcal{T}}, \qquad x \mapsto 1 \otimes x,$$

is the canonical embedding map. Using (2.39) for the choice $y = 1$ and f equal the unit in $\mathcal{T}$, we note the relation

$$\iota(x) \cdot g = \sum_{(x)} (-1)^{|g||x''|} x'(g) \cdot \iota(x'') \ , \qquad x \in \mathcal{G} \ , \ g \in \mathcal{T} \ , \tag{2.40}$$

that we use below. The smash product $\bar{\mathcal{T}} = \mathcal{T} \rtimes \mathcal{G}$ can be alternatively and equivalently defined [29, 45] using the universal elements A and B, by the following relations

$$\big\{(id \otimes \iota)\Delta(x)\big\} \cdot A = A \cdot \big\{(id \otimes \iota)\Delta(x)\big\}, \tag{2.41}$$
$$\big\{(id \otimes \iota)\Delta(x)\big\} \cdot B = B \cdot \big\{(id \otimes \iota)\Delta(x)\big\}, \tag{2.42}$$

where $x \in \mathcal{G}$ and the universal elements A and B are considered as elements of $\mathcal{G} \otimes \bar{\mathcal{T}}$, i.e. in $\mathcal{G} \otimes \mathcal{T} \otimes \mathcal{G}$ where we identify A with $A \otimes 1 \in \mathcal{G} \otimes \mathcal{T} \otimes \mathcal{G}$ and B with $B \otimes 1 \in \mathcal{G} \otimes \mathcal{T} \otimes \mathcal{G}$.

Finally, the *gauge invariant* subalgebra $\mathcal{A}$ of the handle algebra is defined as the subalgebra of $\mathcal{G}$-invariant elements

$$\mathcal{A} = \{f \in \mathcal{T} \subset \bar{\mathcal{T}} \mid \iota(x) f = (-1)^{|f||x|} f \iota(x), \ \forall\, x \in \mathcal{G}\}. \tag{2.43}$$

We note that the above definition of $\mathcal{A}$ is equivalent to

$$\mathcal{A} = \{f \in \mathcal{T} \mid x(f) = \epsilon(x) f, \ \forall\, x \in \mathcal{G}\}$$

and it is clear that $x(f \cdot g) = \epsilon(x) f \cdot g$, for $f, g \in \mathcal{A}$, as follows from (2.38) and using the super Hopf algebra axioms on the co-unit. Therefore $\mathcal{A}$ forms indeed a subalgebra in $\mathcal{T}$.

We show now that $\mathcal{A}$ contains an important subalgebra, the one generated by the two centres $Z(\mathcal{T}^{(a)})$ and $Z(\mathcal{T}^{(b)})$. The crucial observation here is that central elements in a Hopf algebra H are invariants under the adjoint action, i.e. if $z \in Z(H)$ then $\sum_{(x)} x' z S(x'') = \epsilon(x) z$ for all $x \in H$. The same applies for the super Hopf algebras where the adjoint action is now defined as in (2.36), i.e. with the sign factors[4]. We thus have that under the $\mathcal{G}$ action (2.36) on the subalgebra $\mathcal{T}^{(a)}$ the following equalities hold for all $x \in \mathcal{G}$ and $z \in Z(\mathcal{T}^{(a)})$:

$$x(z) = \epsilon(x) z, \tag{2.44}$$

and similarly for the $b-$cycle centre $Z(\mathcal{T}^{(b)})$. We thus get that

$$Z(\mathcal{T}^{(a)}) \subsetneq \mathcal{A}, \qquad Z(\mathcal{T}^{(b)}) \subsetneq \mathcal{A} \tag{2.45}$$

[4]However, we note that as central elements are even the signs in (2.36) and in (2.43) do not play a role.

and all products of elements from the two centres belong to $\mathcal{A}$ too. However, the two centres do not in general generate the algebra $\mathcal{A}$, as it can be seen on the example of $\overline{U}_q\, gl(1|1)$ in next sections.

The Fock representation of the handle algebra $\mathcal{T}$ can be extended to a representation of $\bar{\mathcal{T}}$ and then to one of $\mathcal{A}$. For any element $\iota(x)$, $x \in \mathcal{G}$, we impose that on the representation space $\mathfrak{R}$ it acts in the following way (compare with [45, Prop. 12])

$$D(\iota(x))|0\rangle = \epsilon(x)|0\rangle \ , \tag{2.46}$$

while the action on other vectors is obtained using the commutation relations (2.40). Indeed, recall that any vector in the Fock module can be written as $D(g)|0\rangle$ for some $g \in \mathcal{T}^{(a)}$. Therefore we can write

$$D(\iota(x))\big(D(g)|0\rangle\big) = D\big(\iota(x)\cdot g\big)|0\rangle = \sum_{(x)}(-1)^{|g||x''|}D\big(x'(g)\big)\cdot \epsilon(x'')|0\rangle = D\big(x(g)\big)|0\rangle. \tag{2.47}$$

Here, we first used the requirement that D is an algebra map, then (2.40) in the second equality, and then the Hopf algebra co-unit axiom for the last equality – however a comment is necessary for odd elements g: the sign factors $(-1)^{|g||x''|}$ are actually irrelevant due to the fact that ϵ is an even map, in particular $\epsilon(x'') = 0$ for odd x'', while the sign is $+1$ for even x''. This finally proves (2.47), and therefore D indeed constitutes a representation of $\bar{\mathcal{T}}$. For brevity, we will use the same notation D for both $\mathcal{T}$ and $\bar{\mathcal{T}}$.

Of course, we can define a representation of $\mathcal{A}$ as a restriction of D to $\mathcal{A}$. However, we need a much smaller space – the subspace of $\mathcal{G}$-invariants in $\mathfrak{R}$ that can be formally defined as

$$\mathrm{Inv}(\mathfrak{R}) := \mathrm{Hom}_{\mathcal{G}}(\mathbb{C}, \mathfrak{R}). \tag{2.48}$$

This subspace corresponds in [29] to the "flatness" restriction on D. In the semisimple case, such a restriction can be constructed using an appropriate projector. In the non-semisimple case, such a projector generally does not exist. We however do not need to follow this way as the space $\mathrm{Inv}(\mathfrak{R})$ of gauge-invariant states, i.e. those that $D(\iota(x))|f\rangle = \epsilon(x)|f\rangle$, can be constructed directly from the (gauge-invariant) vacuum $|0\rangle$ by applying all possible gauge-invariant operators, and we know that these are all in $\mathcal{A}$. We thus consider a "truncation" of the representation D to a representation $D_{(\mathcal{A})}$ of the gauge-invariant subalgebra $\mathcal{A}$ generated from the vacuum by $\mathcal{A}$:

$$\mathfrak{R}_{(\mathcal{A})} := D(\mathcal{A})|0\rangle. \tag{2.49}$$

This clearly defines a representation of $\mathcal{A}$, which is a subspace in $\mathfrak{R}$. Assume now $g \in \mathcal{A}$ then $D(g)|0\rangle$ is a gauge-invariant state – indeed, this follows from (2.47) because $x(g) = \epsilon(x)g$. In other words we have shown that $\mathfrak{R}_{(\mathcal{A})} \subset \mathrm{Inv}(\mathfrak{R})$. Moreover, we claim that the subspace $\mathfrak{R}_{(\mathcal{A})}$ contains all gauge-invariant states, i.e.

$$\mathfrak{R}_{(\mathcal{A})} = \mathrm{Inv}(\mathfrak{R}). \tag{2.50}$$

This follows from the fact that $|0\rangle$ is a cyclic vector generating the whole module $\mathfrak{R}$ under the action of $\mathcal{T}$, and similarly all the gauge-invariant states are generated from this cyclic vector by the centraliser of $\mathcal{G}$, which is $\mathcal{A}$ by definition. We note the importance of the *cyclic* vector: in the semisimple case, $\mathcal{A}$ acts on the multiplicity space of $\mathcal{G}$-invariants via an irreducible representation, and thus it would be enough to use any non-zero gauge-invariant state to produce the whole space of $\mathcal{G}$-invariants via the action of $\mathcal{A}$ on it; in the non-semisimple case, the action of $\mathcal{A}$ on the multiplicity space (2.48) is not necessarily irreducible but it is indecomposable, and thus from a gauge-invariant state we might generate only a proper subspace in $\mathrm{Inv}(\mathfrak{R})$, however from a cyclic vector the action of $\mathcal{A}$ generates the whole space of $\mathcal{G}$-invariants.

It is clear that $\mathfrak{R}_{(\mathcal{A})}$ contains an important subspace generated by the a-cycle centre:

$$D\big(Z(\mathcal{T}^{(a)})\big)|0\rangle \subset \mathfrak{R}_{(\mathcal{A})}.$$

From the action (2.32) of the b-cycle centre on the vacuum, it is also clear that the algebra generated by both centres $Z(\mathcal{T}^{(a)})$ and $Z(\mathcal{T}^{(b)})$ (a subalgebra in $\mathcal{A}$) generates the same subspace $D\big(Z(\mathcal{T}^{(a)})\big)|0\rangle$. We will however see in our non-semisimple example in Section 4 that the gauge-invariant algebra $\mathcal{A}$ is much bigger than the algebra generated by $Z(\mathcal{T}^{(a)})$ and $Z(\mathcal{T}^{(b)})$. Assume now $a \in \mathcal{A}$ is an element that is not necessarily written as a product of elements from $Z(\mathcal{T}^{(a)})$ and $Z(\mathcal{T}^{(b)})$. It is however can be written as, recall relations (2.34) and that $\mathcal{T}$ has dimension $\dim(\mathcal{G})^2$,

$$a = \sum_{k,n=1}^{\dim(\mathcal{G})} f_{k,n}\, g_k^{(a)} \cdot g_n^{(b)} \ ,$$

for some numbers $f_{k,n}$. Applying such a general element on the vacuum and using (2.32) we get

$$a|0\rangle = \sum_{k,n=1}^{\dim(\mathcal{G})} f_{k,n}\, \epsilon\big(g_n^{(b)}\big) g_k^{(a)}|0\rangle \ ,$$

i.e. we have

$$a|0\rangle \ \in \ D\big(\mathcal{T}^{(a)}\big)|0\rangle \ .$$

But we assumed that $a \in \mathcal{A}$ or $a|0\rangle$ is a $\mathcal{G}$-invariant, and the only operators from $D\big(\mathcal{T}^{(a)}\big)$ that produce $\mathcal{G}$-invariants from $|0\rangle$ are invariants under the adjoint action, or operators from $D\big(Z(\mathcal{T}^{(a)})\big)$. We thus conclude that

$$\mathfrak{R}_{(\mathcal{A})} = D\big(Z(\mathcal{T}^{(a)})\big)|0\rangle. \tag{2.51}$$

In other words, as a vector space $\mathfrak{R}_{(\mathcal{A})}$ is isomorphic to the centre of $\mathcal{G}$.

We will use this representation $\mathfrak{R}_{(\mathcal{A})}$ for our formulation of the (projective) action of the mapping class group of the torus.

2.6 Representation of mapping class group of the torus

In this section, we describe the realisation of the $SL(2,\mathbb{Z})$ group through elements of the gauge-invariant subalgebra $\mathcal{A}$ of the handle algebra $\mathcal{T}$. Then, we define our projective action of $SL(2,\mathbb{Z})$ on the subspace $\mathfrak{R}_{(\mathcal{A})}$ in $\mathfrak{R}$ generated by $\mathcal{A}$ from the gauge-invariant vacuum. (This projective representation can be interpreted as the space of Chern-Simons observables.) In order to define such a representation, we first recall some facts about the mapping class group of the torus.

The first homotopy group $\pi_1(\mathbb{T}^2)$ of the torus is generated by the elements a, b associated to the corresponding cycles on $\mathbb{T}^2$, which are subjected to the following relation

$$ba^{-1}b^{-1}a = e. \tag{2.52}$$

This relation is interpreted as a lack of punctures or discs removed from the torus. On the group $\pi_1(\mathbb{T}^2)$, one can define two automorphisms α and β which act as

$$\begin{aligned} &\alpha(a) = a, && \alpha(b) = ba, \\ &\beta(a) = b^{-1}a, && \beta(b) = b, \end{aligned} \tag{2.53}$$

and they can be interpreted as Dehn twists along the a- and b-cycles. Recall that $\mathrm{Aut}\big(\pi_1(\mathbb{T}^2)\big)$ is $SL(2,\mathbb{Z})$. We can relate those automorphisms to the standard generators σ, τ of $SL(2,\mathbb{Z})$ as follows

$$\sigma = \alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta, \tag{2.54}$$

$$\tau = \alpha^{-1}. \tag{2.55}$$

It is easy to see that they satisfy the expected relations

$$\sigma^4 = id \qquad\qquad (\sigma\tau)^3 = \sigma^2. \tag{2.56}$$

The main idea now is to use a quantised version of the automorphisms α, β. We have seen in the previous sections that in defining a quantum theory we associate the universal elements A and B to the a-and b-cycles respectively. In fact, the handle algebra $\mathcal{T}$ admits a pair of automorphisms $\alpha, \beta\colon\ \mathcal{T} \to \mathcal{T}$ which realise a "quantum" version of the action (2.53):[5]

$$\begin{aligned} &(id \otimes \alpha)(A) = A, && (id \otimes \beta)(A) = (\boldsymbol{v} \otimes 1) \cdot B^{-1}A, \\ &(id \otimes \alpha)(B) = (\boldsymbol{v}^{-1} \otimes 1) \cdot BA, && (id \otimes \beta)(B) = B, \end{aligned} \tag{2.57}$$

where $\boldsymbol{v}$ is the ribbon element of $\mathcal{G}$ introduced in (2.21). That α and β are automorphisms, i.e. respect the relations (2.27)-(2.29), is proven along the same lines as in the proof of [45, Lem. 6] where the semisimplicity assumption on $\mathcal{G}$ was not actually used but only the general properties of $\boldsymbol{v}$ and of the universal elements A and B that are valid in our case too.

The automorphisms α and β can be expressed as inner automorphisms of the handle algebra, given by the adjoint actions

$$\alpha(x) = \big(\hat{v}(a)\big)^{-1} \cdot x \cdot \hat{v}(a), \qquad \beta(x) = \big(\hat{v}(b)\big)^{-1} \cdot x \cdot \hat{v}(b), \qquad x \in \mathcal{T}, \tag{2.58}$$

of the following elements of the handle algebra $\mathcal{T}$

$$\begin{aligned} \hat{v}(a) &= (\mu \otimes id)\big((\boldsymbol{v}^{-1} \otimes 1) \cdot A\big), \\ \hat{v}(b) &= (\mu \otimes id)\big((\boldsymbol{v}^{-1} \otimes 1) \cdot B\big). \end{aligned} \tag{2.59}$$

[5]The appearance of the ribbon element, when compared to the classical equations above, reflects the quantum nature of the automorphisms α and β.

The proof of (2.58) essentially repeats[6] the one of [45, Lem. 9], and so we omit it. We will interpret the elements $\hat{v}(a)$ and $\hat{v}(b)$ as the "quantum" Dehn twists operators along the $a-$ and $b-$cycles of the torus, correspondingly.

For the further analysis it will be important to note that the elements (2.59) actually belong to the gauge-invariant subalgebra $\mathcal{A}$. Indeed, recall the standard result due to [51]: let H be a unimodular finite-dimensional Hopf algebra over $\mathbb{C}$ and $K \in H \otimes H$ such that $K\Delta(x) = \Delta(x)K$ for all $x \in H$, and let $\phi\colon H \to \mathbb{C}$ be a linear map such that

$$\phi(xy) = \phi(S^2(y)x) \tag{2.60}$$

then $(\phi \otimes id)(K)$ is in the centre of H. Applying this to $K = (\boldsymbol{v}^{-1} \otimes 1) \cdot M$ and $\phi = \mu$ (the equation (2.60) holds for the integral μ, see [49]) we then get that

$$(\mu \otimes id)\big((\boldsymbol{v}^{-1} \otimes 1) \cdot M\big) \;\in\; Z(\mathcal{G})$$

and therefore both the elements $\hat{v}(a)$ and $\hat{v}(b)$ are in the centres $Z\big(\mathcal{T}^{(a)}\big)$ and $Z\big(\mathcal{T}^{(b)}\big)$, respectively. Using the result in (2.45), we conclude that both $\hat{v}(a)$ and $\hat{v}(b)$ belong to $\mathcal{A}$. We note however that these elements are not necessarily in the centre of $\mathcal{A}$.

Using the special elements $\hat{v}(a)$ and $\hat{v}(b)$ in $\mathcal{A}$, we can make a statement, which in the following chapters will be treated very concretely in the cases of a simple "toy" model based on a finite cyclic group, and then for the $\overline{U}_q\, gl(1|1)$ case at a root of unity. Using the quantum Dehn twist operators, we can define the elements that correspond to the actions of the $SL(2,\mathbb{Z})$ group:

$$\mathscr{S} := \hat{v}(b)\hat{v}(a)\hat{v}(b), \tag{2.61}$$

$$\mathscr{T} := \big(\hat{v}(a)\big)^{-1}. \tag{2.62}$$

Recall that both $\hat{v}(a)$ and $\hat{v}(b)$ belong to $\mathcal{A}$, therefore the elements $\mathscr{S}$ and $\mathscr{T}$ are in the gauge-invariant subalgebra $\mathcal{A}$ too.

While $\mathscr{S}$ and $\mathscr{T}$ provide a modular group action on the elements of the handle algebra $\mathcal{T}$ they do not necessarily furnish a projective representation of $SL(2,\mathbb{Z})$ on its representation space $\mathfrak{R}$. However, $\mathscr{S}$ and $\mathscr{T}$ can be realised as operators on the subspace $\mathfrak{R}_{(\mathcal{A})}$ of gauge-invariant states introduced in (2.49), recall also our result in (2.50). This realisation follows from the above result that $\mathscr{S}, \mathscr{T} \in \mathcal{A}$, and therefore

$$D(\mathscr{S})\colon\; \mathfrak{R}_{(\mathcal{A})} \to \mathfrak{R}_{(\mathcal{A})}, \qquad D(\mathscr{T})\colon\; \mathfrak{R}_{(\mathcal{A})} \to \mathfrak{R}_{(\mathcal{A})}.$$

Moreover, we claim that $\mathscr{S}$ and $\mathscr{T}$ operators provide a projective representation of $SL(2,\mathbb{Z})$, i.e. they satisfy

$$D_{(\mathcal{A})}(\mathscr{S}^4) = id, \tag{2.63}$$

$$D_{(\mathcal{A})}\big((\mathscr{S}\mathscr{T})^3\big) = \lambda D_{(\mathcal{A})}(\mathscr{S}^2), \tag{2.64}$$

where $\lambda \in \mathbb{C}^\times$. This statement is strictly speaking conjectural, it naturally generalises [29, Thm. 28] to not necessarily semisimple algebras, and our conjecture is supported by a non-trivial example we demonstrate in Section 4.

[6]The only difference with the elements defined in [45, Lem. 9] is in normalization factor which we omit so that $\hat{v}(a)$ and $\hat{v}(b)$ are invertible.

2.7 Lyubashenko–Majid $SL(2,\mathbb{Z})$ action on the centre

We recall here another construction of a (projective) $SL(2,\mathbb{Z})$ representation associated with $\mathcal{G}$. Having a ribbon factorisable (super) Hopf algebra $\mathcal{G}$, one can construct an infinite series of mapping class group representations on certain spaces of intertwining operators [52]. In particular for a torus without punctures, we have a (projective) representation of the group $SL(2,\mathbb{Z})$ on the centre of $\mathcal{G}$, see the original reference [46]. We will now review this construction for the case of torus, mainly following the more recent exposition [53].

The construction involves three main ingredients: integral, monodromy matrix, and a ribbon element. Let $Z(\mathcal{G})$ denotes the centre of $\mathcal{G}$. The S- and T-transformations from the modular group acting on $Z(\mathcal{G})$ are defined as

$$\mathscr{S}_Z(z) = (\mu\otimes id)\big\{(S(z)\otimes 1)\cdot M\big\}, \qquad z\in Z(\mathcal{G}). \tag{2.65}$$

$$\mathscr{T}_Z(z) = \boldsymbol{v}^{-1}\, z, \tag{2.66}$$

with the ribbon element $\boldsymbol{v}$ defined in (2.21). These two linear maps provide a projective representation of $SL(2,\mathbb{Z})$:

$$\mathscr{S}_Z^4 = id, \qquad\qquad (\mathscr{S}_Z\mathscr{T}_Z)^3 = \lambda\mathscr{S}_Z{}^2,$$

with some non-zero number λ. It is known that in the case of a modular Hopf algebra $\mathcal{G}$, i.e. in the semisimple case, such a representation of $SL(2,\mathbb{Z})$ is equivalent to the Reshetikhin-Turaev construction [37, 38], where the S-transformation is provided by the closure (taking the quantum trace) of the double braiding of a pair of irreducible representations. It was demonstrated that the Reshetikhin-Turaev construction is equivalent to the handle algebra construction in [29, Thm. 29].

2.8 Conjecture on equivalence of two $SL(2,\mathbb{Z})$ actions

So far, we have defined two $SL(2,\mathbb{Z})$ actions, one based on the handle algebra in Section 2.6 and the Lyubashenko-Majid one in Section 2.7, and both are realized on the same vector space – the centre of $\mathcal{G}$. Recall our result in (2.51). As we just mentioned for the semisimple case, it is known that the two actions agree projectively. Let us make the following conjecture.

Conjecture: *Let $\mathcal{G}$ be a finite-dimensional ribbon factorisable (super) Hopf algebra over $\mathbb{C}$. The two $SL(2,\mathbb{Z})$ representations, one defined in (2.61)-(2.62) on $\mathfrak{R}_{(\mathcal{A})}\cong Z(\mathcal{G})$ and the other in (2.65)-(2.66), are projectively isomorphic.*

We will next demonstrate on two examples the two constructions of (projective) $SL(2,\mathbb{Z})$ actions, one based on the handle algebra and the Lyubashenko-Majid one, and show explicitly that they are indeed equivalent. We begin with a "toy" model based on a cyclic group.

3 Toy model — the cyclic group case

In this section, we demonstrate the construction described in the previous section in the simplest possible case – the choice of the Hopf algebra $\mathcal{G}$ given by (the group algebra) of the finite cyclic group $\mathbb{Z}_p$ with $p\in\mathbb{N}$ elements. We will denote this algebra by A_p. This is a semisimple algebra, while a non-semisimple case is considered in the next section.

The Hopf algebra $A_p := \mathbb{C}\mathbb{Z}_p$ is generated by k with the only relation $k^p = 1$. It has the basis $\{k^n\}_{n=0}^{p-1}$ with the commutative multiplication $k^n k^m = k^{n+m}$, and the group-like co-product Δ, the co-unit ϵ and the antipode S such that

$$\Delta(k^n) = k^n \otimes k^n, \qquad \epsilon(k^n) = 1, \qquad S(k^n) = k^{-n}. \tag{3.1}$$

We will also use the notation $q = e^{2\pi i/p}$. For the algebra A_p, the universal R-matrix is then

$$R = \frac{1}{p} \sum_{n,m=0}^{p-1} q^{-nm} k^n \otimes k^m. \tag{3.2}$$

It is straightforward to check the R-matrix axioms.

The monodromy matrix (2.11) is then

$$M = \frac{1}{p} \sum_{n,m=0}^{p-1} q^{-2nm} k^{2n} \otimes k^{2m}. \tag{3.3}$$

It is however non-degenerate for odd values of p only. Indeed, the monodromy matrix can be rewritten as $M = \sum_{n=0}^{p-1} k^{2n} \otimes e_n$ where we introduced the idempotents

$$e_n = \frac{1}{p} \sum_{m=0}^{p-1} q^{-2nm} k^{2m}.$$

It is clear that e_n, for $0 \le n \le p-1$, form a basis in A_p for odd values of p, while for even p we have $e_{n+p/2} = e_n$. Similarly, $\{k^{2n}\}_{n=0}^{p-1}$ is a basis in A_p for odd p only. Therefore, the monodromy matrix takes the form (2.12) for odd values of p only, and A_p is thus factorisable. By this reason, we will assume below that p is odd.

For A_p, the integral (2.16) takes the well-known form:

$$\mu(k^n) = \mathcal{N} \delta_{n,0}, \tag{3.4}$$

where we use the normalisation (2.19) that gives

$$\mathcal{N} = \sqrt{p}. \tag{3.5}$$

Moreover, the co-integral (2.18) is given by (using also the normalisation $\mu(\boldsymbol{c}) = 1$)

$$\boldsymbol{c} = \frac{1}{\sqrt{p}} \sum_{n=0}^{p-1} k^n. \tag{3.6}$$

Using the definitions (2.20) and (2.21) one can find that the balancing element $\boldsymbol{g} = 1$ and the expression for the ribbon element $\boldsymbol{v}$ is given by

$$\boldsymbol{v}^{\pm 1} = \frac{i^{\mp\omega(p)}}{\sqrt{p}} \sum_{n=0}^{p-1} q^{\pm n^2} k^{2n}, \tag{3.7}$$

where we used the "Gauss sum" identity

$$\sum_{m=0}^{p-1} q^{m^2} = i^{\omega(p)} \sqrt{p}, \tag{3.8}$$

with

$$\omega(p) = \begin{cases} 1, & p \in 4\mathbb{Z} + 3, \\ 0, & p \in 4\mathbb{Z} + 5. \end{cases}$$

3.1 Representations of A_p

The finite cyclic group has a very simple representation theory, given that it is a commutative and co-commutative algebra. It admits only 1-dimensional irreducible representations $\pi_n \colon A_p \to \mathbb{C}$ which are

$$\pi_n(k) = q^n \ , \qquad n = 0, \dots, p-1 \ . \tag{3.9}$$

On those representations, we have

$$\begin{aligned}(\pi_n \otimes id)R &= k^n, \\ (\pi_n \otimes id)M &= k^{2n}.\end{aligned}$$

Moreover, since the balancing element is trivial and the algebra is non-graded, the quantum trace (2.22) is simply the ordinary trace on 1-dimensional representations

$$\mathrm{str}_q(\pi_n(k^m)) = q^{nm}.$$

3.2 The handle algebra of A_p

In this section, we solve the exchange equations (2.27)-(2.29) which define the handle algebra $\mathcal{T}$ commutation relations. Since these first two of those relations are identical to the relations for the monodromy matrix of our toy model algebra, we use the following Ansatz for the universal elements A and B

$$A = \frac{1}{p} \sum_{n,m=0}^{p-1} q^{-2nm} k^{2n} \otimes \left(k^{(a)}\right)^{2m}, \tag{3.10}$$

$$B = \frac{1}{p} \sum_{n,m=0}^{p-1} q^{-2nm} k^{2n} \otimes \left(k^{(b)}\right)^{2m}, \tag{3.11}$$

where the subalgebra spanned by $\left\{\left(k^{(a)}\right)^n\right\}_{n=0}^{p-1}$ is the algebra isomorphic to A_p associated to the a-cycle, while $\left\{\left(k^{(b)}\right)^n\right\}_{n=0}^{p-1}$ — the one associated to the b-cycle.

Because the algebra A_p is commutative and the R-matrix is symmetric, the third exchange relation (2.29) simplifies to

$$A_{13}B_{23} = (R_{12})^2 B_{23} A_{13}. \tag{3.12}$$

One can show easily that this equation is satisfied when one imposes the following commutation relation on the elements of the handle algebra

$$\left(k^{(a)}\right)^n \left(k^{(b)}\right)^m = q^{\frac{nm}{2}} \left(k^{(b)}\right)^m \left(k^{(a)}\right)^n.$$

Indeed, it is the commutation relations for the elements of the Heisenberg double of A_p (recall the discussion above Section 2.5). We have thus found all the defining relations in $\mathcal{T}$.

3.3 Fock module $\mathfrak{R}$

In the following we explain the construction of the representation D of the handle algebra from Section 2.4. This representation has a cyclic vector, the vacuum defined by the trivial action of the b-cycle elements:

$$D\big((k^{(b)})^m\big)|0\rangle = |0\rangle \ , \tag{3.13}$$

which of course agrees with (2.32). The representation space $\mathfrak{R}$ is spanned by vectors $\{|n\rangle\}_{n=0}^{p-1}$ which are defined via application of the elements of the a-cycle subalgebra to the vacuum:

$$|n\rangle := D\big((k^{(a)})^{2n}\big)|0\rangle, \tag{3.14}$$

which follows from the definition (2.33) and the form of the universal element A. We note that actually all powers of $k^{(a)}$, odd and even, appear here – it is due to the relation $(k^{(a)})^p = 1$ and the condition that p is odd. From here, using the commutation relations in $\mathcal{T}$, one can calculate the action on arbitrary vectors of $\mathfrak{R}$

$$D\big((k^{(b)})^n\big)|m\rangle = q^{-nm}|m\rangle. \tag{3.15}$$

3.4 Gauge-invariance conditions

Now, we want to investigate the gauge-invariance conditions, explained in Section 2.5, and find the gauge-invariant subalgebra $\mathcal{A}$ of the handle algebra. Because of the commutativity of A_p, the equation (2.36) gives

$$k(k^{(i)}) = k^{(i)}, \tag{3.16}$$

for $i = a, b$. From this follow the commutation relations for the elements of the smash product $\bar{\mathcal{T}}$

$$\begin{aligned} \iota(k^n)\big(k^{(a)}\big)^m &= \big(k^{(a)}\big)^m \iota(k^n), \\ \iota(k^n)\big(k^{(b)}\big)^m &= \big(k^{(b)}\big)^m \iota(k^n), \end{aligned} \tag{3.17}$$

for all $n, m = 0, \ldots, p-1$. Because all elements of $\mathcal{T}$ commute with all elements of $\{\iota(x)|x \in \mathcal{G}\}$, the gauge-invariant subalgebra is in fact isomorphic to the handle algebra itself

$$\mathcal{A} = \mathcal{T}. \tag{3.18}$$

Finally, we extend the representation D of the handle algebra $\mathcal{T}$ to the smash product $\bar{\mathcal{T}}$ by

$$D(\iota(k^n))|m\rangle = |m\rangle.$$

where we also used the relations (3.17).

3.5 $SL(2,\mathbb{Z})$ action from the handle algebra

In this section, we construct the projective $SL(2,\mathbb{Z})$ representation via operators on the representation space of the gauge-invariant subalgebra, which is the handle algebra in this case. We obtain the matrix coefficients of the $\mathscr{S}$ and $\mathscr{T}$ transformations and we verify that those transformations indeed satisfy the relations of $SL(2,\mathbb{Z})$.

Using the integral (3.4) and the ribbon element (3.7), we get the explicit formulae for the quantum Dehn twist operators defined by (2.59)

$$\hat{v}(a) = \frac{1}{\sqrt{p}} \sum_{n=0}^{p-1} q^{n^2} \left(k^{(a)}\right)^{-2n}, \qquad \hat{v}(b) = \frac{1}{\sqrt{p}} \sum_{n=0}^{p-1} q^{n^2} \left(k^{(b)}\right)^{-2n}. \tag{3.19}$$

One can directly check that their adjoint actions via the automorphisms α and β given by (2.58) indeed satisfy the equations (2.57). The two Dehn twists are represented on the representation space $\mathfrak{R}$ by

$$D(\hat{v}(a))|n\rangle = \frac{1}{\sqrt{p}} \sum_{m=0}^{p-1} q^{m^2} |n-m\rangle, \tag{3.20}$$

$$D(\hat{v}(b))|n\rangle = i^{\omega(p)} q^{-n^2} |n\rangle, \tag{3.21}$$

where we used the identity (3.8). Using this representation, the $\mathscr{S}$- and $\mathscr{T}$-matrices, defined by (2.61) and (2.62) respectively, are realised as

$$D(\mathscr{S})|m\rangle = \frac{(-1)^{\omega(p)}}{\sqrt{p}} \sum_{n=0}^{p-1} q^{-2nm} |n\rangle, \tag{3.22}$$

$$D(\mathscr{T})|m\rangle = \frac{1}{\sqrt{p}} \sum_{n=0}^{p-1} q^{-n^2} |m+n\rangle. \tag{3.23}$$

From this explicit action, one can easily calculate that

$$D(\mathscr{S}^2)|m\rangle = |-m\rangle, \qquad D(\mathscr{S}^4)|m\rangle = |m\rangle, \tag{3.24}$$

and by iteratively applying the above expressions we get

$$D\big((\mathscr{S}\mathscr{T})^3\big)|m\rangle = |-m\rangle. \tag{3.25}$$

By comparing the expressions, we see that the $SL(2,\mathbb{Z})$ relations are indeed satisfied:

$$(\mathscr{S}\mathscr{T})^3 = \mathscr{S}^2, \qquad \mathscr{S}^4 = id. \tag{3.26}$$

3.6 Lyubashenko–Majid $SL(2,\mathbb{Z})$ action on the centre

As the algebra A_p is commutative, the centre $Z(A_p)$ is A_p. However, we note that the canonical construction of central elements via the map defined in (2.23) gives

$$a_n \equiv z_{\pi_n} = (\pi_n \otimes id)(M) = k^{2n}.$$

By using equation (2.65) one can find the $\mathscr{S}$ transformation on the central elements

$$\mathscr{S}_Z(a_n) = \frac{1}{\sqrt{p}} \sum_{m=0}^{p-1} q^{-2nm} a_m, \tag{3.27}$$

and therefore

$$\mathscr{S}_Z^2(a_n) = a_{-n}, \qquad\qquad \mathscr{S}_Z^4(a_n) = a_n.$$

In addition we have $\mathscr{T}_Z$ transformation as it was defined in the equation (2.66)

$$\mathscr{T}_Z(a_n) = \frac{i^{\omega(p)}}{\sqrt{p}} \sum_{m=0}^{p-1} q^{-(m-n)^2} a_m. \tag{3.28}$$

Therefore

$$(\mathscr{S}_Z\mathscr{T}_Z)(a_n) = \frac{1}{\sqrt{p}} \sum_{m=0}^{p-1} q^{m^2-2nm} a_m, \qquad\qquad (\mathscr{S}_Z\mathscr{T}_Z)^3(a_n) = i^{\omega(p)} a_{-n},$$

where we used again the identity (3.8). Therefore, we get the relation

$$(\mathscr{S}_Z\mathscr{T}_Z)^3 = \lambda {\mathscr{S}_Z}^2, \tag{3.29}$$

for $\lambda = i^{\omega(p)}$, i.e. we have indeed a projective representation of $SL(2,\mathbb{Z})$.

3.7 Equivalence of two actions

In this section, we show that the two $SL(2,\mathbb{Z})$ actions presented in Sections 3.5 and 3.6 agree projectively. In order to do that, we first establish that the centre $Z(A_p) = A_p$ and the representation space $\mathfrak{R}$ of the gauge-invariant algebra $\mathcal{A}$ for $\mathcal{G} = A_p$ are isomorphic as vector spaces, and this of course agrees with our general result established in (2.51).

Explicitly, we have that

$$Z(A_p) \ni a_n \overset{\simeq}{\longmapsto} |n\rangle \in \mathfrak{R}. \tag{3.30}$$

Moreover, if we take into account this isomorphism, we can compare the coefficients of the relevant actions in the two cases. In order to do that, let us define the coefficients of the $\mathscr{S}$- and $\mathscr{T}$-actions as

$$D(\mathscr{S})|m\rangle = \sum_{n=0}^{p-1} (\mathscr{S}_T)^n_m |n\rangle, \qquad\qquad D(\mathscr{T})|m\rangle = \sum_{n=0}^{p-1} (\mathscr{T}_T)^n_m |n\rangle,$$

for the handle algebra and

$$\mathscr{S}_Z(a_m) = \sum_{n=0}^{p-1} (\mathscr{S}_Z)^n_m a_n, \qquad\qquad \mathscr{T}_Z(a_m) = \sum_{n=0}^{p-1} (\mathscr{T}_Z)^n_m a_n,$$

for the centre of A_p in the LM picture. It is easy to read-off that those coefficients are

$$(\mathscr{S}_T)^n_m = (-1)^{\omega(p)} (\mathscr{S}_Z)^n_m = (-1)^{\omega(p)} \frac{1}{\sqrt{p}} q^{-2nm},$$
$$(\mathscr{T}_T)^n_m = i^{-\omega(p)} (\mathscr{T}_Z)^n_m = \frac{1}{\sqrt{p}} q^{-(n-m)^2}.$$

We see therefore that those two actions agree up to multiplicative constants, i.e. they agree projectively, as it was claimed.

4 $\overline{U}_q\, gl(1|1)$ algebra and its representation

In this section, we introduce a restricted version of the quantum enveloping Hopf algebra $\overline{U}_q\, gl(1|1)$ with q being the primitive pth root of unity, where p is an odd integer. To simplify notation, it will be understood that within this section $\mathcal{G} = \overline{U}_q\, gl(1|1)$.

We begin in Section 4.1 with recalling the Hopf algebra structure on $\mathcal{G}$ and compute its (co)integrals. Then in Section 4.2, we introduce the ribbon structure: we give the universal R-matrix, and calculate the corresponding monodromy matrix and the ribbon element. In Section 4.3, we also review known facts about the representation theory of this algebra. Then we construct the corresponding handle algebra $\mathcal{T}$ in Section 4.5 in terms of generators and relations, its gauge-invariant subalgebra $\mathcal{A}$ is studied in Section 4.7, and its Fock module $\mathfrak{R}_{(\mathcal{A})}$ is described in Section 4.8. The $SL(2,\mathbb{Z})$ action from the handle algebra approach is analysed in Section 4.9 where we also establish an equivalence with the modular action in [10]. Finally, in Section 4.11 we compare this action to the Lyubashenko–Majid action of $SL(2,\mathbb{Z})$ analysed in Section 4.10, confirming the conjecture formulated in Section 2.8.

4.1 Definition and (co)integrals

The restricted quantum group for $gl(1|1)$ that will be denoted by $\mathcal{G} = \overline{U}_q\, gl(1|1)$ is a super Hopf algebra generated by k_α, k_β and e_+, e_- with the defining relations

$$\begin{aligned} &k_\alpha^p = k_\beta^p = 1, && k_\alpha k_\beta = k_\beta k_\alpha, \\ &k_\alpha e_\pm = e_\pm k_\alpha, && \{e_\pm, e_\pm\} = 0, \\ &k_\beta e_\pm = q^{\pm 1} e_\pm k_\beta, && \{e_+, e_-\} = \frac{k_\alpha - k_\alpha^{-1}}{q - q^{-1}}, \end{aligned} \tag{4.1}$$

where the parameter is $q = e^{2\pi i/p}$ and we assume p is a positive odd integer. We note that the generator k_α is central. It is a finite-dimensional algebra with the basis

$$\{k_\alpha^n k_\beta^m e_+^r e_-^s \mid 0 \leq n, m \leq p-1\ ,\ 0 \leq r, s \leq 1\}\ . \tag{4.2}$$

The co-product has the form

$$\begin{aligned} &\Delta(k_\alpha) = k_\alpha \otimes k_\alpha, && \Delta(e_+) = e_+ \otimes 1 + k_\alpha^{-1} \otimes e_+, \\ &\Delta(k_\beta) = k_\beta \otimes k_\beta, && \Delta(e_-) = e_- \otimes k_\alpha + 1 \otimes e_-, \end{aligned} \tag{4.3}$$

the co-unit is

$$\epsilon(k_\alpha) = \epsilon(k_\beta) = 1, \qquad \epsilon(e_+) = \epsilon(e_-) = 0, \tag{4.4}$$

and the antipode is

$$\begin{aligned} &S(k_\alpha) = k_\alpha^{-1}, && S(e_+) = -k_\alpha e_+, \\ &S(k_\beta) = k_\beta^{-1}, && S(e_-) = -e_- k_\alpha^{-1}. \end{aligned} \tag{4.5}$$

The right integral as it was defined in (2.16) evaluated on the basis (4.2) of $\mathcal{G}$ has the following form

$$\mu(k_\alpha^n k_\beta^m e_+^r e_-^s) = \mathcal{N} \delta_{n,-1} \delta_{m,0} \delta_{r,1} \delta_{s,1}, \tag{4.6}$$

or alternatively we can write it as

$$\mu = \mathcal{N}(k_\alpha^{-1}e_+e_-)^*, \tag{4.7}$$

where the normalisation $\mathcal{N}$ will be fixed later.

It can be easily checked that the co-integral (2.18) is given by

$$\boldsymbol{c} = \frac{1}{\mathcal{N}} \sum_{n,m=0}^{p-1} k_\alpha^n k_\beta^m e_+ e_-, \tag{4.8}$$

where as usual we normalise it by $\mu(\boldsymbol{c}) = 1$. It is a two-sided co-integral.

4.2 R-matrix and ribbon element

The super Hopf algebra $\mathcal{G}$ is quasi-triangular with the universal R-matrix

$$R = \frac{1}{p^2}\left(1\otimes 1 - (q-q^{-1})e_+\otimes e_-\right) \sum_{n,m=0}^{p-1}\sum_{s,t=0}^{p-1} q^{nt+ms} k_\alpha^n k_\beta^m \otimes k_\alpha^{-s} k_\beta^{-t}. \tag{4.9}$$

This form of R-matrix was motivated by the construction [54] in the case of generic values of q. In addition, it will be useful (for the handle algebra relations) to spell explicitly the inverse of the R-matrix

$$R^{-1} = \frac{1}{p^2}\left(1\otimes 1 + (q-q^{-1})k_\alpha e_+\otimes k_\alpha^{-1}e_-\right) \sum_{n,m=0}^{p-1}\sum_{s,t=0}^{p-1} q^{-nt-ms} k_\alpha^n k_\beta^m \otimes k_\alpha^{-s} k_\beta^{-t}. \tag{4.10}$$

And we also need the monodromy matrix

$$\begin{aligned} M = &\left((q-q^{-1})^{-1}1\otimes 1 + e_-\otimes e_+ - k_\alpha^{-1}e_+\otimes k_\alpha e_- + (q-q^{-1})k_\alpha^{-1}e_-e_+\otimes k_\alpha e_+e_-\right)\\ &\times (q-q^{-1})\frac{1}{p^2}\sum_{n,m,s,t=0}^{p-1} q^{-2nt-2ms} k_\alpha^{2n} k_\beta^{2m}\otimes k_\alpha^{2s}k_\beta^{2t}. \end{aligned} \tag{4.11}$$

Introducing the idempotents

$$e_{n,m} = \frac{1}{p^2}\sum_{s,t=0}^{p-1} q^{-2nt-2ms} k_\alpha^{2s} k_\beta^{2t},$$

the second line in (4.11) can be written as $(q-q^{-1})\sum_{n,m=0}^{p-1} k_\alpha^{2n}k_\beta^{2m}\otimes e_{n,m}$. Then similarly to analysis of M in Section 3, we conclude that M takes the form (2.12), and it is thus non-degenerate. We note that this is true for odd p only, because only then $\{k_\alpha^{2n}k_\beta^{2m}e_+^r e_-^s\}_{n,m,r,s=0}^{p-1,p-1,1,1}$ and $\{e_{n,m}e_+^r e_-^s\}_{n,m,r,s=0}^{p-1,p-1,1,1}$ are bases of $\mathcal{G}$. (This is why we assumed above that p is odd.) Therefore, $\mathcal{G}$ is a factorisable super Hopf algebra.

With the monodromy matrix (4.11) and according to the equation (2.19) we can fix the normalisation for the integral in (4.7) as

$$\mathcal{N} = \frac{ip}{q-q^{-1}}. \tag{4.12}$$

Using the right integral μ from (4.7), we find the co-modulus $\boldsymbol{a} \in \mathcal{G}$ (2.17) to be

$$\boldsymbol{a} = k_\alpha^{-2}, \tag{4.13}$$

which admits a group-like square root, and therefore the balancing element $\boldsymbol{g}$ is just

$$\boldsymbol{g} = k_\alpha^{-1}. \tag{4.14}$$

We note that the element $\boldsymbol{g}$ is central and it satisfies $S^2(x) = \boldsymbol{g}x\boldsymbol{g}^{-1}$ for all $x \in \mathcal{G}$, and this is consistent with the fact that $S^2 = id$ in this case.

Then, using the expression for the ribbon element $\boldsymbol{v}$ from (2.21) we find its explicit form (after making appropriate re-summation)

$$\boldsymbol{v}^{\pm 1} = \frac{1}{p} k_\alpha^{\pm 1}(1 \mp (q - q^{-1})k_\alpha^{\mp 1} e_- e_+) \sum_{n,m=0}^{p-1} q^{\pm 2nm} k_\alpha^{2n} k_\beta^{2m}. \tag{4.15}$$

One can of course directly check the ribbon axioms (2.13). We therefore conclude that $\mathcal{G}$ is a ribbon factorisable super Hopf algebra.

4.3 Representations of $\overline{U}_q\, gl(1|1)$

Here, we briefly review representation theory of $\mathcal{G}$, which has been studied e.g. in [8, 55–57].

The important class of representations consists of 1-dimensional atypical representations, 2-dimensional typical representations and 4-dimensional indecomposable projective representations[7] that we describe below in a basis. The major difference from the previous section when we considered the algebra A_p based on the finite cyclic group is that $\overline{U}_q\, gl(1|1)$ is not semisimple — as we recall below, there are 4-dimensional projective representations which are reducible but indecomposable.

We start describing the so-called atypical representations $\pi_n \colon\ \mathcal{G} \to \mathbb{C}^{1|0}$, for $n = 0, \ldots, p-1$,

$$\begin{aligned} \pi_n(k_\alpha) &= 1, & \pi_n(e_+) &= 0, \\ \pi_n(k_\beta) &= q^n, & \pi_n(e_-) &= 0, \end{aligned} \tag{4.16}$$

All the atypical representations are one-dimensional and clearly irreducible. The co-unit ϵ corresponds to the atypical representation π_0. Moreover, we have a series of the so-called typical representations $\pi_{e,n} \colon\ \mathcal{G} \to \mathrm{End}(\mathbb{C}^{1|1})$ with $e, n = 0, \ldots, p-1$

$$\begin{aligned} \pi_{e,n}(k_\alpha) &= q^e \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \pi_{e,n}(e_+) &= \begin{pmatrix} 0 & 0 \\ [e]_q & 0 \end{pmatrix}, \\ \pi_{e,n}(k_\beta) &= q^n \begin{pmatrix} q^{-1} & 0 \\ 0 & 1 \end{pmatrix}, & \pi_{e,n}(e_-) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{4.17}$$

where $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$ is the q-number. The typical representations $\pi_{e,n}$ are two-dimensional (with 1 even and 1 odd degrees), and when $e \neq 0$ they are irreducible. When $e = 0$, the representation

[7] These are actual linear representations, also often called "projective modules", and should not be confused with *projective* representations from group theory which are linear up to a multiplicative constant.

$\pi_{0,n}$ is not irreducible anymore, but it is still indecomposable: it is built up from two atypical irreducible representations π_n and π_{n-1} connected by the action of e_-. The corresponding subquotient structure can be written diagrammatically as

$$\pi_n \to \pi_{n-1}, \tag{4.18}$$

where the arrow points to a submodule and it corresponds to a "non-invertible" action of the algebra.

Besides the atypical and typical representations, one has as well the projective representations $\pi_{\mathcal{P}_N}: \mathcal{G} \to Hom(\mathbb{C}^{2|2})$, which are defined by the matrix realisations on 4-dimensional vector space with 2 even and 2 odd degrees as follows

$$\begin{aligned}
\pi_{\mathcal{P}_N}(k_\alpha) &= \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, &\qquad \pi_{\mathcal{P}_N}(e_+) &= \begin{pmatrix} 0&0&0&0\\ -q^{-1}&0&0&0\\ 1&0&0&0\\ 0&1&q^{-1}&0 \end{pmatrix},\\
\pi_{\mathcal{P}_N}(k_\beta) &= q^N \begin{pmatrix} q^{-1}&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&q \end{pmatrix}, &\qquad \pi_{\mathcal{P}_N}(e_-) &= \begin{pmatrix} 0&1&q^{-1}&0\\ 0&0&0&q^{-1}\\ 0&0&0&-1\\ 0&0&0&0 \end{pmatrix},
\end{aligned} \tag{4.19}$$

where vectors $(1\,0\,0\,0)^t$ and $(0\,0\,0\,1)^t$ are odd and $(0\,1\,0\,0)^t$ and $(0\,0\,1\,0)^t$ are even, and where ^{t} denotes the transposition.

The representations $\pi_{\mathcal{P}_N}$ are reducible but indecomposable, and come from the tensor product of two typical representations $\pi_{-e,N} \otimes \pi_{e,1}$. They are built up from 4 atypical representations $\pi_{N+1}, \pi_N, \pi_N, \pi_{N-1}$ which constitute the module according to the subquotient diagram

$$\pi_{\mathcal{P}_N} \quad = \quad \begin{tikzcd} & \pi_N \arrow[dl] \arrow[dr] & \\ \pi_{N+1} \arrow[dr] & & \pi_{N-1} \arrow[dl] \\ & \pi_N \arrow[uu, "\sigma"'] & \end{tikzcd} \tag{4.20}$$

where the arrows are meant to be actions of $e_\pm$ (and here the map σ should be ignored for a moment, it will be explained later). Explicitly, the following vectors of the 4-dimensional module constitute the modules of the atypical representations in the diagram

$$\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix} \in \pi_{N-1}, \qquad \begin{pmatrix} 0\\-q^{-1}\\1\\0 \end{pmatrix} \in \text{bottom } \pi_N, \qquad \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \in \pi_{N+1}, \qquad \begin{pmatrix} 0\\1\\q\\0 \end{pmatrix} \in \text{top } \pi_N.$$

It is worthwhile to note that the Casimir element of $\mathcal{G}$ evaluated on the projective representation $\pi_{\mathcal{P}_N}$ maps the top atypical representation to the bottom one, and it is zero otherwise. It is not realised by an invertible matrix.

For the purposes of the next section, we want to find a matrix σ that maps the bottom atypical sub-representation to the top one and it is zero otherwise, i.e. σ satisfies the following relations

$$\sigma\begin{pmatrix}0\\-q^{-1}\\1\\0\end{pmatrix}=\begin{pmatrix}0\\1\\q\\0\end{pmatrix},\qquad \sigma\begin{pmatrix}0\\1\\q\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\qquad \sigma\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\qquad \sigma\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

which determines σ up to a multiplicative constant. It is realised as a matrix

$$\sigma=\frac{q}{2}\begin{pmatrix}0&0&0&0\\0&-1&q^{-1}&0\\0&-q&1&0\\0&0&0&0\end{pmatrix}. \tag{4.21}$$

Finally, let us recall the definition of the quantum supertrace. The ordinary supertrace of an $n\times n$ matrix $\mathbf{X}$ with the coefficients $[\mathbf{X}]^i_j=X^i_j$ is defined as

$$\mathrm{str}(\mathbf{X})=\sum_{i=1}^{n}(-1)^{|i|}X^i_i, \tag{4.22}$$

where $|i|$ denotes the grading of the diagonal element X^i_i. As applied to two-dimensional matrices on $\mathbb{C}^{1|1}$, the supertrace is explicitly

$$\mathrm{str}\begin{pmatrix}x^1_1&x^1_2\\x^2_1&x^2_2\end{pmatrix}=x^1_1-x^2_2,$$

and for four-dimensional matrices on $\mathbb{C}^{2|2}$

$$\mathrm{str}\begin{pmatrix}x^1_1&x^1_2&x^1_3&x^1_4\\x^2_1&x^2_2&x^2_3&x^2_4\\x^3_1&x^3_2&x^3_3&x^3_4\\x^4_1&x^4_2&x^4_3&x^4_4\end{pmatrix}=x^1_1-x^2_2-x^3_3+x^4_4.$$

Then, the quantum supertrace is defined by inserting the inverse balancing element $\boldsymbol{g}^{-1}$:

$$\mathrm{str}_q(\pi(X))=\mathrm{str}(\pi(\boldsymbol{g}^{-1}X)), \tag{4.23}$$

where π is, for our purposes, a representation of $\mathcal{G}$, i.e. $\pi=\pi_n,\pi_{e,n},\pi_{\mathcal{P}_N}$.

4.4 The centre of $\mathcal{G}$

In this section, we construct a basis in the centre $Z(\mathcal{G})$ of $\mathcal{G}$ using the description of projective representations in the previous section.

First, we recall that central elements can be constructed using the so-called Drinfeld map

$$\phi\mapsto(\phi\otimes id)(M)$$

for ϕ satisfying (2.60). Examples of such ϕ are the quantum traces str_q over representations of $\mathcal{G}$, recall (2.22). Using this construction for the typical and atypical representations, we obtain the central elements

$$c_{e,n} = (\mathrm{str}_q \otimes id)\big\{(\pi_{e,n} \otimes id)M\big\}, \qquad 1 \le e \le p-1,\ 0 \le n \le p-1. \tag{4.24}$$

$$a_n = (\mathrm{str}_q \otimes id)\big\{(\pi_n \otimes id)M\big\}, \qquad 0 \le n \le p-1, \tag{4.25}$$

p^2 elements in total. Their span however gives only a proper subalgebra in $Z(\mathcal{G})$. To construct the missing central elements, we follow an approach in [58] that uses the so-called pseudo-traces. For this, one has to consider a direct sum of all projective indecomposables from the same (categorical) block which is not semisimple. We recall that each block corresponds to a two-sided ideal in the algebra and vice versa, and in our case we have just one such non-semisimple block (in contrast to [58] where one had to consider $p-1$ of them). We thus have only one missing central element and it is given by

$$b = (\mathrm{str}_q \otimes id)\big\{(\pi_\sigma \otimes id)M\big\}, \tag{4.26}$$

where we introduced a special map $\pi_\sigma\colon \mathcal{G} \to \mathrm{End}(\oplus_N \mathcal{P}_N)$ as

$$\pi_\sigma(-) := \bigoplus_{N=0}^{p-1} \sigma \circ \pi_{\mathcal{P}_N}(-) \tag{4.27}$$

and the linear map σ is defined in (4.20)-(4.21). (The quantum trace str_q precomposed with such a map is what we call pseudo-traces, see more details in [58].)

Explicitly, the central elements introduced above are given by the following expressions

$$c_{e,n} = q^e(q-q^{-1})\left\{(1-q^{-2e})e_+e_- - \frac{k_\alpha - k_\alpha^{-1}}{q-q^{-1}}\right\} k_\alpha^{2n-1}k_\beta^{2e}, \tag{4.28}$$

$$a_n = k_\alpha^{2n}, \tag{4.29}$$

$$b = -2(q-q^{-1})^2 \sum_{t=0}^{p-1} k_\alpha^{2t} e_+e_-. \tag{4.30}$$

We first note that these are linearly independent elements in the centre of $\mathcal{G}$. To show that any central element is a linear combination of these ones, we first calculate dimension of the centre by analysing bimodule endomorphisms of the regular representation along the lines in [59], and conclude that the dimension

$$\dim\big(Z(\mathcal{G})\big) = p^2 + 1 \tag{4.31}$$

agrees with the number of the central elements given above.

4.5 The handle algebra of $\overline{U}_q\, gl(1|1)$

In this section, we describe the handle algebra of $\mathcal{G}$ in terms of generators and relations.

We start by stating the form of the universal elements A and B corresponding to the a- and b-cycles, which solve the exchange equations (2.27)-(2.28):

$$
\begin{aligned}
A = &\Big[(q-q^{-1})^{-1} + e_- \otimes e_+^{(a)} - k_\alpha^{-1} e_+ \otimes k_\alpha^{(a)} e_-^{(a)} + (q-q^{-1}) k_\alpha^{-1} e_- e_+ \otimes k_\alpha^{(a)} e_+^{(a)} e_-^{(a)}\Big] \\
&\times \frac{1}{p^2}(q-q^{-1}) \sum_{n,m,s,t=0}^{p-1} q^{-2nt-2ms} k_\alpha^{2n} k_\beta^{2m} \otimes \big(k_\alpha^{(a)}\big)^{2s} \big(k_\beta^{(a)}\big)^{2t}, \\
B = &\Big[(q-q^{-1})^{-1} + e_- \otimes e_+^{(b)} - k_\alpha^{-1} e_+ \otimes k_\alpha^{(b)} e_-^{(b)} + (q-q^{-1}) k_\alpha^{-1} e_- e_+ \otimes k_\alpha^{(b)} e_+^{(b)} e_-^{(b)}\Big] \\
&\times \frac{1}{p^2}(q-q^{-1}) \sum_{n,m,s,t=0}^{p-1} q^{-2nt-2ms} k_\alpha^{2n} k_\beta^{2m} \otimes \big(k_\alpha^{(b)}\big)^{2s} \big(k_\beta^{(b)}\big)^{2t}.
\end{aligned}
$$

where the elements

$$\big\{\big(k_\alpha^{(a)}\big)^n \big(k_\beta^{(a)}\big)^m \big(e_+^{(a)}\big)^r \big(e_-^{(a)}\big)^s\big\}_{n,m,r,s=0}^{p-1,p-1,1,1}$$

span the subalgebra $\mathcal{T}^{(a)}$ isomorphic to $\mathcal{G}$ associated to the a-cycle, and

$$\big\{\big(k_\alpha^{(b)}\big)^n \big(k_\beta^{(b)}\big)^m \big(e_+^{(b)}\big)^r \big(e_-^{(b)}\big)^s\big\}_{n,m,r,s=0}^{p-1,p-1,1,1}$$

— the one associated to the b-cycle.

One can show that the third exchange relation (2.29) implies the following "mixed" commutation relations

$$
\begin{aligned}
&\big(k_\beta^{(a)}\big)^{2a}\big(k_\beta^{(b)}\big)^{2b} = \big(k_\beta^{(b)}\big)^{2b}\big(k_\beta^{(a)}\big)^{2a}, &&\big(k_\alpha^{(a)}\big)^{2a}\big(k_\alpha^{(b)}\big)^{2b} = \big(k_\alpha^{(b)}\big)^{2b}\big(k_\alpha^{(a)}\big)^{2a}, \\
&\big(k_\beta^{(a)}\big)^{2a}\big(k_\alpha^{(b)}\big)^{2b} = q^{2ab}\big(k_\alpha^{(b)}\big)^{2b}\big(k_\beta^{(a)}\big)^{2a}, &&\big(k_\alpha^{(a)}\big)^{2a}\big(k_\beta^{(b)}\big)^{2b} = q^{2ab}\big(k_\beta^{(b)}\big)^{2b}\big(k_\alpha^{(a)}\big)^{2a}, \\
&\big(k_\beta^{(a)}\big)^{2a} e_-^{(b)} = q^{-a} e_-^{(b)}\big(k_\beta^{(a)}\big)^{2a}, &&\big[\big(k_\beta^{(a)}\big)^{2a}, e_+^{(b)}\big] = q^{a}[a]_q(q-q^{-1}) e_+^{(a)}\big(k_\beta^{(a)}\big)^{2a}, \\
&\big(k_\beta^{(b)}\big)^{2a} e_+^{(a)} = q^{2a} e_+^{(a)}\big(k_\beta^{(b)}\big)^{2a}, &&\big(k_\beta^{(b)}\big)^{2a} e_-^{(a)} = q^{-a} e_-^{(a)}\big(k_\beta^{(b)}\big)^{2a} + \\
& &&- q^{-2a}[a]_q(q-q^{-1})\big(k_\alpha^{(a)}\big)\big(k_\alpha^{(b)}\big)^{-1} e_-^{(b)}\big(k_\beta^{(b)}\big)^{2a}, \\
&\big[\big(k_\alpha^{(a)}\big)^{a}, e_+^{(b)}\big] = 0, &&\big[\big(k_\alpha^{(b)}\big)^{a}, e_+^{(a)}\big] = 0, \\
&\big[\big(k_\alpha^{(a)}\big)^{a}, e_-^{(b)}\big] = 0, &&\big[\big(k_\alpha^{(b)}\big)^{a}, e_-^{(a)}\big] = 0,
\end{aligned}
\tag{4.32}
$$

and the following anti-commutation relations

$$
\begin{aligned}
&\{e_+^{(a)}, e_+^{(b)}\} = 0, &&\{e_+^{(a)}, e_-^{(b)}\} = k_\alpha^{(b)}(q-q^{-1})^{-1}, \\
&\{e_-^{(a)}, e_-^{(b)}\} = 0, &&\{e_-^{(a)}, e_+^{(b)}\} = \Big(k_\alpha^{(a)} - \big(k_\alpha^{(a)}\big)^{-1} - k_\alpha^{(a)}\big(k_\alpha^{(b)}\big)^{-2}\Big)(q-q^{-1})^{-1}.
\end{aligned}
\tag{4.33}
$$

We claim that the above relations, together with the $\overline{U}_q\, gl(1|1)$ relations for the generators of the subalgebras $\mathcal{T}^{(a)}$ and $\mathcal{T}^{(b)}$ due to the isomorphisms noted above, constitute the complete set of defining relations for $\mathcal{T}$.

4.6 The action of $\mathcal{G}$ on $\mathcal{T}$

Now, we want to investigate the left action of $\mathcal{G}$ on $\mathcal{T}$, with the end-goal of constructing the smash product $\bar{\mathcal{T}}$. Using (2.36) we obtain

$$
\begin{aligned}
&k_\alpha(k_\alpha^{(i)}) = k_\alpha^{(i)}, && k_\alpha(k_\beta^{(i)}) = k_\beta^{(i)}, && k_\alpha(e_+^{(i)}) = e_+^{(i)}, && k_\alpha(e_-^{(i)}) = e_-^{(i)},\\
&k_\beta(k_\alpha^{(i)}) = k_\alpha^{(i)}, && k_\beta(k_\beta^{(i)}) = k_\beta^{(i)}, && k_\beta(e_+^{(i)}) = qe_+^{(i)}, && k_\beta(e_-^{(i)}) = q^{-1}e_-^{(i)},\\
&e_+(k_\alpha^{(i)}) = 0, && e_+(k_\beta^{(i)}) = \big[e_+^{(i)}, k_\beta^{(i)}\big], && e_+(e_+^{(i)}) = 0, && e_+(e_-^{(i)}) = \frac{k_\alpha^{(i)} - (k_\alpha^{(i)})^{-1}}{q-q^{-1}},\\
&e_-(k_\alpha^{(i)}) = 0, && e_-(k_\beta^{(i)}) = \big[e_-^{(i)}, k_\beta^{(i)}\big]\big(k_\alpha^{(i)}\big)^{-1}, && e_-(e_+^{(i)}) = \frac{1-(k_\alpha^{(i)})^{-2}}{q-q^{-1}}, && e_-(e_-^{(i)}) = 0,
\end{aligned}
$$

for $i = a, b$, and with the obvious choice of algebra isomorphisms $\kappa^{(i)}\colon\ \mathcal{G} \to \mathcal{T}^{(i)}$

$$\kappa^{(i)}(k_\alpha) = k_\alpha^{(i)}, \quad \kappa^{(i)}(k_\beta) = k_\beta^{(i)}, \quad \kappa^{(i)}(e_\pm) = e_\pm^{(i)}. \tag{4.34}$$

This leads to the following (anti-)commutation relations for the elements of the smash product algebra $\bar{\mathcal{T}}$

$$
\begin{aligned}
&\iota(k_\alpha)\big(k_\alpha^{(i)}\big)^{2n} = \big(k_\alpha^{(i)}\big)^{2n}\iota(k_\alpha), && \iota(k_\beta)\big(k_\alpha^{(i)}\big)^{2n} = \big(k_\alpha^{(i)}\big)^{2n}\iota(k_\beta),\\
&\iota(k_\alpha)\big(k_\beta^{(i)}\big)^{2n} = \big(k_\beta^{(i)}\big)^{2n}\iota(k_\alpha), && \iota(k_\beta)\big(k_\beta^{(i)}\big)^{2n} = \big(k_\beta^{(i)}\big)^{2n}\iota(k_\beta),\\
&\iota(k_\alpha)e_+^{(i)} = e_+^{(i)}\iota(k_\alpha), && \iota(k_\beta)e_+^{(i)} = e_+^{(i)}(\iota(k_\beta)+1),\\
&\iota(k_\alpha)e_-^{(i)} = e_-^{(i)}\iota(k_\alpha), && \iota(k_\beta)e_-^{(i)} = e_-^{(i)}(\iota(k_\beta)-1),
\end{aligned}
\tag{4.35}
$$

and

$$
\begin{aligned}
\iota(e_+)\big(k_\alpha^{(i)}\big)^{2n} &= \big(k_\alpha^{(i)}\big)^{2n}\iota(e_+),\\
\iota(e_+)\big(k_\beta^{(i)}\big)^{2n} &= \big(k_\beta^{(i)}\big)^{2n}\iota(e_+) - q^{-n}[n]_q(q-q^{-1})\big(k_\beta^{(i)}\big)^{2n}e_+^{(i)},\\
\iota(e_+)e_+^{(i)} &= -e_+^{(i)}\iota(e_+),\\
\iota(e_+)e_-^{(i)} &= -e_-^{(i)}\iota(e_+) + \frac{k_\alpha^{(i)} - (k_\alpha^{(i)})^{-1}}{q-q^{-1}},
\end{aligned}
\tag{4.36}
$$

and

$$
\begin{aligned}
\iota(e_-)\big(k_\alpha^{(i)}\big)^{2n} &= \big(k_\alpha^{(i)}\big)^{2n}\iota(e_-),\\
\iota(e_-)\big(k_\beta^{(i)}\big)^{2n} &= \big(k_\beta^{(i)}\big)^{2n}\iota(e_-) + q^{n}[n]_q(q-q^{-1})\big(k_\alpha^{(i)}\big)^{-1}\big(k_\beta^{(i)}\big)^{2n}e_-^{(i)}\iota(k_\alpha),\\
\iota(e_-)e_+^{(i)} &= -e_+^{(i)}\iota(e_-) + \frac{1-(k_\alpha^{(i)})^{-2}}{q-q^{-1}}\iota(k_\alpha),\\
\iota(e_-)e_-^{(i)} &= -e_-^{(i)}\iota(e_-),
\end{aligned}
\tag{4.37}
$$

for $i = a, b$. It can be checked that the above (anti-)commutation relations are reproduced from the equations (2.41)-(2.42) that are explicitly given in our case by the system of equations:

$$
\begin{aligned}
&(1\otimes\iota(k_\alpha))A = A(1\otimes\iota(k_\alpha)),\\
&(1\otimes\iota(k_\beta))A = A(1\otimes\iota(k_\beta)),\\
&(k_\alpha^{-1}\otimes\iota(e_+) + e_+\otimes\iota(1))A = A(k_\alpha^{-1}\otimes\iota(e_+) + e_+\otimes\iota(1)),\\
&(1\otimes\iota(e_-) + e_-\otimes\iota(k_\alpha))A = A(1\otimes\iota(e_-) + e_-\otimes\iota(k_\alpha)),
\end{aligned}
\tag{4.38}
$$

for the a-cycle, while for the b-cycle they are

$$\begin{aligned}
&(1\otimes\iota(k_\alpha))B = B(1\otimes\iota(k_\alpha)),\\
&(1\otimes\iota(k_\beta))B = B(1\otimes\iota(k_\beta)),\\
&(k_\alpha^{-1}\otimes\iota(e_+)+e_+\otimes\iota(1))B = B(k_\alpha^{-1}\otimes\iota(e_+)+e_+\otimes\iota(1)),\\
&(1\otimes\iota(e_-)+e_-\otimes\iota(k_\alpha))B = B(1\otimes\iota(e_-)+e_-\otimes\iota(k_\alpha)).
\end{aligned}\tag{4.39}$$

4.7 The gauge-invariant subalgebra $\mathcal{A}$

Here, we study the gauge-invariant subalgebra $\mathcal{A}$. In order to investigate it, we begin with an arbitrary vector in $\mathcal{A}$ of the form

$$\begin{aligned}
f = \sum_{\substack{n_1,m_1,\\ n_2,m_2=0}}^{p-1}\sum_{\substack{r_1,s_1,\\ r_2,s_2=0}}^{1} & f_{r_1,s_1,r_2,s_2}(n_1,m_1,n_2,m_2)\big(k_\alpha^{(a)}\big)^{2n_1}\big(k_\beta^{(a)}\big)^{2m_1}\big(e_+^{(a)}\big)^{r_1}\big(e_-^{(a)}\big)^{s_1}\times\\
&\times\big(k_\alpha^{(b)}\big)^{2n_2}\big(k_\beta^{(b)}\big)^{2m_2}\big(e_+^{(b)}\big)^{r_2}\big(e_-^{(b)}\big)^{s_2}.
\end{aligned}\tag{4.40}$$

While the indices n_1,m_1,n_2,m_2 of the coefficients $f_{r_1,s_1,r_2,s_2}(n_1,m_1,n_2,m_2)$ are *a priori* integers, we extend them to half-integers by equating the indices $n\pm\frac{1}{2}$ with $n+\frac{p\pm1}{2}$. We will use this convention in this and the following sections. Then, the gauge-invariance conditions

$$\iota(x)f = (-1)^{|f||x|}f\iota(x),\qquad x=k_\alpha,k_\beta,e_+,e_-,\tag{4.41}$$

translate to a set of conditions for the coefficients

$$\begin{aligned}
&f_{1,1,1,1}(n_1,m_1,n_2-\tfrac{1}{2},m_2)-f_{1,1,1,1}(n_1,m_1,n_2+\tfrac{1}{2},m_2)+q^{-m_2}[m_2]_q(q-q^{-1})^2f_{1,1,0,0}(n_1,m_1,n_2,m_2)\\
&=-q^{-m_1}[m_1]_q(q-q^{-1})^2f_{0,1,1,0}(n_1,m_1,n_2,m_2),
\end{aligned}$$

$$\begin{aligned}
&f_{1,1,1,1}(n_1-\tfrac{1}{2},m_1,n_2,m_2)-f_{1,1,1,1}(n_1+\tfrac{1}{2},m_1,n_2,m_2)+q^{-m_1}[m_1]_q(q-q^{-1})^2f_{0,0,1,1}(n_1,m_1,n_2,m_2)\\
&=q^{-m_2}[m_2]_q(q-q^{-1})^2f_{1,0,0,1}(n_1,m_1,n_2,m_2),
\end{aligned}$$

$$\begin{aligned}
&f_{1,1,0,0}(n_1-\tfrac{1}{2},m_1,n_2,m_2)-f_{1,1,0,0}(n_1+\tfrac{1}{2},m_1,n_2,m_2)+q^{-m_1}[m_1]_q(q-q^{-1})^2f_{0,0,0,0}(n_1,m_1,n_2,m_2)\\
&=f_{1,0,0,1}(n_1,m_1,n_2+\tfrac{1}{2},m_2)-f_{1,0,0,1}(n_1,m_1,n_2-\tfrac{1}{2},m_2),
\end{aligned}$$

$$\begin{aligned}
&f_{0,0,1,1}(n_1,m_1,n_2-\tfrac{1}{2},m_2)-f_{0,0,1,1}(n_1,m_1,n_2+\tfrac{1}{2},m_2)+q^{-m_2}[m_2]_q(q-q^{-1})^2f_{0,0,0,0}(n_1,m_1,n_2,m_2)\\
&=f_{0,1,1,0}(n_1-\tfrac{1}{2},m_1,n_2,m_2)-f_{0,1,1,0}(n_1+\tfrac{1}{2},m_1,n_2,m_2),
\end{aligned}$$

$$\begin{aligned}
&f_{1,1,1,1}(n_1,m_1,n_2,m_2)-f_{1,1,1,1}(n_1+1,m_1,n_2,m_2)-q^{-m_1}[m_1]_q(q-q^{-1})^2f_{0,0,1,1}(n_1+\tfrac{1}{2},m_1,n_2,m_2)\\
&=-q^{m_2-2m_1}[m_2]_q(q-q^{-1})^2f_{0,1,1,0}(n_1,m_1,n_2+\tfrac{1}{2},m_2),
\end{aligned}$$

$$\begin{aligned}
&f_{1,1,1,1}(n_1,m_1,n_2,m_2)-f_{1,1,1,1}(n_1,m_1,n_2+1,m_2)+q^{-m_2}[m_2]_q(q-q^{-1})^2f_{1,1,0,0}(n_1,m_1,n_2+\tfrac{1}{2},m_2)\\
&=q^{m_1-2m_2}[m_1]_q(q-q^{-1})^2f_{1,0,0,1}(n_1+\tfrac{1}{2},m_1,n_2,m_2),
\end{aligned}$$

$$f_{0,0,1,1}(n_1,m_1,n_2,m_2) - f_{0,0,1,1}(n_1,m_1,n_2+1,m_2) + q^{-m_2}[m_2]_q(q-q^{-1})^2 f_{0,0,0,0}(n_1,m_1,n_2+\frac{1}{2},m_2)$$
$$= q^{2(m_1-m_2)}\left(f_{1,0,0,1}(n_1+1,m_1,n_2,m_2) - f_{1,0,0,1}(n_1,m_1,n_2,m_2)\right),$$

$$f_{1,1,0,0}(n_1,m_1,n_2,m_2) - f_{1,1,0,0}(n_1+1,m_1,n_2,m_2) + q^{-m_1}[m_1]_q(q-q^{-1})^2 f_{0,0,0,0}(n_1+\frac{1}{2},m_1,n_2,m_2)$$
$$= q^{2(m_2-m_1)}\left(f_{0,1,1,0}(n_1,m_1,n_2,m_2) - f_{0,1,1,0}(n_1,m_1,n_2+1,m_2)\right),$$

for $n_1, n_2, m_1, m_2 = 0, \dots p-1$, and all the other coefficients are zero. The equations above have been obtained in the following way: one can commute elements $\iota(x)$ through the elements from the expansion (4.40)

$$e_{r_1,s_1,r_2,s_2}(n_1,m_1,n_2,m_2) = \left(k_\alpha^{(a)}\right)^{2n_1}\left(k_\beta^{(a)}\right)^{2m_1}\left(e_+^{(a)}\right)^{r_1}\left(e_-^{(a)}\right)^{s_1}\times$$
$$\times\left(k_\alpha^{(b)}\right)^{2n_2}\left(k_\beta^{(b)}\right)^{2m_2}\left(e_+^{(b)}\right)^{r_2}\left(e_-^{(b)}\right)^{s_2},$$

and this produces terms coming from the non-trivial commutation relations (4.35)-(4.37). Then in the sum (4.40), as the elements $e_{r_1,s_1,r_2,s_2}(n_1,m_1,n_2,m_2)$ are linearly independent, the coefficients in front of them have to vanish independently from one another, and this leads to a set of equations on the coefficients $f_{r_1,s_1,r_2,s_2}(n_1,m_1,n_2,m_2)$. First, the equations corresponding to the commutation with $\iota(k_\beta)$ imply that all the coefficients except $f_{1,1,1,1}$, $f_{1,1,0,0}$, $f_{0,0,1,1}$, $f_{1,0,0,1}$, $f_{0,1,1,0}$ and $f_{0,0,0,0}$ are zero. In particular, we see that $\mathcal{A}$ is even. Then, the first four equations above were obtained from the commutation with $\iota(e_+)$, while the remaining four were obtained in the case with $\iota(e_-)$.

We now make an important observation: $\mathcal{A}$ contains all elements $f \in Z(\mathcal{T}^{(i)})$ which are central within the i-cycle subalgebra $\mathcal{T}^{(i)}$, or such that

$$[f, k_\alpha^{(i)}] = [f, k_\beta^{(i)}] = [f, e_\pm^{(i)}] = 0.$$

This of course follows from our general result in (2.45). One can also see this via a direct calculation. As said above f should be even, and we further assume that f can be written as a sum of products of even elements corresponding to the two cycles a and b, and we show below that such an assumption gives non-trivial solutions. From the assumption, it follows that $r_1+s_1 = 0 \bmod 2$ and $r_2+s_2 = 0 \bmod 2$. In other words, we take the following Ansatz for the coefficients from (4.40):

$$f_{r_1,s_1,r_2,s_2}(n_1,m_1,n_2,m_2) = f^{(a)}_{r_1,s_1}(n_1,m_1) f^{(b)}_{r_2,s_2}(n_2,m_2) \tag{4.42}$$

with

$$f^{(i)}_{0,1}(n,m) = f^{(i)}_{1,0}(n,m) = 0, \qquad i = a,b. \tag{4.43}$$

So, in particular $f_{1,0,0,1}(n_1,m_1,n_2,m_2) = f_{0,1,1,0}(n_1,m_1,n_2,m_2) = 0$. Then, the gauge-invariance equations reduce to the following simple equation

$$f^{(i)}_{1,1}(n+\frac{1}{2},m) - f^{(i)}_{1,1}(n-\frac{1}{2},m) = q^{-m}[m]_q(q-q^{-1})^2 f^{(i)}_{0,0}(n,m). \tag{4.44}$$

We now recall the description of the centre in Section 4.4, and check that (4.43) and (4.44) are satisfied if and only if the element

$$f^{(i)} = \sum_{n,m=0}^{p-1}\sum_{r,s=0}^{1} f^{(i)}_{r,s}(n,m)\left(k_\alpha^{(i)}\right)^{2n}\left(k_\beta^{(i)}\right)^{2m}\left(e_+^{(i)}\right)^{r}\left(e_-^{(i)}\right)^{s}$$

belongs to the centre of $\mathcal{T}^{(i)}$. We establish this result using the basis of the centre provided by (4.24)-(4.26). We thus conlcude that the centres of the both cycle subalgebras are indeed contained in the gauge-invariant subalgebra $\mathcal{A}$.

The subalgebra in $\mathcal{A}$ generated by the two centres of $\mathcal{T}^{(i)}$ turns out to be only a proper subalgebra. Using a symbolic algebra computer program, we obtained all solutions to the gauge-invariance equations (the eight equations below (4.41)) for values of p ranging from 3 to 13. Based on those results, we claim that

$$\dim \mathcal{A} = 2p^4 + 4. \tag{4.45}$$

We see that only $(p^2+1)^2$ linearly independent elements out of $2p^4+4$ are generated by central elements of $\mathcal{T}^{(a)}$ and $\mathcal{T}^{(b)}$. In other words, there are still many elements in $\mathcal{A}$ that do not satisfy the assumption (4.43). For an example of such elements, we have

$$\sum_{n,m=0}^{p-1} \big(k_\alpha^{(a)}\big)^{2n} e_\pm^{(a)} \big(k_\alpha^{(b)}\big)^{2m} e_\mp^{(b)} \;\in\; \mathcal{A},$$

which clearly cannot be obtained as a linear combination of products of elements from $Z(\mathcal{T}^{(a)})$ and $Z(\mathcal{T}^{(b)})$. We do not give a basis in $\mathcal{A}$. However our aim is to describe the Fock representation of $\mathcal{A}$, and for this we actually do not need to know an explicit basis – it is turned out that only $Z(\mathcal{T}^{(a)})$ contributes to gauge-invariant states as it is explained below, of course in agreement with the general result in (2.51).

4.8 Fock representation

Here, we investigate the Fock-type representation of $\mathcal{A}$ and find an explicit basis in it.

We begin with the representation D of the handle algebra $\mathcal{T}$. We define the vacuum vector $|0\rangle \in \mathfrak{R}$ such that

$$D\big(k_\alpha^{(b)}\big)|0\rangle = D\big(k_\beta^{(b)}\big)|0\rangle = |0\rangle, \qquad\qquad D\big(e_+^{(b)}\big)|0\rangle = D\big(e_-^{(b)}\big)|0\rangle = 0, \tag{4.46}$$

recall (2.32). The vectors of the representation space

$$\mathfrak{R} = \big\{|n,m,r,s\rangle\big\}_{n,m,r,s=0}^{p-1,p-1,1,1}$$

are defined by the action of the a-cycle elements on the vacuum vector as follows

$$|n,m,r,s\rangle := D\Big(\big(k_\alpha^{(a)}\big)^{2n}\big(k_\beta^{(a)}\big)^{2m}\big(e_+^{(a)}\big)^r\big(e_-^{(a)}\big)^s\Big)|0\rangle. \tag{4.47}$$

The representation D extends to a representation of $\bar{\mathcal{T}}$ by

$$D(\iota(k_\alpha))|0\rangle = D(\iota(k_\beta))|0\rangle = |0\rangle, \qquad\qquad D(\iota(e_\pm))|0\rangle = 0. \tag{4.48}$$

We now turn to the representation $\mathfrak{R}_{(\mathcal{A})}$ of $\mathcal{A}$ defined by (2.49). The representation space $\mathfrak{R}_{(\mathcal{A})}$ is in fact isomorphic as a vector space to the centre $Z(\mathcal{G})$ of the Hopf algebra $\mathcal{G}$, recall our result in (2.51). One can actually check this result by a direct calculation. Indeed, using the general form (4.40) of an element in $\mathcal{A}$, we get

$$\begin{aligned} D(f)|0\rangle &= \sum_{n_1,m_1=0}^{p-1} \sum_{r_1,s_1=0}^{1} \left(\sum_{n_2,m_2=0}^{p-1} f_{r_1,s_1,0,0}(n_1,m_1,n_2,m_2) \right) |n_1,m_1,r_1,s_1\rangle = \\ &=: \sum_{n_1,m_1=0}^{p-1} \sum_{r_1,s_1=0}^{1} \tilde{f}_{r_1,s_1}(n_1,m_1)|n_1,m_1,r_1,s_1\rangle. \end{aligned}$$

One can show that when the coefficients $f_{r_1,s_1,r_2,s_2}(n_1,m_1,n_2,m_2)$ satisfy the gauge-invariance equations – the eight equations below (4.41) – the appropriate coefficients $\tilde{f}_{r_1,s_1}(n_1,m_1)$ do satisfy the equations (4.43)-(4.44), which are satisfied if and only if the corresponding vector belongs to the vector space $D\big(Z(\mathcal{T}^{(a)})\big)|0\rangle$. We thus conclude that the vector space of solutions is isomorphic to the centre of $\mathcal{G}$.

Recall that the centre of $\mathcal{G}$ and its basis are described in Section 4.4. The embedding map $\kappa^{(a)}$ from (4.34) applied to the central elements from (4.28)-(4.30) gives the gauge-invariant vectors

$$v_{e,n} := D\big(\kappa^{(a)}(c_{e,n})\big)|0\rangle, \qquad x_n := D\big(\kappa^{(a)}(a_n)\big)|0\rangle, \qquad w := D\big(\kappa^{(a)}(b)\big)|0\rangle. \tag{4.49}$$

From the above discussion, or from (2.51), we thus have

$$\text{basis in } \mathfrak{R}_{(\mathcal{A})} = \big\{\, v_{e,n}, w, x_n \,\big\}_{e=1,n=0}^{p-1,p-1}. \tag{4.50}$$

A straightforward calculation gives

$$v_{e,n} = (q-q^{-1})^2[e]_q|n-\tfrac{1}{2},e,1,1\rangle - q^e\big(|n,e,0,0\rangle - |n-1,e,0,0\rangle\big), \tag{4.51}$$

$$w = -2(q-q^{-1})^2 \sum_{t=0}^{p-1} |t,0,1,1\rangle, \tag{4.52}$$

$$x_n = |n,0,0,0\rangle. \tag{4.53}$$

And we follow here the convention where the vectors corresponding to half-integer values are identified with those corresponding to the integer ones according to

$$|n \pm \tfrac{1}{2}, m, r, s\rangle := |n + \tfrac{p \pm 1}{2}, m, r, s\rangle.$$

Moreover, the indices e,n of the elements $v_{e,n}, x_n$ are taken modulo p, and in what follows we use

$$v_{e\pm p,n} := v_{e,n}, \qquad v_{e,n\pm p} := v_{e,n}, \qquad x_{n\pm p} := x_n.$$

4.9 $SL(2,\mathbb{Z})$ action from the handle algebra

In this section, we construct the realisation of the $SL(2,\mathbb{Z})$ group as operators on the representation space of the gauge-invariant subalgebra of the handle algebra. In order to do that, we use the definitions of the mapping class group operators (2.59) corresponding to the Dehn twists along the cycles of the torus. In the end, we obtain matrix coefficients of the $\mathscr{S}$ and $\mathscr{T}$ transformations, and check explicitly the $SL(2,\mathbb{Z})$ relations.

Proceeding as in the case of the toy model, by a direct evaluation of (2.59) we get the following expressions

$$\hat{v}(i) = -\frac{i}{p}\sum_{n,m=0}^{p-1} q^{m(2n+1)}\big(k_\alpha^{(i)}\big)^{2n}\big(k_\beta^{(i)}\big)^{2m}\big(1+(q-q^{-1})k_\alpha^{(i)}e_+^{(i)}e_-^{(i)}\big), \qquad i=a,b. \tag{4.54}$$

The quantum Dehn twist operators evaluated on the representation D are then given by the matrix coefficients

$$\begin{aligned}
D(\hat{v}(b))|n,m,0,0\rangle &= -iq^{m(1-2n)}|n,m,0,0\rangle,\\
D(\hat{v}(b))|n,m,0,1\rangle &= -iq^{-2nm}|n,m,0,1\rangle,\\
D(\hat{v}(b))|n,m,1,0\rangle &= -iq^{m(1-2n)}|n,m,1,0\rangle,\\
D(\hat{v}(b))|n,m,1,1\rangle &= -iq^{-2nm}\left(|n,m,1,1\rangle-\frac{q^{2m}}{q-q^{-1}}|n-\tfrac{1}{2},m,0,0\rangle\right),
\end{aligned} \tag{4.55}$$

and

$$\begin{aligned}
D(\hat{v}(a))|n,m,0,0\rangle =& -\frac{i}{p}\sum_{s,t=0}^{p-1} q^{t(2s+1)}\,(|n+s,m+t,0,0\rangle+\\
&+(q-q^{-1})|n+s+\tfrac{1}{2},m+t,1,1\rangle\Big),\\
D(\hat{v}(a))|n,m,0,1\rangle =& -\frac{i}{p}\sum_{s,t=0}^{p-1} q^{t(2s+1)}|n+s,m+t,0,1\rangle,\\
D(\hat{v}(a))|n,m,1,0\rangle =& -\frac{i}{p}\sum_{s,t=0}^{p-1} q^{t(2s+1)}|n+s+1,m+t,1,0\rangle,\\
D(\hat{v}(a))|n,m,1,1\rangle =& -\frac{i}{p}\sum_{s,t=0}^{p-1} q^{t(2s+1)}|n+s+1,m+t,1,1\rangle.
\end{aligned} \tag{4.56}$$

$\mathscr{S}$-transformation

On the representation D, the $\mathscr{S}$-transformation (2.61) is given explicitly by

$$\begin{aligned}
D(\mathscr{S})|n,m,0,0\rangle &= \frac{i}{p}(q-q^{-1})q^m\sum_{s,t=0}^{p-1} q^{-2nt-2ms}|s,t,1,1\rangle,\\
D(\mathscr{S})|n,m,0,1\rangle &= \frac{i}{p}q^{-m}\sum_{s,t=0}^{p-1} q^{-2(n-\frac{1}{2})t-2ms}|s,t,0,1\rangle,\\
D(\mathscr{S})|n,m,1,0\rangle &= \frac{i}{p}q^{2m}\sum_{s,t=0}^{p-1} q^{-2nt-2ms}|s,t,1,0\rangle,\\
D(\mathscr{S})|n,m,1,1\rangle &= \frac{i}{p}\sum_{s,t=0}^{p-1} q^{-2(n-\frac{1}{2})t-2ms}\left[q^m(-1+q^{-2t})|s,t,1,1\rangle-\frac{1}{q-q^{-1}}|s,t,0,0\rangle\right].
\end{aligned}$$

Then, the action of $\mathscr{S}$ on the gauge-invariant vectors (4.50) in $\mathfrak{R}_{(\mathcal{A})}$ is

$$D_{(\mathcal{A})}(\mathscr{S})v_{e,n} = \frac{i}{p}\sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} q^{-2s(n-\frac{1}{2})-2e(t-\frac{1}{2})}v_{s,t} - \frac{i}{p}(q^e - q^{-e})\sum_{t=0}^{p-1} q^{-2et}x_t, \tag{4.57}$$

$$D_{(\mathcal{A})}(\mathscr{S})w = 2i(q-q^{-1})\sum_{t=0}^{p-1} x_t, \tag{4.58}$$

$$D_{(\mathcal{A})}(\mathscr{S})x_n = \frac{i}{p(q-q^{-1})}\sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} \frac{q^{-2ns}}{[s]_q}v_{s,t} - \frac{i}{2p(q-q^{-1})}w. \tag{4.59}$$

and so it can be calculated that

$$\begin{aligned}
&D_{(\mathcal{A})}(\mathscr{S}^2)v_{e,n} = -v_{-e,1-n}, && D_{(\mathcal{A})}(\mathscr{S}^4)v_{e,n} = v_{e,n},\\
&D_{(\mathcal{A})}(\mathscr{S}^2)w = w, && D_{(\mathcal{A})}(\mathscr{S}^4)w = w,\\
&D_{(\mathcal{A})}(\mathscr{S}^2)x_n = x_{-n}, && D_{(\mathcal{A})}(\mathscr{S}^4)x_n = x_n,
\end{aligned}$$

where e, n indices are taken modulo p and therefore we set $v_{-e,n} := v_{p-e,n}$, etc. One sees that the fourth power of $\mathscr{S}$ is an identity on the subset of gauge-invariant vectors

$$D_{(\mathcal{A})}(\mathscr{S}^4)y = y, \qquad \forall y = v_{e,n}, w, x_n. \tag{4.60}$$

We note the similarity between our S-transformation in (4.57)-(4.59) and the one spelled out in [10, Sec. 3.5.4]. Mikhaylov describes the action of the S-transformation on $p^2 + p$ states, denoted by $|L_{n,m}\rangle$, $|L_m\rangle$ and $|L_{P,m}\rangle$. But these are not linearly independent. By inspection one can see that his representation space is spanned by $p^2 + 1$ linearly independent states, just as ours. More specifically, one can identify our vectors $v_{n,m}$, x_m, w with the respective states $|L_{n,m}\rangle$, $|L_m\rangle$ and the sum $\sum_m |L_{P,m}\rangle$, up to some normalization. In this basis, Mikhaylov's action of the S-transformation can be seen to agree with the formulas we displayed above.

$\mathscr{T}$-transformation

On the representation space $\mathfrak{R}$, $\mathscr{T}$-transformation (2.62) is given explicitly by

$$\begin{aligned}
D(\mathscr{T})|n,m,0,0\rangle &= \frac{i}{p}\sum_{s,t=0}^{p-1} q^{-t(2s-1)}\Big(|n+s,m+t,0,0\rangle+\\
&\qquad - (q-q^{-1})|n+s-\frac{1}{2},m+t,1,1\rangle\Big),\\
D(\mathscr{T})|n,m,0,1\rangle &= \frac{i}{p}\sum_{s,t=0}^{p-1} q^{-t(2s-1)}|n+s,m+t,0,1\rangle,\\
D(\mathscr{T})|n,m,1,0\rangle &= \frac{i}{p}\sum_{s,t=0}^{p-1} q^{-t(2s-1)}|n+s-1,m+t,1,0\rangle,\\
D(\mathscr{T})|n,m,1,1\rangle &= \frac{i}{p}\sum_{s,t=0}^{p-1} q^{-t(2s-1)}|n+s-1,m+t,1,1\rangle.
\end{aligned}$$

Then, the action of $\mathscr{T}$ on gauge-invariant vectors is

$$D_{(\mathcal{A})}(\mathscr{T})v_{e,n} = \frac{i}{p}\sum_{\substack{s,t=0,\\ s\neq -e}}^{p-1} q^{-2st}v_{e+s,n+t} + \frac{i}{p}(q^e - q^{-e})\sum_{t=0}^{p-1} q^{2e(t-n+\frac{1}{2})}x_t, \tag{4.61}$$

$$D_{(\mathcal{A})}(\mathscr{T})w = iw, \tag{4.62}$$

$$D_{(\mathcal{A})}(\mathscr{T})x_n = -\frac{i}{p(q-q^{-1})}\sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} \frac{q^{-s(2t-1)}}{[s]_q}v_{s,n+t} + \frac{i}{2p(q-q^{-1})}w + \frac{i}{p}\sum_{t=0}^{p-1} x_t. \tag{4.63}$$

It can be calculated that in fact, together with the $\mathscr{S}$-transformation, the $\mathscr{T}$-transformation defined in this way provides an action of $SL(2,\mathbb{Z})$ on the sub-space of gauge-invariant vectors

$$D_{(\mathcal{A})}\big((\mathscr{S}\mathscr{T})^3\big)y = D_{(\mathcal{A})}(\mathscr{S}^2)y, \qquad \forall y = v_{e,n}, w, x_n. \tag{4.64}$$

Let us once again compare with the formulas for the T-transformation in [10]. At first sight the two sets of formulas look quite different even after Mikhaylov's formulas are written in a proper basis, as we described after (4.60). But the two representations turn out to be equivalent via the conjugation with the modular S-matrix. Put differently, our T-transformation was defined through a Dehn twist along the a-cycle. But equivalently, one can also use the Dehn twist along the b-cycle. These two Dehn twists are related by a modular S-transformation and, in our terminology, it is the Dehn twist along the b-cycle that is described by the formulas in [10]. In conclusion, the representation of the modular group we obtained through the general formalism we developed in Section 2 is equivalent to the one of Mikhaylov when the gauge group G of our Chern-Simons theory is $G = GL(1|1)$.

4.10 Lyubashenko–Majid $SL(2,\mathbb{Z})$ action on the centre

In this section, we consider the $SL(2,\mathbb{Z})$ action of LM type and compute the $\mathscr{S}$- and $\mathscr{T}$-transformations on the basis elements of $Z(\mathcal{G})$ constructed in Section 4.4. Moreover, we check that these transformations provide a projective $SL(2,\mathbb{Z})$ action indeed.

$\mathscr{S}$-transformation

Using equation (2.65), we find the LM-type $\mathscr{S}$-transformation

$$\mathscr{S}_Z(c_{e,n}) = -\frac{i}{p}\sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} q^{-2e(t-\frac{1}{2})-2s(n-\frac{1}{2})}c_{s,t} + \frac{i}{p}(q^e - q^{-e})\sum_{t=0}^{p-1} q^{-2et}a_t, \tag{4.65}$$

$$\mathscr{S}_Z(b) = -2i(q-q^{-1})\sum_{t=0}^{p-1} a_t, \tag{4.66}$$

$$\mathscr{S}_Z(a_n) = -\frac{i}{p(q-q^{-1})}\sum_{\substack{s,t=0,\\ s\neq 0}}^{p-1} \frac{q^{-2ns}}{[s]_q}c_{s,t} + \frac{i}{2p(q-q^{-1})}b. \tag{4.67}$$

Iterative application of the formulae (4.65)-(4.67) then leads to

$$\begin{aligned} &\mathscr{S}_Z^2(c_{e,n}) = -c_{-e,-n+1}, && \mathscr{S}_Z^4(c_{e,n}) = c_{e,n}, \\ &\mathscr{S}_Z^2(b) = b, && \mathscr{S}_Z^4(b) = b, \\ &\mathscr{S}_Z^2(a_n) = a_{-n}, && \mathscr{S}_Z^4(a_n) = a_n, \end{aligned}$$

where we recall that e, n indices are taken modulo p and so we have $c_{-e,n} := c_{p-e,n}$, etc. We thus see that the fourth power of the $\mathscr{S}$-transformation is an identity on the centre of $\mathcal{G}$:

$$\mathscr{S}_Z^4 = id. \tag{4.68}$$

$\mathscr{T}$-transformation

Using equation (2.66) we find the action of the $\mathscr{T}$-transformation on the central elements of $\mathcal{G}$

$$\mathscr{T}_Z(c_{e,n}) = \frac{1}{p} \sum_{\substack{s,t=0,\\ s\neq -e}}^{p-1} q^{-2st} c_{e+s,n+t} + \frac{1}{p} \sum_{t=0}^{p-1} q^{2e(t+\frac{1}{2})} (q^e - q^{-e}) a_{n+t}, \tag{4.69}$$

$$\mathscr{T}_Z(b) = b, \tag{4.70}$$

$$\mathscr{T}_Z(a_n) = -\frac{1}{p(q-q^{-1})} \sum_{\substack{s,t=0,\\ s\neq 0}}^{p-1} \frac{q^{-s(2t-1)}}{[s]_q} c_{s,n+t} + \frac{1}{p} \sum_{t=0}^{p-1} a_t + \frac{1}{2p(q-q^{-1})} b. \tag{4.71}$$

Iteratively applying the above $\mathscr{T}$-transformation, as well as the $\mathscr{S}$-transformation considered above, we can establish that on the centre of $\mathcal{G}$ we have the following identity

$$(\mathscr{S}_Z \mathscr{T}_Z)^3 = -i \mathscr{S}_Z^2. \tag{4.72}$$

Therefore, we see that on the centre of the quantum group we have a projective $SL(2, \mathbb{Z})$ action.

4.11 Equivalence of two actions

In this section, we show that the two $SL(2, \mathbb{Z})$ actions presented in the sections above agree projectively.

We first note that the centre $Z(\mathcal{G})$ and the representation space $\mathfrak{R}_{(\mathcal{A})}$ of the gauge-invariant subalgebra $\mathcal{A}$ are isomorphic as vector spaces, in agreement with (2.51). Explicitly, we have the correspondence, recall definitions in (4.49):

$$Z(\mathcal{G}) \ni c_{e,n} \longmapsto v_{e,n} \in \mathfrak{R}_{(\mathcal{A})}, \tag{4.73}$$

$$Z(\mathcal{G}) \ni a_n \longmapsto x_n \in \mathfrak{R}_{(\mathcal{A})}, \tag{4.74}$$

$$Z(\mathcal{G}) \ni b \longmapsto w \in \mathfrak{R}_{(\mathcal{A})}. \tag{4.75}$$

Moreover, if we take into account the above isomorphism between $Z(\mathcal{G})$ and $\mathfrak{R}_{(\mathcal{A})}$, we can compare the coefficients of the relevant actions in the two cases. In order to do that, let us define the coefficients of the $\mathscr{S}$-action for the handle algebra:

$$
\begin{aligned}
D_{(\mathcal{A})}(\mathscr{S})v_{n,m} &= \sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} (\mathscr{S}_{\mathcal{T}})_{n,m}^{s,t} v_{s,t} + \sum_{t=0}^{p-1} (\mathscr{S}_{\mathcal{T}})_{n,m}^{t} x_t + (\mathscr{S}_{\mathcal{T}})_{n,m}^{\bullet} w,\\
D_{(\mathcal{A})}(\mathscr{S})x_m &= \sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} (\mathscr{S}_{\mathcal{T}})_{m}^{s,t} v_{s,t} + \sum_{t=0}^{p-1} (\mathscr{S}_{\mathcal{T}})_{m}^{t} x_t + (\mathscr{S}_{\mathcal{T}})_{m}^{\bullet} w,\\
D_{(\mathcal{A})}(\mathscr{S})w &= \sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} (\mathscr{S}_{\mathcal{T}})_{\bullet}^{s,t} v_{s,t} + \sum_{t=0}^{p-1} (\mathscr{S}_{\mathcal{T}})_{\bullet}^{t} x_t + (\mathscr{S}_{\mathcal{T}})_{\bullet}^{\bullet} w,
\end{aligned}
$$

and for the $\mathscr{T}$-action as

$$
\begin{aligned}
D_{(\mathcal{A})}(\mathscr{T})v_{n,m} &= \sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} (\mathscr{T}_{\mathcal{T}})_{n,m}^{s,t} v_{s,t} + \sum_{t=0}^{p-1} (\mathscr{T}_{\mathcal{T}})_{n,m}^{t} x_t + (\mathscr{T}_{\mathcal{T}})_{n,m}^{\bullet} w,\\
D_{(\mathcal{A})}(\mathscr{T})x_m &= \sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} (\mathscr{T}_{\mathcal{T}})_{m}^{s,t} v_{s,t} + \sum_{t=0}^{p-1} (\mathscr{T}_{\mathcal{T}})_{m}^{t} x_t + (\mathscr{T}_{\mathcal{T}})_{m}^{\bullet} w,\\
D_{(\mathcal{A})}(\mathscr{T})w &= \sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} (\mathscr{T}_{\mathcal{T}})_{\bullet}^{s,t} v_{s,t} + \sum_{t=0}^{p-1} (\mathscr{T}_{\mathcal{T}})_{\bullet}^{t} x_t + (\mathscr{T}_{\mathcal{T}})_{\bullet}^{\bullet} w.
\end{aligned}
$$

And similarly for the transformations on $Z(\mathcal{G})$:

$$
\begin{aligned}
\mathscr{S}_Z(c_{n,m}) &= \sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} (\mathscr{S}_Z)_{n,m}^{s,t} c_{s,t} + \sum_{t=0}^{p-1} (\mathscr{S}_Z)_{n,m}^{t} a_t + (\mathscr{S}_Z)_{n,m}^{\bullet} b,\\
\mathscr{S}_Z(a_m) &= \sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} (\mathscr{S}_Z)_{m}^{s,t} c_{s,t} + \sum_{t=0}^{p-1} (\mathscr{S}_Z)_{m}^{t} a_t + (\mathscr{S}_Z)_{m}^{\bullet} b,\\
\mathscr{S}_Z(b) &= \sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} (\mathscr{S}_Z)_{\bullet}^{s,t} c_{s,t} + \sum_{t=0}^{p-1} (\mathscr{S}_Z)_{\bullet}^{t} a_t + (\mathscr{S}_Z)_{\bullet}^{\bullet} b,
\end{aligned}
$$

and

$$\begin{aligned}
\mathscr{T}_Z(c_{n,m}) &= \sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} (\mathscr{T}_Z)^{s,t}_{n,m} c_{s,t} + \sum_{t=0}^{p-1} (\mathscr{T}_Z)^{t}_{n,m} a_t + (\mathscr{T}_Z)^{\bullet}_{n,m} b,\\
\mathscr{T}_Z(a_m) &= \sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} (\mathscr{T}_Z)^{s,t}_{m} c_{s,t} + \sum_{t=0}^{p-1} (\mathscr{T}_Z)^{t}_{m} a_t + (\mathscr{T}_Z)^{\bullet}_{m} b,\\
\mathscr{T}_Z(b) &= \sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} (\mathscr{T}_Z)^{s,t}_{\bullet} c_{s,t} + \sum_{t=0}^{p-1} (\mathscr{T}_Z)^{t}_{\bullet} a_t + (\mathscr{T}_Z)^{\bullet}_{\bullet} b.
\end{aligned}$$

It can be read-off that the non zero coefficients are

$$\begin{aligned}
&(\mathscr{S}_{\mathcal{T}})^{s,t}_{n,m} = -(\mathscr{S}_Z)^{s,t}_{n,m} = \frac{i}{p} q^{-2s(m-\frac{1}{2})-2n(t-\frac{1}{2})}, && (\mathscr{T}_{\mathcal{T}})^{s,t}_{n,m} = i(\mathscr{T}_Z)^{s,t}_{n,m} = \frac{i}{p} q^{-2(s-n)(t-m)},\\
&(\mathscr{S}_{\mathcal{T}})^{t}_{n,m} = -(\mathscr{S}_Z)^{t}_{n,m} = -\frac{i}{p}(q^n - q^{-n}) q^{-2nt}, && (\mathscr{T}_{\mathcal{T}})^{t}_{n,m} = i(\mathscr{T}_Z)^{t}_{n,m} = \frac{i}{p}(q^n - q^{-n}) q^{2n(t-m+\frac{1}{2})},\\
&(\mathscr{S}_{\mathcal{T}})^{s,t}_{m} = -(\mathscr{S}_Z)^{s,t}_{m} = \frac{i}{p(q-q^{-1})} \frac{q^{-2ms}}{[s]_q}, && (\mathscr{T}_{\mathcal{T}})^{s,t}_{m} = i(\mathscr{T}_Z)^{s,t}_{m} = -\frac{i}{p(q-q^{-1})} \frac{q^{-2s(t-m-\frac{1}{2})}}{[s]_q},\\
&(\mathscr{S}_{\mathcal{T}})^{\bullet}_{m} = -(\mathscr{S}_Z)^{\bullet}_{m} = -\frac{i}{2p(q-q^{-1})}, && (\mathscr{T}_{\mathcal{T}})^{t}_{m} = i(\mathscr{T}_Z)^{t}_{m} = \frac{i}{p},\\
&(\mathscr{S}_{\mathcal{T}})^{t}_{\bullet} = -(\mathscr{S}_Z)^{t}_{\bullet} = 2i(q-q^{-1}), && (\mathscr{T}_{\mathcal{T}})^{\bullet}_{m} = i(\mathscr{T}_Z)^{\bullet}_{m} = \frac{i}{2p(q-q^{-1})},\\
& && (\mathscr{T}_{\mathcal{T}})^{\bullet}_{\bullet} = i(\mathscr{T}_Z)^{\bullet}_{\bullet} = i.
\end{aligned}$$

Comparing the coefficients of $\mathscr{S}$ and $\mathscr{T}$ from the handle algebra to the ones $\mathscr{S}_Z$ and $\mathscr{T}_Z$ from the LM construction, we see that they indeed agree projectively:

$$\mathscr{S}_Z = -\mathscr{S}_{\mathcal{T}}, \qquad\qquad \mathscr{T}_Z = -i\mathscr{T}_{\mathcal{T}}. \tag{4.76}$$

5 Outlook

In this work, we considered the quantisation of $GL(1|1)$ Chern-Simons theory at odd integer level on a torus $\Sigma = \Sigma_{1,0} = \mathbb{T}^2$ with no punctures. While the general framework of combinatorial quantisation allows to consider an arbitrary simplicial decomposition of Σ, we only considered the minimal decomposition of the torus with a single 2-cell, two 1-cells and one 0-cell. There are a number of extensions that we shall address in forthcoming work.

To begin with, we will replace the torus $\mathbb{T}^2$ by a Riemann surface $\Sigma = \Sigma_{g,n}$ of arbitrary genus g and with any number n of punctures. The first step is then to choose some simplicial decomposition. The minimal choice would involve $(n+1)$ number of 2-cells, $(2g+n)$ 1-cells and a single 0-cell. If we adopt this choice, the monodromy (or loop) algebra we have discussed in this work is the only building block that is used in the combinatorial quantisation. Of course, one needs as many of these algebras as there are 1-cells and they satisfy exchange relations that must reflect the topology of our surface, generalising what we saw here for the torus. For more general simiplicial decomposition with more than one 0-cell, one needs a second building block, the holonomy (or link) algebra. It is a close relative of the $GL(1|1)$ quantum group, i.e. of the Hopf-dual for $GL(1|1)$. Once introduced, link and loop algebras must be combined into a larger algebraic structure in which they satisfy a system of exchange relations which are determined by the simplicial decomposition and by the R-matrix of $\mathcal{G}$. All this will be discussed in detail in forthcoming work. There we will also show that the spaces of Chern-Simons states are actually independent of the simplicial decomposition so that the minimal choice can always be adopted.

The construction of representations of the modular group $SL(2,\mathbb{Z})$ that was our main focus above also possesses a natural extension to $\Sigma = \Sigma_{g,n}$. In fact, for higher genus and in the presence of punctures, the modular group gets replaced by the (pure) mapping class group of the n-punctured surface. The fundamental generators are the Dehn twists along non-contractible curves on Σ. To construct representations of the mapping class group, we can follow precisely the constructions we have described in this work. All we need to prescribe are the corresponding Chern-Simons observables that are associated with the non-contractible curves on Σ. In this step we can use the same formula as for the two non-trivial cycles of the torus, see (2.59). In some sense, one key result of the present work was to show that this prescription is equivalent to Lyubachenko–Majid's construction for the torus as well as to Mikhaylov's representation [10] of the modular group, at least for $GL(1|1)$ at integer level. Once this is established, the inherent factorisability of the combinatorial prescription provides a canonical extension to punctured surfaces of higher genus. Constructing the corresponding representation of the mapping class group is one of the main goals of our future work. Again, our construction will be restricted to $GL(1|1)$ Chern-Simons theory at integer level.

There are two additional extensions we are planning to describe in forthcoming papers. One of them is to go beyond the case of integer levels. In other words, we want to admit deformation parameters q which are no longer given by a root of unity. Very little is known from other approaches about such an extension. So, it seems worthwhile to look at it in the case of the torus $\Sigma = \mathbb{T}^2$ first. Once the theory for the torus is developed, the combinatorial approach provides a straightforward extension to other surfaces.

The final step is then to go beyond $GL(1|1)$. As we have mentioned in the introduction, for 2-dimensional supergroup WZNW models the quantisation resembles that of the $GL(1|1)$ model whenever the gauge group G is of type I. Given the usual duality between WZW models and Chern-Simons theory, we expect the same to be true for the 3-dimensional model. One of the more immediate goals therefore is to develop the combinatorial quantisation of supergroup Chern-Simons theory for gauge supergroups G of type I, at least as long as the level is integer. The integer level is important here as it reduces the quantum symmetry to a *finite-dimensional* super Hopf algebra – the case where our general construction in Section 2 is applicable. Carrying out these extensions, we hope to construct a plethora of new representations of mapping class groups for 2-dimensional surfaces $\Sigma_{g,n}$ or arbitrary genus g.

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