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Magnetized orbifolds and localized flux

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Abstract

Magnetized orbifolds play an important role in compactifications of string theories and higher-dimensional field theories to four dimensions. Magnetic flux leads to chiral fermions, it can be a source of supersymmetry breaking and it is an important ingredient of moduli stabilization. Flux quantization on orbifolds is subtle due to the orbifold singularities. Generically, Wilson line integrals around these singularities are non-trivial, which can be interpreted as localized flux. As a consequence, flux densities on orbifolds can take the same values as on tori. We determine the transition functions for the flux vector bundle on the orbifold $T^2/\mathbb{Z}_2$ and the related twisted boundary conditions of zero-mode wave functions. We also construct "untwisted" zero-mode functions that are obtained for singular vector fields related to the Green's function on a torus, and we discuss the connection between zeros of the wave functions and localized flux. Twisted and untwisted zero-mode functions are related by a singular gauge transformation.

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Contents

1 Introduction

Orbifold compactifications play an important role in string theory [1,2]. They partially break supersymmetry and lead to chiral fermion spectra in four dimensions. The same effects can be achieved by compactifications on magnetized tori [3, 4]. Particularly interesting are magnetized D-branes wrapping tori or toroidal orbifolds, which were studied in compactifications of type-I string theory [5–7]. The interplay of these ideas led to a class of four-dimensional chiral gauge theories, constructed as type-I or type-II string vacua with D-branes and orientifolds, which capture the main features of the Standard Model and its supersymmetric extension (for reviews see, for example [8–10]).

In view of the complexity of string compactifications, especially the need to stabilize all moduli fields of the theory, also compactifications of higher-dimensional field theories have been considered as an intermediate step towards a solution of the full problem. In particular orbifold grand unified models (GUTs) in five and six dimensions can successfully account for the doublet-triplet splitting, the breaking of GUT gauge groups and flavour physics [11–16], as well as the stabilization of moduli [17–19]. Orbifold GUTs are strongly inspired by heterotic string compactifications (for reviews see, for example [20, 21]).

A complementary approach are compactifications on magnetized tori, which were thoroughly studied in [22] and extended to magnetized orbifolds in [23–27]. Since magnetic flux can provide both, chiral fermions and supersymmetry breaking, one can obtain extensions of the Standard Model where supersymmetry is broken at a high scale that is related to the size of the compact dimensions [28]. Moreover, magnetized orbifold compactifications have interesting implications for flavour physics [29–33], and in this simple setup the interplay of flux and nonperturbative effects at the orbifold fixed points allows to stabilize all moduli in Minkowski or de Sitter vacua [23, 34]. Particularly interesting is the effect of flux on quantum corrections to scalar masses. In models of gauge-Higgs unification magnetic flux can keep Wilson line scalars at zero mass due to symmetries of the higher-dimensional theory [35–37].

Compared to compactifications on magnetized tori, flux compactifications on orbifolds are subtle due to the orbifold singularities. In particular, there is a puzzle concerning the allowed flux densities. For instance, it has been argued that for a quantized flux density $f = 2\pi M$, $M \in \mathbb{Z}$ on a torus T^2, the allowed flux density on an orbifold $T^2/\mathbb{Z}_2$ is $f = 4\pi M$ [4, 23, 27]. Considering a closed path on the orbifold and using naively Stokes' theorem, this follows immediately from the fact that the area of the orbifold is half the area of the torus. On the other hand, zero-mode wave functions have been constructed on magnetized orbifolds for flux densities $f = 2\pi M$ without any signs of inconsistency [24, 25]. One of the main goals of this paper is to clarify this puzzle. This will be achieved by carefully discussing the flux vector bundle on the orbifold. As we shall see, the non-trivial transition functions on the orbifold will lead to non-trivial Wilson line integrals around orbifold fixed points, which makes flux densities $f = 2\pi M$ indeed consistent.

The non-trivial Wilson lines around orbifold fixed points can be interpreted as localized magnetic flux. Localized flux has previously been considered in connection with fixed-point anomalies [38, 39] as well as localized Fayet-Illiopoulos terms [40]. In the latter case zero-mode wave functions of charged bulk fields have been constructed by means of torus Green's functions whose singularities are localized at orbifold fixed points. We shall extend this construction to magnetized orbifolds and show that these "untwisted" zero-mode functions are closely related to the standard "twisted" zero-mode functions with boundary conditions of Scherk-Schwarz type [41]. As we shall see, untwisted wave functions can be mapped to twisted wave functions by means of a singular gauge transformation.

The paper is organized as follows. In Section 2 we first briefly review 1-cycles on orbifolds. We then discuss the vector bundle for magnetic flux in Landau gauge and derive the non-trivial transition functions and the related twisted boundary conditions. Subsequently, we consider the singular vector fields obtained from the Green's function of the bosonic string on the torus. Section 3 is devoted to zero-modes on the orbifold. We first consider the regular flux vector bundle and briefly recall the derivation of "twisted" wave functions in terms of Jacobi theta-functions. We then discuss the pattern of zeros of the wave functions for odd and even flux densities. Finally, untwisted zero-mode functions are constructed for singular vector fields and compared with the corresponding twisted wave functions. Our results are summarized in Section 4. In the appendices we collect some formulae for Jacobi theta-functions and 6d gamma-matrices, which are used in the calculations presented in the main text.

2 Gauge theories on orbifolds

In our discussion of gauge theories on orbifolds 1-cycles around orbifold fixed points play a crucial role. We therefore briefly recall their relation to the standard torus one-cycles. We then discuss the vector bundles related to magnetic flux and Wilson lines and compute the corresponding transition functions. They are compared with singular vector fields obtained from the torus Green's function, which are invariant under torus translations and for which the transition functions are trivial.

2.1 One-cycles on orbifolds

Consider a six-dimensional theory compactified on a torus T^2 to four-dimensional Minkowski space. The torus is obtained from the covering space $\mathbb{R}^2$ by modding out a two-dimensional lattice,

$$y \sim t(y) = y + \lambda \,, \tag{1}$$

where $\lambda = n_1\lambda_1 + n_2\lambda_2$ is a linear combination of two lattice vectors $\lambda_{1,2}$ with integer coefficients. They correspond to two basic translations $t_{1,2}$ and define the fundamental domain of the torus. The dimensionless coordinates y are related to physical coordinates by $y = (y_1, y_2) = (x^5, x^6)/L$, where L denotes a fixed physical length scale.

The shape of the torus is parametrized by two real moduli $\tau_{1,2}$ in the two-dimensional metric $(g_2)_{mn}$,

$$(g_2)_{mn} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & \tau_1^2 + \tau_2^2 \end{pmatrix} \,, \tag{2}$$

and the physical volume of the torus is $V_{T^2} = L^2$. A basis of 1-cycles and the fundamental domain of the torus are depicted in Fig. 1. Modding out a rotational $\mathbb{Z}_2$ symmetry,

$$y \sim p(y) = -y \,, \tag{3}$$

one obtains the orbifold $T^2/\mathbb{Z}_2$. Its fundamental domain can be chosen as $y_1 \in [0, 1/2)$, $y_2 \in [0, 1)$, or alternatlively as $y_1 \in [0, 1)$, $y_2 \in [0, 1/2)$. Hence, its volume is $V_{T^2/\mathbb{Z}_2} = L^2/2$. The transformations $\{t_1, t_2, p\}$ generate the so-called space group. Modding out its action from the covering space $\mathbb{R}^2$ yields the orbifold $T^2/\mathbb{Z}_2$. The space group does not act freely, but there are fixed points located at

$$\zeta_1 = (0,0) \;\;,\;\; \zeta_2 = (1/2,0) \;\;,\;\; \zeta_3 = (0,1/2) \;\;,\;\; \zeta_4 = (1/2,1/2) \,. \tag{4}$$

The orbifold has the topology of a sphere with four points removed. At each fixed point there is a conical singularity with deficit angle π, corresponding to singular curvature.

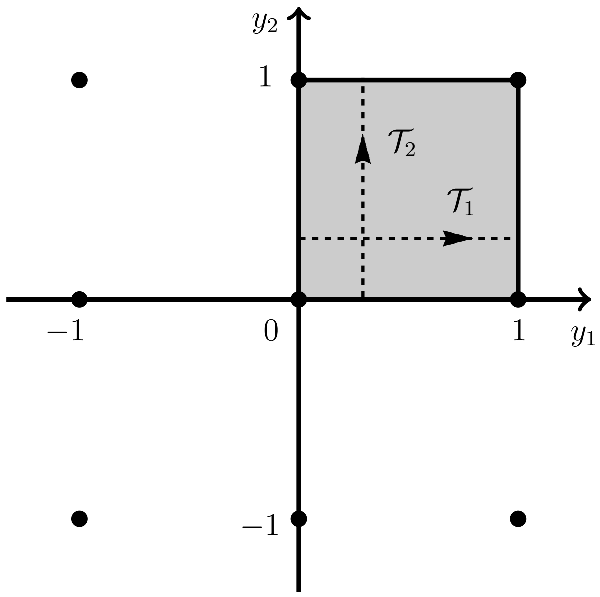


Figure 1: A torus lattice $\mathbb{Z}^2$ and its fundamental domain (gray); the basis of 1-cycles $\mathcal{T}_{1,2}$ is depicted by dashed arrows.

The bulk away from the fixed points is flat.

As shown in [27], it is often convenient to decompose the bulk 1-cycles $\mathcal{T}_{1,2}$ in terms of the "canonical" 1-cycles $\mathcal{C}_i$, $i = 1, \ldots, 4$, encircling the orbifold fixed point, see Fig. 2(a). The $\mathbb{Z}_2$ operator p corresponds to the 1-cycle $\mathcal{C}_1$. The torus cycles $\mathcal{T}_{1,2}$ can be projected to the fundamental domain of the orbifold, see Fig. 2(b), and then deformed continuously, see Fig. 2(c), yielding $\mathcal{T}_1 \sim \mathcal{C}_3 + \mathcal{C}_4 \sim -(\mathcal{C}_1 + \mathcal{C}_2)$ and $\mathcal{T}_2 \sim \mathcal{C}_1 + \mathcal{C}_3$, where a minus sign indicates a reversed orientation. The geometry described above has important consequences for gauge fields on the orbifold, as we shall see in the following sections.

2.2 Gauge fields on a torus

Consider now a $U(1)$ vector field $A = A_m dy_m$ and a charged complex matter field ϕ on the covering space. The field theory on the torus is obtained by modding out a two-dimensional lattice from space-time as well as field space. Vector and matter fields are required to be invariant under lattice translations $t = n_1 t_1 + n_2 t_2$ up to a gauge transformation,

$$A_m(t(y)) = A_m(y) - \frac{1}{q}\partial_m \Lambda_t(y)\,, \quad \phi(t(y)) = e^{i\Lambda_t(y)}\phi(y)\,, \tag{5}$$

such that the covariant derivative $D_m\phi = (\partial_m + iqA_m)\phi$ transforms like ϕ. The periodicity conditions for the matter field are left invariant under the so-called large gauge

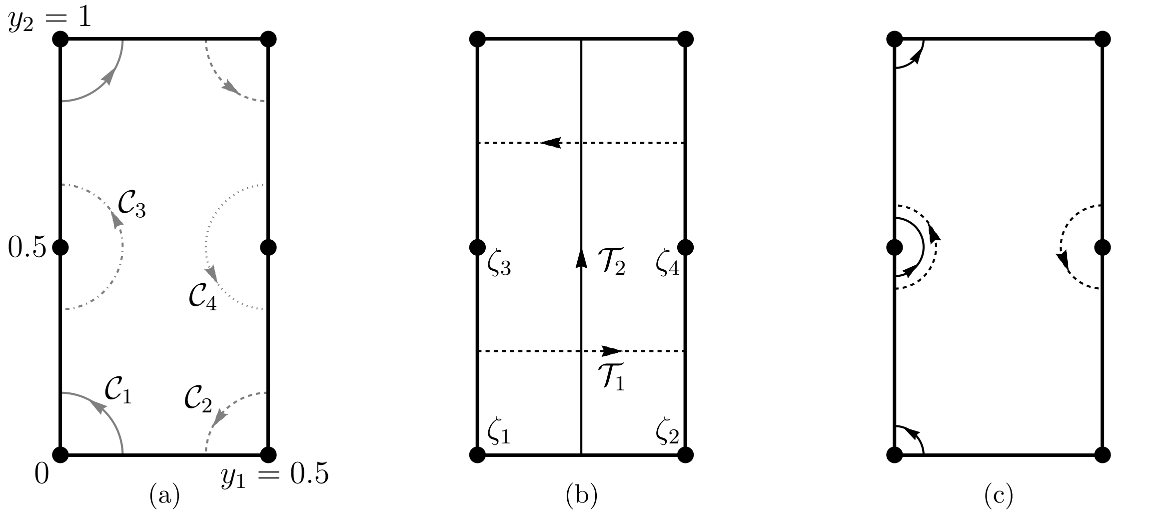


Figure 2: The fundamental domain of $T^2/\mathbb{Z}_2$; the black dots denote the orbifold fixed points. The canonical basis of orbifold 1-cycles is presented in (a). (b) shows the bulk 1-cycles $\mathcal{T}_{1,2}$, and (c) illustrates the decompositions $\mathcal{T}_1 \sim \mathcal{C}_3 + \mathcal{C}_4$ and $\mathcal{T}_2 \sim \mathcal{C}_1 + \mathcal{C}_3$. From [27].

transformations,

$$\phi(t(y)) \to e^{i\Lambda_{(k_1,k_2)}}\phi(y)\,, \quad \Lambda_{(k_1,k_2)} = 2\pi(k_1 y_1 + k_2 y_2)\,, \quad k_{1,2} \in \mathbb{Z}\,, \tag{6}$$

under which the vector field shifts as

$$A_m(y) \to A_m(y) - \frac{2\pi}{q} k_m\,. \tag{7}$$

On the covering space constant gauge fields

$$A = \alpha_m dy_m\,, \quad \alpha_m \in [0, 2\pi)\,, \tag{8}$$

are unphysical, they can be removed by gauge transformations. On the torus, however, they cannot be removed by the remaining large gauge transformations. Hence, Wilson lines corresponding to 1-cycles $\mathcal{T} = n_1\mathcal{T}_1 + n_2\mathcal{T}_2$,

$$W_{\mathcal{T}} = \exp\left[-iq\int_{\mathcal{T}} A\right] = e^{-iq(n_1\alpha_1 + n_2\alpha_2)}\,, \tag{9}$$

do have a physical meaning and play an important role.

Since the torus T^2 is not simply connected, vector fields and matter fields are represented by fibre bundles (for a review see, for example [42]). For the monopole field on a sphere this has been thoroughy discussed in [43], and magnetic fields on a

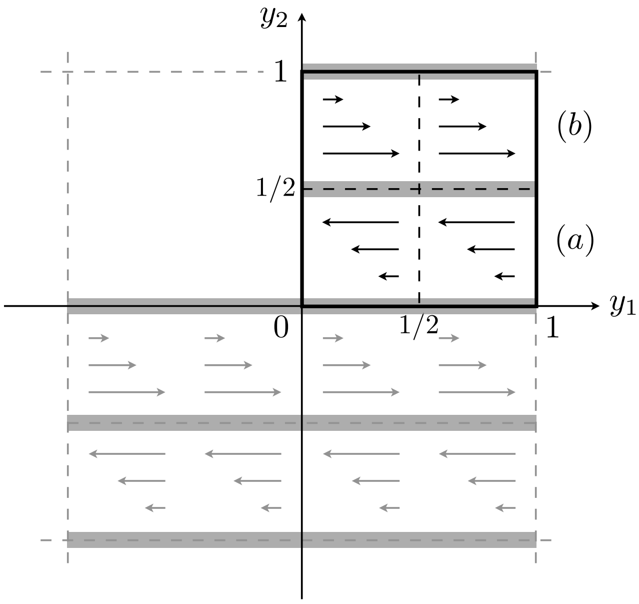


Figure 3: Three copies of the fundamental domain of the torus in the y_1–y_2-plane with a magnetic vector field in Landau gauge. The overlap of the two patches of the bundle contains the circles $y_2 = 0$ and $y_2 = 1/2$.

torus have been considered in [44]. Two patches covering the torus are shown in Fig. 3[1], together with a vector field in Landau gauge,

$$A_1 = \begin{cases} -fy_2 \,, & 0 \le y_2 < \frac{1}{2} \quad (a) \\ -f(y_2 - 1) \,, & \frac{1}{2} \le y_2 < 1 \quad (b) \end{cases} \,, \quad A_2 = 0 \,, \tag{10}$$

leading to the constant magnetic field

$$F = dA = fv \,, \tag{11}$$

where $v = dy_1 \wedge dy_2$. The overlap of the two patches contains the circles $y_2 = 1/2$ and $y_2 = 0$. The transition function that relates fields at the same point in the two patches is given by[2]

$$\begin{aligned} \phi_b = S_{ba}\phi_a \,, \quad S_{ba} = e^{i\Lambda_{ba}} \,, \quad S_{ba} = S_{ab}^{-1} \,, \\ A_m^b = A_m^a + \frac{i}{q} S_{ba}^{-1} \partial_m S_{ba} = A_m^a - \frac{1}{q} \partial_m \Lambda_{ba} \,. \end{aligned} \tag{12}$$

[1]Here we follow [44]. Strictly speaking, a patch should not contain a non-contractable cycle. However, in Landau gauge, covering the torus with more patches would only introduce additional trivial transition functions.

[2]We follow the notation and conventions of [43], with the replacement $e \to -q$.

At $y_2 = 1/2$, this implies

$$A_1^b(y_1, \tfrac{1}{2}) - A_1^a(y_1, \tfrac{1}{2}) = -\frac{1}{q}\partial_1 \Lambda_{ba} = f\,, \tag{13}$$

which yields $\Lambda_{ba} = -qfy_1$ and therefore the transition function

$$S_{ba} = e^{i\Lambda_{ba}} = e^{-iqfy_1}\,. \tag{14}$$

From the required single-valuedness of the transition function,

$$S_{ba}(y_1 + 1) = S_{ba}(y_1)\,, \tag{15}$$

one obtains the quantization condition for the magnetic flux,

$$qf = 2\pi M\,, \quad M \in \mathbb{Z}\,. \tag{16}$$

At $y_2 = 0$, the vector field in the patches (a) and (b) is the same, $A_1^a(y_1, 0) = A_1^b(y_1, 0) = 0$. The transition function at $y_2 = 0$ is therefore trivial, $S_{ab} = 1$.

Starting at $y_2 = 1/2$ in patch (b) and going around the torus in y_2-direction via patch (b) and patch (a) until $y_2 = 1/2$ in patch (a), the vector field changes from $A_1 = f/2$ to $A_1 = -f/2$. This necessitates a non-trivial transition function $S_{ba}(y_1)$, which in the literature on magnetized tori is usually treated as twisted boundary condition,

$$\phi(y_1, y_2 + 1) = S_{ba}^{-1}(y_1)\phi(y_1, y_2) = e^{iqfy_1}\phi(y_1, y_2)\,, \tag{17}$$

i.e. the twist factor corresponds to the transition function S_{ab} on the torus.

Constant vector fields $A_m = \alpha_m$ can be chosen to be the same in both patches. Hence, the transition functions are trivial and all values $\alpha_m \in [0, 2\pi)$ are allowed. On the covering space a constant vector field can be removed by a gauge transformation. Writing $A_m(y) = \alpha_m + A'_m(y)$, one has

$$\begin{aligned} A_m(y) \to A'_m(y) &= A_m(y) - \frac{1}{q}\partial_m \Lambda_{(\alpha)}(y)\,, \\ \phi(y) \to \phi'(y) &= e^{i\Lambda_{(\alpha)}(y)}\phi(y)\,, \quad \Lambda_{(\alpha)}(y) = q(\alpha_1 y_1 + \alpha_2 y_2)\,. \end{aligned} \tag{18}$$

For a translation t by a lattice vector λ, the boundary condition (5) changes to

$$\phi(y + \lambda) = e^{i(q(\alpha_1 y_1 + \alpha_2 y_2) + \Lambda_t)}\phi(y)\,. \tag{19}$$

This means that the effect of a constant background field can be represented by the term $\exp(iq(\alpha_1 y_1 + \alpha_2 y_2))$ in a twisted boundary condition.

2.3 Regular gauge fields on orbifolds

For vector fields on the covering space which are odd under reflections up to a gauge transformation,

$$A_m(p(y)) = -A_m(y) - \frac{1}{q}\partial_m \Lambda_p(y)\,, \quad \phi(p(y)) = e^{i\Lambda_p(y)}\phi(y)\,, \quad p(y) = -y\,, \tag{20}$$

one can mod out a $\mathbb{Z}_2$ symmetry, which leads to a field theory on the orbifold $T^2/\mathbb{Z}_2$ (see, for example [45]). The vector field (10) is odd under reflections. Hence, the projection to the orbifold does not require an additional gauge transformation and we have $\Lambda_p = 0$.

It is instructive to compare Wilson lines in the bulk with Wilson lines around fixed points. For the line integral between two points we define

$$W_{QP} = \exp\left[-iq\int_P^Q A\right]. \tag{21}$$

The bulk Wilson line shown in Fig. 4a is then given by (see, for example [43]),

$$W_{AFEDCBA} = W^{(a)}_{AF} S_{ab}(F) W^{(b)}_{FE} W^{(b)}_{ED} W^{(b)}_{DC} S_{ba}(C) W^{(a)}_{CB} W^{(a)}_{BA}\,. \tag{22}$$

Here the superscripts denote the relevant patches. Using Eqs. (14) and (21) one obtains

$$W_{AFEDCBA} = e^{iqf(y_{1F}-y_{1C}+\epsilon)} e^{-iqf(y_{2E}-y_{2A})\epsilon} = e^{-iq\Delta F}\,, \tag{23}$$

where $\Delta F = f\epsilon\delta$, with $\epsilon = \overline{AB} = \overline{ED}$ and $\delta/2 = \overline{AF} = \overline{FE} = \overline{CD} = \overline{BC}$. This is the expected result that the line integral is given by the enclosed flux according to Stokes' theorem. To obtain this result it is crucial to take the transition function S_{ba} and S_{ab} into account. The Wilson line integral around the fixed point shown in Fig. 4b can be calculated in the same way,

$$W_{ADCBA} = W^{(a)}_{AD} S_{ab}(D) W^{(b)}_{DC} W^{(b)}_{CB} S_{ba}(B) W^{(a)}_{BA}\,. \tag{24}$$

At the boundary $y_1 = 1/2$ between the patches (a) and (b) the vector field does not change. Hence, the transition function is trivial, $S_{ba}(B) = 1$. Using Eqs. (14) and (21) one now finds

$$W_{ADCBA} = e^{iqf/2} e^{-iq\Delta F}\,, \tag{25}$$

where again $\Delta F = f\epsilon\delta$, with $\epsilon = \overline{AB}$ and $\delta/2 = \overline{AD} = \overline{CD}$. The result differs from the naive expectation by a factor which does not vanish in the limit where the enclosed

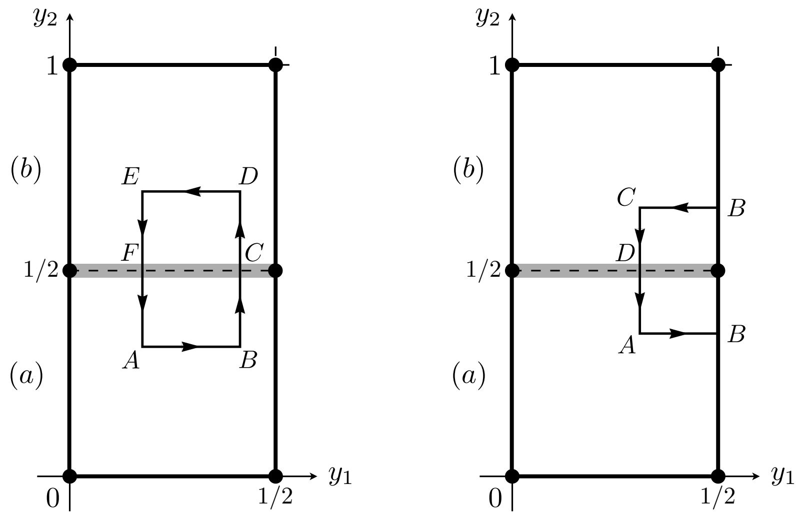


Figure 4: a) (left) Wilson line integral in the bulk; b) (right) Wilson line integral around the fixed point at ζ_4.

area goes to zero,

$$W_4 = \lim_{\epsilon,\delta\to 0} W_{ADCBA} = e^{iqf/2}\,. \tag{26}$$

The flux quantization (16) then implies $W_4 = \pm 1$. One easily verifies that the line integrals around the other fixed points are $W_i = 1$, $i = 1, 2, 3$. The flux quantization condition can therefore be expressed as

$$e^{iqf/2} = W_1 W_2 W_3 W_4 \equiv W = \pm 1\,. \tag{27}$$

Hence, for $W = 1$, the bulk flux $F = \int_{T^2/\mathbb{Z}_2} dA$ satisfies $qF = qf/2 = 2\pi M$, $M \in \mathbb{Z}$, whereas for $W = -1$ one has $qF = \pi + 2\pi M$.

This is an interesting result. Originally, the magnetic flux density on the orbifold was assumed to be twice as large as on the torus, $qf \in 4\pi\mathbb{Z}$, since the area of the orbifold is half the area of the torus. The bulk flux on the orbifold is then quantized as the one on the torus, $qF = qf/2 \in 2\pi\mathbb{Z}$ [4, 23]. This immediately follows from Stokes' theorem if one assumes that the surface integral receives no contribution from the fixed points. On the contrary, it has been argued that a flux density $qf \in 2\pi\mathbb{Z}$ is consistent also on the orbifold since it allows normalizable wave functions [24, 25]. Eq. (27) shows how this is indeed possible due to the effect of non-trivial Wilson line integrals around the fixed points. Such Wilson line integrals can be interpreted as localized flux [27]. In general, one has at each fixed point

$$W_i = e^{-iqF_i}\,, \quad F_i = \frac{\pi}{q}(\delta_{(W_i,-1)} + 2k_i)\,, \quad k_i \in \mathbb{Z}\,. \tag{28}$$

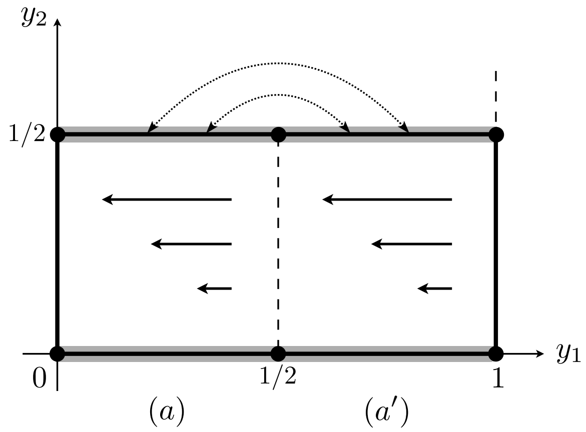


Figure 5: Possible choice of fundamental domain on the magnetized orbifold $T^2/\mathbb{Z}_2$. At the boundary $y_2 = 1/2$ points connected by dotted lines are identified.

From Eq. (27) one then obtains

$$q(F + \sum_i F_i) = \pi(\delta_{(W,-1)} + 2M + \sum_i(\delta_{(W_i,-1)} + 2k_i)) \,. \tag{29}$$

Since $(\delta_{(W,-1)} + \sum_i \delta_{(W_i,-1)}) \in 2\mathbb{Z}$, the total flux of bulk and fixed points always satisfies the torus quantization condition

$$q(F + \sum_i F_i) \in 2\pi\mathbb{Z} \,. \tag{30}$$

The bulk flux alone, however, can be odd, $qF \in \pi\mathbb{Z}$.

For completeness, let us point out that the above discussion is independent of the choice of the fundamental domain of the orbifold. Fig. 5 shows the second choice for a fundamental domain of the orbifold with a field configuration projected from the torus,

$$A_1 = -fy_2 \,, \quad 0 \le y_1 < 1 \,, \quad 0 \le y_2 < \tfrac{1}{2} \quad (a), (a') \,. \tag{31}$$

We introduce patches (a) and (a'), separated by the boundaries $y_1 = 0$ and $y_1 = 1/2$. They correspond to the patches (a) and (b) in Fig. 3. The transition functions at these boundaries are obviously trivial. On the orbifold the y_1-intervals $[0, 1/2]$ and $[1/2, 1]$ at the boundaries are identified. At $y_2 = 0$ and $y_2 = 1/2$ one has

$$y_1 \sim s(y_1) = 1 - y_1 \,, \quad y_1 \in [0, 1/2] \,. \tag{32}$$

Since the identification involves a reflection (see Fig. 5), the corresponding transition

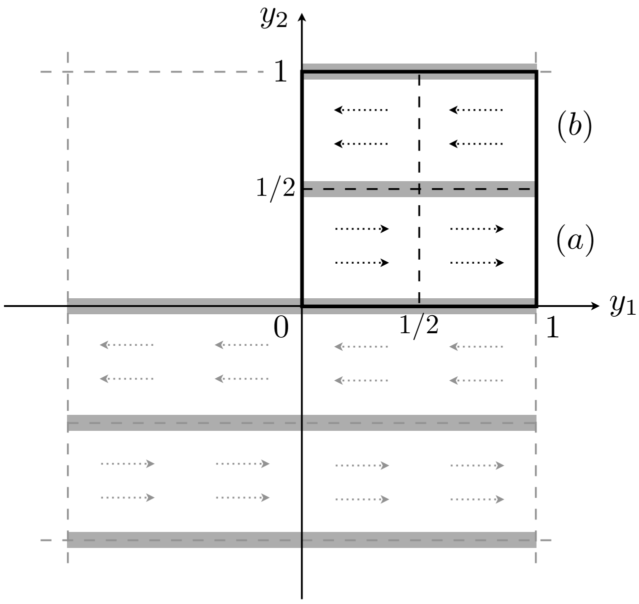


Figure 6: Three copies of the fundamental domain of the torus in the y_1–y_2-plane with constant vector field. Two patches of the bundle are seperated by the circles $y_2 = 0$ and $y_2 = 1/2$.

functions are obtained from

$$A_1^{a'}(s(y)) = -A_1^a(y) - \frac{1}{q}\partial_1\Lambda_{a'a}(y)\,. \tag{33}$$

At $y_2 = 0$, the vector field vanishes and the transition function is therefore trivial. At $y_2 = 1/2$, one has $A_1^{a'}(s(y)) + A_1^a(y) = -f$, which yields $\Lambda_{a'a} = -qfy_1$, and therefore the transition function

$$S_{a'a} = e^{-iqfy_1}\,. \tag{34}$$

This transition function is identical to the transition function S_{ba} given in Eq. (14). This has to be the case, as the patches (a') and (b) both describe the "back" of the "orbifold pillow", as a comparison of Figs. 3 and 5 shows.

Let us now consider "Wilson lines", i.e. constant vector fields, on the orbifold. Fig. 6 shows a field configuration on the torus, which is odd under reflection so that it can be projected on the orbifold,

$$A_1 = \begin{cases} \alpha_1\,, & 0 \le y_2 < \frac{1}{2} \quad (a) \\ -\alpha_1\,, & \frac{1}{2} \le y_2 < 1 \quad (b) \end{cases}\,, \quad A_2 = 0\,. \tag{35}$$

Analogously to the discussion below Eq. (10), the transition function at $y_2 = 1/2$ is

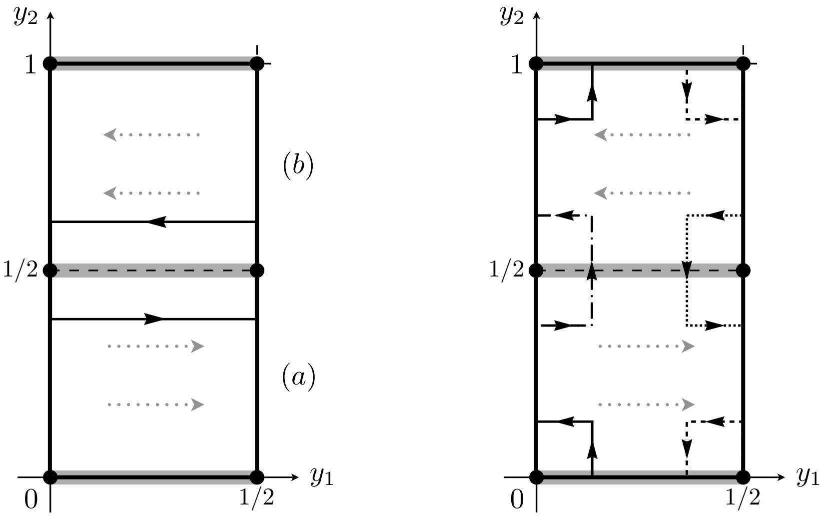


Figure 7: Wilson line integrals on orbifold with constant vector field. Left: bulk Wilson line $W_{\mathcal{T}_1}$; right: fixed-point Wilson lines W_i.

determined by

$$A_1^b(y_1, \tfrac{1}{2}) - A_1^a(y_1, \tfrac{1}{2}) = -\frac{1}{q}\partial_1 \Lambda_{ba} = -2\alpha_1 \,, \tag{36}$$

from which one obtains

$$S_{ba}(y_1, \tfrac{1}{2}) = e^{i\Lambda_{ba}(y_1, y_2=1/2)} = e^{2iq\alpha_1 y_1} \,. \tag{37}$$

The required single-valuedness of the transition function, $S_{ba}(y_1+1) = S_{ba}(y_1)$, implies

$$\alpha_1 = \frac{\pi k_1}{q} \,, \quad k_1 \in \mathbb{Z} \,. \tag{38}$$

This is the well-known fact that Wilson lines on orbifolds are discrete. The same holds for α_2. In the transition from (b) to (a) the vector field changes by $2\pi k_1/q$. Hence, $S_{ab}(y_1, 0) = S_{ab}^{-1}(y_1, 1/2) = S_{ba}(y_1, 1/2)$.

It is now straightforward to compute bulk Wilson lines and Wilson lines around fixed points (see Fig. 7). For k_1 odd, one finds

$$\begin{aligned} W_1 = W_3 = 1 \,, \quad W_2 = W_4 = -1 \,, \\ W_{\mathcal{T}_1} = W_3 W_4 = W_1^{-1} W_2^{-1} = -1 \,, \\ W_{\mathcal{T}_2} = W_1 W_3 = W_2^{-1} W_4^{-1} = 1 \,. \end{aligned} \tag{39}$$

Note that the projections of the torus Wilson lines $W_{\mathcal{T}_{1,2}}$ now factorize into products

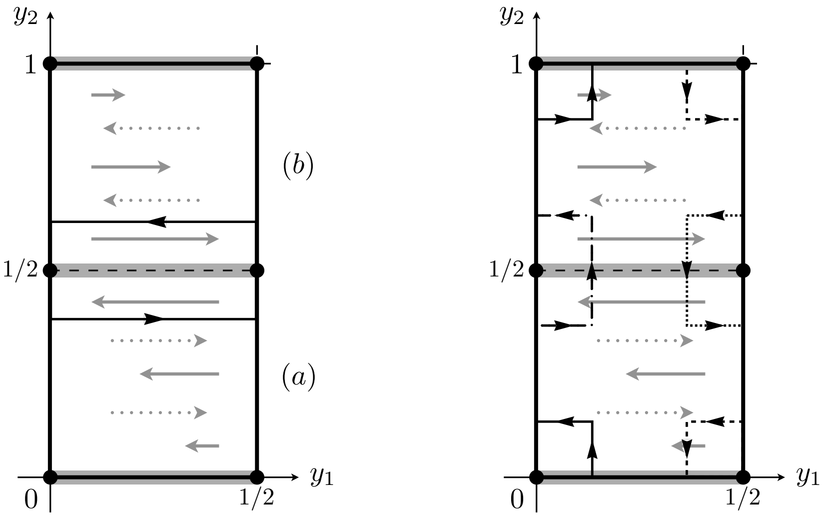


Figure 8: Wilson line integrals on orbifold with magnetic flux and constant vector field. Left: bulk Wilson line $W_{\mathcal{T}_1}$; right: fixed-point Wilson lines W_i.

of fixed-point Wilson line integrals. For a constant vector fields $A^a = \alpha_2 dy_2$ and $A^b = -\alpha_2 dy_2$ on the patches (a) and (b), respectively (see Fig.6), one obtains the transition functions $S_{ba} = e^{2iq\alpha_2 y_2}$ at $y_2 \approx 1/2$ and $S_{ab} = e^{-2iq\alpha_2 y_2}$ at $y_2 \approx 0$. The transition functions at $y_1 \approx 1/2$ and $y_1 \approx 0$ are trivial. The Wilson line integrals now read (k_2 odd)

$$\begin{aligned} W_1 &= W_2 = 1\,, \quad W_3 = W_4 = -1\,, \\ W_{\mathcal{T}_1} &= W_3 W_4 = W_1^{-1} W_2^{-1} = 1\,, \\ W_{\mathcal{T}_2} &= W_1 W_3 = W_2^{-1} W_4^{-1} = -1\,. \end{aligned} \tag{40}$$

For k_1 and k_2 odd, the Wilson line integrals W_i are given by the product of the factors given in Eqs. (39) and (40).

Our particular interest concerns the Wilson line integrals in the case of non-vanishing magnetic flux together with constant background fields, see Fig. 8. The transition functions can be read off from Eqs. (14) and (37),

$$\begin{aligned} y_2 \approx \tfrac{1}{2}: &\quad S_{ba} = e^{i(2\pi(k_1 y_1 + k_2 y_2) - q f y_1)}\,, \\ y_2 \approx 0: &\quad S_{ba} = e^{2i\pi k_1 y_1}\,, \end{aligned} \tag{41}$$

where $qf = 2\pi M$, $M \in \mathbb{Z}$. Knowing the transition functions one easily determines the fixed-point Wilson line integrals W_i for different values of M, k_1 and k_2. The results are summarized in Table 1.

In the case of even bulk flux the number of negative Wilson lines W_i is even. Which Wilson lines are negative depends on the values of k_1 and k_2. This pattern has already been discussed in [27]. For odd bulk flux [24, 25], an odd number of Wilson lines has

M	(k_1,k_2)	W_1	W_2	W_3	W_4
even	$(0,0)$	$+$	$+$	$+$	$+$
	$(1,0)$	$+$	$-$	$+$	$-$
	$(0,1)$	$+$	$+$	$-$	$-$
	$(1,1)$	$+$	$-$	$-$	$+$
odd	$(0,0)$	$+$	$+$	$+$	$-$
	$(1,0)$	$+$	$-$	$+$	$+$
	$(0,1)$	$+$	$+$	$-$	$+$
	$(1,1)$	$+$	$-$	$-$	$-$

Table 1: Wilson line integrals around the orbifold fixed points ζ_i, $i = 1,\ldots,4$, for different transition functions determined by k_1, k_2 and M.

to be negative. Again it depends on the values of k_1 and k_2, which Wilson lines are negative. In this way the standard quantization condition, which holds on the torus, is restored for the total flux, the sum of bulk and localized fluxes.

2.4 Singular gauge fields

So far we have considered regular gauge fields defined on the orbifold by means of transition functions, which are related to twisted boundary conditions on the covering space. It is remarkable that due to the singular fixed points, magnetic fields on orbifolds can also be described by means of singular gauge fields without the need to introduce transition functions. This possibility has previously been discussed in connection with localized Fayet-Iliopoulos terms [40].

Consider the Green's function of the bosonic string on a torus [46] with the singularity located at the position of one of the orbifold fixed points,

$$G(z-\zeta,\tau) = \frac{c}{2}\ln|\vartheta_1(z-\zeta,\tau)|^2 - \frac{c\pi}{\tau_2}(\mathrm{Im}(z-\zeta))^2\,. \tag{42}$$

Here $z = y_1+\tau y_2$, $\zeta = \rho+\tau\eta$, and ϑ_1 is the Jacobi theta-function listed in Appendix A. The Green's function satisfies the differential equation

$$\partial\overline{\partial}G(z-\zeta,\tau) = \pi c\;\delta^2(z-\zeta) - \frac{c\pi}{2\tau_2}\,, \tag{43}$$

where we have used[3]

$$\partial = \partial_z = \frac{i}{2\tau_2}(\bar\tau\partial_1-\partial_2)\,,\quad \overline{\partial} = \partial_{\bar z} = -\frac{i}{2\tau_2}(\tau\partial_1-\partial_2)\,,\quad \delta^2(z-\zeta) = \frac{1}{2\tau_2}\delta^2(y-\zeta)\,. \tag{44}$$

[3]For simplicity, we use the same symbol ζ for $\rho+\tau\eta$ and (ρ,η).

Defining the vector field [40]

$$A = A_z dz + A_{\bar{z}} d\bar{z}\,, \quad A_z = i\partial G\,,\ A_{\bar{z}} = -i\bar{\partial} G\,, \tag{45}$$

one obtains

$$F = dA = F_{z\bar{z}} dz \wedge d\bar{z} = \tfrac{1}{2} F_{mn} dy_m \wedge dy_n\,, \quad F_{z\bar{z}} = \partial A_{\bar{z}} - \bar{\partial} A_z = -2i\partial\bar{\partial} G\,, \tag{46}$$

and therefore

$$\begin{aligned} F_{mn} &= -2i\epsilon_{mn}\tau_2 F_{z\bar{z}} = -4\epsilon_{mn}\tau_2 \partial\bar{\partial} G \\ &= \epsilon_{mn}(-2\pi c\ \delta^2(y-\zeta) + 2\pi c)\,. \end{aligned} \tag{47}$$

Clearly, the vector field A describes a constant bulk flux density c, which is related to a flux density of opposite sign localized at an orbifold fixed point. From the discussion of flux quantization in the previous section we know $qc \in \mathbb{Z}$.

In the vicinity of the orbifold fixed point, $z \simeq \zeta$, the vector field is singular,

$$\vartheta_1(z-\zeta,\tau) \propto (z-\zeta)\,, \quad \bar{\partial} G(z-\zeta,\tau) \simeq \frac{c}{2}\frac{z-\zeta}{|z-\zeta|^2}\,, \tag{48}$$

and with

$$A_z = \frac{i}{2\tau_2}(\bar{\tau} A_1 - A_2)\,, \quad A_{\bar{z}} = -\frac{i}{2\tau_2}(\tau A_1 - A_2)\,, \tag{49}$$

one obtains

$$A_m = c\tau_2 \epsilon_{mn} \frac{(y-\zeta)_n}{|y-\zeta|^2}\,. \tag{50}$$

This is precisely the vortex field introduced in [27] to account for localized flux.

The Green's function $G(z,\tau)$ is even and invariant under lattice translations,

$$G(z,\tau) = G(-z,\tau)\,, \quad G(z,\tau) = G(z+1,\tau)\,, \quad G(z,\tau) = G(z+\tau,\tau)\,. \tag{51}$$

Since $2\zeta_i$ is a lattice vector for all fixed points, $\zeta_1 = (0,0)$, $\zeta_2 = (1/2,0)$, $\zeta_3 = (0,1/2)$ and $\zeta_4 = (1/2,1/2)$, $G(z-\zeta_i)$ is invariant under reflections at the origin, $G(z-\zeta_i,\tau) = G(-z-\zeta_i,\tau)$. Hence, the vector field $A_z(z-\zeta;c)$ is invariant under lattice translations and odd under reflection at the origin. It can therefore be projected to the orbifold. Because of the invariance under lattice translations it is not necessary to intoduce patches and transition functions for the singular vector field.

Using Eq. (50) one obtains for the Wilson line integral around the 1-cycle $\mathcal{C}_i$ for the vector field $A_m(y-\zeta_i;c_i)$,

$$W_i = \exp\left(-iq\oint_{\mathcal{C}_i} A\right) = e^{-iq\pi c_i}\,, \tag{52}$$

which implies $W_i = \pm 1$ for $c_i = k_i/q$, $k_i \in \mathbb{Z}$. Here we have taken into account that the orbifold fixed point ζ_i has a deficit angle π.

3 Wave functions

We now turn to 6d Weyl fermions in a background $U(1)$ gauge field,

$$\mathcal{L}_f = i\bar{\Psi}(x)\Gamma^a e_a^M D_M \Psi(x)\,, \quad \Gamma^7\Psi = -\Psi\,. \tag{53}$$

Here $\Gamma^0,\ldots,\Gamma^6$ are gamma-matrices in six dimensions (see Appendix B), e_a^M is the inverse vielbein, $\Gamma^7 = \Gamma^0 \cdot \ldots \cdot \Gamma^6$ and $D_M = \partial_M + iqA_M$ is the gauge covariant derivative. For the torus metric (2), the inverse zweibein ($a = 5,6$; $m = 1,2$) is given by[4]

$$(e_a^m) = \frac{1}{\sqrt{\tau_2}}\begin{pmatrix} \tau_2 & 0 \\ -\tau_1 & 1 \end{pmatrix}\,, \tag{54}$$

where we have used the definitions

$$e_m^a \delta_{ab} e_n^b = g_{mn}\,, \quad e_a^m e_m^b = \delta_{ab}\,. \tag{55}$$

The 6d Weyl fermion Ψ contains two 4d Weyl fermions of opposite chirality. For gamma-matrices in Weyl representation, one has

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}\,, \quad \gamma_5\psi_L = -\psi_L\,, \quad \gamma_5\psi_R = \psi_R\,. \tag{56}$$

On the orbifold, we impose chiral boundary conditions,

$$\psi_L(x^\mu, y_m) = \psi_L(x^\mu, -y_m)\,, \quad \psi_R(x^\mu, y_m) = -\psi_R(x^\mu, -y_m)\,, \tag{57}$$

which correspond to one possible embedding of the orbifold twist into the $SU(2)_R$ symmetry of the 6d theory. $\Gamma^a e_a^m D_m$ is the mass operator of the fermion fields in the

[4]Lower and upper indices label rows and columns, respectively.

effective 4d theory, and from the 6d Dirac equation

$$\Gamma^a e_a^M D_M \Psi(x,y) = 0 \tag{58}$$

one obtains a coupled system of equations for the two 4d Weyl fermions ψ_L and ψ_R,

$$\begin{aligned} i\gamma^\mu \partial_\mu \psi_L &= -\frac{i}{\sqrt{\tau_2}}(\tau D_1 - D_2)\psi_R = 2\sqrt{\tau_2}(\bar{\partial} + iqA_{\bar{z}})\psi_R\,, \\ i\gamma^\mu \partial_\mu \psi_R &= -\frac{i}{\sqrt{\tau_2}}(\bar{\tau} D_1 - D_2)\psi_L = -2\sqrt{\tau_2}(\partial + iqA_z)\psi_L\,. \end{aligned} \tag{59}$$

Here we have assumed a background gauge field in the compact dimensions, which contains a constant part and magnetic flux part, $A_m = \alpha_m + A_m^{\text{flux}}$. The constant part can be removed by a field redefinition, see Eq. (18). This changes the boundary conditions to[5]

$$\psi_{L,R}(y+\lambda) = e^{iq(\alpha_1 y_1 + \alpha_2 y_2)} S_{ba}^{-1}(y)\psi_{L,R}(y)\,, \tag{60}$$

where S_{ba} is the transition function of the vector bundle.

For comparison, a complex scalar Φ with charge q satisfies the equation of motion

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu + (g_2)^{mn} D_m D_n)\Phi = 0\,. \tag{61}$$

Eqs. (59) and (61) are a convenient starting point to construct mass spectra and wave-functions.

3.1 Twisted wave functions

For the magnetic vector field in Landau gauge, $A_1 = -fy_2$, Eqs. (59) can be rewritten as

$$i\gamma^\mu\partial_\mu\psi_L = -i\sqrt{2qf}a^\dagger\psi_R\,, \quad i\gamma^\mu\partial_\mu\psi_R = i\sqrt{2qf}a\ \psi_L\,, \tag{62}$$

where we have introduced the differential operators ($qf > 0$)

$$a = -(2qf\tau_2)^{-1/2}(\bar{\tau}(\partial_1 - iqfy_2) - \partial_2)\,, \quad a^\dagger = (2qf\tau_2)^{-1/2}(\tau(\partial_1 - iqfy_2) - \partial_2)\,. \tag{63}$$

The operators a and $a^\dagger$ satisfy the commutation relation $[a, a^\dagger] = 1$, and they can therefore be interpreted as annihilation and creation operators.[6] An orthonormal set

[5]In the following we drop the dependence on the 4d space-time coordinates for simplicity.

[6]For the symmetric gauge, $A_1 = -fy_2/2, A_2 = fy_1/2$, and $\tau = i$, one obtains, in the conventions of [37]: $a = i(\partial_z + qf\bar{z})/\sqrt{2qf}, a^\dagger = i(\partial_{\bar{z}} - qfz)/\sqrt{2qf}$.

of mode functions is given by ($qf = 2\pi M$, $M \in \mathbb{N}$)

$$\begin{aligned} \xi_{n,j} &= \frac{i^n}{\sqrt{n!}} \left(a^\dagger\right)^n \xi_j \,, \quad a\, \xi_j = 0 \,, \\ a\, \xi_{n,j} &= i\sqrt{n}\; \xi_{n-1,j} \,, \quad a^\dagger \xi_{n,j} = -i\sqrt{n+1}\; \xi_{n+1,j} \,, \end{aligned} \tag{64}$$

where $j = 1, \ldots, M$ labels the degeneracy of the ground state, with the corresponding mode functions ξ_j. Expanding the chiral fermions ψ_L and ψ_R in terms of the mode functions $\xi_{n,j}$,

$$\psi_L(x,y) = \sum_{n,j} \psi_{Ln,j}(x)\xi_{n,j}(y) \,, \quad \psi_R(x,y) = \sum_{n,j} \psi_{Rn,j}(x)\xi_{n,j}(y) \,, \tag{65}$$

one obtains from Eqs. (62) and (64)

$$\begin{aligned} & i\gamma^\mu \partial_\mu \psi_{L0,j} = 0 \,, \quad i\gamma^\mu \partial_\mu \psi_{Ln+1,j} = -\sqrt{2qf(n+1)}\; \psi_{Rn,j} \,, n \geq 0 \,, \\ & i\gamma^\mu \partial_\mu \psi_{Rn,j} = -\sqrt{2qf(n+1)} \psi_{Ln+1,j} \,, n \geq 0 \,. \end{aligned} \tag{66}$$

The fermions $\psi_{L0,j}$ are the M expected zero-modes, and the pair of chiral fermions $(\psi_{Ln+1,j}, \psi_{Rn,j})$ form 4d Dirac fermions with masses $m_{n,j} = \sqrt{2qf(n+1)}$.

Correspondingly, for the complex scalar Φ one finds (cf. (61))

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu - 2qf(a^\dagger a + \tfrac{1}{2}))\Phi = 0 \,. \tag{67}$$

After a mode expansion, $\Phi = \sum_{n,j} \Phi_{n,j}\xi_{n,j}$, one obtains the scalar mass spectrum $M^2_{n,j} = 2qf(n+1)$, $n \geq 0$.

We are particularly interested in the fermionic zero-modes $\psi_{L0,j}$. Their mode functions are determined by the equation

$$a\, \xi_j \propto (\bar{\tau} D_1 - D_2)\xi_j = 0 \,, \tag{68}$$

where $D_{1,2}$ are now the covariant derivatives in the flux background. From Eq. (60) and the transition functions (41) one obtains the twisted boundary conditions ($qf = 2\pi M, M \in \mathbb{N}$)

$$\begin{aligned} \xi_{n,j}(y + \lambda_1) &= e^{-\pi i k_1} \xi_{n,j}(y) \,, \\ \xi_{n,j}(y + \lambda_2) &= e^{-\pi i (k_2 - 2M y_1)} \xi_{n,j}(y) \,. \end{aligned} \tag{69}$$

Clearly, the mode functions $\xi_{n,j}$ depend on the number of flux quanta M as well as k_1 and k_2, which determine the boundary conditions.

For comparison with the untwisted wave functions, which we will construct in the following section, we briefly recall the derivation of the zero-mode functions, following

[22]. The boundary conditions (69) are satisfied by functions ξ which can be written as

$$\xi(y) = e^{-i\pi(k_1y_1+k_2y_2)} \sum_n f_n(y_2) e^{2\pi i n y_1} \,, \tag{70}$$

where the coefficient functions $f_n(y_2)$, $n \in \mathbb{Z}$, fulfill the recurrence relation

$$f_n(y_2+1) = f_{n-M}(y_2) \,. \tag{71}$$

The zero-mode equation (68), $a\ \xi = 0$, yields first-order differential equations for the functions $f_n(y_2)$, whose solutions are given by

$$f_n(y_2) = c_n e^{-i\pi\bar\tau M y_2^2 - i\pi(\bar\tau k_1 - k_2)y_2 + 2\pi i\bar\tau n y_2} \,. \tag{72}$$

Because of the recurrence relation (71), only M of the constants c_n are independent. Writing $n = lM + j$, with $l \in \mathbb{Z}$ and $j = 0, \ldots, M-1$, one has

$$c_n = \mathcal{N}_j e^{-\frac{i\pi\bar\tau}{M}(lM+j)^2 + \frac{i\pi}{M}(\bar\tau k_1 - k_2)(lM+j)} \,, \tag{73}$$

where the $\mathcal{N}_j$ are normalization constants. Hence, there are indeed M independent zero-mode functions ξ_j on the torus. Combining Eqs. (70), (72) and (73), and replacing the sum over n by a double sum over j and l, one obtains the wave functions ξ_j as infinite sums over l, which can be conveniently expressed in terms of Jacobi theta-functions (see Appendix B) [22],

$$\begin{aligned} \xi_j(y) &= \mathcal{N}_j e^{-i\pi M\bar\tau y_2^2 - i\pi k_1 \bar z} \sum_l e^{-i\pi M\bar\tau(l+j/M)^2 + 2\pi i(l+j/M)(M\bar z + (\bar\tau k_1 - k_2)/2)} \\ &= \mathcal{N}_j e^{-i\pi M\bar\tau y_2^2 - i\pi k_1 \bar z}\ \vartheta \begin{bmatrix} j/M \\ (k_1\bar\tau - k_2)/2 \end{bmatrix} (M\bar z, -M\bar\tau) \,, \quad \bar z = y_1 + \bar\tau y_2 \,. \end{aligned} \tag{74}$$

Since $\tau_2 = -\mathrm{Im}\bar\tau > 0$, these zero-mode functions are normalizable for $M > 0$.

Under reflection the zero-mode ξ_j turns into another zero mode $\xi_{j'}$. One finds the relation

$$\xi_j(-y) = e^{\frac{2\pi i}{M}(j' - k_1/2)k_2} \frac{\mathcal{N}_j}{\mathcal{N}_{j'}} \xi_{j'}(y) \,, \quad j' = mM + k_1 - j \,, \quad m \in \mathbb{Z} \,, \tag{75}$$

with m chosen such that $j, j' = 1, \ldots, M-1$. From ξ_j and $\xi_{j'}$ one can form even and odd linear combinations,

$$\begin{aligned} \xi_j^\eta(y) &\propto \xi_j(y) + \eta\xi_j(-y) \,, \quad \eta = +, - \,, \\ \xi_j^\eta(y) &= \eta\xi_j^\eta(-y) \,. \end{aligned} \tag{76}$$

	j	j'	$k_2 = 0$ $\eta = +$	$k_2 = 0$ $\eta = -$	$k_2 = 1$ $\eta = +$	$k_2 = 1$ $\eta = -$
$k_1 = 0$ M even	$0, 1, \ldots, \frac{M}{2}$	$0, M{-}1, \ldots, \frac{M}{2}$	$\frac{M}{2} + 1$	$\frac{M}{2} - 1$	$\frac{M}{2}$	$\frac{M}{2}$
$k_1 = 0$ M odd	$0, 1, \ldots, \frac{M-1}{2}$	$0, M{-}1, \ldots, \frac{M+1}{2}$	$\frac{M+1}{2}$	$\frac{M-1}{2}$	$\frac{M+1}{2}$	$\frac{M-1}{2}$
$k_1 = 1$ M even	$0, 1, 2, \ldots, \frac{M}{2}$	$1, 0, M{-}1, \ldots, \frac{M}{2} + 1$	$\frac{M}{2}$	$\frac{M}{2}$	$\frac{M}{2}$	$\frac{M}{2}$
$k_1 = 1$ M odd	$0, 1, 2 \ldots, \frac{M+1}{2}$	$1, 0, M{-}1, \ldots, \frac{M+1}{2}$	$\frac{M+1}{2}$	$\frac{M-1}{2}$	$\frac{M-1}{2}$	$\frac{M+1}{2}$

Table 2: Zero-modes j and j', which are related by reflection, with the choice $j' \geq j$. The last four columns give the number of even and odd linear combinations that depend on the values of M, k_1 and k_2. Their sum is always M.

For $j \neq j'$ one obtains one even and one odd zero-mode. In the case $j = j'$ the zero-mode is either even or odd. From Eq. (75) one easily derives the relations between j and j' for given M, k_1 and k_2 (see Table 2). For $k_1 = 0$ the $j = 0$ mode is even; moreover, for M even the $j = M/2$ mode is even ($k_2 = 0$) or odd ($k_2 = 1$). For $k_1 = 1$ and M odd the $(M+1)/2$ mode is even ($k_2 = 0$) or odd ($k_2 = 1$). This leads to the numbers of even and odd zero-modes listed in Table 2. These results have previously been obtained in [25].

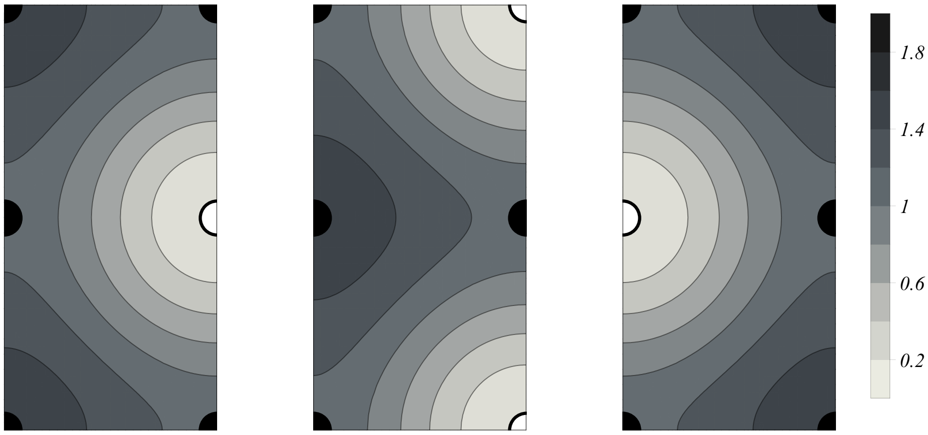
The field theory on the orbifold is defined by the chiral boundary conditions (57) which require that 4d left-handed fermions are linear combinations of even mode functions. For the zero-modes these are[7]

$$\begin{aligned}
\xi_j^+(y) &= \mathcal{N}_j e^{-i\pi M\bar{\tau}y_2^2}\Bigg(e^{-i\pi k_1\bar{z}}\ \vartheta\begin{bmatrix} j/M \\ (k_1\bar{\tau} - k_2)/2 \end{bmatrix}(M\bar{z}, -M\bar{\tau}) \\
&\qquad + e^{i\pi k_1\bar{z}}\ \vartheta\begin{bmatrix} j/M \\ (k_1\bar{\tau} - k_2)/2 \end{bmatrix}(-M\bar{z}, -M\bar{\tau})\Bigg) \\
&= \mathcal{N}_j e^{-i\pi M\bar{\tau}y_2^2}\sum_l e^{-i\pi M\bar{\tau}(l+j/M)^2 + i\pi(\bar{\tau}k_1 - k_2)(l+j/M)}\cos\left[2\pi\left(lM + j - \frac{k_1}{2}\right)\bar{z}\right],
\end{aligned} \tag{77}$$

where $\mathcal{N}_j$ are adjusted normalization constants on the orbifold.

Let us now consider some examples. For $M = 0$, one has the standard orbifold result without flux, a single constant mode function for $k_1 = k_2 = 0$, and there are no non-vanishing mode functions otherwise. For $M = 1$, one obtains a single zero-mode $j = 0$ for each pair (k_1, k_2). For $k_1 = k_2 = 1$, the zero-mode is odd, otherwise it is even.

[7]Note, that these zero-mode functions are the complex conjugate of the wave functions given in [27].



(a) $k_1 = 0, k_2 = 0$ (b) $k_1 = 1, k_2 = 0$ (c) $k_1 = 0, k_2 = 1$

Figure 9: Absolute square of even zero-mode functions for $M = 1$ and different values of k_1 and k_2. The white circles indicate where the mode functions vanish.

Using properties of the Jacobi theta-functions (see Appendix A) one has

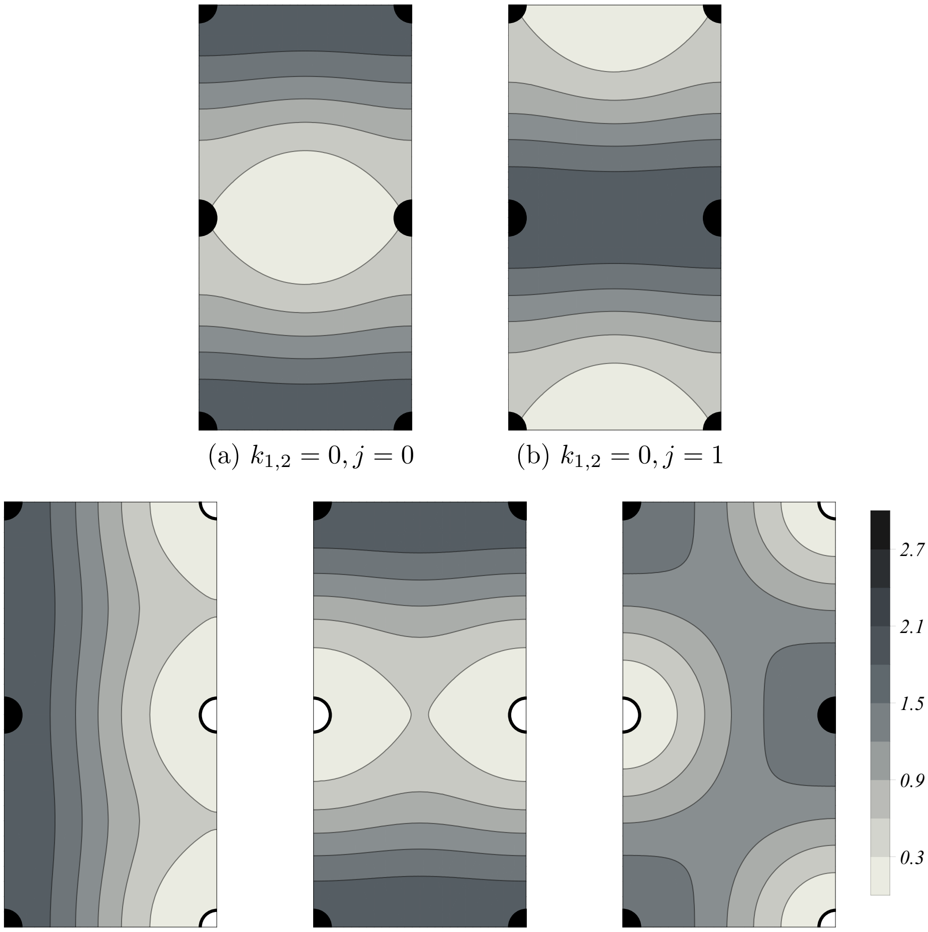
$$\begin{aligned}\xi_0^+(y) &= \mathcal{N}_0 e^{-i\pi\bar{\tau}y_2^2} e^{-i\pi k_1 \bar{z}}\ \vartheta\begin{bmatrix} 0 \\ (k_1\bar{\tau} - k_2)/2 \end{bmatrix}(\bar{z}, -\bar{\tau}) \\ &= \mathcal{N}_0 e^{-i\pi\bar{\tau}y_2^2} e^{-i\pi k_1 \bar{z}}\ \vartheta(\bar{z} + (k_1\bar{\tau} - k_2)/2, -\bar{\tau})\,.\end{aligned} \tag{78}$$

The three even zero-mode functions are shown in Figure 9. For each (k_1, k_2) combination the zero-mode function vanishes at one fixed point.

For $M = 2$ there are two zero-modes, $j = 0, 1$. In the case $k_1 = k_2 = 0$ both of them are even. Their mode functions read

$$\begin{aligned}\xi_j^+(y) &= \mathcal{N}_j e^{-2\pi i\bar{\tau}y_2^2}\ \vartheta\begin{bmatrix} j/2 \\ 0 \end{bmatrix}(2\bar{z}, -2\bar{\tau}) \\ &= \mathcal{N}_j e^{-2\pi i\bar{\tau}y_2^2} e^{-i\pi\bar{\tau}j^2/2 + 2\pi i j\bar{z}}\ \vartheta(2\bar{z} - j\bar{\tau}, -2\bar{\tau})\,, \quad j = 0, 1\,.\end{aligned} \tag{79}$$

The two mode functions are depicted in Fig. 10(a,b). Note that the functions are non-zero at all fixed points. In the remaining cases there is always one even and one odd mode functions. For the even mode functions one obtains from Eq. (77), after some



(a) $k_{1,2} = 0, j = 0$ (b) $k_{1,2} = 0, j = 1$

(c) $k_1 = 1, k_2 = 0$ (d) $k_1 = 0, k_2 = 1$ (e) $k_1 = 1, k_2 = 1$

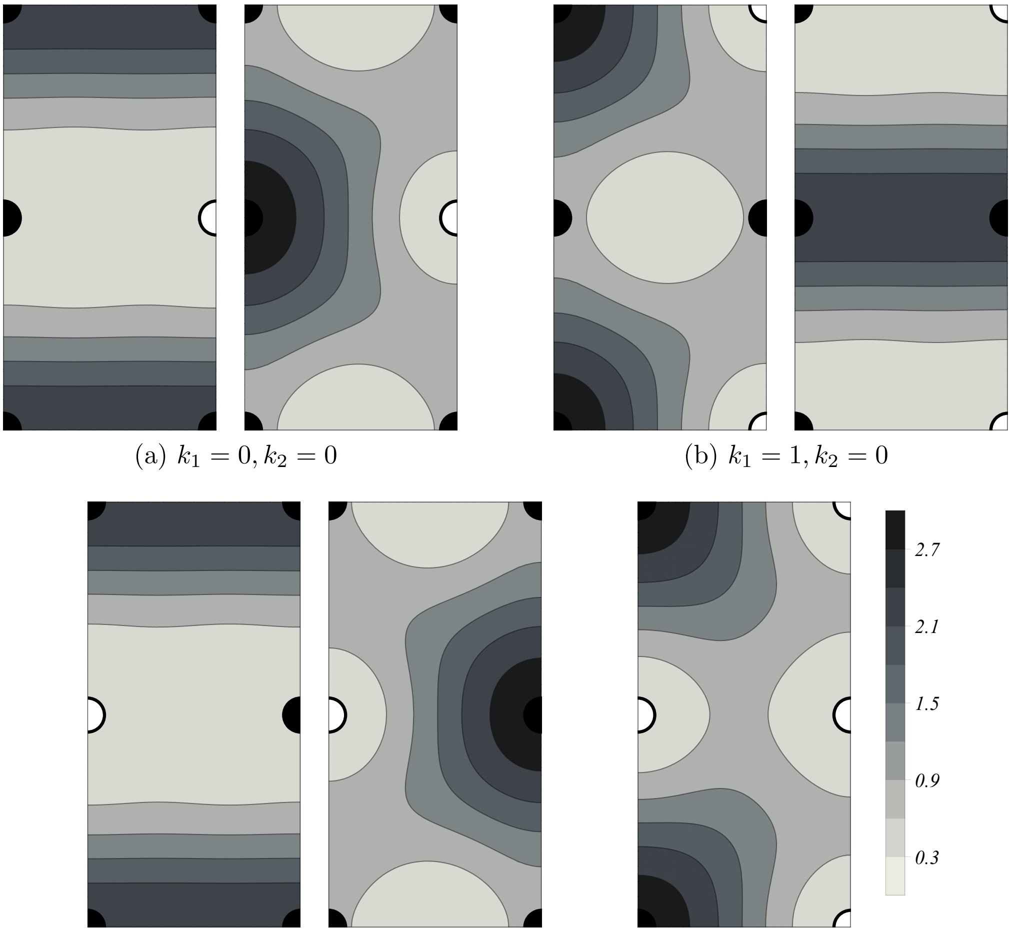
Figure 10: Absolute square of even zero-mode functions for $M = 2$ and different values of k_1 and k_2. The white circles indicate where the mode functions vanish.

manipulations,

$$\begin{aligned}
\xi_0^+(y) &= \mathcal{N}_0 e^{-2\pi i\bar\tau y_2^2} \left(e^{-i\pi\bar z}\ \vartheta(2\bar z + \bar\tau/2, -2\bar\tau) + e^{i\pi\bar z}\ \vartheta(2\bar z - \bar\tau/2, -2\bar\tau)\right) , \\
&\qquad\qquad\qquad\qquad\qquad\qquad k_1 = 1, k_2 = 0 \, , \\
\xi_0^+(y) &= \mathcal{N}_0 e^{-2\pi i\bar\tau y_2^2}\ \vartheta(2\bar z - 1/2, -2\bar\tau) \, , \ k_1 = 0, k_2 = 1 \, , \\
\xi_0^+(y) &= \mathcal{N}_0 e^{-2\pi i\bar\tau y_2^2} \left(e^{-i\pi\bar z}\ \vartheta(2\bar z + \bar\tau/1 - 1/2, -2\bar\tau) + e^{i\pi\bar z}\ \vartheta(2\bar z - \bar\tau/2 + 1/2, -2\bar\tau)\right) , \\
&\qquad\qquad\qquad\qquad\qquad\qquad k_1 = 1, k_2 = 1 \, .
\end{aligned} \tag{80}$$

The even functions are shown in Fig. 10(c,d,e). They all vanish at two fixed points.

It is straightforward to continue the discussion to larger values of M. For $M = 3$ one obtains a pattern similar to the one for $M = 1$. For $(k_1, k_2) = (0,0), (1,0), (0,1)$ one finds two zero-modes, $j = 0$ and $j = 1$ (see Fig. 11(a,b,c)). The zeros of the mode



(a) $k_1 = 0, k_2 = 0$ (b) $k_1 = 1, k_2 = 0$ (c) $k_1 = 0, k_2 = 1$ (d) $k_1 = 1, k_2 = 1$

Figure 11: Absolute squre of even zero-mode functions for $M = 3$ and different values of k_1 and k_2. The white circles indicate where the mode functions vanish.

functions appear at the same fixed points as for $M = 1$. An interesting new aspect is that now an even mode function also exists for $(k_1, k_2) = (1, 1)$ with zeros at three fixed points (see Fig. 11(d)).

Finally, the mode functions for $M = 4$ are shown in Fig. 12. The pattern completely agrees with the case $M = 2$, especially with respect to the distribution of zeros at fixed points. The only difference is that for each combination of k_1, k_2 the number of zero modes has increased by one. The mode functions agree with those obtained in [27].

3.2 Untwisted wave functions

In [27] it has been pointed out that the zeros of the zero-mode functions are related to negative Wilson line integrals around the orbifold fixed points, which might be interpreted in terms of localized flux, envoking Stokes' theorem. Such localized flux

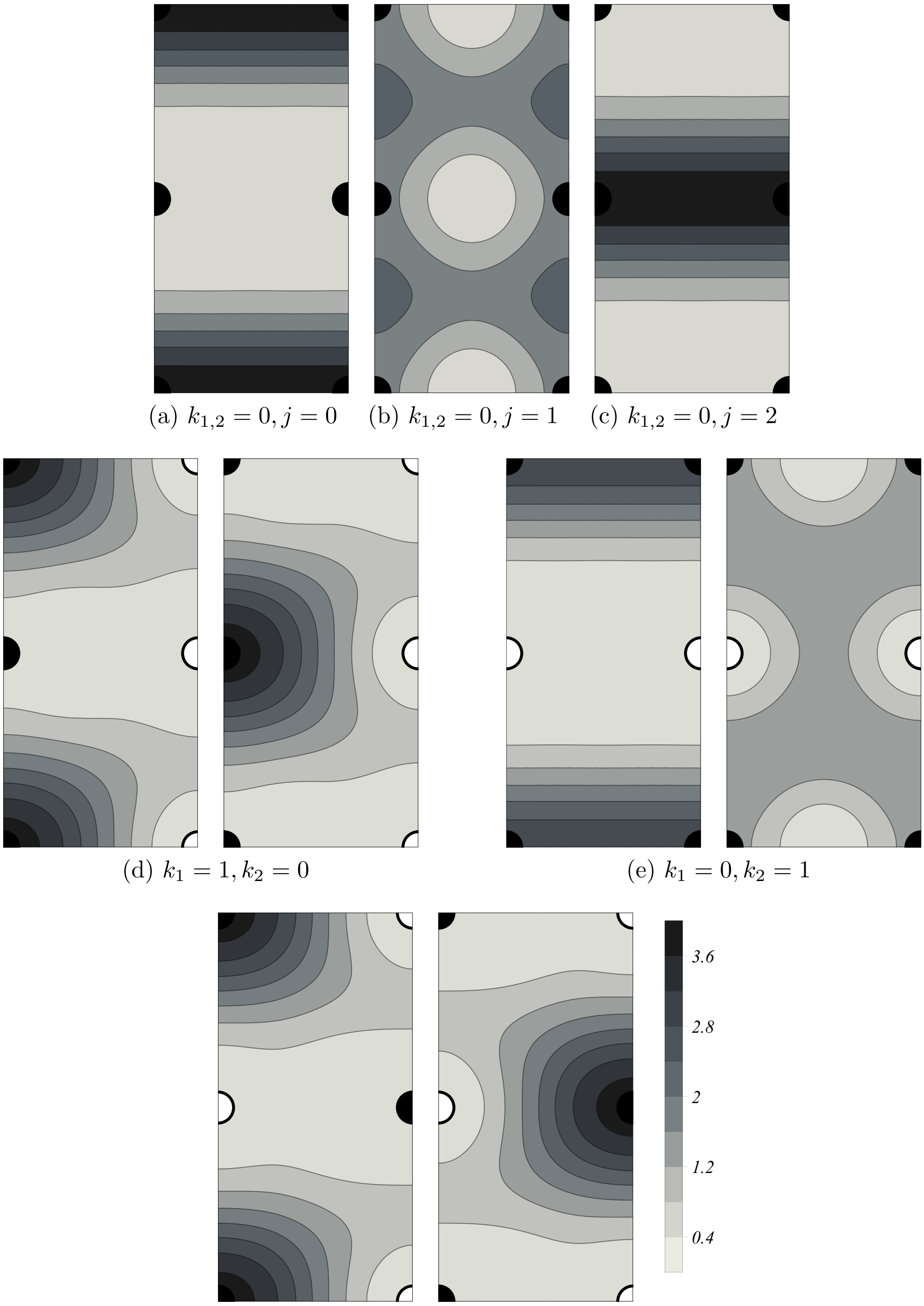


Figure 12: Absolute square of even zero-mode functions for $M = 4$ and different values of k_1 and k_2. The white circles indicate where the mode functions vanish.

has previously been discussed in connection with fixed point anomalies [38, 39] and also with respect to localized Fayet-Iliopoulos terms [40]. In the latter case a consistent description of the localized flux is obtained by means of the Green's function on a torus, whose gradient yields a singular vector field, as discussed in Section 2.4.

From the torus Green's function one obtains a localized flux together with a bulk flux (see (47)),

$$F_{12} = -2\pi c\, \delta^2(y-\zeta) + 2\pi c\,. \tag{81}$$

Hence, in our convention for the bulk flux, $qf = qF_{12} = 2\pi M > 0$, one has $c = M/q$. The torus Green's function $G(z-\zeta,\tau)$ has been used to describe zero-modes of a charged bulk field [40]. Indeed, for a vector field $A_z = i\partial G$, the wave function

$$\chi_\zeta(z) \propto e^{qG(z-\zeta,\tau)} \tag{82}$$

solves the field equation

$$(\partial + iqA_z)\chi_\zeta(z) = 0\,. \tag{83}$$

Using Eqs. (44) and (49) it is apparent that Eq. (83) is nothing but the field equation (68), with the regular vector field of Section 2.3 replaced by the singular vector field $A_z = i\partial G$. Using (42), the zero-mode function (82) can be written as

$$\chi_\zeta(z) \propto |\vartheta_1(z-\zeta,\tau)|^M e^{-\frac{\pi M}{\tau_2}(\mathrm{Im}(z-\zeta))^2}\,, \tag{84}$$

where ζ denotes the Green's function's singularity. The asympotic behaviour of the wave function close to the singularity is given by

$$\chi_\zeta(z) \underset{z\to\zeta}{\propto} |z-\zeta|^M\,. \tag{85}$$

Clearly, the wave function is normalizable for $M > 0$.

Wave functions for bulk flux $M = 1$ can be obtained from Green's functions $G(z-\zeta_i)$, where ζ_i are the four orbifold fixed points. From Eq. (84), and using relations among theta-functions listed in Appendix A, one obtains for the fixed points ζ_2, ζ_3 and ζ_4:

$$\chi_\zeta(z) = \begin{cases} |e^{-i\pi\bar\tau y_2^2 - i\pi\bar z}\vartheta(\bar z + \bar\tau/2, -\bar\tau)|\,, & \zeta_2 = 1/2\ (k_1 = 1, k_2 = 0) \\ |e^{-i\pi\bar\tau y_2^2}\vartheta(\bar z - 1/2, -\bar\tau)|\,, & \zeta_3 = \tau/2\ (k_1 = 0, k_2 = 1) \\ |e^{-i\pi\bar\tau y_2^2}\vartheta(\bar z + \bar\tau/2, -\bar\tau)|\,, & \zeta_4 = (1+\tau)/2\ (k_1 = 0, k_2 = 0)\,. \end{cases} \tag{86}$$

These wave functions are precisely the modulus of the $M = 1$ zero-mode functions given in Eq. (78) (see Fig. 9). This connection is not unexpected since both sets of functions are zero modes for the same bulk flux, $M = 1$, just for different choices of the vector field. By construction, the zeros are obvious for the untwisted wave functions, see Eq. (85). The three fixed points ζ_2, ζ_3 and ζ_4 correspond to three pairs (k_1, k_2), which are given in brackets in Eq. (86). The wave function χ_{ζ_1} corresponds to the odd zero mode among the untwisted wave functions.

At first sight the appearance of the modulus in Eq. (86) is surprizing. As a consequence, the untwisted wave functions transform trivially under translations by the lattice vectors λ_1 and λ_2. This, however, is expected. Like the Green's function, the vector field is invariant under translations by lattice vectors. Therefore, no non-trivial transition functions occur, and the boundary conditions are trivial. The situation is different for the regular vector field, where the non-trivial transition functions lead to twisted boundary conditions, which can be satisfied by appropriate phase factors.

The case $M = 2$ can be treated in a similar way. Two bulk flux quanta can be obtained by localizing one quantum at two fixed points. The vector field is now the sum $A_z = i(\partial G(z - \zeta_1, \tau) + \partial G(z - \zeta_2, \tau))$, and the wave function is the product

$$\begin{aligned} \chi_{\zeta_1,\zeta_2} &= \chi_{\zeta_1}(z)\chi_{\zeta_2}(z) \propto e^{q(G(z-\zeta_1,\tau)+G(z-\zeta_2,\tau))} \\ &\propto |\vartheta_1(z-\zeta_1,\tau)\vartheta_1(z-\zeta_2,\tau)| e^{-\frac{\pi}{\tau_2}((\mathrm{Im}(z-\zeta_1))^2+(\mathrm{Im}(z-\zeta_2))^2)} \,. \end{aligned} \tag{87}$$

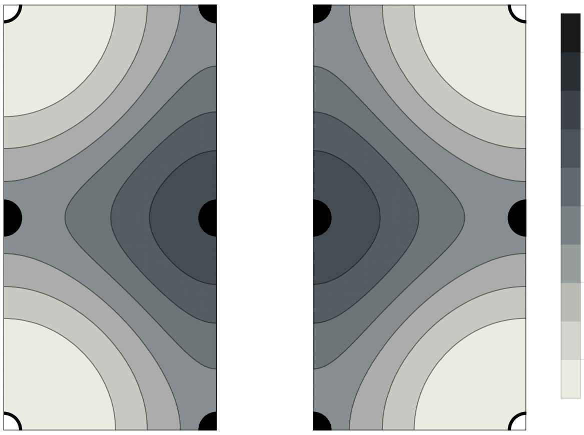
Using Eq. (111) the product of theta-functions can be witten as

$$\begin{aligned} &\vartheta_1(z-\zeta_1,\tau)\ \vartheta_1(z-\zeta_2,\tau) \\ &\qquad = \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(z-\zeta_1+1/2,\tau)\ \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(z-\zeta_2+1/2,\tau) \\ &\qquad = \sum_{m=0}^{1}\vartheta\begin{bmatrix}(m+1)/2\\0\end{bmatrix}(2z-\zeta_1-\zeta_2+1,2\tau)\ \vartheta\begin{bmatrix}m/2\\0\end{bmatrix}(\zeta_1-\zeta_2,2\tau)\,. \end{aligned} \tag{88}$$

The second theta-function in the last line does not depend on z. Hence, the product of ϑ_1-functions can be expressed as a linear combination of z-dependent theta-functions. A look at Fig. 10 suggests to consider the combination of fixed points (ζ_2, ζ_4), (ζ_3, ζ_4) and (ζ_2, ζ_3). After some algebra, one then obtains from Eqs. (87) and (88):

$$\chi_{\zeta,\zeta'} \propto e^{-2\pi\tau_2 y_2^2}\begin{cases} |e^{-i\pi\bar{z}}\ \vartheta(2\bar{z}+\bar{\tau}/2,-2\bar{\tau}) + e^{i\pi\bar{z}}\ \vartheta(2\bar{z}-\bar{\tau}/2,-2\bar{\tau})|\,, \\ \qquad \zeta_2,\zeta_4 \quad (k_1=1,k_2=0) \\ |\vartheta(2\bar{z}-1/2,-2\bar{\tau})|\,, \quad \zeta_3,\zeta_4 \quad (k_1=0,k_2=1) \\ |e^{-i\pi\bar{z}}\ \vartheta(2\bar{z}+(\bar{\tau}-1)/2,-2\bar{\tau}) + e^{i\pi\bar{z}}\ \vartheta(2\bar{z}-(\bar{\tau}-1)/2,-2\bar{\tau})|\,, \\ \qquad \zeta_2,\zeta_3 \quad (k_1=1,k_2=1)\,. \end{cases} \tag{89}$$

These wave functions are again the modulus of the $M = 2$ twisted zero-mode functions listed in Eq. (80). There are three more untwisted zero-mode functions corresponding to the pairs of fixed points (ζ_1, ζ_2), (ζ_1, ζ_3) and (ζ_1, ζ_4). They correspond to the three



(a) $\chi^2_{\zeta_1}$ (b) $\chi^2_{\zeta_2}$

Figure 13: Untwisted zero-mode functions with two flux quanta at a fixed point.

odd twisted zero modes that are not shown in Fig. 10.

Further $M = 2$ untwisted zero-mode functions are obtained by localizing two flux quanta at one fixed point. The corresponding mode functions read

$$\begin{aligned}\chi^2_{\zeta_i} &\propto e^{qG(z-\zeta_i,\tau)} \\ &\propto |\vartheta_1(z-\zeta_i,\tau)^2| e^{-\frac{2\pi}{\tau_2}(\mathrm{Im}(z-\zeta_i))^2}\,, \quad i = 1,\dots,4\,.\end{aligned} \tag{90}$$

Using Eq. (88) the product of ϑ_1-functions can be written as

$$\begin{aligned}\vartheta_1(z-\zeta_i,\tau)^2 = \vartheta\begin{bmatrix}0\\0\end{bmatrix}(0,2\tau)\;\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(2z-2\zeta_i+1,\tau) \\ +\,\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(0,2\tau)\;\vartheta\begin{bmatrix}0\\0\end{bmatrix}(2z-2\zeta_i,\tau)\,.\end{aligned} \tag{91}$$

The wave functions $\chi^2_{\zeta_i}$ can easily be expressed in terms of two $k_1 = k_2 = 0$ mode functions ξ^+_j, see Eq. (79). For $\chi^2_{\zeta_1}$ and $\chi^2_{\zeta_2}$, for instance, one finds

$$\chi^2_{\zeta_1} = |a\xi^+_0 - b\xi^+_1|\,, \quad \chi^2_{\zeta_2} = |a\xi^+_0 + b\xi^+_1|\,, \tag{92}$$

where $a = \vartheta_1(-1/2,-2\bar{\tau})/\mathcal{N}_0$ and $b = \vartheta(0,-2\bar{\tau})/\mathcal{N}_1$. Analogous expressions can be given for $\chi^2_{\zeta_3}$ and $\chi^2_{\zeta_4}$. The wave functions $\chi^2_{\zeta_1}$ and $\chi^2_{\zeta_2}$ are shown in Fig. 13. By construction, they vanish at $z = \zeta_1$ and $z = \zeta_2$, respectively.

The construction of untwisted zero-mode functions described above can be extended to larger values of M in a straightforward way. These wave functions have characteristic zeros which may be interesting in physical applications. However, the combinatorics becomes more involved because of the many possibilities to distribute flux quanta over the fixed points, and the extension to larger M goes beyond the scope of this paper.

3.3 Singular gauge transformations

The regular and singular gauge fields discussed in Section 2 yield the same bulk flux, and the corresponding zero-mode functions constructed in the previous subsections are closely related. As we shall now show, these two descriptons of a gauge theory on orbifolds can indeed be directly mapped into each other by means of a singular gauge transformation.

The regular gauge field in Section 2.3 is given by Eq. (31),

$$A^{(r)} = A_1^{(r)} dy_1 + A_2^{(r)} dy_2 \,, \quad A_1^{(r)} = -2\pi M y_2 \,, \quad A_2^{(r)} = 0 \,. \tag{93}$$

Alternatively, the singular gauge field

$$A^{(s)} = A_z^{(s)} dz + A_{\bar{z}}^{(s)} d\bar{z} = A_1^{(s)} dy_1 + A_2^{(s)} dy_2 \tag{94}$$

is defined by means of the Green's function $G(z-\zeta,\tau)$, where ζ denotes one of the orbifold singularities (see (42), (45)). With $c = M/q$ and $\zeta = \rho + \tau\eta$, complex and real components of the singular vector field are given by

$$\begin{aligned} qA_z^{(s)} &= iq\partial G = iM\partial \left(\frac{1}{2} \ln |\vartheta_1(z-\zeta,\tau)|^2 - \frac{\pi}{\tau_2} (\mathrm{Im}(z-\zeta))^2 \right) \\ &= iM\partial \left(\frac{1}{2} \ln \vartheta_1(z-\zeta,\tau) - \frac{\pi}{\tau_2} (\mathrm{Im}(z-\zeta))^2 \right) = qA_{\bar{z}}^{(s)\,*} \,, \end{aligned} \tag{95}$$

$$qA_1^{(s)} = q(A_z^{(s)} + A_{\bar{z}}^{(s)}) = -2\pi M(y_2 - \eta) + iM\partial_1 \ln \frac{\vartheta_1}{|\vartheta_1|} \,, \tag{96}$$

$$qA_2^{(s)} = q(\tau A_z^{(s)} + \bar{\tau} A_{\bar{z}}^{(s)}) = -2\pi M \tau_1 (y_2 - \eta) + iM\partial_2 \ln \frac{\vartheta_1}{|\vartheta_1|} \,. \tag{97}$$

Comparing Eq. (93) and Eqs. (96) and (97) it is clear that regular and singular gauge fields are related by a singular gauge transformation,

$$A^{(r)} = A^{(s)} - \frac{1}{q} d\Lambda \,, \tag{98}$$

where Λ is given by

$$\Lambda = 2\pi M \eta y_1 - \pi M \tau_1 (y_2 - \eta)^2 + iM \ln \frac{\vartheta_1}{|\vartheta_1|} \,. \tag{99}$$

The local gauge parameter Λ is ill-defined at the singularity $z = \zeta$ of the Green's function. Away from this point Λ is real. The crucial point of this transformation is that it changes the factor $|\vartheta_1|^M$ appearing in untwisted wave functions to the factor ϑ_1^{*M} appearing in twisted wave functions,

$$\chi \propto |\vartheta_1|^M \quad \longrightarrow \quad \xi = e^{i\Lambda}\chi \propto \vartheta_1^{*M}\,. \tag{100}$$

Note that ϑ_1^* is a function of $\bar{\tau}$, like the holomorphic part of the zero-mode functions ξ_j.

Unwisted wave functions transform trivially under lattice translations,

$$\chi(y_1+1, y_2) = \chi(y_1, y_2)\,, \quad \chi(y_1, y_2+1) = \chi(y_1, y_2)\,. \tag{101}$$

The boundary conditions of twisted wave functions are then determined by the transformation of Λ under lattice translations. From Eqs. (99), (117) and (118) one obtains

$$\begin{aligned} \Lambda(y_1+1, y_2) &= -\pi M(1-2\eta) + \Lambda(y_1, y_2)\,, \\ \Lambda(y_1, y_2+1) &= -\pi M(1+2\rho-2y_1) + \Lambda(y_1, y_2)\,. \end{aligned} \tag{102}$$

This agrees precisely with the twisted boundary conditions (69), with the identification up to mod 2,

$$k_1 = M(1-2\eta)\,, \quad k_2 = M(1-2\rho)\,. \tag{103}$$

Hence, the location of the singularity at $\zeta = \rho + \tau\eta$ determines the constant Wilson line factors $k_{1,2}$ of the regular vector field. The generalization of this result to untwisted wave functions with contributions from different singularities at $\zeta_i = \rho_i + \tau\eta_i$ is obvious. Up to mod 2, k_1 and k_2 are now given by

$$k_1 = \sum_i M_i(1-2\eta_i)\,, \quad k_2 = \sum_i M_i(1-2\rho_i)\,, \tag{104}$$

where $-M_i$ are the localized fluxes at the fixed points ζ_i. This result is indeed consistent with the explicit examples discussed in the previous section. For $M = 1$, (k_1, k_2) is given by $(1, 0)$ for ζ_2, $(0, 1)$ for $\zeta_3 = \tau/2$ and $(0, 0)$ for $\zeta_4 = (1+\tau)/2$; this agrees with the list in Eq. (86). For $M = 2$, with both flux quanta at the same fixed point, one obviously has $k_1 = k_2 = 0$, which is consistent with Eq. (92). Finally, for $M = 2$, with flux quanta localized at different fixed points (ζ, ζ'), (k_1, k_2) is given by $(1, 0)$ for (ζ_2, ζ_4), $(0, 1)$ for (ζ_3, ζ_4) and $(1, 1)$ for (ζ_2, ζ_3). This is in agreement with Eq. (89).

Note that the above procedure, mapping singular to regular gauge fields, is restricted to the bulk, excluding the orbifold fixed points. It is tempting to conjecture that integer localized fluxes correspond to localised fermion zero-modes. In this way not only the mod 2 parity but the entire localized flux would be a physical quantity. It would then influence the fields localized at the orbifold singularities without modifying the bulk content. However, despite being of general interest, this question goes beyond the scope of our investigations.

4 Summary and Outlook

We have studied in detail $U(1)$ gauge fields on the orbifold $T^2/\mathbb{Z}_2$. One of the main goals has been to clarify the quantization condition for magnetic flux. Contrary to the naive expectation $qF = 2\pi M$, $M \in \mathbb{Z}$, we showed that also flux values $qF = \pi + 2\pi M$, $M \in \mathbb{Z}$, are allowed, confirming results in [24, 25]. This is an effect of the orbifold fixed points ζ_i. They can have non-trivial Wilson lines W_i around them, which can be interpreted as localized flux, $qF_i = \pi(\delta_{(W_i,-1)} + 2k_i)$, $k_i \in \mathbb{Z}$. The total flux of bulk and fixed points then satisfies the standard quantization condition $q(F + \sum_i F_i) \in 2\pi\mathbb{Z}$. To obtain these results it is crucial to treat the flux background as a vector bundle on the orbifold.

Localized flux can be used to construct normalized zero-modes of charged bulk fields [40]. We used this method to systematically construct zero-mode wave functions for different flux densities. The background gauge field is now singular. It is obtained from torus Green's functions whose singularities are located at orbifold singularities. The localized flux densities can vary and are related to the bulk flux density. The zero-mode wave functions vanish at the fixed points where flux is localized. Since the Green's function is invariant under lattice translations and reflection at the origin, the corresponding untwisted wave functions satisfy trivial boundary conditions. For comparison, we also recalled the construction of the standard twisted wave functions for regular background fields. These fields are not invariant under lattice tranlations and have non-trivial transition functions. Hence, the corresponding wave functions satisfy twisted boundary conditions.

For small values of magnetic flux we showed that there is a one-to-one correspondence between twisted and untwisted zero-mode functions, and it is a matter of convenience which basis to use. It is satisfactory to see explicitly how untwisted wave functions can be mapped to twisted wave functions by means of singular gauge transformations. An advantage of the untwisted wave functions is the geometric origin of the wave function zeros, which may be phenomenologically interesting. It appears straightforward to extend the construction of untwisted wave functions to large magnetic flux as well as to other orbifolds.

Magnetized orbifolds play an important role in compactifications of type-I string theories. It appears interesting to analyze the role of localized flux in these constructions and to obtain a better understanding of the relation to field theory compactifications. This may be particularly valuable in view of the challenging problem of supersymmetry breaking.

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A Jacobi theta-functions

For convenience we list a number of relations among Jacobi theta-functions which were used in calculations presented in the previous sections. We follow the conventions of [46].

The basic theta-function is given by ($n \in \mathbb{Z}$)

$$\vartheta(z,\tau) = \sum_n e^{\pi i \tau n^2 + 2\pi i n z} \,. \tag{105}$$

A useful extension is the theta-function with characteristics,

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\tau) = \sum_n e^{\pi i \tau (n+\alpha)^2} e^{2\pi i (n+\alpha)(z+\beta)} \,. \tag{106}$$

It satisfies the relation

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\tau) = \vartheta\begin{bmatrix}\alpha\\ 0\end{bmatrix}(z+\beta,\tau) \tag{107}$$

and is related to the basic theta function by

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\tau) = e^{i\pi\tau\alpha^2 + 2\pi i \alpha(z+\beta)}\vartheta(z,\tau) \,. \tag{108}$$

The theta-function with characteristics includes as special cases

$$\vartheta(z,\tau) = \vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(z,\tau) = \vartheta\begin{bmatrix}1\\ 0\end{bmatrix}(z,\tau) \,, \tag{109}$$

$$-\vartheta_1(z,\tau) = \vartheta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(z,\tau) \,. \tag{110}$$

An important "addition formula" is given by [22, 47]

$$\begin{aligned}\vartheta\begin{bmatrix}\alpha\\ 0\end{bmatrix}(z_1, M\tau)\vartheta\begin{bmatrix}\beta\\ 0\end{bmatrix}(z_2, N\tau) = \\ \sum_{m=0}^{M+N-1} \vartheta\begin{bmatrix}(M\alpha + N\beta + mM)/(M+N)\\ 0\end{bmatrix}(z_1+z_2, (M+N)\tau) \\ \times\vartheta\begin{bmatrix}(\alpha - \beta + m)/(M+N)\\ 0\end{bmatrix}(Nz_1 - Mz_2, MN(M+N)\tau) \,.\end{aligned} \tag{111}$$

Further relations read

$$\vartheta(z+1,\tau) = \vartheta(z,\tau)\,, \tag{112}$$
$$\vartheta(z+\tau,\tau) = e^{-i\pi\tau-2\pi iz}\vartheta(z,\tau)\,, \tag{113}$$
$$\vartheta(z,\tau) = \vartheta(-z,\tau)\,, \tag{114}$$
$$-\vartheta_1(z,\tau) = e^{i\pi\tau/4+i\pi(z+1/2)}\vartheta(z+(\tau+1)/2,\tau)\,, \tag{115}$$
$$\vartheta((1+\tau)/2,\tau) = 0\,, \tag{116}$$
$$\vartheta_1(z+1,\tau) = e^{i\pi}\vartheta_1(z,\tau)\,, \tag{117}$$
$$\vartheta_1(z+\tau,\tau) = e^{i\pi(1-\tau-2z)}\vartheta_1(z,\tau)\,, \tag{118}$$
$$-\vartheta_1(z,\tau)^* = e^{-i\pi\bar\tau/4-i\pi(\bar z+1/2)}\vartheta(\bar z+(\bar\tau+1)/2,-\bar\tau)\,, \tag{119}$$

B Gamma-matrices

For completeness, and in order to avoid confusion, we list our conventions for the 6d gamma-matrices in the following. We start from the Wess-Bagger conventions in four dimensions [48],

$$\begin{gathered}\{\gamma_\mu,\gamma_\nu\} = -2\eta_{\mu\nu}\,, \quad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)\,,\\ \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3\,, \quad \gamma_5\psi_L = -\psi_L\,, \quad \gamma_5\psi_R = \psi_R\,.\end{gathered} \tag{120}$$

This is extended to six dimensions using $\eta_{MN} = \mathrm{diag}(-1,1,1,1,1,1)$ and

$$\{\Gamma_M,\Gamma_N\} = -2\eta_{MN}\,. \tag{121}$$

Explicitly, we use the representation

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} 0 & i\gamma^5 \\ i\gamma^5 & 0 \end{pmatrix}, \quad \Gamma^6 = \begin{pmatrix} 0 & -\gamma^5 \\ \gamma^5 & 0 \end{pmatrix}, \tag{122}$$

which implies

$$\Gamma_7 = -\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^5\Gamma^6 = \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix}. \tag{123}$$

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