

The Double Pentaladder Integral to All Orders

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ABSTRACT: We compute dual-conformally invariant ladder integrals that are capped off by pentagons at each end of the ladder. Such integrals appear in six-point amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills theory. We provide exact, finite-coupling formulas for the basic double pentaladder integrals as a single Mellin integral over hypergeometric functions. For particular choices of the dual conformal cross ratios, we can evaluate the integral at weak coupling to high loop orders in terms of multiple polylogarithms. We argue that the integrals are exponentially suppressed at strong coupling. We describe the space of functions that contains all such double pentaladder integrals and their derivatives, or coproducts. This space, a prototype for the space of Steinmann hexagon functions, has a simple algebraic structure, which we elucidate by considering a particular discontinuity of the functions that localizes the Mellin integral and collapses the relevant symbol alphabet. This function space is endowed with a coaction, both perturbatively and at finite coupling, which mixes the independent solutions of the hypergeometric differential equation and constructively realizes a coaction principle of the type believed to hold in the full Steinmann hexagon function space.

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1 Introduction

Despite substantial progress, our understanding of particle scattering in perturbative quantum field theory remains incomplete. One might think that this is to be expected, and that perturbation theory inherently limits us to order-by-order progress in the number of loops. However, the last decade has seen the development of powerful new methods to address scattering at the multi-loop level in both gauge and gravity theories — see, for example, refs. [1–14] and refs. [12, 15–19]. Many of these methods are expected to function up to any desired order in perturbation theory. In the case of planar $\mathcal{N} = 4$ super-Yang Mills (SYM) theory [20, 21] there is even an all-orders geometric formulation [7].

Rather, our understanding is incomplete because most of these methods supply, not scattering amplitudes, but integrands depending on loop momenta. Evaluating the multi-loop Feynman integrals produced by these methods is a substantial endeavor in its own right, with two loops only just beginning to yield to systematic analysis [22–40]. In general, perturbative scattering amplitudes are complicated transcendental functions of momentum invariants. If we want to understand these amplitudes to all orders, then we need to understand how to compute these functions order by order, and further, how to sum them into all-orders expressions.

It is not obvious that this is possible in general. However, in the planar limit of $\mathcal{N} = 4$ SYM we have unique evidence that it should be, due to the presence of integrability [41]. Integrability has been used to compute the theory’s cusp anomalous dimension for finite coupling [42] and has been instrumental in the Pentagon Operator Product Expansion, which calculates finite-coupling amplitudes in an expansion around a kinematic limit [43–49]. Notably, the perturbative expansion of these formulas has a finite radius of convergence in the coupling. The kinematic dependence of four- and five-particle amplitudes in planar $\mathcal{N} = 4$ SYM is also captured to all loop orders by the BDS ansatz [50], which is uniquely dictated by the theory’s dual conformal symmetry [1, 51–55]. While this symmetry does not uniquely fix the form of amplitudes involving more than five particles, it does restrict the problem to a special class of dual conformally invariant (DCI) integrals [51, 56–58], and by extension restricts the form and kinematic dependence of these amplitudes at finite coupling.

As a consequence, a great deal is known about the space of functions that can contribute to six- and seven-particle perturbative amplitudes in planar $\mathcal{N} = 4$ SYM. When these

amplitudes are normalized by the BDS ansatz, they can be written in terms of dual superconformal invariants (that encode the helicity structure) multiplied by multiple polylogarithms that have known kinematic dependence and branch cuts only in physical channels [59–63]. Additional physical constraints come from the Steinmann relations, which imply that double discontinuities of amplitudes must be zero when taken in overlapping channels [64–66]. These relations are obeyed by the polylogarithmic part of the amplitude when it is normalized by the BDS-like ansatz (which contains only two-particle kinematic invariants) [67–69]. We refer to this space of multiple polylogarithms as the space of Steinmann hexagon functions (\mathcal{H}) and Steinmann heptagon functions for six- and seven-point kinematics, respectively. These function spaces have proven sufficient to describe maximally helicity violating (MHV) and next-to-MHV (NMHV) amplitudes at six points through six loops [60, 61, 68–74], and at seven points through four loops [63, 75]. While the fact that only multiple polylogarithms show up in these amplitudes remains conjectural, there exists evidence that it holds to all loop orders [5]. Also, the specific arguments of the polylogarithms in the six-point case are consistent with a recent all-orders analysis of the Landau equations [76].

In this article, we will focus on a particular class of DCI integrals inside the space of Steinmann hexagon functions. These integrals have a ‘double pentaladder’ (hereafter just ‘pentaladder’) topology, meaning they take the form of a ladder integral capped on each end by a pentagon with three external massless legs, for a total of six massless legs. Starting at two loops, there are two integrals with this topology, denoted by $\Omega^{(L)}$ and $\tilde{\Omega}^{(L)}$, corresponding to two inequivalent numerator factors that render the pentagon integration infrared finite. These integrals constitute the most nontrivial part of the amplitude at two loops, and contribute to the amplitude at all loop orders [3]. Moreover, members of these classes of integrals are known to be related to each other at adjacent loop orders by a pair of second-order differential equations [70, 77].

Armed with these differential equations, we consider finite-coupling versions of $\Omega^{(L)}$ and $\tilde{\Omega}^{(L)}$ by summing over the loop order weighted by $(-g^2)^L$, as was done previously for a related box ladder integral [78]. While these quantities are *not* the full finite-coupling six-point amplitude, they do constitute well-defined contributions to it that sum up an infinite class of Feynman integrals. By exploiting the symmetries that preserve the dual coordinates on each side of their ladders, variables can be found that simplify the differential equations these integrals obey. Remarkably, after performing a separation of variables, we obtain compact representations of the finite-coupling versions of $\Omega^{(L)}$ and $\tilde{\Omega}^{(L)}$ in terms of a single Mellin integral over products of hypergeometric functions. These representations are valid for any value of the coupling. Factoring the second-order differential operators into first-order operators, we are led to consider two additional classes of integrals, $\mathcal{O}^{(L)}$ and $\mathcal{W}^{(L)}$, that inherit this finite-coupling description. In order to generate more Steinmann hexagon functions, we go on to consider the enveloping space of polylogarithmic functions that is generated by taking all possible derivatives of these integrals at arbitrarily high loop order. We refer to this space of functions as the Ω space. It is graded by an integer weight, where for example $\Omega^{(L)}$ has weight $2L$.

Surprisingly, the nontrivial part of the Ω space is entirely encoded in the discontinuity of these integrals with respect to the channel carrying momentum along the ladder. After taking this discontinuity, the finite-coupling representation of each integral can be rewritten as a contour integral over a branch cut that collapses to a pole in the weak coupling expansion. Perturbatively, the dependence on the kinematic variables reduces to powers of logarithms in one variable and single-valued harmonic polylogarithms (SVHPLs) [79] in the other two variables. This simplicity allows us to recursively construct the function space corresponding to this discontinuity to arbitrary weight. Promoting this space to the full Ω space also turns out to be incredibly simple, since the kernel of the discontinuity operation within that subspace contains only two functions at each weight.

Using similar methods, we also resum the pentabox ladder integrals, which are capped by a pentagon on one end of the ladder and an off-shell box on the other end. These integrals contribute to seven- and higher-particle amplitudes in planar $\mathcal{N} = 4$ SYM. There is an analogous enveloping space of polylogarithmic functions associated with these integrals, which can be easily constructed by taking a kinematic limit of the Ω space.

These new finite-coupling representations give us formidable control over the original L -loop integrals. In various kinematical limits, they lead to explicit formulae for the integrals to high loop orders. They also give us a handle on the structure of the Ω space, which (as a space of multiple polylogarithms) is endowed with a Hopf algebra and an associated coaction [80–86]. In particular, the relevant discontinuities of the $\Omega^{(L)}$, $\tilde{\Omega}^{(L)}$, $\mathcal{O}^{(L)}$, and $\mathcal{W}^{(L)}$ integrals are related to each other by first-order differential operators, and this system of differential equations is encoded in the coaction. The coaction can thus be realized as a 4×4 matrix that acts on the vector of the discontinuities of these integrals. (The coaction on the integrals themselves maps to a slightly larger space of functions, and must be described by a larger matrix.)

The coaction on the discontinuity of these integrals can also be defined at finite coupling (that is, nonperturbatively) in the form of a matrix product of path-ordered exponentials. By construction, this nonperturbative coaction satisfies a coaction principle [87–89], meaning that the first entry of the coaction always maps to the original space of discontinuity functions, while the second entry can map to a larger space, in general. We expect that this structure can be lifted to the full Ω space. A similar coaction principle also seems to be at work in perturbative string theory [90, 91], ϕ^4 theory [89], QED [92], and the full space of Steinmann hexagon functions (where data currently exist through six loops) [69, 74]. The finite-coupling structure of the Ω space lends weight to the conjecture that \mathcal{H} is endowed with a similar structure to all orders. In many ways, the Ω space thus serves as an instructive toy model for the full space of hexagon functions, as well as for quantum field theory more generally.

In studying these integrals, we hope to inaugurate a new approach to Feynman integrals that goes beyond order-by-order progress in perturbation theory. The $\Omega^{(L)}$ and $\tilde{\Omega}^{(L)}$ integrals can now be described analytically to any order, as well as at finite coupling — the already substantial all-orders understanding of this class of integrands is now complemented by a thorough understanding of the type of functions to which they integrate. Similar types of

functions can be expected to appear in planar $\mathcal{N} = 4$ more generally, at least in the MHV and NMHV sectors. We hope that other infinite families of integrals can be identified and characterized in a similar manner, eventually extending such an all-orders description to scattering amplitudes themselves.

The remainder of this paper is organized as follows. In section 2 we define the $\Omega^{(L)}$ and $\tilde{\Omega}^{(L)}$ integrals. We also define the related pentabox and box ladder integrals, for which the pentagon at one or both ends of the ladder is replaced by an off-shell box. We then introduce the differential equations these integrals satisfy. In section 3 we leverage the symmetries of the $\Omega^{(L)}$ integrals, as well as their single-valuedness in the region where all cross ratios are close to 1, to resum them into a one-fold Mellin integral over hypergeometric functions. We do the same for $\tilde{\Omega}^{(L)}$, and introduce a family of related functions. In section 4 we show that the finite-coupling representations of these integrals may be equivalently recast as an infinite series, corresponding to the Taylor expansion around a particular kinematic limit. By further expanding the coefficients of this series at weak coupling, we may resum it away from the limit for certain two-dimensional slices of the space of kinematics, in terms of multiple polylogarithms. Then, in section 4.6 we analyze our integrals at strong coupling, finding evidence that they become exponentially suppressed for a large chunk of the Euclidean region.

Section 5 describes the Ω space of functions appearing in the coaction of the basic integrals. It showcases a space of functions relevant to the six-point scattering amplitude that can be constructed explicitly to all loop orders. First we consider the discontinuity of the functions with respect to the channel carrying momentum along the ladder. This discontinuity is simpler to analyze, yet it contains nearly all the information about the full space. In particular, the discontinuity space can be efficiently reconstructed from its coaction, which we formulate nonperturbatively. We conclude in section 6 with a discussion of these results and possible directions for future work.

This paper includes three appendices: Appendix A collects relations between different sets of kinematic variables; appendix B describes some “extended Steinmann relations” that have been found in the full space of hexagon functions \mathcal{H} ; and appendix C describes analogous relations for the spaces Ω and Ω_c , as well as coproduct relations between the various integrals, and how a curious “double coproduct” operator acts on the Ω space. We also provide three ancillary files. Two of them, `omega1vwL0-8.m` and `omegauv0L0-8.m`, give the integral $\Omega^{(L)}$ on the surfaces $u = 1$ and $w = 0$, respectively, through eight loops in terms of multiple polylogarithms. The third, `omegacdiscwt0-12.m`, gives the c -discontinuity of all the functions in the Ω space through weight 12.

2 Finite Dual Conformal Invariant Integrals

2.1 Dual conformal symmetry

In addition to the superconformal symmetry that follows from its Lagrangian formulation, $\mathcal{N} = 4$ SYM develops a dual conformal symmetry in the planar limit [51–55]. This new symmetry is associated with conformal transformations acting on the dual (or region) coordinates

$x_i^{\alpha\dot{\alpha}}$, defined via

$$p_i^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}}, \quad (2.1)$$

where $p_i^{\alpha\dot{\alpha}}$ is the momentum of the i^{th} scattering particle, and $x_{n+1}^{\alpha\dot{\alpha}} \equiv x_1^{\alpha\dot{\alpha}}$. Planarity implies that these coordinates can only appear in integrals via the squared differences

$$x_{ij}^2 \equiv (x_i - x_j)^2 = \det(x_i^{\alpha\dot{\alpha}} - x_j^{\alpha\dot{\alpha}}), \quad (2.2)$$

where $x_{i,i+1}^2 = 0$ when leg i is massless.

The planar loop integrand also depends on dual coordinates x_r, x_s , etc., associated with the interior region of each loop. After dividing out by the tree-level MHV superamplitude, the loop integrand multiplied by the integration measure becomes dual conformal invariant [55, 93]. In particular, such an object must be invariant under the dual conformal inversion operator I ,

$$I[x_i^{\alpha\dot{\alpha}}] = \frac{x_i^{\alpha\dot{\alpha}}}{x_i^2} \quad \Rightarrow \quad I[x_{ij}^2] = \frac{x_{ij}^2}{x_i^2 x_j^2}, \quad I[d^4 x_r] = \frac{d^4 x_r}{(x_r^2)^4}. \quad (2.3)$$

As a consequence, external dual coordinates should appear the same number of times in the numerator and denominator of the integrand, while the dual loop coordinates should appear four more times in the denominator.

For example, the two-loop pentaladder integrals can be written in terms of the integral

$$\mathcal{I}_{dpl}^{(2)} \propto \int \frac{d^4 x_r d^4 x_s}{i\pi^2 i\pi^2} \frac{x_{ar}^2 x_{bs}^2}{(x_{1r}^2 x_{2r}^2 x_{3r}^2 x_{4r}^2) x_{rs}^2 (x_{4s}^2 x_{5s}^2 x_{6s}^2 x_{1s}^2)}, \quad (2.4)$$

where, in addition to dual coordinates associated with each loop, we have introduced a pair of points $x_a^{\alpha\dot{\alpha}}$ and $x_b^{\alpha\dot{\alpha}}$ that solve the null-separation conditions $x_{a1}^2 = x_{a2}^2 = x_{a3}^2 = x_{a4}^2 = 0$ and $x_{b4}^2 = x_{b5}^2 = x_{b6}^2 = x_{b1}^2 = 0$ [77]. This choice suppresses the integrand in each of the limits where the denominator vanishes, rendering the integral infrared (IR) finite.

These conditions each admit two parity-conjugate solutions, namely

$$x_{a1}^{\alpha\dot{\alpha}} = \frac{\lambda_1^\alpha \left(\lambda_{3\beta} x_3^{\beta\dot{\alpha}} \right) - \lambda_3^\alpha \left(\lambda_{1\beta} x_1^{\beta\dot{\alpha}} \right)}{\langle 13 \rangle}, \quad x_{a2}^{\alpha\dot{\alpha}} = \frac{\left(x_3^{\alpha\dot{\beta}} \tilde{\lambda}_{3\dot{\beta}} \right) \tilde{\lambda}_1^{\dot{\alpha}} - \left(x_1^{\alpha\dot{\beta}} \tilde{\lambda}_{1\dot{\beta}} \right) \tilde{\lambda}_3^{\dot{\alpha}}}{[13]}, \quad (2.5)$$

$$x_{b1}^{\alpha\dot{\alpha}} = \frac{\lambda_4^\alpha \left(\lambda_{6\beta} x_6^{\beta\dot{\alpha}} \right) - \lambda_6^\alpha \left(\lambda_{4\beta} x_4^{\beta\dot{\alpha}} \right)}{\langle 46 \rangle}, \quad x_{b2}^{\alpha\dot{\alpha}} = \frac{\left(x_6^{\alpha\dot{\beta}} \tilde{\lambda}_{6\dot{\beta}} \right) \tilde{\lambda}_4^{\dot{\alpha}} - \left(x_4^{\alpha\dot{\beta}} \tilde{\lambda}_{4\dot{\beta}} \right) \tilde{\lambda}_6^{\dot{\alpha}}}{[46]}, \quad (2.6)$$

where $\langle ij \rangle \equiv \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta$ and $[ij] \equiv \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_j^{\dot{\beta}}$. These solutions seem to give us four possible choices of numerator in the pentaladder integral (2.4). However, only two of these choices give rise to different integrals, because replacing both $x_a^{\alpha\dot{\alpha}}$ and $x_b^{\alpha\dot{\alpha}}$ with their parity conjugates gives rise to integrals that differ only by terms that vanish after integration [77].

Choosing pairs (x_{a1}, x_{b1}) or (x_{a2}, x_{b2}) that are related by the cyclic shift $i \rightarrow i + 3$ gives rise to the integral $\Omega^{(2)}$ (up to a kinematic prefactor required to make it DCI, given below). On the other hand, choosing the pair (x_{a1}, x_{b2}) gives rise to the integral $\tilde{\Omega}^{(2)}$ (again, up to a kinematic prefactor), which is parity conjugate to the integral (x_{a2}, x_{b1}) .

These integrals can also be expressed in terms of momentum twistors [94, 95]

$$Z_i^R = (\lambda_i^\alpha, x_i^{\beta\dot{\alpha}} \lambda_{i\beta}), \quad (2.7)$$

where $R = (\alpha, \dot{\alpha})$ is a combined $SU(2, 2)$ index. Momentum twistors live in the projective space \mathbb{CP}^3 , as they are invariant under the independent little group rescalings, $Z_i \rightarrow t_i Z_i$ for each i . They are related to the squared differences defined in eq. (2.2) by

$$x_{ij}^2 = \frac{\langle i-1, i, j-1, j \rangle}{\langle i-1, i \rangle \langle j-1, j \rangle}, \quad (2.8)$$

where the four-bracket $\langle ijkl \rangle \equiv \langle Z_i Z_j Z_k Z_l \rangle = \epsilon_{RSTU} Z_i^R Z_j^S Z_k^T Z_l^U$ is invariant under $SU(2, 2)$, but is not projectively invariant. The spinor products $\langle ij \rangle$ are not DCI, but cancel out in projectively-invariant, DCI ratios. Using eq. (2.8), the expression for $\Omega^{(2)}$ can be written in terms of momentum twistors as

$$\begin{aligned} \Omega^{(2)} = & \int \frac{d^4 Z_{AB}}{i\pi^2} \frac{d^4 Z_{CD}}{i\pi^2} \frac{\langle AB13 \rangle}{(\langle AB61 \rangle \langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle)} \\ & \times \frac{\langle CD46 \rangle \langle 1256 \rangle \langle 2345 \rangle \langle 6134 \rangle}{\langle ABCD \rangle (\langle CD34 \rangle \langle CD45 \rangle \langle CD56 \rangle \langle CD61 \rangle)}, \end{aligned} \quad (2.9)$$

which has been made projectively invariant and DCI by the inclusion of the kinematic factor $\langle 1256 \rangle \langle 2345 \rangle \langle 6134 \rangle$. The modified index structure of the integration variables, from single dual indices to pairs of momentum twistor labels, encodes the fact that points in dual space map to lines in momentum twistor space. We have additionally used the replacements $x_a \rightarrow Z_1 Z_3, x_b \rightarrow Z_4 Z_6$, which selects out $x_{a1}^{\alpha\dot{\alpha}}$ and $x_{b1}^{\alpha\dot{\alpha}}$ from eqs. (2.5) and (2.6). Notice that this integral is invariant under the dihedral transformation that exchanges momentum twistors (legs) $1 \leftrightarrow 3$ and $4 \leftrightarrow 6$ while leaving 2 and 5 invariant.

To write $\tilde{\Omega}^{(2)}$ in the language of momentum twistors, we use the fact that parity maps Z_i to the ray orthogonal to Z_{i-1}, Z_i , and Z_{i+1} . In particular, mapping Z_i and Z_j to their parity conjugates sends the four-bracket $\langle Z_i Z_j Z_k Z_l \rangle \rightarrow \langle (i-1i+1) \cap (j-1j+1) Z_k Z_l \rangle$, where $(i-1i+1) \cap (j-1j+1)$ denotes the intersection of the hyperplanes spanned by $\{Z_{i-1}, Z_i, Z_{i+1}\}$, and $\{Z_{j-1}, Z_j, Z_{j+1}\}$, respectively. Using this map to send $x_{b1}^{\alpha\dot{\alpha}}$ to $x_{b2}^{\alpha\dot{\alpha}}$ in eq. (2.9), we arrive at the expression

$$\begin{aligned} \tilde{\Omega}^{(2)} = & \int \frac{d^4 Z_{AB}}{i\pi^2} \frac{d^4 Z_{CD}}{i\pi^2} \frac{\langle AB13 \rangle}{(\langle AB61 \rangle \langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle)} \\ & \times \frac{(\langle D345 \rangle \langle C561 \rangle - \langle C345 \rangle \langle D561 \rangle) \langle 1246 \rangle \langle 2346 \rangle}{\langle ABCD \rangle (\langle CD34 \rangle \langle CD45 \rangle \langle CD56 \rangle \langle CD61 \rangle)}, \end{aligned} \quad (2.10)$$

where we have made use of the identity

$$\langle ij(klm) \cap (nop) \rangle = \langle iklm \rangle \langle jnop \rangle - \langle jklm \rangle \langle inop \rangle, \quad (2.11)$$

and have replaced the previous kinematic factor by $\langle 1246 \rangle \langle 2346 \rangle$. (We have additionally multiplied by an overall minus sign to stay consistent with the definition in the literature [70].) Unlike $\Omega^{(2)}$, which is parity even, $\tilde{\Omega}^{(2)}$ has both a parity even and parity odd part. Like $\Omega^{(2)}$, it is symmetric under the dihedral transformation $1 \leftrightarrow 3, 4 \leftrightarrow 6$.

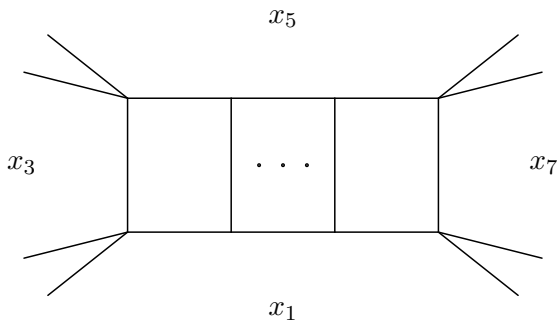


Figure 1. The eight- or higher-point L -loop ladder integral, labelled by dual coordinates.

2.2 The Box Ladder, Pentabox Ladder, and (Double) Pentaladder Integrals

Before introducing the L -loop generalizations of $\Omega^{(2)}$ and $\tilde{\Omega}^{(2)}$, let us first consider the simpler ‘box ladder’ integrals shown in fig. 1. The box ladder integrals involve only four dual coordinates, but none are null separated, so the first all-massless scattering amplitude to which they could contribute would be an eight-point amplitude.

The representative of this class at one loop is just the DCI four-mass box integral,

$$\mathcal{I}_l^{(1)}(x_1, x_3, x_5, x_7) = \int \frac{d^4 x_r}{i\pi^2} \frac{x_{15}^2 x_{37}^2}{x_{1r}^2 x_{3r}^2 x_{5r}^2 x_{7r}^2}. \quad (2.12)$$

The L -loop integral can be defined iteratively by integrating over the appropriate dual coordinate in the $(L-1)$ -loop integral,

$$\mathcal{I}_l^{(L)}(x_1, x_3, x_5, x_7) = \int \frac{d^4 x_r}{i\pi^2} \frac{x_{15}^2 x_{37}^2}{x_{1r}^2 x_{3r}^2 x_{5r}^2 x_{7r}^2} \mathcal{I}_l^{(L-1)}(x_1, x_3, x_5, x_r), \quad (2.13)$$

weighted by the appropriate propagator factors.

The box ladder integrals depend on only two cross ratios, conventionally expressed in terms of the variables z and \bar{z} defined by

$$\frac{x_{13}^2 x_{57}^2}{x_{15}^2 x_{37}^2} = \frac{1}{(1-z)(1-\bar{z})}, \quad \frac{x_{17}^2 x_{35}^2}{x_{15}^2 x_{37}^2} = \frac{z\bar{z}}{(1-z)(1-\bar{z})}, \quad (2.14)$$

where $\bar{z} = z^*$ on the Euclidean sheet where the cross ratios are real and positive.

In general, for a sequence of L -loop ladder integrals $\mathcal{I}^{(L)}$, we define the finite-coupling, or resummed, version by

$$\mathcal{I}(g^2) = \sum_{L=0}^{\infty} (-g^2)^L \mathcal{I}^{(L)}, \quad (2.15)$$

i.e. we just drop the (L) superscript. Typically the integral can be normalized so that $\mathcal{I}^{(L)}$ is a pure function, that is, an iterated integral with no rational prefactor, for $L \geq 1$, while the tree quantity $\mathcal{I}^{(0)}$ is rational.

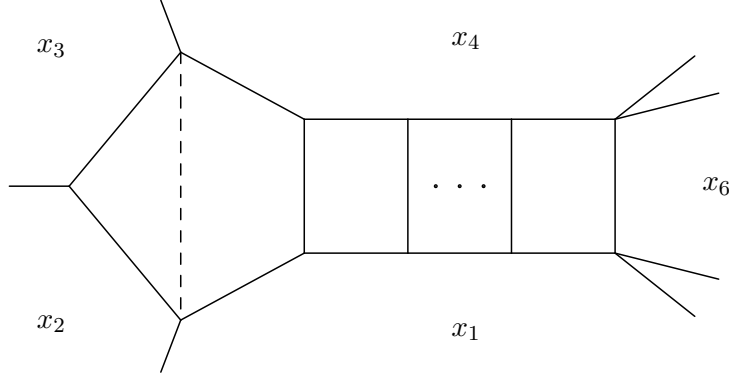


Figure 2. The seven- or higher-point pentabox ladder integral, labelled by dual coordinates, where the ladder is formed out of $L - 1$ loops. The dashed line represents a numerator factor that renders the integral DCI.

The box ladder integrals have long been known to all loop orders [96, 97], and can be written as [96, 98, 99]

$$\mathcal{I}_l^{(L)}(x_1, x_3, x_5, x_7) = \frac{(1-z)(1-\bar{z})}{z-\bar{z}} f^{(L)}(z, \bar{z}), \quad (2.16)$$

where

$$f^{(L)}(z, \bar{z}) = \sum_{r=0}^L \frac{(-1)^r (2L-r)!}{r!(L-r)!L!} \ln^r(z\bar{z}) (\text{Li}_{2L-r}(z) - \text{Li}_{2L-r}(\bar{z})). \quad (2.17)$$

This class of integrals has been evaluated at finite coupling, i.e. resummed to all orders [78]:

$$f(z, \bar{z}) = \sum_{L=0}^{\infty} (-g^2)^L f^{(L)}(z, \bar{z}) = \int_{2g}^{\infty} \frac{\zeta d\zeta}{\sqrt{\zeta^2 - 4g^2}} 2 \cos\left(\frac{1}{2}\sqrt{\zeta^2 - 4g^2} \ln \frac{1}{z\bar{z}}\right) \frac{\sinh[(\pi - \phi)\zeta]}{i \sinh(\pi\zeta)}. \quad (2.18)$$

Here we have changed normalization by multiplying the result given in eq. (21) of [78] by $t\mu/i$, and changed integration variables from z to ζ to avoid confusion with our variables z and \bar{z} (in our variables, $\phi = \arg z$). Finally, we have taken $\kappa^2 \rightarrow 4g^2$ in order to match our coupling and normalization conventions. This result will provide a cross-check of the method used to obtain our main finite-coupling results in section 3.

We can promote the ladder integrals to the pentabox integrals shown in fig. 2 by attaching a pentagon to the end of one of the ladder integrals. This is done by carrying out a single integration on the ladder integral of the form

$$\mathcal{I}_{pl}^{(L)}(x_1, x_2, x_3, x_4, x_6) = \frac{x_{14}^2 x_{26}^2 x_{36}^2}{x_{6a}^2} \int \frac{d^4 x_r}{i\pi^2} \frac{x_{ar}^2}{x_{1r}^2 x_{2r}^2 x_{3r}^2 x_{4r}^2 x_{6r}^2} \mathcal{I}_l^{(L-1)}(x_1, x_r, x_4, x_6), \quad (2.19)$$

where the point x_a should be chosen to be null-separated from the dual variables x_1, x_2, x_3 , and x_4 . Choosing x_{a1} or x_{a2} in eq. (2.5) gives the same result, i.e. this integral is parity-even.

The pentabox ladder integrals involve five dual coordinates. Since two pairs of coordinates are not null separated, the first all-massless scattering amplitude they can appear in is a seven-point amplitude.

The pentabox ladder integrals can be alternatively defined by attaching $L - 1$ boxes to the one-loop pentagon integral,

$$\mathcal{I}_{pl}^{(1)}(x_1, x_2, x_3, x_4, x_6) = \frac{x_{14}^2 x_{26}^2 x_{36}^2}{x_{6a}^2} \int \frac{d^4 x_r}{i\pi^2} \frac{x_{ar}^2}{x_{1r}^2 x_{2r}^2 x_{3r}^2 x_{4r}^2 x_{6r}^2}, \quad (2.20)$$

where the boxes are added iteratively by the integration

$$\mathcal{I}_{pl}^{(L)}(x_1, x_2, x_3, x_4, x_6) = \frac{x_{14}^2 x_{26}^2 x_{36}^2}{x_{6a}^2} \int \frac{d^4 x_r}{i\pi^2} \frac{x_{ar}^2}{x_{1r}^2 x_{2r}^2 x_{3r}^2 x_{4r}^2 x_{6r}^2} \mathcal{I}_{pl}^{(L-1)}(x_1, x_2, x_3, x_4, x_r). \quad (2.21)$$

The pentabox ladder integrals depend on the cross-ratios

$$u = \frac{x_{16}^2 x_{24}^2}{x_{26}^2 x_{14}^2}, \quad v = \frac{x_{46}^2 x_{13}^2}{x_{36}^2 x_{14}^2}. \quad (2.22)$$

Our main interest in this paper is in the (double) pentaladder integrals, which involve six dual coordinates, all null separated from their neighbors, so that these integrals will appear in all-massless six-point amplitudes.

There are two classes of pentaladder integrals that can be defined, corresponding to the two inequivalent numerator choices highlighted in the last section. The diagram for the first class of integrals, $\Omega^{(L)}$, is shown in fig. 3. The dashed lines in this diagram indicate the numerator factors $x_{a1}^{\alpha\dot{\alpha}}$ and $x_{b1}^{\alpha\dot{\alpha}}$, although we could have equivalently chosen $x_{a2}^{\alpha\dot{\alpha}}$ and $x_{b2}^{\alpha\dot{\alpha}}$ (which, in our convention, would have swapped these dashed lines for wavy lines). The diagram for $\tilde{\Omega}^{(L)}$ differs by the exchange of just one of these numerator factors for its parity conjugate—or, graphically, by the exchange of one of the dashed lines for a wavy line.

These pentaladder integrals can be most easily defined in momentum twistor space, as repeated insertions of a box into $\Omega^{(2)}$ and $\tilde{\Omega}^{(2)}$. For example, the three loop integrals may be obtained from eqs. (2.9) and (2.10) by the replacement

$$\begin{aligned} \Omega^{(2)} \rightarrow \Omega^{(3)} &: \frac{1}{\langle ABCD \rangle} \rightarrow \int \frac{d^4 Z_{EF}}{i\pi^2} \frac{\langle 6134 \rangle}{\langle ABEF \rangle \langle EF61 \rangle \langle EF34 \rangle \langle EFCD \rangle}, \end{aligned} \quad (2.23)$$

with an obvious generalization to higher loops.

Six-point DCI integrals can in general depend on the three cross ratios

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad v = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2}, \quad w = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2}. \quad (2.24)$$

As at two loops, $\Omega^{(L)}$ and $\tilde{\Omega}^{(L)}$ are both symmetric under the simultaneous exchange of legs $1 \leftrightarrow 3$ and $4 \leftrightarrow 6$. This transformation exchanges u and v , but we will have to be careful about signs when transforming the parity-odd part of $\tilde{\Omega}^{(L)}$. The distinction between these integrals in terms of their numerator factors follows the same rule as at two loops—namely, $\Omega^{(L)}$ picks out pairs of points x_a and x_b that are related by the cyclic shift $i \rightarrow i + 3$ in their dual indices, while for $\tilde{\Omega}^{(L)}$ these points are related by $i \rightarrow i + 3$ plus parity conjugation.

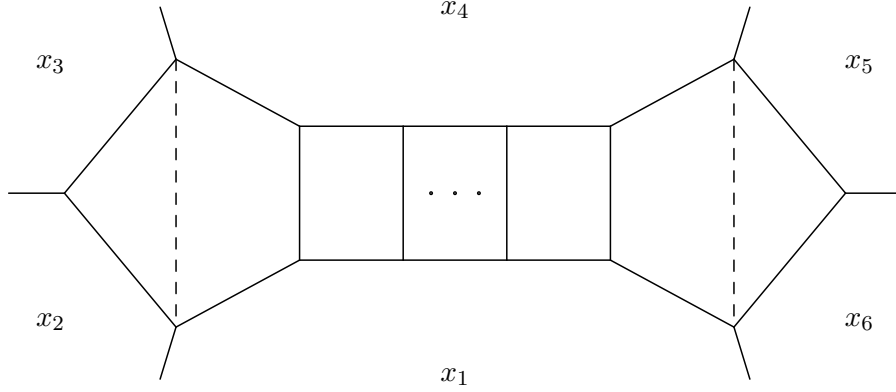


Figure 3. The six-point integral $\Omega^{(L)}$, labelled by dual coordinates, where the ladder is formed out of $L - 2$ loops. The dashed lines represent numerator factors that render the integral DCI.

2.3 Climbing the ladders with differential equations

These ladder integrals all share one crucial attribute: they satisfy a set of differential equations that relate adjacent loop orders [77]. For the box ladders, this differential equation is quite simple to write down, and has been known for some time:

$$[z\partial_z\bar{z}\partial_{\bar{z}} - g^2] f(z, \bar{z}, g^2) = 0. \quad (2.25)$$

We have rearranged the traditional presentation of this relation, which relates $f^{(L)}$ to $f^{(L-1)}$, to the all-loop expression $f(z, \bar{z}, g^2)$ in order to emphasize that it really is valid for finite coupling.

A similar, if slightly more complicated, differential equation applies to the pentabox ladders. We define the quantity

$$\Psi^{(L)}(u, v) = (1 - u - v) \mathcal{I}_{pl}^{(L)}(x_1, x_2, x_3, x_4, x_6), \quad (2.26)$$

and its finite-coupling version $\Psi(u, v, g^2)$ via eq. (2.15). Then $\Psi(u, v, g^2)$ obeys the differential equation

$$[(1 - u - v)uv\partial_u\partial_v + g^2] \Psi(u, v, g^2) = 0. \quad (2.27)$$

Each of the (double) pentaladders also satisfies a differential equation, which we first give at fixed loop order L . The differential equation relating $\Omega^{(L)}$ and $\Omega^{(L-1)}$ was derived in momentum-twistor space, and reads [77]

$$\frac{\langle 1234 \rangle \langle 2345 \rangle}{\langle 6134 \rangle} Z_1 \cdot \frac{\partial}{\partial Z_2} \left(\frac{1}{\langle 2345 \rangle} Z_6 \cdot \frac{\partial}{\partial Z_1} \Omega^{(L)} \right) = -\Omega^{(L-1)}. \quad (2.28)$$

The corresponding relation between $\tilde{\Omega}^{(L)}$ and $\tilde{\Omega}^{(L-1)}$ is

$$\frac{\langle 1234 \rangle \langle 2346 \rangle}{\langle 6134 \rangle} Z_1 \cdot \frac{\partial}{\partial Z_2} \left(\frac{1}{\langle 2346 \rangle} Z_6 \cdot \frac{\partial}{\partial Z_1} \tilde{\Omega}^{(L)} \right) = -\tilde{\Omega}^{(L-1)}. \quad (2.29)$$

This equation was given in ref. [70] for $L = 2$, but the same derivation holds for any $L > 2$, except that the sign of the right-hand side needs to be flipped; we also redefine $\tilde{\Omega}^{(1)} \rightarrow -\tilde{\Omega}^{(1)}$ with respect to ref. [70]. Note that $\Omega^{(1)}$ and $\tilde{\Omega}^{(1)}$ are one-loop hexagon integrals with double numerator insertions [70, 93].

We define the finite-coupling versions of $\Omega^{(L)}$ and $\tilde{\Omega}^{(L)}$ by,

$$\Omega = \sum_L (-g^2)^L \Omega^{(L)}, \quad \tilde{\Omega} = \sum_L (-g^2)^L \tilde{\Omega}^{(L)}. \quad (2.30)$$

The finite-coupling analogs of the above differential equations will be discussed in section 3, after we introduce some new kinematic variables which dramatically simplify them.

3 Ladders at Finite Coupling

3.1 Separated form of the differential equations

The ladder in the $\Omega^{(L)}$ and $\tilde{\Omega}^{(L)}$ integrals is framed by the dual coordinates x_1 and x_4 , as can be seen in fig. 3. We will exploit the symmetries that preserve these two points in order to write a finite-coupling expression for these integrals. The same technique will also be applied to other systems containing the same ladder.

Thanks to dual conformal symmetry we can put x_1 and x_4 at zero and infinity, respectively. It is then easy to see that the symmetry preserving their location consists of $\text{SO}(4)$ rotations and scale transformations. Writing the $\text{SO}(4)$ algebra as a product of two $\text{SU}(2)$'s, this indicates that the ladders are controlled by a $\text{SU}(2)_L \times \text{SU}(2)_R \times \text{GL}(1)$ symmetry. The idea will be to find variables which transform as simply as possible under each factor of this group.

Let us first parametrize the hexagon kinematics explicitly in this frame using the embedding formalism. Each dual coordinate $x_i^\mu \equiv \sigma_{\alpha\dot{\alpha}}^\mu x_i^{\alpha\dot{\alpha}}$ (where σ are the usual Pauli matrices) is encoded as a null six-vector $X_i \equiv (x_i^\mu, X_i^+, X_i^-)$, with respect to the metric $X_i \cdot X_j \equiv X_i^+ X_j^- + X_i^- X_j^+ - x_i \cdot x_j$:

$$X_i = \begin{pmatrix} 0 & p_2^\mu & -p_4^\mu & 0 & p_5^\mu & -p_1^\mu \\ 0 & 0 & -p_2 \cdot p_4 & 1 & -p_1 \cdot p_5 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.1)$$

where i labels the columns of the matrix. Here we have put the points X_1 and X_4 at 0 and ∞ , respectively. Because the six external momenta in the ladder integral are massless, in addition to $X_i^2 = 0$ for each i , we also have that $X_i \cdot X_{i+1} = 0$. This forces many components to vanish, and implies in addition that $p_i^2 = 0$. Note that the p_i are related to, but not equal to, the momenta in the original frame.

The ladder integrals depend only on the cross ratios (2.24), which now evaluate to

$$\begin{aligned} u &\equiv \frac{X_1 \cdot X_3 X_4 \cdot X_6}{X_1 \cdot X_4 X_3 \cdot X_6} = \frac{p_2 \cdot p_4}{(p_1 + p_2) \cdot p_4}, & v &\equiv \frac{X_2 \cdot X_4 X_5 \cdot X_1}{X_2 \cdot X_5 X_4 \cdot X_1} = \frac{p_1 \cdot p_5}{(p_1 + p_2) \cdot p_5}, \\ w &\equiv \frac{X_3 \cdot X_5 X_6 \cdot X_2}{X_3 \cdot X_6 X_5 \cdot X_2} = \frac{p_1 \cdot p_2 p_4 \cdot p_5}{(p_1 + p_2) \cdot p_4 (p_1 + p_2) \cdot p_5}. \end{aligned} \quad (3.2)$$

Notice that these variables are invariant under separate rescalings of p_4 and p_5 , and are also invariant under a common rescaling of p_1 and p_2 .

Let us now look at the action of the $SU(2)_L \times SU(2)_R \times GL(1)$ symmetry on one endpoint of the ladder, that is, on X_2 and X_3 , which according to eq. (3.1) depend on p_2 and p_4 . The $GL(1)$ scale transformations act simply by rescaling p_2 . Using the spinor helicity factorization for null vectors, $p_i = \lambda_{i\alpha} \tilde{\lambda}_{i\dot{\alpha}}$, the first $SU(2)$ factor acts on holomorphic spinors $\lambda_{2\alpha}$ and $\lambda_{4\alpha}$, and the second on anti-holomorphic spinors $\tilde{\lambda}_{2\dot{\alpha}}$ and $\tilde{\lambda}_{4\dot{\alpha}}$. This motivates introducing the following variables:

$$x \equiv \frac{\langle 14 \rangle \langle 25 \rangle}{\langle 15 \rangle \langle 24 \rangle}, \quad y \equiv \frac{[14][25]}{[15][24]}, \quad z \equiv \frac{p_2 \cdot p_4 p_2 \cdot p_5}{p_1 \cdot p_4 p_1 \cdot p_5}, \quad (3.3)$$

with $\langle 14 \rangle = \epsilon_{\alpha\beta} \lambda_1^\alpha \lambda_4^\beta$, $[14] = -\epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_1^{\dot{\alpha}} \tilde{\lambda}_4^{\dot{\beta}}$, so that $\langle 14 \rangle [41] = 2p_1 \cdot p_4$, and similarly for the other spinor products. Each of x and y is invariant under one of the $SU(2)$'s but not the other, and scale transformations act only on z .

To find the change of variables between (u, v, w) and (x, y, z) , first note from eq. (3.2) that

$$\frac{1-u}{u} = \frac{p_1 \cdot p_4}{p_2 \cdot p_4}, \quad \frac{1-v}{v} = \frac{p_2 \cdot p_5}{p_1 \cdot p_5}, \quad (3.4)$$

which readily yields

$$xy = \frac{(1-u)(1-v)}{uv}, \quad z = \frac{u(1-v)}{v(1-u)}. \quad (3.5)$$

Using the Schouten identity in eq. (3.3) one similarly finds

$$(1-x)(1-y) = \frac{w}{uv}. \quad (3.6)$$

Solving eqs. (3.5) and (3.6) for x and y in terms of u, v, w , we get

$$x = 1 + \frac{1-u-v-w+\sqrt{\Delta}}{2uv}, \quad y = 1 + \frac{1-u-v-w-\sqrt{\Delta}}{2uv}, \quad (3.7)$$

with $\Delta = (1-u-v-w)^2 - 4uvw$ as usual. The choice of sign in front of $\sqrt{\Delta}$ in eq. (3.7) is somewhat arbitrary; parity flips this sign and exchanges $x \leftrightarrow y$.

The differential equations (2.28) and (2.29) are expressed in terms of momentum twistors, so it is useful to write x, y, z in terms of ratios of momentum-twistor four-brackets. In appendix A we recall the momentum-twistor representations of the cross ratios u, v, w , and also of the variables y_u, y_v, y_w . Both u, v, w and x, y, z are rational functions of y_u, y_v, y_w ; see eqs. (A.5) and (A.7)–(A.9). Using eqs. (A.5) and (A.6) in eq. (3.7), the momentum-twistor representations of x, y, z are

$$x = \frac{\langle 1246 \rangle \langle 1356 \rangle}{\langle 1236 \rangle \langle 1456 \rangle}, \quad y = \frac{\langle 1345 \rangle \langle 2346 \rangle}{\langle 1234 \rangle \langle 3456 \rangle}, \quad z = \frac{\langle 1236 \rangle \langle 1246 \rangle \langle 1345 \rangle \langle 3456 \rangle}{\langle 1234 \rangle \langle 1356 \rangle \langle 1456 \rangle \langle 2346 \rangle}. \quad (3.8)$$

In this representation, it is easy to show that the dihedral flip $Z_i \leftrightarrow Z_{4-i}$ that leaves $\Omega^{(L)}$ and $\tilde{\Omega}^{(L)}$ invariant transforms the above variables as $x \leftrightarrow y$ and $z \rightarrow 1/z$. Also notice that

under the cyclic transformation $Z_i \rightarrow Z_{i+3}$, x is exchanged with y , while z is left invariant, allowing us to identify this transformation with parity. This transformation also sends the dual coordinates $x_i \rightarrow x_{i+3}$. Inspecting fig. 3, we see that $\Omega^{(L)}$ is invariant under parity because the left and right numerator factors transform into each other under $i \rightarrow i + 3$. In contrast, $\tilde{\Omega}^{(L)}$ has both parity-even and parity-odd parts.

The x, y, z variables simplify the momentum-twistor differential operators appearing in eqs. (2.28) and (2.29). Using the Schouten identity for four-brackets and the chain rule, we have

$$Z_6 \cdot \frac{\partial}{\partial Z_1} = \frac{\langle 1346 \rangle \langle 2345 \rangle}{\langle 1234 \rangle \langle 1345 \rangle} (y\partial_y + z\partial_z), \quad (3.9)$$

$$Z_1 \cdot \frac{\partial}{\partial Z_2} = \frac{\langle 1346 \rangle}{\langle 2346 \rangle} (y\partial_y - z\partial_z). \quad (3.10)$$

Using these relations, the differential equation (2.28) for Ω becomes

$$-\frac{\langle 2345 \rangle \langle 1346 \rangle}{\langle 1345 \rangle \langle 2346 \rangle} (y\partial_y - z\partial_z)(y\partial_y + z\partial_z)\Omega^{(L)} = -\Omega^{(L-1)} \quad (3.11)$$

which can be expressed in terms of y and z only as

$$\frac{1-y}{y} \left[(y\partial_y)^2 - (z\partial_z)^2 \right] \Omega^{(L)} = -\Omega^{(L-1)}. \quad (3.12)$$

At finite coupling, using the definition (2.30), the Ω ladders thus satisfy the equations:

$$\left[\frac{1-y}{y} \left((y\partial_y)^2 - (z\partial_z)^2 \right) - g^2 \right] \Omega(x, y, z, g^2) = 0, \quad (3.13)$$

$$\left[\frac{1-x}{x} \left((x\partial_x)^2 - (z\partial_z)^2 \right) - g^2 \right] \Omega(x, y, z, g^2) = 0, \quad (3.14)$$

where we have also used the fact that $\Omega(x, y, z, g^2)$ is even under parity, $x \leftrightarrow y$, to add the second equation. This form of the equations will be very convenient because they now take on a separated form, thanks to switching to the x, y, z variables.

The corresponding differential equations for $\tilde{\Omega}(x, y, z, g^2)$, derived in a similar way from eq. (2.29), are

$$\left[\frac{1-y}{y} \left((y\partial_y)^2 - (z\partial_z)^2 \right) - \frac{1}{y} (y\partial_y + z\partial_z) - g^2 \right] \tilde{\Omega}(x, y, z, g^2) = 0, \quad (3.15)$$

$$\left[\frac{1-x}{x} \left((x\partial_x)^2 - (z\partial_z)^2 \right) - \frac{1}{x} (x\partial_x - z\partial_z) - g^2 \right] \tilde{\Omega}(x, y, z, g^2) = 0. \quad (3.16)$$

In particular, expressing (2.29) in terms of the x, y, z variables leads to (3.15), whereas (3.16) follows from the latter by a flip transformation, as discussed under (3.8). Notice from the momentum-twistor forms (2.28) and (2.29) that the second-order part of the differential operator is exactly the same for Ω and $\tilde{\Omega}$, because the $\langle 2345 \rangle$ and $\langle 2346 \rangle$ factors cancel in the second-order terms. The extra linear term for $\tilde{\Omega}$ arises when the operator $Z_1 \cdot \partial/\partial Z_2$ acts on the factor $\langle 2345 \rangle / \langle 2346 \rangle$.

3.2 Pentaladders

We now turn to the solution of the differential equations (3.13) and (3.14) for Ω , and (3.15) and (3.16) for $\tilde{\Omega}$. We begin by diagonalizing $z\partial_z$ using a Mellin representation. (A related double Mellin representation of the box ladder integral has been obtained using integrability [98].) We seek separated solutions for $\Omega(x, y, z, g^2)$ of the form

$$z^{i\nu/2} F_\nu(x, y). \quad (3.17)$$

Equation (3.14) then gives

$$\left[(1-x)(x\partial_x)^2 + \frac{1}{4}(1-x)\nu^2 - xg^2 \right] F_\nu(x, y) = 0, \quad (3.18)$$

while eq. (3.13) gives the identical equation for F in y . The four independent solutions to the pair of differential equations can be labeled by the signs of ν :

$$F_{\pm\nu}^{j(\nu)}(x) F_{\pm\nu}^{j(\nu)}(y), \quad j(\nu) \equiv i\sqrt{\nu^2 + 4g^2}, \quad (3.19)$$

where F_ν^j are hypergeometric functions, normalized to $F_\nu^j(1) = 1$:

$$F_\nu^j(x) \equiv \frac{\Gamma(1 + \frac{i\nu+j}{2})\Gamma(1 + \frac{i\nu-j}{2})}{\Gamma(1 + i\nu)} x^{i\nu/2} {}_2F_1\left(\frac{i\nu+j}{2}, \frac{i\nu-j}{2}, 1 + i\nu, x\right). \quad (3.20)$$

Below when discussing the box ladders we will find that $j(\nu) - 1$ is the $\text{SO}(4)$ spin, which suggests viewing the differential equations (3.13) and (3.14) intuitively as two relations among the three Casimir invariants of $\text{SU}(2)_L \times \text{SU}(2)_R \times \text{GL}(1)$.

To find the physically relevant combination of the solutions (3.19), we impose the fact that Ω must be smooth in the entire positive octant $u, v, w > 0$. In particular, consider the neighborhood of the point $(u, v, w) = (1, 1, 1)$, where x and y are both small. The function should admit a regular Taylor series expansion. However only the combination xy is regular: x/y depends in a complicated way on the angle of approach. From the behavior of eq. (3.20) near the origin, we see that requiring the leading term in this limit to be a power of xy leaves only two acceptable solutions: $F_{+\nu}^{j(\nu)}(x)F_{+\nu}^{j(\nu)}(y)$ and $F_{-\nu}^{j(\nu)}(x)F_{-\nu}^{j(\nu)}(y)$.

To get a further constraint, we note from eqs. (3.5) and (3.6) that:

$$\frac{w}{uv} = (1-x)(1-y), \quad (x-1) + (y-1) = \frac{1-u-v-w}{uv}, \quad (3.21)$$

which imply that $(1-x)$ and $(1-y)$ can both switch sign in the positive octant. They can only switch simultaneously when $1-u-v-w$ switches sign. However, the individual hypergeometric functions contain singular logarithms of the form $\ln(1-x)$ in their Taylor series.¹ Smoothness in the positive octant thus requires that they combine into the regular combination $\ln[(1-x)(1-y)] = \ln \frac{w}{uv}$. This singles out a unique linear combination of the

¹These logarithms can be identified using the relations (3.36), (3.46), (3.47) and (3.48). While $F_\nu^j(x)$ is finite as $x \rightarrow 1$, its derivative behaves like $-g^2 \ln(1-x)$.

above two functions. We conclude that an integral representation of the following form must hold:

$$\Omega(u, v, w, g^2) = \int_{-\infty}^{\infty} d\nu c(\nu, g^2) z^{i\nu/2} \frac{F_{+\nu}^{j(\nu)}(x)F_{+\nu}^{j(\nu)}(y) - F_{-\nu}^{j(\nu)}(x)F_{-\nu}^{j(\nu)}(y)}{\sinh(\pi\nu)}, \quad (3.22)$$

where we have integrated over the dilatation eigenvalue ν with a yet undetermined coefficient $c(\nu, g^2)$. The insertion of the explicit factor of $1/\sinh(\pi\nu)$ is motivated by the following consideration: Regularity of the Taylor series at $(u, v, w) = (1, 1, 1)$ implies that the singularities of $\frac{c(\nu, g^2)}{\sinh(\pi\nu)}$ can be at most single poles at imaginary integers, since in this limit $x, y \rightarrow 0$ and the integral can be done by residues (closing the contour below the real axis on the first term, and above in the second term). Thus $c(\nu, g^2)$ is an entire function.

To fix this function, we find a boundary condition at large ν . At large ν , the integral simplifies dramatically, because one of the indices on the hypergeometric functions, $i\nu - j$, goes to zero, so using eq. (3.5) we get simply

$$z^{i\nu/2} F_{+\nu}^j(x)F_{+\nu}^j(y) \rightarrow \left(\frac{u(1-v)}{v(1-u)}\right)^{i\nu/2} \left(\frac{(1-u)(1-v)}{uv}\right)^{i\nu/2} = \left(\frac{1-v}{v}\right)^{i\nu}. \quad (3.23)$$

We see that for the above term, the large ν region dominates² the limit $v \rightarrow +\infty$. Choosing the phase, say,

$$\log\left(\frac{1-v}{v}\right) \rightarrow \log\left(\frac{v-1}{v}\right) - i\pi, \quad (3.24)$$

the integrand numerator acquires a factor $e^{\pi\nu}$, canceling the same factor from the $\sinh(\pi\nu)$ in the denominator and making it marginally convergent. Thus

$$\lim_{v \rightarrow +\infty} \Omega(u, v, w, g^2) = \int_{\sim 1}^{\infty} d\nu 2c(\nu, g^2)e^{-i\nu/v}. \quad (3.25)$$

If c approaches a constant, we get linear growth in v as $v \rightarrow \infty$. Now, at tree level ($g^2 = 0$), we have that $\Omega(u, v, w, 0) = 1 - u - v$. (We didn't define $\Omega^{(0)}$ as an integral but it can be found by acting on the known one-loop integral $\Omega^{(1)}$ with the differential operator (3.14).) Hence the tree-level quantity does grow linearly in the limit. On the other hand, the loop corrections $\Omega^{(L)}$ with $L \geq 1$ grow at most logarithmically in this limit, since they are uniform transcendental functions. We conclude that the missing coefficient must be independent of the coupling, and equal to $c(\nu, g^2) = 1/(2i)$.

Our final result is therefore:

$$\boxed{\Omega(u, v, w, g^2) = \int_{-\infty}^{\infty} \frac{d\nu}{2i} z^{i\nu/2} \frac{F_{+\nu}^{j(\nu)}(x)F_{+\nu}^{j(\nu)}(y) - F_{-\nu}^{j(\nu)}(x)F_{-\nu}^{j(\nu)}(y)}{\sinh(\pi\nu)}} \quad (3.26)$$

where x, y, z are defined in eqs. (3.5) and (3.7), $j(\nu) = i\sqrt{\nu^2 + 4g^2}$, and the hypergeometric functions F are defined in eq. (3.20). This formula is the main result of this section.

²Similarly, for the second term in eq. (3.22) the limit of large ν governs the $u \rightarrow +\infty$ limit. For our purposes, it will be sufficient to focus on the $v \rightarrow +\infty$ limit.

As a simple check, at tree level, $g^2 = 0$, the hypergeometric functions become trivial and $F_\nu^{j(\nu)}(x) \rightarrow x^{i\nu/2}$. From eq. (A.10), $\sqrt{xy/z} = (1-u)/u$ and $\sqrt{xyz} = (1-v)/v$, so we get

$$\Omega(u, v, w, 0) = \int_{-\infty}^{\infty} \frac{d\nu}{2i \sinh(\pi\nu)} \left[\left(\frac{1-v}{v} \right)^{i\nu} - \left(\frac{1-u}{u} \right)^{-i\nu} \right]. \quad (3.27)$$

Performing the integral by closing the contour in the lower half-plane and summing over residues at $\nu = -ik$, we reproduce the result $\Omega(u, v, w, 0) = 1 - u - v$. In general, it is straightforward to expand the integral (3.26) around $(u, v, w) = (1, 1, 1)$, since x and y are both small and so only the residues at a finite number of poles contribute. The residues give hypergeometric functions and only a finite number of terms in their Taylor expansions are needed. The expansion coefficients can therefore be obtained exactly in g^2 . At one loop, we have resummed the series around $(1, 1, 1)$, finding as expected:

$$\Omega^{(1)}(u, v, w) \equiv \frac{-d}{dg^2} \Omega(u, v, w, g^2) \Big|_{g^2=0} = \text{Li}_2(1-u) + \text{Li}_2(1-v) + \text{Li}_2(1-w) + \ln u \ln v - 2\zeta_2. \quad (3.28)$$

Section 4 discusses how to perform such expansions at higher loop orders.

The pentalladder integral $\tilde{\Omega}$ with mixed numerators can be analyzed in an identical fashion. The only change is that the solutions to the differential equation (3.16) with the separated form (3.17) are now $F_{+\nu}^{j(\nu)'}(x)$ and $F_{-\nu-2i}^{j(\nu)'}(x)$ with

$$F_\nu^{j'}(x) \equiv \frac{\Gamma(\frac{i\nu+j}{2})\Gamma(\frac{i\nu-j}{2})}{\Gamma(i\nu)} x^{i\nu/2} {}_2F_1\left(\frac{i\nu+j}{2}, \frac{i\nu-j}{2}, i\nu, x\right) \quad (3.29)$$

instead of eq. (3.20). These functions are to be multiplied by the solutions to (3.15), which are found (by letting $x \rightarrow y$ and $\nu \rightarrow -\nu$) to be $F_{-\nu}^{j(\nu)'}(y)$ and $F_{\nu-2i}^{j(\nu)'}(y)$. As in the case of Ω , imposing regularity in the positive octant fixes a unique combination of these solutions. It first requires the pairings $F_{+\nu}^{j(\nu)'}(x)F_{+\nu-2i}^{j(\nu)'}(y)$ and $F_{-\nu-2i}^{j(\nu)'}(x)F_{-\nu}^{j(\nu)'}(y)$, and then it imposes a relative minus sign between the two.

We also require that the large- v limit is saturated by its tree-level expression. The one-loop result [93] is

$$\tilde{\Omega}^{(1)}(u, v, w) = -\ln\left(\frac{u}{w}\right) \ln v + \frac{1-y_v}{1-y_u y_v} \ln\left(\frac{u}{v}\right) \ln w, \quad (3.30)$$

normalized with the opposite sign as eq. (E.4) of ref. [70]. We note that this is not a pure transcendental function, although $\tilde{\Omega}^{(L)}$ is pure for $L \geq 2$. The integral $\tilde{\Omega}^{(L)}$ has only been defined so far for $L \geq 1$ but we can define it at $L = 0$ by applying the differential equation (3.15), which gives us, after simplification using formulas in appendix A:

$$\tilde{\Omega}^{(0)}(u, v, w) = -(1-u)(1-v) - v(1-u) \frac{1-y_v}{1-y_u} \left(1 + \frac{\ln w}{1-w}\right). \quad (3.31)$$

This expression has a strange form for a “tree” object, since it contains a logarithm. However, hitting $\tilde{\Omega}^{(0)}$ once more with the differential operator gives zero, confirming that it is indeed the leading term of the $\tilde{\Omega}$ family.

In the limit of large v (choosing the branch of the square root where $x \rightarrow 1$, corresponding to $y_v \rightarrow 0$, $y_u \rightarrow \infty$ with $y_u y_v \rightarrow y$), we have $\frac{1-y_v}{1-y_u} \rightarrow 0$. Hence $\tilde{\Omega}^{(0)}$ is dominated by the first term, which grows linearly like $v(1-u)$. The one-loop result (3.30) and the higher loop orders grow at most logarithmically, so the large- v limit is again tree-level exact. Matching the integral representation with this limit, our final result for $\tilde{\Omega}$ is

$$\tilde{\Omega}(u, v, w, g^2) = g^2 \int_{-\infty}^{\infty} \frac{d\nu}{2i} z^{i\nu/2} \frac{F_{+\nu}^{j(\nu)'}(x) F_{+\nu-2i}^{j(\nu)'}(y) - F_{-\nu-2i}^{j(\nu)'}(x) F_{-\nu}^{j(\nu)'}(y)}{\sinh(\pi\nu)}. \quad (3.32)$$

We note the asymmetry between x and y , which originates from the differential equations (3.15) and (3.16): $\tilde{\Omega}$ is not invariant under parity.

3.3 Differential relations among pentaladder integrals

It turns out that the functions Ω and $\tilde{\Omega}$ are not independent, but rather they appear as derivatives of one another. These relations can be understood as properties of the hypergeometric functions entering eqs. (3.26) and (3.32).

It will prove useful to first introduce other auxiliary integrals, $\mathcal{O}(u, v, w, g^2)$, which we call the odd ladder because it has odd parity and its perturbative coefficients $\mathcal{O}^{(L)}$ all have odd weight, and an even companion $\mathcal{W}(u, v, w, g^2)$. The odd ladder will be a generalization of the one-loop six-dimensional integral $\tilde{\Phi}_6$ studied in ref. [100]. (See also ref. [101].) The latter integral satisfies

$$\sqrt{\Delta} \partial_w \Omega^{(2)} = -\tilde{\Phi}_6, \quad (3.33)$$

and it is the first parity odd function in the space of hexagon functions.

By analogy with eq. (3.33), we now define

$$\mathcal{O}^{(L-1)} \equiv \sqrt{\Delta} \partial_w \Omega^{(L)} = (-x \partial_x + y \partial_y) \Omega^{(L)}, \quad (3.34)$$

using eq. (A.13) and identifying $\mathcal{O}^{(1)} = -\tilde{\Phi}_6$. The corresponding finite-coupling definitions of \mathcal{O} and its even companion \mathcal{W} are

$$\mathcal{O} \equiv \frac{1}{g^2} (x \partial_x - y \partial_y) \Omega, \quad \mathcal{W} \equiv (x \partial_x + y \partial_y) \Omega. \quad (3.35)$$

This normalization of \mathcal{W} will prove convenient in section 5.

Given the finite-coupling solution for Ω , eq. (3.26), we can obtain one for the odd ladder integral simply by acting with the differential operator on the products $F_{\pm\nu}^j(x) F_{\pm\nu}^j(y)$ in the integrand. The hypergeometric function satisfies

$$x \frac{d}{dx} F_{\nu}^{j(\nu)}(x) = \frac{i\nu}{2} F_{\nu}^{j(\nu)}(x) + g^2 F_{\nu-2i}^{j(\nu)'}(x), \quad (3.36)$$

which can be verified, for example, using the hypergeometric function series representation (4.2). Note that here we used that $-\nu^2 - [j(\nu)]^2 = 4g^2$.

When we apply eq. (3.36) to eq. (3.35), using eq. (3.26), the $i\nu/2$ terms cancel, and we are left with:

$$\begin{aligned} \mathcal{O}(u, v, w, g^2) = \int_{-\infty}^{\infty} \frac{d\nu}{2i} \frac{z^{i\nu/2}}{\sinh(\pi\nu)} & \left[F_{+\nu-2i}^{j(\nu)'}(x) F_{+\nu}^{j(\nu)}(y) - F_{+\nu}^{j(\nu)}(x) F_{+\nu-2i}^{j(\nu)'}(y) \right. \\ & \left. - F_{-\nu-2i}^{j(\nu)'}(x) F_{-\nu}^{j(\nu)}(y) + F_{-\nu}^{j(\nu)}(x) F_{-\nu-2i}^{j(\nu)'}(y) \right]. \end{aligned} \quad (3.37)$$

Similarly,

$$\begin{aligned} \mathcal{W}(u, v, w, g^2) = g^2 \int_{-\infty}^{\infty} \frac{d\nu}{2i} \frac{z^{i\nu/2}}{\sinh(\pi\nu)} & \left[F_{+\nu}^{j(\nu)'}(x) F_{+\nu}^{j(\nu)}(y) + F_{+\nu}^{j(\nu)}(x) F_{+\nu-2i}^{j(\nu)'}(y) \right. \\ & \left. - F_{-\nu-2i}^{j(\nu)'}(x) F_{-\nu}^{j(\nu)}(y) - F_{-\nu}^{j(\nu)}(x) F_{-\nu}^{j(\nu)'}(y) \right]. \end{aligned} \quad (3.38)$$

We are now in a position to discuss several first-order differential relations among the integrals. If we take the difference between eq. (3.13) and eq. (3.14) so as to cancel the z derivative, then the x and y derivatives factorize as $(x\partial_x)^2 - (y\partial_y)^2 = (x\partial_x + y\partial_y)(x\partial_x - y\partial_y)$, which shows that Ω can also be written as a first derivative of the odd ladder integral,

$$\Omega = \frac{(1-x)(1-y)}{x-y} (x\partial_x + y\partial_y) \mathcal{O}. \quad (3.39)$$

This equation generalizes a relation found between $\Omega^{(1)}$ and a derivative of $\tilde{\Phi}_6$ in ref. [100].

However, we also find empirically, to high orders in perturbation theory, that the x and y derivatives of \mathcal{O} also contain the parity-even part of $\tilde{\Omega}$, which we call $\tilde{\Omega}_e$, and the z derivative of \mathcal{O} generates the parity-odd part, $\tilde{\Omega}_o$, where

$$\tilde{\Omega} = \tilde{\Omega}_e + \tilde{\Omega}_o. \quad (3.40)$$

In particular, we find that

$$x\partial_x \mathcal{O} = -\tilde{\Omega}_e + \frac{x}{1-x} \Omega, \quad (3.41)$$

$$y\partial_y \mathcal{O} = \tilde{\Omega}_e - \frac{y}{1-y} \Omega, \quad (3.42)$$

which gives back eq. (3.39) and also

$$\tilde{\Omega}_e = \frac{xy}{x-y} \left[(1-x)\partial_x + (1-y)\partial_y \right] \mathcal{O}. \quad (3.43)$$

The parity-odd relation is just

$$\tilde{\Omega}_o = -z\partial_z \mathcal{O}. \quad (3.44)$$

These two empirical relations combine to

$$\tilde{\Omega} = \left(\frac{xy}{x-y} \left[(1-x)\partial_x + (1-y)\partial_y \right] - z\partial_z \right) \mathcal{O}. \quad (3.45)$$

In appendix C.3, we provide additional relations of a similar nature.

We would like to derive these differential relations from the finite-coupling integral representations. In order to do so, it is useful to consider the derivatives of $F_{\nu-2i}^{j(\nu)'}(x)$ and $F_{\nu}^{j(\nu)'}(x)$. If we apply the differential operator $(1-x)x d/dx$ to the left- and right-hand sides of eq. (3.36), we can use the second-order differential equation (3.18) satisfied by F_{ν}^j to simplify the left-hand side, obtaining a formula for the derivative of $F_{\nu-2i}^{j(\nu)'}(x)$:

$$\frac{d}{dx} F_{\nu-2i}^{j(\nu)'}(x) = -\frac{i\nu}{2x} F_{\nu-2i}^{j(\nu)'}(x) + \frac{1}{1-x} F_{\nu}^{j(\nu)}(x). \quad (3.46)$$

Furthermore, only two out of the three functions $F_{+\nu-2i}^{j(\nu)'}(x)$, $F_{+\nu}^{j(\nu)'}(x)$ and $F_{+\nu}^{j(\nu)}(x)$ can be linearly independent; indeed, hypergeometric identities can be used to show that

$$F_{+\nu-2i}^{j(\nu)'}(x) = F_{+\nu}^{j(\nu)'}(x) - \frac{i\nu}{g^2} F_{+\nu}^{j(\nu)}(x). \quad (3.47)$$

Combining eqs. (3.46) and (3.47), we obtain an equation for the derivative of $F_{\nu}^{j(\nu)'}(x)$:

$$\frac{d}{dx} F_{\nu}^{j(\nu)'}(x) = \frac{i\nu}{2x} F_{\nu}^{j(\nu)'}(x) + \frac{1}{1-x} F_{\nu}^{j(\nu)}(x). \quad (3.48)$$

The physical significance of the four functions we have introduced is now clear: they form a complete basis for products of F and F' with arguments x and y that are smooth in the Euclidean region. More precisely, we can form the four-vector

$$V_i(\nu, g^2) = \left\{ \mathcal{W}, \Omega, \tilde{\Omega}_e, \mathcal{O} \right\}_{\nu}, \quad i = 1, 2, 3, 4, \quad (3.49)$$

where the subscript ν means to focus on the ν -integrand (i.e. to drop $\int d\nu/(2i)$ in each of the integrals in eq. (3.38), eq. (3.26), eq. (3.32) and eq. (3.37)). Because this basis is complete, all the x , y and z derivatives of the V_i can be expressed as linear combinations of the V_i , allowing the total differential to be written in matrix form:

$$dV_i(\nu, g^2) = (dM_{ij}(\nu, g^2))V_j(\nu, g^2). \quad (3.50)$$

Computing the derivatives, we find the following explicit form for the matrix M :

$$M = \begin{pmatrix} \frac{i\nu}{2} \ln z & -g^2 \ln c - \frac{\nu^2}{2} \ln(xy) & g^2 \ln(xy) & 0 \\ \frac{1}{2} \ln(xy) & \frac{i\nu}{2} \ln z & 0 & \frac{g^2}{2} \ln \frac{x}{y} \\ -\frac{1}{2} \ln c & 0 & \frac{i\nu}{2} \ln z & \frac{g^2}{2} \ln \frac{1-x}{1-y} + \frac{\nu^2}{4} \ln \frac{x}{y} \\ 0 & -\ln \frac{1-x}{1-y} & -\ln \frac{x}{y} & \frac{i\nu}{2} \ln z \end{pmatrix} \quad (3.51)$$

where we have abbreviated $c \equiv (1-x)(1-y)$.

The matrix M provides us with first-order differential relations between integrals. Since the variable ν is to be integrated over, to find these relations we should take combinations of x and y derivatives that are independent of ν . For example, the x and y derivatives of the

second row of M give us back the definitions eq. (3.35), whereas the x and y derivatives of the fourth row give the two nontrivial relations eq. (3.39) and eq. (3.43). Finally, from the first and third row we find two additional first-order relations:

$$\frac{(1-x)(1-y)}{x-y} (x\partial_x - y\partial_y) \mathcal{W} = g^2 \Omega, \quad (3.52)$$

$$2(1-x)(1-y) (x\partial_x + y\partial_y) \tilde{\Omega}_e = (x+y-2xy)\mathcal{W} - g^2(x-y)\mathcal{O}. \quad (3.53)$$

The first follows readily from the factorized form of the Ω differential equation. Using eq. (3.44), the second-order eq. (3.15) for $\tilde{\Omega}$ could also be readily rederived from the matrix M .

This discussion shows that Ω and $\tilde{\Omega}$ naturally fit inside a common system. In section 5 we will use the matrix M to define an enlarged set of transcendental functions, which will be closed under the action of taking any derivative.

3.4 Pentabox ladders

Consider now the similar integral in fig. 2 where we chop off the pentagon on the right and replace it by a box. The two light-like-separated dual coordinates of the right pentagon are replaced by a single point $q = x_6$ which is not light-like separated from x_1 or x_4 .

At one loop, for example, the integral we consider is given in eq. (2.20), or in terms of seven momentum twistors,

$$\Psi^{(1)} = \int \frac{d^4 Z_{AB}}{\pi^2} \frac{\langle AB13 \rangle \langle 56(712) \cap (234) \rangle}{\langle AB71 \rangle \langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB56 \rangle}. \quad (3.54)$$

Using the embedding formalism as in eq. (3.1), and again putting the sides of the ladder at 0 and ∞ , we parametrize the external kinematics as

$$X_i = \begin{pmatrix} 0 & p_2^\mu & -p_4^\mu & 0 & q^\mu \\ 0 & 0 & -p_2 \cdot p_4 & 1 & q^2/2 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad (3.55)$$

where $q^2 \neq 0$ reflects the external masses on one side of the ladder. The two cross-ratios are now:

$$u \equiv \frac{x_{16}^2 x_{24}^2}{x_{26}^2 x_{14}^2} = \frac{q^2}{(q-p_2)^2}, \quad v \equiv \frac{x_{46}^2 x_{13}^2}{x_{36}^2 x_{14}^2} = \frac{p_2 \cdot p_4}{(p_2 - q) \cdot p_4}. \quad (3.56)$$

Scale transformations act by rescaling q , which leaves invariant the variable

$$\frac{(1-u)(1-v)}{uv} = \frac{2p_2 \cdot q}{q^2} \frac{q \cdot p_4}{p_2 \cdot p_4}. \quad (3.57)$$

Therefore we make an ansatz for Ψ as a sum of terms

$$z^{i\nu/2} F(x), \quad x \equiv \frac{(1-u)(1-v)}{uv}, \quad z \equiv \frac{u(1-v)}{v(1-u)}. \quad (3.58)$$

Using the chain rule, we can rewrite u and v partial derivatives in terms of x and z derivatives,

$$-(1-v)v\partial_v = x\partial_x + z\partial_z, \quad -(1-u)u\partial_u = x\partial_x - z\partial_z. \quad (3.59)$$

Applying also the identity $(1-u-v)/[(1-u)(1-v)] = -(1-x)/x$, the differential equation (2.27) for $\Psi(u, v, g^2)$ (defined via eq. (2.26)) becomes

$$\left[\frac{1-x}{x} \left((x\partial_x)^2 - (z\partial_z)^2 \right) - g^2 \right] \Psi(x, z, g^2) = 0, \quad (3.60)$$

which has exactly the same form as eq. (3.14) for $\Omega(x, y, z, g^2)$! Thus for the ansatz (3.58) we get exactly the equation for $F(x)$ given in eq. (3.18); that is, the partial wave decomposition of Ψ gives a sum of $F_{\pm\nu}^{j(\nu)}$ terms. As in the double pentagon case, here we could again argue that the relative coefficient is fixed by analyticity around $u, v = 1$ and $u + v = 1$, up to an entire function, itself fixed from the $v \rightarrow \infty$ limit.

In fact the equivalence of the two problems was already understood from the differential equation in ref. [77]: we can get to the pentabox ladders by taking the $w \rightarrow 0$ limit of the result (3.26), where $y \rightarrow 1$ and thus $F_{\pm\nu}^j(y) \rightarrow 1$. This gives immediately:

$$\Psi(u, v, g^2) = \Omega(u, v, 0, g^2) = \int_{-\infty}^{\infty} \frac{d\nu}{2i} z^{i\nu/2} \frac{F_{+\nu}^{j(\nu)}(x) - F_{-\nu}^{j(\nu)}(x)}{\sinh(\pi\nu)}. \quad (3.61)$$

3.5 Box ladders

We can conduct a similar analysis for the box ladders depicted in fig. 1. In this case a finite-coupling expression was already derived in ref. [78]. We will show that we reproduce this expression, which we presented in our conventions in eq. (2.18). We consider the ladder whose long sides are labelled by x_1 and x_5 , as in fig. 1, so again we set these points to zero and infinity. The data then are the ratio x_3^2/x_7^2 and the angle between x_3 and x_7 , which are the norm and phase of the complex variable z defined in eq. (2.14). The conventional cross ratios u and v are defined by

$$u = \frac{x_{13}^2 x_{57}^2}{x_{15}^2 x_{37}^2} = \frac{z\bar{z}}{(1-z)(1-\bar{z})}, \quad v = \frac{x_{35}^2 x_{17}^2}{x_{15}^2 x_{37}^2} = \frac{1}{(1-z)(1-\bar{z})}; \quad (3.62)$$

note that $u/v = z\bar{z}$. The sequence of ladders obeys the differential equation:

$$[z\partial_z \bar{z}\partial_{\bar{z}} - g^2] f(z, \bar{z}, g^2) = 0, \quad (3.63)$$

as previously presented in eq. (2.25). The one-loop case is

$$f^{(1)}(z, \bar{z}) = 2(\text{Li}_2(z) - \text{Li}_2(\bar{z})) - \ln(z\bar{z})(\text{Li}_1(z) - \text{Li}_1(\bar{z})). \quad (3.64)$$

The differential equation then requires $f^{(0)} = (z - \bar{z})/[(1-z)(1-\bar{z})]$. Again we make an ansatz for f as a sum of terms of the form:

$$(z\bar{z})^{i\nu/2} F(\phi), \quad e^{i\phi} \equiv \sqrt{z/\bar{z}}; \quad (3.65)$$

that is, $\phi = \arg z$ when z is complex, and the differential equation becomes

$$(\nu^2 - \partial_\phi^2 + 4g^2)F(\phi) = 0. \quad (3.66)$$

The general solution is a combination of $F_{\pm j}(\phi) = e^{\pm ij\phi}$, with $j(\nu) = i\sqrt{\nu^2 + 4g^2}$ as before. Because $j(\nu)$ is conjugate to ϕ , the angle between x_3 and x_7 , it should be interpreted as SO(4) spin.³

To find the correct combination of solutions $F_{\pm j}(\phi)$, we wish to impose that the loop corrections to f vanish when $z = \bar{z}$, as a consequence of the factor $(z - \bar{z})$ removed in eq. (2.16). Working in the Euclidean region $\bar{z} = z^*$, this occurs for two different values of the phase: $\phi = 0, \pi$. As we will see below, the proper interpretation of the vanishing at $\phi = 0$ turns out to be subtle, because there is a singularity at $z = \bar{z} = 1$. In fact the restriction to $\phi = 0$ at tree level results in a nonvanishing distribution supported at that point. However, f vanishes identically at $\phi = \pi$. This means that for each value of ν only the combination of solutions $F_{\pm j}(\phi)$ that vanishes at $\phi = \pi$ contributes, so that we can write:

$$f(z, \bar{z}, g^2) = \int_{-\infty}^{\infty} d\nu (z\bar{z})^{i\nu/2} \frac{\sin[(\pi - \phi)j(\nu)]}{\sin(\pi j(\nu))} c(\nu, g^2). \quad (3.67)$$

The symmetries of the integral imply that $c(\nu, g^2)$ is an even function of ν . Now consider single-valuedness as $(z\bar{z}) \rightarrow 0$. There the ν -integral will be done by residues in the lower-half plane and each residue should correspond to an integer spin j in order to be single-valued. This implies that the only singularities of $\frac{c(\nu, g^2)}{\sin(\pi j)}$ are single poles at integer spin, and therefore $c(\nu, g^2)$ is an entire function.

To fix the asymptotics of $c(\nu, g^2)$ we consider the limit $\bar{z} \rightarrow 1$ (with z otherwise fixed). We get a power divergence at tree level, but only logs at loop level; in the above expression this divergence will come from large positive ν , where $j \approx i\nu$:

$$\frac{1}{\bar{z} - 1} \sim \int_{\sim 1}^{\infty} d\nu c(\nu, g^2) e^{i\nu(\bar{z}-1)} \Rightarrow c(\nu, g^2) \rightarrow \frac{1}{i}. \quad (3.68)$$

Because $c(\nu, g^2)$ is entire, this behavior determines it uniquely:

$$f(z, \bar{z}, g^2) = \int_{-\infty}^{\infty} d\nu (z\bar{z})^{i\nu/2} \frac{\sin[(\pi - \phi)j(\nu)]}{i \sin(\pi j(\nu))}, \quad j(\nu) = i\sqrt{\nu^2 + 4g^2}. \quad (3.69)$$

We can now understand the vanishing at $\phi = 0$ more precisely. The sine factors cancel in this limit and the integral produces a delta-function $\delta(|z| - 1)$ that is independent of g^2 . This reproduces precisely the singular behavior of the tree-level function, and otherwise it vanishes for generic $|z|$.

We remark that eq. (3.69) resembles an integrability-based representation of the box ladder integral [98], in which the variables analogous to ν are spectral parameters.

³More precisely, comparing with the Gegenbauer polynomials with spin ℓ , $C_\ell^{(1)}(\cos \phi) = \frac{\sin((\ell+1)\phi)}{\sin \phi}$, with the denominator corresponding to the $(z - \bar{z})$ factored out in eq. (2.16), we see that $j(\nu) - 1$ should be identified with SO(4) spin.

Let us compare with eq. (2.18). Setting $\nu = \sqrt{\zeta^2 - 4g^2}$, $\zeta = -ij(\nu)$ in this integral, we get

$$f(z, \bar{z}) = \int_0^\infty d\nu 2 \cos\left(\frac{1}{2}\nu \ln \frac{u}{v}\right) \frac{\sin[(\pi - \phi)j]}{i \sin(\pi j)}, \quad (3.70)$$

which is precisely the same as eq. (3.69). We conclude that the method works for the box ladders as well, although it involves an additional subtlety because of the tree level singularity at $z, \bar{z} \rightarrow 1$.

4 Sum Representation and Perturbative Evaluation as Polylogarithms

The integral representations presented in the previous section capture the Ω integrals fully at finite coupling. If one is interested in extracting numerical values, or in finding $\Omega^{(L)}$ at a particular loop order, it is useful to derive alternate representations in terms of infinite sums. In this section we will derive a representation of this sort, and use it to efficiently find polylogarithmic expressions for $\Omega^{(L)}$ in specific limits.

4.1 Sum representation

In eq. (3.26), we may change the sign of the integration variable ν in the term containing $F_{-\nu}^j$, and rewrite the integral as

$$\Omega(u, v, w, g^2) = \mathcal{P} \left(\int_{-\infty}^{+\infty} \frac{d\nu}{2i} (z^{i\nu/2} + z^{-i\nu/2}) \frac{F_{+\nu}^j(x) F_{+\nu}^j(y)}{\sinh \pi\nu} \right), \quad (4.1)$$

where \mathcal{P} denotes the Cauchy principal value, which is necessary because now the integrand has a pole on the integration contour, at $\nu = 0$. This simply amounts to the prescription of including half the contribution of this pole to the integral.

Using the power series definition of the ${}_2F_1$ Gauss hypergeometric function⁴,

$${}_2F_1(a, b, c, x) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n) \Gamma(b+n)}{\Gamma(a) \Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+n)} \frac{x^n}{\Gamma(n+1)}, \quad (4.2)$$

we can deduce that $F_\nu^j(x)$ will produce poles when $(i\nu \pm j)/2 + n = -k$, or equivalently when

$$\nu = i \left(\frac{g^2}{k+n} + k+n \right), \quad k \geq 0, \quad (4.3)$$

namely only in the upper-half ν -plane. We may thus choose to evaluate eq. (4.1) by closing the contour in the lower-half plane, picking up poles at $\nu = -ik$ from the $\sinh(\pi\nu)$ factor in the denominator. Redefining

$$F_\nu^j(x) = x^{i\nu/2} \hat{F}_\nu^j(x), \quad (4.4)$$

⁴From the definition, it is evident that the function is symmetric in $a \leftrightarrow b$.

we thus arrive at the following series representation of the all-loop Ω ,

$$\Omega(u, v, w, g^2) = - \sum_{k=1}^{\infty} [(-\sqrt{xyz})^k + (-\sqrt{xy/z})^k] \hat{F}_{-ik}^j(x) \hat{F}_{-ik}^j(y) - \hat{F}_0^{2ig}(x) \hat{F}_0^{2ig}(y). \quad (4.5)$$

By virtue of eq. (4.2), as well as the following argument transformation formula,

$${}_2F_1(a, b, c, x) = (1-x)^{c-a-b} {}_2F_1(c-a, c-b, c, x), \quad (4.6)$$

we may express the functions \hat{F} as

$$\hat{F}_{-ik}^j(x) = \frac{\Gamma(1 + \frac{k+j}{2})\Gamma(1 + \frac{k-j}{2})}{\Gamma(1+k)} + g^2 \sum_{n=1}^{\infty} \frac{\Gamma(\frac{k+j}{2} + n)\Gamma(\frac{k-j}{2} + n)}{\Gamma(1+k+n)\Gamma(n+1)} x^n, \quad (4.7)$$

$$\hat{F}_{-ik}^j(x) = (1-x) \sum_{n=0}^{\infty} \frac{\Gamma(\frac{k+j}{2} + 1 + n)\Gamma(\frac{k-j}{2} + 1 + n)}{\Gamma(1+k+n)\Gamma(n+1)} x^n, \quad (4.8)$$

where $j = \sqrt{k^2 - 4g^2}$. With the help of these formulas, and the sum representation (4.5), we may easily obtain kinematic expansions of Ω around $x = y = 0$ to the desired order. Through eq. (A.10), these expansions are equivalent to expansions in (u, v, w) around $(1, 1, 1)$.

In a similar way, we can expand the integrals \mathcal{O} , $\tilde{\Omega}_o$, $\tilde{\Omega}_e$ and \mathcal{W} around $x = y = 0$. For this purpose, we also need the expansion of $F_{\nu-2i}^{j(\nu)'}$. We define

$$x^{-i\nu/2} F_{\nu-2i}^{j(\nu)'}(x) \Big|_{\nu=-ik} \equiv \hat{F}_{-ik}^{j'}(x) = \sum_{n=1}^{\infty} \frac{\Gamma(\frac{k+j}{2} + n)\Gamma(\frac{k-j}{2} + n)}{\Gamma(1+k+n)\Gamma(n)} x^n. \quad (4.9)$$

Note that $\hat{F}_{-ik}^{j'}$ only differs from \hat{F}_{-ik}^j by a factor of n in the n^{th} term, and an overall factor of $1/g^2$. The two series expansions are related by

$$x \frac{d}{dx} \hat{F}_{-ik}^j(x) = g^2 \hat{F}_{-ik}^{j'}(x), \quad (4.10)$$

$$(1-x) \frac{d}{dx} \left[x^k \hat{F}_{-ik}^{j'}(x) \right] = x^k \hat{F}_{-ik}^j(x), \quad (4.11)$$

the latter result following from eq. (4.8).

Then the series expansion of the all-orders odd ladder integral \mathcal{O} is:

$$\begin{aligned} \mathcal{O}(u, v, w, g^2) = & - \sum_{k=1}^{\infty} [(-\sqrt{xyz})^k + (-\sqrt{xy/z})^k] \left[\hat{F}_{-ik}^{j'}(x) \hat{F}_{-ik}^j(y) - \hat{F}_{-ik}^j(x) \hat{F}_{-ik}^{j'}(y) \right] \\ & - \hat{F}_0^{2ig'}(x) \hat{F}_0^{2ig}(y) + \hat{F}_0^{2ig}(x) \hat{F}_0^{2ig'}(y). \end{aligned} \quad (4.12)$$

The expansion of the odd part of $\tilde{\Omega}$ can be found by applying $-z\partial_z$ to eq. (4.12):

$$\tilde{\Omega}_o(u, v, w, g^2) = \sum_{k=1}^{\infty} \frac{k}{2} [(-\sqrt{xyz})^k - (-\sqrt{xy/z})^k] \left[\hat{F}_{-ik}^{j'}(x) \hat{F}_{-ik}^j(y) - \hat{F}_{-ik}^j(x) \hat{F}_{-ik}^{j'}(y) \right]. \quad (4.13)$$

The expansion of the even part of $\tilde{\Omega}$ can be found by expanding the all orders result (3.32) for $\tilde{\Omega}$ and then subtracting off the odd part (4.13). The result is

$$\begin{aligned} \tilde{\Omega}_e(u, v, w, g^2) = & - \sum_{k=1}^{\infty} [(-\sqrt{xyz})^k + (-\sqrt{xy/z})^k] \left\{ \frac{k}{2} \left[\hat{F}_{-ik}^{j'}(x) \hat{F}_{-ik}^j(y) + \hat{F}_{-ik}^j(x) \hat{F}_{-ik}^{j'}(y) \right] \right. \\ & \left. + g^2 \hat{F}_{-ik}^{j'}(x) \hat{F}_{-ik}^{j'}(y) \right\} - g^2 \hat{F}_0^{2ig'}(x) \hat{F}_0^{2ig'}(y). \end{aligned} \quad (4.14)$$

Finally, the expansion of the even ladder integral \mathcal{W} is given by

$$\begin{aligned} \mathcal{W}(u, v, w, g^2) = & -g^2 \left\{ \sum_{k=1}^{\infty} [(-\sqrt{xyz})^k + (-\sqrt{xy/z})^k] \left[\hat{F}_{-ik}^{j'}(x) \hat{F}_{-ik}^j(y) + \hat{F}_{-ik}^j(x) \hat{F}_{-ik}^{j'}(y) \right] \right. \\ & \left. + \hat{F}_0^{2ig'}(x) \hat{F}_0^{2ig}(y) + \hat{F}_0^{2ig}(x) \hat{F}_0^{2ig'}(y) \right\} \\ & - \sum_{k=1}^{\infty} k [(-\sqrt{xyz})^k + (-\sqrt{xy/z})^k] \hat{F}_{-ik}^j(x) \hat{F}_{-ik}^j(y). \end{aligned} \quad (4.15)$$

4.2 Weak coupling expansion

Let us now discuss how to perform the weak coupling expansion of eq. (4.5), as well as its kinematic resummation to multiple polylogarithms, at least in some limits. We begin with the rightmost term in eq. (4.5), coming from the $\nu = 0$ residue where $j = 2ig$. The F_0^{2ig} functions appearing in this term are given by the all-order relation

$$\hat{F}_0^{2ig}(x) = \frac{\pi g}{\sinh \pi g} {}_2F_1(ig, -ig, 1, x), \quad (4.16)$$

where the factor involving Γ functions in eq. (3.20) was eliminated with the help of the reflection formula,

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin \pi x}. \quad (4.17)$$

For its weak-coupling expansion, it will be more convenient to use the representation (4.7), together with the identity

$$\Gamma(n+\epsilon) = \Gamma(1+\epsilon)\Gamma(n) \sum_{i=0}^{\infty} \epsilon^i \underbrace{Z_{1,\dots,1}}_{i \text{ times}}(n-1), \quad (4.18)$$

where

$$Z_{m_1, \dots, m_j}(n) = \sum_{n \geq i_1 > i_2 > \dots > i_j > 0} \frac{1}{i_1^{m_1} i_2^{m_2} \dots i_j^{m_j}} \quad (4.19)$$

are Euler-Zagier sums, or particular values of Z -sums [102]. Inserting eq. (4.18) twice into the series expansion for the ${}_2F_1$ in eq. (4.16), we find

$$\hat{F}_0^{2ig}(x) = \frac{\pi g}{\sinh \pi g} \left(1 + g^2 \sum_{n=1}^{\infty} \frac{x^n}{n^2} \sum_{j,k=0}^{\infty} (-1)^j (ig)^{j+k} \underbrace{Z_{1,\dots,1}}_j(n-1) \underbrace{Z_{1,\dots,1}}_k(n-1) \right). \quad (4.20)$$

After reexpressing the product of Euler-Zagier sums as a linear combination thereof, with the help of the quasi-shuffle (also known as stuffle) algebra relations, for example

$$Z_1(n)Z_1(n) = 2Z_{1,1}(n) + Z_2(n), \quad (4.21)$$

eq. (4.20) may be immediately evaluated at any loop order in terms of harmonic polylogarithms (HPLs) [103] with argument x ,

$$H_{m_1, m_2, \dots, m_j}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^{m_1}} Z_{m_2, \dots, m_j}(n-1). \quad (4.22)$$

Let us now look at the remaining terms in eq. (4.5). Since \hat{F} is symmetric in $j \leftrightarrow -j$, see for example eq. (4.7), the choice of branch when we expand j in the coupling is immaterial, and we may pick

$$j(k) = k\sqrt{1 - \frac{4g^2}{k^2}} = k \sum_{l=0}^{\infty} \binom{1/2}{l} \left(\frac{-4g^2}{k^2}\right)^l. \quad (4.23)$$

Separating the contribution that is small at weak coupling,

$$\epsilon(k, g^2) = \frac{j-k}{2}, \quad (4.24)$$

we may again use the identities (4.17) and (4.18), this time for $\epsilon(k, g^2)$, in order to rewrite eq. (4.7) as

$$\begin{aligned} \hat{F}_{-ik}^j(x) = & \frac{\pi\epsilon}{\sin \pi\epsilon} \left[\sum_{i=0}^{\infty} \epsilon^i \underbrace{Z_{1, \dots, 1}(k)}_i \right. \\ & \left. + g^2 \sum_{n=1}^{\infty} \frac{x^n}{n(k+n)} \sum_{i,j=0}^{\infty} (-1)^i \epsilon^{i+j} \underbrace{Z_{1, \dots, 1}(n-1)}_i \underbrace{Z_{1, \dots, 1}(k+n-1)}_j \right]. \end{aligned} \quad (4.25)$$

This formula allows us to obtain the weak coupling expansion of \hat{F}_{-ik}^j most efficiently, by first expanding in $\epsilon(k, g^2)$, and then in g with the help of eqs. (4.23)–(4.24). In this manner, it is evident that the most complicated sums in eq. (4.5) will always be of the form

$$\sum_{k,n,m} \frac{(-r)^k}{k^p} \frac{x^n}{n(k+n)} Z_{1, \dots, 1}(n-1) Z_{1, \dots, 1}(k+n-1) \frac{y^m}{m(k+m)} Z_{1, \dots, 1}(m-1) Z_{1, \dots, 1}(k+m-1), \quad (4.26)$$

for $r = \sqrt{xyz}$ or $r = \sqrt{xy/z}$ and p any positive integer. The lengths of the strings of 1's in this expression are arbitrary.

4.3 Kinematic resummation of $\Omega^{(L)}(1, v, w)$ and $\Omega^{(L)}(u, v, 0)$

To our knowledge, no algorithm currently exists for directly evaluating these kinds of sums in terms of multiple polylogarithms. It would be very interesting to develop one based on

our understanding of hexagon functions. However, it turns out that it is indeed possible to resum eq. (4.5) in the limit $y \rightarrow 0$ and $z \rightarrow \infty$, with x and $r = \sqrt{xyz}$ held fixed. Inspecting eq. (A.10), we see that this limit corresponds to the following two-dimensional subspace of hexagon kinematics,

$$u = 1, \quad v = \frac{1}{1+r}, \quad w = \frac{1-x}{1+r}. \quad (4.27)$$

In this subspace, only the first term in eq. (4.25) survives in $\hat{F}_{-ik}^j(y=0)$. We let $(-r)^k = x^k \times (-r/x)^k$, replace the summation variable n with $n' = n + k$ in the other $\hat{F}_{-ik}^j(x)$ factor, and exchange the order of summation. Then the most complicated sums take the form

$$\sum_{n'=1}^{\infty} \frac{x^{n'}}{n'} Z_{1,\dots,1}(n'-1) \sum_{k=1}^{n'-1} \frac{(-r/x)^k}{k^p} Z_{1,\dots,1}(k) \frac{1}{(n'-k)} Z_{1,\dots,1}(n'-k-1). \quad (4.28)$$

Crucially, the rightmost sum can be done with the help of algorithm B of ref. [102]. This algorithm has been implemented in the `nestedsums` library [104] within the `GiNaC` framework, and by interfacing it with `Mathematica` we are able to replace all sums in k of the form (4.28) with Z -sums with outer summation index $n' - 1$ (possibly accompanied by rational factors $(r/x)^{n'}$). With the help of the quasi-shuffle algebra relations, we may rewrite their products with the leftmost Z -sum in eq. (4.28) as linear combinations of Z -sums, similarly to what we did for \hat{F}_0^{2ig} in eq. (4.20). Finally, we evaluate the remaining sum over n' in terms of multiple polylogarithms with the help of their sum representation,

$$\text{Li}_{m_1,\dots,m_j}(x_1,\dots,x_j) = \sum_{i=1}^{\infty} \frac{x_1^i}{i^{m_1}} Z(i-1; m_2, \dots, m_j; x_2, \dots, x_j), \quad (4.29)$$

where

$$Z(n; m_1, \dots, m_j; x_1, \dots, x_j) \equiv \sum_{n \geq i_1 > i_2 > \dots > i_j > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_j^{i_j}}{i_j^{m_j}}. \quad (4.30)$$

Very similar techniques have been used to evaluate [105, 106] the leading, and part of the subleading, contribution to the hexagon Wilson loop OPE near the collinear limit, as well as to resum [107] all single-particle gluon bound states contributing to the double scaling limit, $v \rightarrow 0$ with u, w fixed.

The systematic procedure we have described works in principle at any loop order, subject to limitations in computer power. We have used it to obtain explicit expressions for $\Omega^{(L)}(1, v, w)$ through $L = 8$ loops. We quote here the first two loop orders,

$$\Omega^{(1)}(1, v, w) = -2H_{1,1}(-r) + \text{Li}_{1,1}\left(-r, -\frac{x}{r}\right) - 2H_2(-r) + H_2(x) - 2\zeta_2, \quad (4.31)$$

$$\begin{aligned} \Omega^{(2)}(1, v, w) &= H_{2,2}(-r) - 2H_{3,1}(-r) - 2(H_{2,2}(-r) + 2H_{2,1,1}(-r)) - H_{2,2}(x) \\ &+ \text{Li}_{2,2}\left(-r, -\frac{x}{r}\right) + 2\text{Li}_{2,2}\left(x, -\frac{r}{x}\right) + \text{Li}_{3,1}\left(-r, -\frac{x}{r}\right) + \text{Li}_{3,1}\left(x, -\frac{r}{x}\right) \\ &+ 2\text{Li}_{2,1,1}\left(-r, 1, -\frac{x}{r}\right) + \text{Li}_{2,1,1}\left(-r, -\frac{x}{r}, -\frac{r}{x}\right) + 2\text{Li}_{2,1,1}\left(x, -\frac{r}{x}, 1\right) \\ &- \text{Li}_{2,1,1}\left(x, -\frac{r}{x}, -\frac{x}{r}\right) + H_4(-r) + 2\zeta_2(H_2(x) - H_2(-r)) - 6\zeta_4, \end{aligned} \quad (4.32)$$

where

$$r = \frac{1}{v} - 1, \quad x = 1 - \frac{w}{v}. \quad (4.33)$$

Results through eight loops are contained in the ancillary file `omega1vwL0-8.m` provided with this paper.

In precisely the same fashion, we may also resum $\Omega^{(L)}(u, v, 0)$, which, as discussed around eq. (3.61), is equivalent to the dual conformal pentabox ladder $\Psi^{(L)}(u, v)$ defined in (2.26) and shown in fig. 2. Starting from the sum representation (4.5), as already mentioned the limit in question amounts to letting $y \rightarrow 1$. In this limit, the way F_ν^j are normalized implies that $\hat{F}_{-ik}^j(y) \rightarrow 1$, also for $k = 0$. Up to two loops we obtain

$$\begin{aligned} \Psi^{(1)}(u, v) = & -H_{1,1} \left(-\sqrt{\frac{x}{z}} \right) - H_{1,1}(-\sqrt{xz}) + \text{Li}_{1,1} \left(-\sqrt{\frac{x}{z}}, -\sqrt{xz} \right) \\ & + \text{Li}_{1,1} \left(-\sqrt{xz}, -\sqrt{\frac{x}{z}} \right) - \zeta_2 - H_2 \left(-\sqrt{\frac{x}{z}} \right) - H_2(-\sqrt{xz}) + H_2(x), \end{aligned} \quad (4.34)$$

$$\begin{aligned} \Psi^{(2)}(u, v) = & -H_{2,1,1} \left(-\sqrt{\frac{x}{z}} \right) - H_{2,1,1}(-\sqrt{xz}) - H_{2,2}(x) + \text{Li}_{2,2} \left(x, -\sqrt{\frac{z}{x}} \right) \\ & + \text{Li}_{2,2} \left(x, -\frac{1}{\sqrt{xz}} \right) + \text{Li}_{2,2} \left(-\sqrt{\frac{x}{z}}, -\sqrt{xz} \right) + \text{Li}_{2,2} \left(-\sqrt{xz}, -\sqrt{\frac{x}{z}} \right) \\ & + \text{Li}_{2,1,1} \left(x, -\sqrt{\frac{z}{x}}, 1 \right) - \text{Li}_{2,1,1} \left(x, -\sqrt{\frac{z}{x}}, -\sqrt{\frac{x}{z}} \right) + \text{Li}_{2,1,1} \left(x, -\frac{1}{\sqrt{xz}}, 1 \right) \\ & - \text{Li}_{2,1,1} \left(x, -\frac{1}{\sqrt{xz}}, -\sqrt{xz} \right) + \text{Li}_{2,1,1} \left(-\sqrt{\frac{x}{z}}, 1, -\sqrt{xz} \right) - \frac{7\zeta_4}{4} \\ & + \text{Li}_{2,1,1} \left(-\sqrt{xz}, 1, -\sqrt{\frac{x}{z}} \right) + \zeta_2 H_2(x) - \zeta_2 H_2 \left(-\sqrt{\frac{x}{z}} \right) \\ & - \zeta_2 H_2(-\sqrt{xz}) + H_4 \left(-\sqrt{\frac{x}{z}} \right) + H_4(-\sqrt{xz}), \end{aligned} \quad (4.35)$$

where

$$x = \frac{(1-u)(1-v)}{uv}, \quad z = \frac{u(1-v)}{v(1-u)}, \quad \sqrt{\frac{x}{z}} = \frac{1-u}{u}, \quad \sqrt{xz} = \frac{1-v}{v}. \quad (4.36)$$

The polylog arguments are all rational in u, v in this case. Here as well we have carried out the computation up to $L = 8$; the resulting expressions may be found in the ancillary file `omegauv0L0-8.m`.

4.4 The line $(1, 1, w)$

The line $(u, v, w) = (1, 1, w)$ corresponds to taking $y \rightarrow 0$ at fixed x and z , with $w = 1 - x$ on this line. Examining the series expansions found in section 4.1, we see that only the $k = 0$

terms involving \hat{F}_0^{2ig} and $\hat{F}_0^{2ig'}$ survive, since $\sqrt{xyz} \rightarrow 0$ and $\sqrt{xy/z} \rightarrow 0$. Also, $\hat{F}_0^{2ig'}(0) = 0$. Therefore both the even and odd parts of $\tilde{\Omega}$ vanish on this line,

$$\tilde{\Omega}_e(1, 1, w) = \tilde{\Omega}_o(1, 1, w) = 0. \quad (4.37)$$

The vanishing of $\tilde{\Omega}_o$ is also a consequence of its antisymmetry under $u \leftrightarrow v$.

For Ω , the $k = 0$ term in eq. (4.5) gives, using eq. (4.16),

$$\begin{aligned} \Omega(1, 1, w, g^2) &= -\hat{F}_0^{2ig}(0)\hat{F}_0^{2ig}(x) \\ &= -\left(\frac{\pi g}{\sinh \pi g}\right)^2 {}_2F_1(ig, -ig, 1, x) \\ &= -\left(\frac{\pi g}{\sinh \pi g}\right)^2 \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n-1} (k^2 + g^2)}{(n!)^2} x^n \\ &= -1 + \sum_{L=1}^{\infty} (g^2)^L \left[-\underbrace{H_{2, \dots, 2}}_L(x) + \sum_{m=1}^L (-1)^m (2 - 4m) \zeta_{2m} \underbrace{H_{2, \dots, 2}}_{L-m}(x) \right], \end{aligned} \quad (4.38)$$

with $x = 1 - w$. At $x = 0$, the term in eq. (4.38) with $m = L$ supplies the value of Ω at $(u, v, w) = (1, 1, 1)$:

$$\Omega(1, 1, 1, g^2) = -\left(\frac{\pi g}{\sinh \pi g}\right)^2 = -1 + \sum_{L=1}^{\infty} (-g^2)^L (2 - 4L) \zeta_{2L}. \quad (4.39)$$

At $x = 1$, we have

$$\begin{aligned} \Omega(1, 1, 0, g^2) &= -\left(\frac{\pi g}{\sinh \pi g}\right)^2 {}_2F_1(ig, -ig, 1, 1) \\ &= -\frac{\pi g}{\sinh \pi g} = -1 - \sum_{L=1}^{\infty} (-g^2)^L \frac{2^{2L-1} - 1}{2^{2L-2}} \zeta_{2L}. \end{aligned} \quad (4.40)$$

Note that the HPLs obey

$$w \frac{d}{dw} \left[(1-w) \frac{d}{dw} \underbrace{H_{2, \dots, 2}}_N(1-w) \right] = \underbrace{H_{2, \dots, 2}}_{N-1}(1-w). \quad (4.41)$$

Therefore $\Omega(1, 1, w, g^2)$ satisfies the differential equation

$$w \frac{d}{dw} \left[(1-w) \frac{d}{dw} \Omega(1, 1, w, g^2) \right] = g^2 \Omega(1, 1, w, g^2). \quad (4.42)$$

The first few perturbative orders for $\Omega^{(L)}(1, 1, w)$ are:

$$\Omega^{(0)}(1, 1, w) = -1, \quad (4.43)$$

$$\Omega^{(1)}(1, 1, w) = H_2(1-w) - 2\zeta_2, \quad (4.44)$$

$$\Omega^{(2)}(1, 1, w) = -H_{2,2}(1-w) + 2\zeta_2 H_2(1-w) - 6\zeta_4, \quad (4.45)$$

$$\Omega^{(3)}(1, 1, w) = H_{2,2,2}(1-w) - 2\zeta_2 H_{2,2}(1-w) + 6\zeta_4 H_2(1-w) - 10\zeta_6. \quad (4.46)$$

The one- and two-loop formulae can be recovered from eqs. (4.31) and (4.32) by letting $r \rightarrow 0$, which leaves only the HPLs with argument $x = 1 - w$.

Similarly, the odd ladder integral becomes

$$\begin{aligned}
\mathcal{O}(1, 1, w, g^2) &= -\hat{F}_0^{2ig}(0)\hat{F}_0^{2ig'}(x) \\
&= \frac{1}{g^2} x \frac{d}{dx} \Omega(1, 1, 1 - x, g^2) \\
&= \sum_{L=1}^{\infty} (g^2)^{L-1} \left[-\underbrace{H_{1,2,\dots,2}}_L(x) + \sum_{m=1}^L (-1)^m (2 - 4m) \zeta_{2m} \underbrace{H_{1,2,\dots,2}}_{L-m}(x) \right], \\
&= -\frac{1}{g^2} (1 - w) \frac{d}{dw} \Omega(1, 1, w, g^2), \tag{4.47}
\end{aligned}$$

so it sits in the middle of the differential equation (4.42). On the line $(1, 1, w)$, the even ladder integral is simply related to the odd one at one higher loop:

$$\mathcal{W}(1, 1, w, g^2) = g^2 \mathcal{O}(1, 1, w, g^2). \tag{4.48}$$

As we will see in subsection 4.6, the $(1, 1, w)$ limit offers us insight into the strong-coupling analysis of the integrals. In addition, we can study the radius of convergence in the g plane of the perturbative expansion for $\Omega(1, 1, w, g^2)$ using eq. (4.38). The same arguments that lead to eq. (4.3) show that the hypergeometric function ${}_2F_1(ig, -ig, 1, x)$ has poles at $g = \pm i$, which are the poles in the g plane closest to the origin. They also match the location of the closest poles of the prefactor $\pi g / \sinh(\pi g)$. Therefore the radius of convergence of the perturbative expansion of $\Omega(1, 1, w, g^2)$ is unity for all w . We can check this result at $w = 1$ and $w = 0$ by observing that the ratio of successive loop orders in eqs. (4.39) and (4.40) goes to -1 as $L \rightarrow \infty$.

We remark here that the resummed integral $\Omega(u, v, w, g^2)$ appears correctly weighted in the full BDS-like normalized MHV amplitude $\mathcal{E}(u, v, w, g^2)$, when g^2 is identified with the standard coupling parameter in planar $\mathcal{N} = 4$ SYM, $g^2 = N_c g_{\text{YM}}^2 / (4\pi)^2$, where g_{YM} is the Yang-Mills coupling and the gauge group is $SU(N_c)$. In order to establish that it is correctly weighted, one can use the ‘‘rung rule’’ for performing two-particle cuts in planar $\mathcal{N} = 4$ SYM [108, 109]. This rule provides the normalization of the $\Omega^{(L)}$ terms within the L -loop integrand, relative to the normalization at one lower loop. Therefore it makes sense to compare the unit radius of convergence for $\Omega(1, 1, w, g^2)$ with the radius of convergence for amplitudes. The latter is not firmly established [68, 69, 71], but it appears to be closer to $1/16$, the value for the cusp anomalous dimension [42], and much smaller than 1. It would be interesting to use our finite-coupling representation (3.26) to investigate the perturbative radius of convergence of $\Omega(u, v, w, g^2)$ for more general kinematics than just the line $(1, 1, w)$.

4.5 The line $(1, v, 1)$

Next we consider the line $(u, v, w) = (1, v, 1)$. From eqs. (4.33) and (A.10), this corresponds to letting $y \rightarrow 0$ with x and yz fixed, and then letting $r = \sqrt{xyz} = -x$, where $v = 1/(1 - x)$,

$x = 1 - 1/v$. Applying this substitution to the series representation of the ladder integrals in eqs. (4.5), (4.12)–(4.15), and using $F_{-ik}^{j'}(0) = 0$, yields

$$\Omega(1, 1/(1-x), 1, g^2) = -\hat{F}_0^{2ig}(0)\hat{F}_0^{2ig}(x) - \sum_{k=1}^{\infty} x^k \hat{F}_{-ik}^j(0)\hat{F}_{-ik}^j(x), \quad (4.49)$$

$$\mathcal{O}(1, 1/(1-x), 1, g^2) = -\hat{F}_0^{2ig}(0)\hat{F}_0^{2ig'}(x) - \sum_{k=1}^{\infty} x^k \hat{F}_{-ik}^j(0)\hat{F}_{-ik}^{j'}(x), \quad (4.50)$$

$$\tilde{\Omega}_e(1, 1/(1-x), 1, g^2) = -\sum_{k=1}^{\infty} \frac{k}{2} x^k \hat{F}_{-ik}^j(0)\hat{F}_{-ik}^{j'}(x), \quad (4.51)$$

$$\tilde{\Omega}_o(1, 1/(1-x), 1, g^2) = -\tilde{\Omega}_e(1, 1/(1-x), 1, g^2), \quad (4.52)$$

$$\begin{aligned} \mathcal{W}(1, 1/(1-x), 1, g^2) &= -g^2 \left[\hat{F}_0^{2ig}(0)\hat{F}_0^{2ig'}(x) + \sum_{k=1}^{\infty} x^k \hat{F}_{-ik}^j(0)\hat{F}_{-ik}^{j'}(x) \right] \\ &\quad - \sum_{k=1}^{\infty} kx^k \hat{F}_{-ik}^j(0)\hat{F}_{-ik}^j(x). \end{aligned} \quad (4.53)$$

Hence $\tilde{\Omega} = \tilde{\Omega}_e + \tilde{\Omega}_o$ vanishes on the line $(1, v, 1)$, and using eqs. (4.10) and (4.11) we have

$$(1-x) \frac{d}{dx} \mathcal{O}(x) = \Omega(x), \quad (4.54)$$

$$x \frac{d}{dx} \Omega(x) = \mathcal{W}(x). \quad (4.55)$$

So it is enough to specify the loop expansion of $\Omega(1, v, 1)$ below.

Now Ω , $\tilde{\Omega}_e$, \mathcal{O} and \mathcal{W} are symmetric under $u \leftrightarrow v$ ($z \leftrightarrow 1/z$), while $\tilde{\Omega}_o$ is anti-symmetric. Therefore, on the line $(u, 1, 1)$, with $u = 1/(1-x)$, we can also use the above formulas, except that the sign of $\tilde{\Omega}_o$ is reversed so that in $\tilde{\Omega}$ it doubles $\tilde{\Omega}_e$ instead of cancelling it:

$$\tilde{\Omega}_o(1/(1-x), 1, 1, g^2) = \tilde{\Omega}_e(1/(1-x), 1, 1, g^2) = \tilde{\Omega}_e(1, 1/(1-x), 1, g^2), \quad (4.56)$$

$$\tilde{\Omega}(1/(1-x), 1, 1, g^2) = 2\tilde{\Omega}_e(1, 1/(1-x), 1, g^2). \quad (4.57)$$

On the line $(1, v, 1)$, the first few orders of explicit results for $\Omega^{(L)}$ are:

$$\Omega^{(1)}(1, v, 1) = -H_2 - H_{1,1} - 2\zeta_2, \quad (4.58)$$

$$\Omega^{(2)}(1, v, 1) = H_4 + H_{2,2} - 6\zeta_4, \quad (4.59)$$

$$\Omega^{(3)}(1, v, 1) = -2H_6 - H_{4,2} - H_{3,3} - H_{2,4} - H_{2,2,2} + 6\zeta_4 H_2 - 10\zeta_6, \quad (4.60)$$

$$\begin{aligned} \Omega^{(4)}(1, v, 1) = & 5H_8 + 2(H_{6,2} + H_{5,3} + H_{4,4} + H_{3,5} + H_{2,6}) \\ & + H_{4,2,2} + H_{3,3,2} + H_{3,2,3} + H_{2,4,2} + H_{2,3,3} + H_{2,2,4} + H_{2,2,2,2} \\ & - 6\zeta_4(H_4 + H_{2,2}) + 10\zeta_6 H_2 - 14\zeta_8, \end{aligned} \quad (4.61)$$

$$\begin{aligned} \Omega^{(5)}(1, v, 1) = & -14H_{10} - 5(H_{8,2} + H_{7,3} + H_{6,4} + H_{5,5} + H_{4,6} + H_{3,7} + H_{2,8}) \\ & - 2(H_{6,2,2} + H_{5,3,2} + H_{5,2,3} + H_{4,4,2} + H_{4,3,3} + H_{4,2,4} + H_{3,5,2} + H_{3,4,3} \\ & + H_{3,3,4} + H_{3,2,5} + H_{2,6,2} + H_{2,5,3} + H_{2,4,4} + H_{2,3,5} + H_{2,2,6}) \\ & - H_{4,2,2,2} - H_{3,3,2,2} - H_{3,2,3,2} - H_{3,2,2,3} - H_{2,4,2,2} - H_{2,3,3,2} \\ & - H_{2,3,2,3} - H_{2,2,4,2} - H_{2,2,3,3} - H_{2,2,2,4} - H_{2,2,2,2,2} \\ & + 6\zeta_4(2H_6 + H_{4,2} + H_{3,3} + H_{2,4} + H_{2,2,2}) \\ & - 10\zeta_6 H_{2,2} + 14\zeta_8 H_2 - 18\zeta_{10}, \end{aligned} \quad (4.62)$$

where $x = 1 - 1/v$ is the implicit argument of $H_{\vec{w}} = H_{\vec{w}}(x)$. These formulae can be obtained from the results obtained on $(1, v, w)$ by letting $r \rightarrow -x$, which collapses the multiple polylogarithms, for example,

$$\text{Li}_{2,1,1}\left(x, -\frac{r}{x}, -\frac{x}{r}\right) \rightarrow \text{Li}_{2,1,1}(x, 1, 1) = H_{2,1,1}(x), \quad (4.63)$$

using eq. (4.22).

In the coefficients of the non- ζ terms in eqs. (4.58)–(4.62), one can see the emergence of the Catalan numbers,

$$C_n = \frac{(2n)!}{(n+1)!n!} = 2 \frac{(2n-1)!}{(n+1)!(n-1)!} = 1, 2, 5, 14, 42, 132, 429, \dots \quad (4.64)$$

Although it is not really apparent yet, the coefficients of the $\zeta_{2(k+1)}$ terms for $k > 1$ are controlled by a $2k$ -fold convolution of the Catalan numbers. Define

$$C_{n,k} \equiv \frac{k}{2n-k} \binom{2n-k}{n}, \quad (4.65)$$

which satisfies

$$C_{n,k} = \sum_{i_1+i_2+\dots+i_k=n} C_{i_1-1} C_{i_2-1} \dots C_{i_k-1}, \quad (4.66)$$

with $C_n \equiv 0$ for $n < 0$. It also obeys $C_{n+1,1} = C_{n+1,2} = C_n$. Also note from eqs. (4.59)–(4.62) that the Catalan number required is related to the *depth*, i.e. the length of the (compressed) HPL weight vector. All weight vectors having the same depth appear with the same coefficient, which is nonzero if all entries are ≥ 2 . (The only exception is at $L = 1$, which contains $H_{1,1}$.)

We find that, for $L > 1$, $\Omega^{(L)}(1, v, 1)$ is given by

$$\Omega^{(L)}(1, v, 1) = (-1)^L \left[X_0^{(L)} + \sum_{k=2}^L (2-4k) \zeta_{2k} X_k^{(L-k)} \right], \quad (4.67)$$

where the no- ζ term is

$$X_0^{(L)} = \underbrace{H_{2,2,\dots,2}(x)}_L + \sum_{m=1}^{L-1} C_{L-m} \sum_{\vec{w} \in g_{m,2L}} H_{\vec{w}}(x), \quad (4.68)$$

and the ζ terms are

$$X_k^{(0)} = (-1)^k, \quad (4.69)$$

$$X_k^{(L)} = (-1)^k \underbrace{H_{2,2,\dots,2}(x)}_L + \sum_{m=1}^{L-k+1} C_{L+k-1-m,2(k-1)} \sum_{\vec{w} \in g_{m,2L}} H_{\vec{w}}(x) \quad L > 0. \quad (4.70)$$

Here $g_{m,n}$ is the set of weight vectors $\vec{w} = (w_1, w_2, \dots, w_m)$ of depth m and weight $\sum_{i=1}^m w_i = n$, with all $w_i \geq 2$.

Note that the first term of $X_0^{(L)}$ also appears in eq. (4.38) for $\Omega(1, 1, w)$. Also, using $C_{n+1,2} = C_n$ we see that $X_2^{(L)} = X_0^{(L)}$; that is, the ζ_4 terms are controlled by $X_2^{(L-2)} = X_0^{(L-2)}$, which is exactly the same function describing the no- ζ terms at two lower loops. The ζ_6 terms are the first to require a true convolution of the Catalan numbers, i.e. $C_{n,4}$.

We have checked eq. (4.67) exactly through six loops, and through 13 loops via the series expansion (4.49) to order x^{10} . At 13 loops, the full answer contains 75,025 non- ζ terms, 10,946 ζ_4 terms, 4,136 ζ_6 terms, 1,351 ζ_8 terms, 246 ζ_{10} terms, 13 ζ_{12} terms, and one each of the $\zeta_{14}, \zeta_{16}, \dots, \zeta_{26}$ terms.

Note that the non-Catalan term in $\Omega^{(L)}(1, v, 1)$ is equal to the much simpler expression for $\Omega^{(L)}(1, 1, w)$, after identifying $1-x = w = 1/v$. This is simply the $k=0$ term in eq. (4.49), while the terms containing the Catalan numbers come from the $k > 0$ terms.

4.6 Strong-coupling behavior

A remarkable feature of the finite-coupling $\Omega, \tilde{\Omega}$ integrals (3.26) and (3.32) is that we can evaluate them outside the radius of convergence of the weak-coupling region we used to derive them, all the way to strong coupling. Here we provide evidence that the functions become exponentially suppressed as $g \rightarrow \infty$ for a large subspace of the Euclidean domain. This is very similar to the observed strong-coupling behavior of the box ladder integrals for general kinematics [78].

For simplicity, let us begin with the $(1, 1, w)$ line, which we also analyzed in section 4.4 where we focused on weak coupling. At $w = 1$, the exponential suppression is clear from eq. (4.39):

$$\Omega(1, 1, 1, g^2) = - \left(\frac{\pi g}{\sinh \pi g} \right)^2 \sim -4\pi^2 g^2 \exp(-2\pi g), \quad \text{as } g \rightarrow \infty, \quad (4.71)$$

and similarly for $w = 0$, from eq. (4.40):

$$\Omega(1, 1, 0, g^2) = -\frac{\pi g}{\sinh \pi g} \sim -2\pi g \exp(-\pi g), \quad \text{as } g \rightarrow \infty. \quad (4.72)$$

Generally, going from $w = 1$ to $w = 0$ the absolute value of the function increases monotonically between these two limits. To study in detail the behavior between the endpoints, the problem is reduced to the asymptotic analysis of the (normalized) hypergeometric functions $F_0^{2ig}(x)$ defined in eq. (3.20), which enters on the first line of (4.38). Fortunately, a very detailed saddle point analysis of this precise class of hypergeometric functions has been carried out in ref. [110], see in particular Theorems 3.1 and 3.2, which are valid in the region $x < 0$ and $0 < x < 1$ respectively. Focusing on the region $0 < x < 1$, we can write their result as:

$$\lim_{g \rightarrow \infty} F_\nu^{j(\nu)}(x) = \sqrt{\pi g} \left(\frac{1}{x} - 1\right)^{1/4} e^{-g\phi(x)}, \quad \phi(x) = \arccos(2x - 1), \quad (4.73)$$

which in fact holds for any fixed value of $|\nu| \ll g$, up to relative corrections of order $1/g$. The angle ϕ varies continuously and monotonically from $\phi(0^+) = \pi$ to $\phi(1^-) = 0$. Including the second hypergeometric factor in (4.38), which gives $F_0^{2ig}(0) = \frac{\pi g}{\sinh(\pi g)}$, we thus get:

$$\Omega(1, 1, w, g^2 \rightarrow \infty) = -2(\pi g)^{3/2} \left(\frac{1}{x} - 1\right)^{1/4} \times e^{-g(\phi(x)+\pi)} \times (1 + \mathcal{O}(1/g)) \quad (4.74)$$

where $x = 1 - w$. The dependence of the exponent on w could also be obtained simply by solving the hypergeometric differential equation at leading order in g .⁵ We observe that the exponent smoothly interpolates between eq. (4.71) and eq. (4.72). The prefactors do not quite go smoothly, but this can be understood as a breakdown of the saddle point approximation for extreme values of x very close to the endpoints, see ref. [110] for details.

For the other integrals in our basis, it follows from equations (4.47) and (4.48) that $\mathcal{O}(1, 1, w, g^2)$ and $\mathcal{W}(1, 1, w, g^2)$ are also exponentially suppressed for all $x < 1$ ($w > 0$), whereas we recall that $\tilde{\Omega}$ vanishes identically on this line.

In more general kinematics, we can analyze the integral representation in (3.26), which we reproduce here for convenience:

$$\Omega(u, v, w, g^2) = \int_{-\infty}^{\infty} \frac{d\nu}{2i} z^{i\nu/2} \frac{F_{+\nu}^{j(\nu)}(x)F_{+\nu}^{j(\nu)}(y) - F_{-\nu}^{j(\nu)}(x)F_{-\nu}^{j(\nu)}(y)}{\sinh(\pi\nu)}. \quad (4.75)$$

By plotting the integrand for various values of x, y, z we find that it is generally dominated by the region near the origin, where $|\nu| \sim 1 \ll g$. In this regime we can use the limit in eq. (4.73) to deduce the following exponential suppression of the integrand:

$$F_{+\nu}^{j(\nu)}(x)F_{+\nu}^{j(\nu)}(y) \sim e^{-g(\phi(x)+\phi(y))}, \quad (4.76)$$

where we have focused on the exponent. Thus, assuming that the region $|\nu| \sim 1$ indeed dominates as suggested by the numerics, Ω is itself suppressed by at least the same factor:

$$\left| \Omega(u, v, w, g^2 \rightarrow \infty) \right| \lesssim e^{-g(\phi(x)+\phi(y))}. \quad (4.77)$$

⁵We thank Bob Cahn for this observation.

Remarkably, the true behavior of Ω for generic x, y appears to be even more strongly suppressed. This can be seen from the fact that the asymptotic expansion (4.73) to all orders in $1/g$ turns out to contain only even powers of ν , so there is a cancellation between the two terms in the numerator of (4.75), causing the dominant behavior to come from subleading exponential corrections to (4.73). While these could in principle be analyzed using formulas in ref. [110], we will simply conclude this subsection with the observation that Ω is exponentially suppressed at strong coupling.

5 The Ω Space

In this section we analyze the Ω integrals in general kinematics, working perturbatively in the coupling. We define the Ω space of functions to be that containing all iterated derivatives (more precisely, all iterated $\{n-1, 1\}$ coproducts) of the \mathcal{W} , Ω , $\tilde{\Omega}$ and \mathcal{O} integrals to arbitrary loop orders. We are interested in studying the Ω space primarily as a model for the full space of Steinmann hexagon functions [69, 74]. If we can characterize the functions that appear in the Ω integrals to all orders, we will have encompassed a substantial slice of the full space of Steinmann hexagon functions, and this may give hints as to their overall structure. We will achieve this by first showing that a certain discontinuity of the Ω integrals is simple, then using this insight to build the full Ω space.

5.1 Coproduct formalism and hexagon function space

Like MHV and NMHV amplitudes in planar $\mathcal{N} = 4$ sYM, the integrals defined in section 2.2 are expected to evaluate to multiple polylogarithms. This implies that they are endowed with a Hopf algebra. In particular, there is a coproduct operation which breaks functions apart into simpler ones. This yields various concrete representations as iterated integrals.

For a more complete review of how Hopf algebras and the coproduct show up in amplitudes, see ref. [111]. The key property for us will be that derivatives only act on objects appearing in the second entry of the coproduct Δ .⁶

$$\Delta \frac{\partial}{\partial z} F = \left(\text{id} \otimes \frac{\partial}{\partial z} \right) \Delta F. \quad (5.1)$$

In particular, we can define a set of functions F^s by the coproduct action

$$\Delta_{n-1,1} F = \sum_{s \in \mathcal{S}} F^s \otimes \ln s \quad (5.2)$$

for any multiple polylogarithm F of weight n that involves symbol letters in the set \mathcal{S} . The functions F^s are then iterated integrals of weight $n-1$. The derivative of F with respect to

⁶We use the term ‘‘coproduct’’ somewhat loosely; in many cases ‘‘coaction’’ would be more appropriate because the spaces to the left and right of ‘‘ \otimes ’’ are actually different. Note also that here Δ denotes the coproduct, and not the kinematical quantity defined below eq. (3.7).

an underlying kinematic variable z (say) is

$$\partial_z F = \sum_{s \in \mathcal{S}} F^s \partial_z \ln s. \quad (5.3)$$

By repeating this operation n times and integrating along various paths with appropriate boundary conditions, one obtains concrete integral representations of F . The coproduct unifies these representations in a canonical way.

In six-particle kinematics, the traditional symbol alphabet is given by [60, 83]

$$\mathcal{S}_{\text{hex}} = \{u, v, w, 1 - u, 1 - v, 1 - w, y_u, y_v, y_w\}. \quad (5.4)$$

The list of iterated integrals with this alphabet at any given weight is finite, and is called the hexagon function space. However, constructing this space is nontrivial and currently unsolved beyond weight 12, even after imposing the Steinmann conditions [69] and further restricting to the level of the symbol.

For our discussion it will be convenient to parametrize the same space with a new alphabet

$$\mathcal{S}'_{\text{hex}} = \{a, b, c, m_u, m_v, m_w, y_u, y_v, y_w\}, \quad (5.5)$$

where $a = \frac{u}{vw}$, $m_u = \frac{1-u}{u}$, and the others are defined by cyclic permutations of u, v, w , as in eq. (A.16). The letters a, b, c are physically significant due to the Steinmann relations, which state that amplitudes (or individual Feynman integrals) can't have simultaneous discontinuities in overlapping channels. Each contains a single three-particle Mandelstam invariant: $a \propto (x_{25}^2)^2$, $b \propto (x_{36}^2)^2$, $c \propto (x_{14}^2)^2$ (see the definitions of the cross ratios in eq. (2.24)). In the new alphabet the Steinmann relations state simply that a can never appear next to b in the first two entries of the symbol (or b next to c , or a next to c) [74]. We discuss this condition further in appendix B.

From the kinematic relations (A.10) and (A.16) in appendix A, we see that five of the nine hexagon letters can be taken to be simple combinations of the variables x, y, z which simplify the ladders:

$$c = (1-x)(1-y), \quad m_u = \sqrt{\frac{xy}{z}}, \quad m_v = \sqrt{xyz}, \quad y_u y_v = \frac{y}{x}, \quad y_w = \frac{x(1-y)}{y(1-x)}. \quad (5.6)$$

These five letters are equivalent under multiplication to

$$\mathcal{S}_{\text{disc}} = \{x, 1-x, y, 1-y, z\}, \quad (5.7)$$

which are the only letters appearing in the matrix M introduced in eq. (3.51). As we will now show, after taking a discontinuity in c , the ladder integrals collapse to a space with just these five letters, which will enable their complete description.

5.2 The box ladders and their discontinuities

It is helpful to first describe the analogous, simpler, space for the box ladder integrals. The pure functions entering these integrals are given explicitly in terms of classical polylogarithms in eq. (2.17). An alternate representation, which exposes their coproduct structure a bit better, is in terms of single-valued harmonic polylogarithms (SVHPLs) [79].

Recall that the ordinary HPLs [103] with uncompressed arguments \vec{w} , $w_i \in \{0, 1\}$, obey the differential relations,

$$\frac{\partial}{\partial z} H_{0\vec{w}}(z) = \frac{H_{\vec{w}}(z)}{z}, \quad \frac{\partial}{\partial z} H_{1\vec{w}}(z) = \frac{H_{\vec{w}}(z)}{1-z}, \quad (5.8)$$

along with the “initial conditions”

$$H_1(z) = -\ln(1-z), \quad H_{\underbrace{0, \dots, 0}_n}(z) = \frac{1}{n!} \ln^n z. \quad (5.9)$$

The SVHPLs, $\mathcal{L}_{\vec{w}}$, $w_i \in \{0, 1\}$, are functions of z and \bar{z} that are linear combinations of products $\sim H_{\vec{w}}(z)H_{\vec{w}'}(\bar{z})$ and are real analytic in the complex plane minus the punctures at $0, 1, \infty$. They satisfy a similar set of differential relations,

$$\frac{\partial}{\partial z} \mathcal{L}_{0\vec{w}}(z, \bar{z}) = \frac{\mathcal{L}_{\vec{w}}(z, \bar{z})}{z}, \quad \frac{\partial}{\partial z} \mathcal{L}_{1\vec{w}}(z, \bar{z}) = \frac{\mathcal{L}_{\vec{w}}(z, \bar{z})}{1-z}, \quad (5.10)$$

with

$$\mathcal{L}_1(z, \bar{z}) = -\ln|1-z|^2, \quad \mathcal{L}_{\underbrace{0, \dots, 0}_n}(z, \bar{z}) = \frac{1}{n!} \ln^n |z|^2. \quad (5.11)$$

Their symbol alphabet is

$$\mathcal{S}_f = \{z, 1-z, \bar{z}, 1-\bar{z}\}, \quad (5.12)$$

along with the single-valuedness requirement that the first entry is either $z\bar{z}$ or $(1-z)(1-\bar{z})$.

In terms of the $\mathcal{L}_{\vec{w}}$ functions, the box ladders become [99, 112],

$$f^{(L)}(z, \bar{z}) = (-1)^L \left[\underbrace{\mathcal{L}_{0, \dots, 0, 1, 0, 0, \dots, 0}}_{2L} - \underbrace{\mathcal{L}_{0, \dots, 0, 0, 1, 0, \dots, 0}}_{2L} \right]. \quad (5.13)$$

These are depth 1 SVHPLs [113], where the single “1” in the weight vector appears in one of the two middlemost slots.

Now we ask, what is the f space of functions that contains all coproducts of the $f^{(L)}(z, \bar{z})$ as $L \rightarrow \infty$? This is a minimal space in which we can construct all $f^{(L)}$ ’s as iterated integrals. Taking the derivative $z\partial_z$ simply clips off the first “0”. Taking the anti-holomorphic derivative $\bar{z}\partial_{\bar{z}}$ can be a bit complicated for a generic SVHPL, but at depth 1 it just clips off the last “0”. Iterating this procedure, the “1” can slide forward and backward to any location in the string, until it reaches either end, where it is clipped off by taking a $1-z$ or $1-\bar{z}$ coproduct instead of a z or \bar{z} one. In summary, by taking coproducts using the letters in \mathcal{S}_f , we generate

weight n	0	1	2	3	4	5	6	7	8
f_n functions	1	2	3	4	5	6	7	8	9
Disc f_n functions	0	1	2	3	4	5	6	7	8

Table 1. Number of weight n functions in the f space, and in the space of functions after taking a discontinuity in $1 - z$, holding $1 - \bar{z}$ fixed. Only a single function gets lost in the process.

all depth 1 weight n SVHPLs, of which there are n , as well as the single depth 0 function at this weight, $\mathcal{L}_{0,\dots,0}$. The dimension of this space at weight n is $n + 1$, as illustrated in table 1.

The last line in table 1 shows what happens to the dimension if we take the discontinuity in $1 - z$, holding $1 - \bar{z}$ fixed, for all functions in the f space. This discontinuity is associated with the cut in the channel carrying momentum along the ladder [99]. It will be analogous to the c -discontinuity of the pentalladder integrals. The discontinuity of $f^{(L)}$ (defined as the difference of its value for $z > 1$ taken above or below the branch cut) is given by

$$\frac{1}{2\pi i} \text{Disc } f^{(L)}(z, \bar{z}) = \frac{(-1)^L}{L!(L-1)!} \ln(z/\bar{z}) (\ln z \ln \bar{z})^{L-1}. \quad (5.14)$$

It is depth 0, since the discontinuity removed the “1”. That is, the symbol entries belong to

$$\mathcal{S}_{\text{Disc } f} = \{z, \bar{z}\}. \quad (5.15)$$

The discontinuity $\text{Disc } f^{(L)}$ is itself not a single-valued function. Because of this, when we take derivatives to fill out the full $\text{Disc } f$ space, using the fact that discontinuities commute with derivatives, we get all n monomials at weight $n - 1$: $\ln^k z \ln^{n-1-k} \bar{z}$, $k = 0, 1, \dots, n - 1$. Thus the dimension of the space f is reduced by only one in passing to $\text{Disc } f$, even though the symbol alphabet is halved in size.

This means there is very little loss of information in going to $\text{Disc } f$: any function in f can be recovered from its discontinuity at the price of a single boundary condition. Indeed the only combinations of SVHPLs with vanishing $z = 1$ discontinuity are the simple logarithms $\ln^k(z\bar{z})$. The ladder integral $f(z, \bar{z})$ can be characterized as the unique combination of SVHPLs with the discontinuity (5.14) and which vanishes at $z, \bar{z} \rightarrow 0$.

Before we discuss the pentalladders, it is instructive to understand this simplification from the perspective of the integral representation in eq. (3.69). We fix \bar{z} to a value between 0 and 1 while we analytically continue to $z > 1$. The argument of the sine is then complex, $\pi - \phi \equiv \pi - \frac{1}{2i} \ln(z/\bar{z})$, with a real part that approaches $\pm\pi$ depending on whether we approach the cut from above or from below. The discontinuity thus gives a simpler integral with the $\sin(\pi j)$ denominator canceled:

$$\frac{1}{2\pi i} \text{Disc } f(z, \bar{z}, g^2) = - \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} (z\bar{z})^{i\nu/2} \left[\left(\frac{z}{\bar{z}}\right)^{j(\nu)/2} + \left(\frac{z}{\bar{z}}\right)^{-j(\nu)/2} \right]. \quad (5.16)$$

The integrand is now an entire function of ν . This is so despite the branch points of $j(\nu) = i\sqrt{\nu^2 + 4g^2}$ at $\nu = \pm 2ig$, because it is even in j . It is convenient to use this property to shift

the contour slightly below the two branch points. One can then integrate over the two terms separately: closing the contour in the upper-half plane for the first term, where it decays, and below for the second term. The discontinuity thus reduces to a small contour integral over the cut in the first term: letting $s = i\nu/2$,

$$\frac{1}{2\pi i} \text{Disc } f(z, \bar{z}, g^2) = - \oint_{[-g, g]} \frac{ds}{i\pi} z^{s+\sqrt{s^2-g^2}} \bar{z}^{s-\sqrt{s^2-g^2}}. \quad (5.17)$$

It is now easy to see why the discontinuity involves only powers of $\ln z$ and $\ln \bar{z}$ in perturbation theory: both exponents are uniformly small over the integration contour, and so can be series-expanded. Expanding the integrand at small coupling and taking the coefficient of $(-g^2)^L$, we indeed find a pole in $1/s$ whose residue reproduces eq. (5.14).⁷

Notice that an alternative evaluation of eq. (5.16) would have been to move the contour to slightly above the two branch points, closing it in the lower half-plane, and picking up the contribution from the second term in eq. (5.16) instead. In this version, we would have the same formula with $j \rightarrow -j$ and an opposite overall sign from the reversed contour orientation. Thus eq. (5.17) must be odd under $j = 2\sqrt{s^2 - g^2} \rightarrow -j$. Since this symmetry exchanges z and \bar{z} , it is consistent with eq. (5.14) being odd under $z \leftrightarrow \bar{z}$.

5.3 The pentaladder integrals and their c -discontinuity

By investigating the first few loop orders, we observed that the c -discontinuity of the pentaladders is similarly very simple. Here we will derive such a simplification directly from the integral representation of section 3, where the discontinuity will collapse the ν integral onto a small circle as in the preceding example.

The c -discontinuity represents a cut along the channel carrying momentum along the ladder, from switching the sign of x_{14}^2 from spacelike to timelike. In terms of the cross ratios u, v, w in eq. (2.24), it can be implemented by

$$u \rightarrow e^{-i\pi} u, \quad v \rightarrow e^{-i\pi} v, \quad w \rightarrow w. \quad (5.18)$$

Normalizing it so that $\ln c$ has discontinuity 1, the discontinuity is properly defined as

$$\text{Disc}_c \Omega(a, b, c) \equiv \frac{1}{4\pi i} \left[\Omega(a, b, e^{2\pi i} c) - \Omega(a, b, e^{-2\pi i} c) \right]. \quad (5.19)$$

To translate this to the x, y, z variables we use eqs. (3.5) and (3.6):

$$xy = \frac{(1-u)(1-v)}{uv}, \quad (1-x)(1-y) = c, \quad z = \frac{u(1-v)}{v(1-u)}. \quad (5.20)$$

After the continuation we have $u, v < 0$, which corresponds to $x > 1$ and $y > 1$, with x just below the $x > 1$ branch cut in the $e^{2\pi i} c$ term, and similarly for y . Because both u and v

⁷ The integral can be computed exactly as a Bessel function, $g \ln(z/\bar{z})/\sqrt{\ln z \ln \bar{z}} \times I_1(2g\sqrt{\ln z \ln \bar{z}})$, reproducing the resummation of eq. (5.14).

rotate in the same way in eq. (5.18), we also need xy to acquire the phase $2\pi i$. This phase can be put in either x or y , with z not rotating; the result will be physically equivalent due to the single-valuedness constraint enforced in section 3.2. We conclude that a valid continuation path for the $e^{2\pi i}c$ term is to take $x > 1$ and $y > 1$ below the cut, but with a prior rotation of y around the origin.

The discontinuity in the finite-coupling formula (3.26) for Ω is trickier to compute than that for the box ladder f because $x > 1$ ends up outside the radius of convergence of the hypergeometric series. We deal with this complication by using the standard hypergeometric transformation law under $x \rightarrow 1/x$ to express the result in terms of new functions with argument $1/x$. Applied to eq. (3.20), this transformation yields hypergeometric functions with spin and dimension effectively interchanged:

$$\tilde{F}_j^\nu(x) \equiv F_{ij}^{i\nu}(1/x) = \frac{\Gamma(1 - \frac{j+i\nu}{2})\Gamma(1 - \frac{j-i\nu}{2})}{\Gamma(1-j)} x^{j/2} {}_2F_1(-\frac{j+i\nu}{2}, -\frac{j+i\nu}{2}, 1-j, \frac{1}{x}). \quad (5.21)$$

(Note that the transformation $\nu \rightarrow ij$, $j \rightarrow i\nu$ is equivalent to $\nu^2 \rightarrow \nu^2 + 4g^2$, $g^2 \rightarrow -g^2$ in eq. (3.18), which is also equivalent to letting $x \rightarrow 1/x$ in the differential operator. See ref. [114] for further examples of such transformations.)

The analytic continuation of F to $x > 1 - i0$ is then written as

$$F_\nu^j(x) \rightarrow \frac{e^{i\pi(j-i\nu)/2} \sin(\pi \frac{j+i\nu}{2})}{\sin(\pi j)} \tilde{F}_j^\nu(x) + \frac{e^{-i\pi(j+i\nu)/2} \sin(\pi \frac{j-i\nu}{2})}{\sin(\pi j)} \tilde{F}_{-j}^\nu(x), \quad (5.22)$$

while for y we get an additional factor of $e^{-\pi\nu}$, accounting for the rotation around the origin. For the $e^{-2\pi i}c$ term the phases just get reversed, allowing us to compute the discontinuity:

$$\text{Disc}_c F_\nu^j(x) F_\nu^j(y) = \frac{\sin^2(\pi \frac{j+i\nu}{2})}{2\pi \sin(\pi j)} \tilde{F}_j^\nu(x) \tilde{F}_j^\nu(y) - \frac{\sin^2(\pi \frac{j-i\nu}{2})}{2\pi \sin(\pi j)} \tilde{F}_{-j}^\nu(x) \tilde{F}_{-j}^\nu(y). \quad (5.23)$$

When we subtract the $\nu \rightarrow -\nu$ term in the integral (3.26), after noting that $\tilde{F}_j^\nu|_{\nu \rightarrow -\nu} = \tilde{F}_j^\nu$, the trigonometric factors simplify dramatically and we end up with only:

$$\text{Disc}_c \Omega(x, y, z, g^2) = \int_{-\infty}^{\infty} \frac{d\nu}{4\pi} z^{i\nu/2} \left(\tilde{F}_{j(\nu)}^\nu(x) \tilde{F}_{j(\nu)}^\nu(y) + \tilde{F}_{-j(\nu)}^\nu(x) \tilde{F}_{-j(\nu)}^\nu(y) \right). \quad (5.24)$$

This result should be compared with eq. (5.16): the power laws for the box ladders have simply been replaced by hypergeometric functions.

A further simplification, as in the box ladder case, is that the integral can be rewritten as a contour integral. Because the integrand is symmetrical in $j = i\sqrt{\nu^2 + 4g^2} \leftrightarrow -j$, it does not have branch points at $\nu = \pm 2ig$, but to discuss its terms separately we need to choose a branch. We pick the one with $j \sim i\nu$ at large $|\nu|$. By the analysis leading to eq. (4.5), the function $\tilde{F}_{j(\nu)}^\nu(x)$ then only has poles in the lower-half-plane and is analytic in the upper half-plane (for $x > 1$). The function $\tilde{F}_{-j(\nu)}^\nu(x)$ has the opposite properties. Shifting the contour to an imaginary part just below $-2ig$, and closing the contour as in the box ladder case, we

conclude that the c -discontinuity is saturated by the integral over the short cut from $-2ig$ to $2ig$ of the first term:

$$\boxed{\text{Disc}_c \Omega(x, y, z, g^2) = \oint_{[-2ig, 2ig]} \frac{d\nu}{4\pi} z^{i\nu/2} \tilde{F}_{j(\nu)}^\nu(x) \tilde{F}_{j(\nu)}^\nu(y).} \quad (5.25)$$

This formula is the main result of this subsection. An immediate consequence is that the dependence on z in perturbation theory occurs solely through powers of $\ln z$.

More generally, at weak coupling, the integrand of eq. (5.25) can be series expanded in ν and j , which are uniformly small over the contour. Just as for the box ladder discontinuity (5.17), only odd powers of j contribute to the integral. It is helpful to explicitly pick out the odd part and divide it by j . At this point we also recall that the Ω pentaladders were identified in eq. (3.49) as one component of a four-vector. Repeating the calculation for the other integrals, we write

$$\text{Disc}_c \{ \mathcal{W}(g^2), \Omega(g^2), \tilde{\Omega}_e(g^2), \mathcal{O}(g^2) \} = \oint_{[-g, g]} \frac{ds}{i\pi} \sqrt{s^2 - g^2} \text{Disc}_c \mathcal{V}_i(s, g^2) \quad (5.26)$$

where we have set $s = i\nu/2$ for future convenience. The result eq. (5.25) then implies that

$$\text{Disc}_c \mathcal{V}_2(s, g^2) \equiv \frac{z^s}{4\sqrt{s^2 - g^2}} \left(\tilde{F}_j^{-2is}(x) \tilde{F}_j^{-2is}(y) - (j \rightarrow -j) \right)_{j=2\sqrt{s^2 - g^2}}. \quad (5.27)$$

The other entries share the generic form

$$\text{Disc}_c \mathcal{V}_i(s, g^2) = \frac{z^s}{4\sqrt{s^2 - g^2}} \left(X_{i,j}^{-2is} - X_{i,-j}^{-2is} \right)_{j=2\sqrt{s^2 - g^2}}, \quad (5.28)$$

and are given explicitly as

$$X_{1,j}^\nu = g^2 \left(\tilde{F}_j^{\nu'}(x) \tilde{F}_j^\nu(y) + \tilde{F}_j^\nu(x) \tilde{F}_j^{-\nu'}(y) \right), \quad X_{2,j}^\nu = \tilde{F}_j^\nu(x) \tilde{F}_j^\nu(y), \quad (5.29)$$

$$X_{3,j}^\nu = \frac{g^2}{2} \left(\tilde{F}_j^{\nu'}(x) \tilde{F}_j^{-\nu'}(y) + \tilde{F}_j^{-\nu'}(x) \tilde{F}_j^{\nu'}(y) \right), \quad X_{4,j}^\nu = \tilde{F}_j^{\nu'}(x) \tilde{F}_j^\nu(y) - \tilde{F}_j^\nu(x) \tilde{F}_j^{\nu'}(y), \quad (5.30)$$

where $\tilde{F}_j^{\nu'}$ is the hypergeometric function entering the $x \rightarrow 1/x$ transform of eq. (3.29):

$$\tilde{F}_j^{\nu'} \equiv \frac{\Gamma(1 - \frac{j+i\nu}{2}) \Gamma(\frac{-j+i\nu}{2})}{\Gamma(1 - j)} x^{j/2} {}_2F_1\left(1 - \frac{j+i\nu}{2}, \frac{-j+i\nu}{2}, 1 - j, \frac{1}{x}\right). \quad (5.31)$$

We now describe this result explicitly at weak coupling.

5.4 Perturbative expansions and coproducts of c -discontinuities

The discontinuity integrand in eq. (5.26) can be doubly Taylor-expanded in s and g^2 :

$$\tilde{\mathcal{V}}_i(s, g^2) \equiv \text{Disc}_c \mathcal{V}_i(s, g^2) = \sum_{L=1}^{\infty} \sum_{k=0}^{\infty} (-g^2)^{L-1} s^k \tilde{\mathcal{V}}_{i,k}^{(L)}. \quad (5.32)$$

In general, we expect each term to involve powers of $\ln z$, as mentioned, as well as polylogarithms of x and y originating from the expansion of the hypergeometric functions. For example, taking the $g^2 \rightarrow 0$, $s \rightarrow 0$ limit of eq. (5.28) using the methods of section 4, we get

$$\tilde{\mathcal{V}}_{1,0}^{(1)} = 1, \quad \tilde{\mathcal{V}}_{2,0}^{(1)} = \frac{1}{2} \ln(xy), \quad \tilde{\mathcal{V}}_{3,0}^{(1)} = -\frac{1}{2} \ln c, \quad (5.33)$$

and

$$\tilde{\mathcal{V}}_{4,0}^{(1)} = \text{Li}_2(1-x) - \text{Li}_2(1-y) + \frac{1}{2} \ln \frac{x}{y} \ln c, \quad (5.34)$$

where as usual $c = (1-x)(1-y)$. More generally, we find that the functions have uniform transcendental weights

$$\text{weight } \tilde{\mathcal{V}}_{i,k}^{(L)} = 2L + k + \{-2, -1, -1, 0\}. \quad (5.35)$$

A good way to see these weights and to compute the higher-order terms is to use the differential equation that the $\tilde{\mathcal{V}}_{i,k}^{(L)}$ satisfy. Before taking the discontinuity, by using the properties of the hypergeometric functions we found a set of four coupled first-order equations, eqs. (3.50) and (3.51). Not surprisingly, since discontinuities and derivatives commute, $\text{Disc}_c \mathcal{V}_i(s, g^2)$ satisfies precisely the same coupled equations. Using eq. (5.3) we can rewrite the system as a coaction

$$\Delta_{\bullet,1} \left(\tilde{\mathcal{V}}_i(s, g^2) \right) = \left(\tilde{\mathcal{V}}_j(s, g^2) \right) \otimes M_{ij}(s, g^2), \quad (5.36)$$

where we recall the definition

$$M_{ij}(s, g^2) = \begin{pmatrix} s \ln z & -g^2 \ln c + 2s^2 \ln(xy) & g^2 \ln(xy) & 0 \\ \frac{1}{2} \ln(xy) & s \ln z & 0 & \frac{g^2}{2} \ln \frac{x}{y} \\ -\frac{1}{2} \ln c & 0 & s \ln z & \frac{g^2}{2} \ln \frac{x-1}{y-1} - s^2 \ln \frac{x}{y} \\ 0 & -\ln \frac{x-1}{y-1} & -\ln \frac{x}{y} & s \ln z \end{pmatrix}. \quad (5.37)$$

Although the entries of this matrix are all weight one transcendental functions, we see that the relative weights of the $\tilde{\mathcal{V}}_{i,k}^{(L)}$ functions are encoded in the powers of g and s . Namely, if we assign these expansion parameters both transcendental weight minus one, the diagonal terms in the matrix have weight zero, the upper-triangular entries have weight minus one, and the lower-triangular terms have weight one. For fixed L and k , coacting on $\tilde{\mathcal{V}}_{j,k}^{(L)}$ with $M_{ij}(s, g^2)$ will thus increase the weight of the iterated integrals in each entry by one, but the resulting entries should be interpreted as multiplying different powers of s and g in the expansion (5.32).

This coaction can be used to construct the functions $\tilde{\mathcal{V}}_{i,k}^{(L)}$ iteratively, using a single boundary condition, which we will describe shortly. One begins with the transcendental weight 0 vector and coacts on it using the matrix M to get the complete weight-one component

of $\tilde{\mathcal{V}}_i(s, g^2)$:

$$\tilde{\mathcal{V}}_i(s, g^2) \Big|_{\text{weight } 0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \Delta_{0,1} \tilde{\mathcal{V}}_i(s, g^2) = \begin{pmatrix} s(1 \otimes \ln z) \\ \frac{1}{2}(1 \otimes \ln(xy)) \\ -\frac{1}{2}(1 \otimes \ln c) \\ 0 \end{pmatrix}. \quad (5.38)$$

Reading off the second and third element, we reproduce $\tilde{\mathcal{V}}_{2,0}^{(1)} = \frac{1}{2} \ln(xy)$ and $\tilde{\mathcal{V}}_{3,0}^{(1)} = -\frac{1}{2} \ln c$. The first element has a factor of s , so it corresponds to a term with $k = 1$, in particular $\tilde{\mathcal{V}}_{1,1}^{(1)} = \ln z$.

Following this procedure further, one can construct the c -discontinuity of our basis functions to any weight by iteratively coacting with matrix M . At each step one has to supplement the information from the coproduct with one boundary condition, because (being effectively a derivative operator) $\Delta_{\bullet,1}$ kills all constants. A convenient limit can be given at $(x, y, z) = (1, 1, 1)$ (corresponding to $(u, v, w) = (-\infty, -\infty, 1)$). There we find from eq. (5.28) that $\tilde{\mathcal{V}}_2$ and $\tilde{\mathcal{V}}_4$ vanish, while

$$\lim_{x,y,z \rightarrow 1} \tilde{\mathcal{V}}_1(s, g^2) = \frac{2\pi g^2 \sin(\pi j)}{j(\cos(\pi j) - \cos(2\pi s))}, \quad (5.39)$$

$$\lim_{x,y,z \rightarrow 1} \frac{\tilde{\mathcal{V}}_3(s, g^2)}{\tilde{\mathcal{V}}_1(s, g^2)} = -\frac{1}{2} \left(\psi(s + \frac{j}{2}) + \psi(s - \frac{j}{2}) + \psi(-s + \frac{j}{2}) + \psi(-s - \frac{j}{2}) + 4\gamma_E \right), \quad (5.40)$$

where $j = 2\sqrt{s^2 - g^2}$ as above, and γ_E is the Euler-Mascheroni constant. For $\tilde{\mathcal{V}}_3$ we have dropped the singular logarithm $\ln c$ in this limit, in order to focus on the constant piece. Up to weight 4, these can be expanded explicitly as

$$\lim_{x,y,z \rightarrow 1} \left[\tilde{\mathcal{V}}_i(s, g^2) \right] = \begin{pmatrix} 1 + 2g^2\zeta_2 + (8g^2s^2 - 2g^4)\zeta_4 + \dots \\ 0 \\ (4s^2 - 2g^2)\zeta_3 + \dots \\ 0 \end{pmatrix}. \quad (5.41)$$

In general, we find that the top row and ratio of the first and third can be expanded into even and odd zeta values respectively:

$$\lim_{x,y,z \rightarrow 1} \tilde{\mathcal{V}}_1(s, g^2) = 1 - 2 \sum_{k=1}^{\infty} \zeta_{2k} \sum_{L=1}^k (-g^2)^L (2s)^{2k-2L} \frac{(2k-L-1)!}{(2k-2L)!(L-1)!}, \quad (5.42)$$

$$\lim_{x,y,z \rightarrow 1} \frac{\tilde{\mathcal{V}}_3(s, g^2)}{\tilde{\mathcal{V}}_1(s, g^2)} = 2 \sum_{k=1}^{\infty} \zeta_{2k+1} \sum_{L=0}^k (-g^2)^L (2s)^{2k-2L} \frac{(2k-L-1)!k}{(2k-2L)!L!}. \quad (5.43)$$

Referring back to eq. (5.33), we see that the functions there match eq. (5.41), modulo $\ln c$ terms, in the limit $x, y, z \rightarrow 1$ without the addition of any constants, as needed (remember that the first entry in (5.38) should be compared to the $g^0 s^1$ term in (5.41)).

To get $\tilde{\mathcal{V}}_{4,0}^{(1)}$ and the weight-two contributions to the other functions, we now coact with M on the weight-one vector in (5.38), which we have promoted to a vector of full functions. This gives

$$\Delta_{1,1}\tilde{\mathcal{V}}_i(s, g^2) = \begin{pmatrix} s^2(\ln(xy) \otimes \ln(xy) + \ln z \otimes \ln z) - \frac{g^2}{2}(\ln(xy) \otimes \ln c + \ln c \otimes \ln(xy)) \\ \frac{s}{2}(\ln(xy) \otimes \ln z + \ln z \otimes \ln(xy)) \\ -\frac{s}{2}(\ln c \otimes \ln z + \ln z \otimes \ln c) \\ \frac{1}{2}\left(\ln c \otimes \ln \frac{x}{y} - \ln(xy) \otimes \ln \frac{x-1}{y-1}\right) \end{pmatrix}, \quad (5.44)$$

the fourth component of which is indeed the coproduct of eq. (5.34). The other three entries can be promoted to products of logs, which can be matched to the boundary condition (5.41) to give

$$\begin{aligned} \tilde{\mathcal{V}}_{1,2}^{(1)} &= \frac{1}{2} \ln^2(xy) + \frac{1}{2} \ln^2 z, & \tilde{\mathcal{V}}_{2,1}^{(1)} &= \frac{1}{2} \ln z \ln(xy), & \tilde{\mathcal{V}}_{3,1}^{(1)} &= -\frac{1}{2} \ln z \ln c, \\ \tilde{\mathcal{V}}_{1,0}^{(2)} &= \frac{1}{2} \ln(xy) \ln c - 2\zeta_2. \end{aligned} \quad (5.45)$$

The space of functions generated by this procedure turns out to be one we have already encountered—the space of SVHPLs introduced in eqns. (5.10) and (5.11), where z and \bar{z} are now equal to $1-x$ and $1-y$. For instance, we can rewrite all the functions we have computed above as

$$\begin{aligned} \tilde{\mathcal{V}}_{2,0}^{(1)} &= -\frac{1}{2}\mathcal{L}_1, & \tilde{\mathcal{V}}_{3,0}^{(1)} &= -\frac{1}{2}\mathcal{L}_0, & \tilde{\mathcal{V}}_{1,2}^{(1)} &= \mathcal{L}_{1,1} + \frac{1}{2}\ln^2 z, & \tilde{\mathcal{V}}_{2,1}^{(1)} &= -\frac{1}{2}\mathcal{L}_1 \ln z, \\ \tilde{\mathcal{V}}_{3,1}^{(1)} &= -\frac{1}{2}\mathcal{L}_0 \ln z, & \tilde{\mathcal{V}}_{4,0}^{(1)} &= \frac{1}{2}\mathcal{L}_{0,1} - \frac{1}{2}\mathcal{L}_{1,0}, & \tilde{\mathcal{V}}_{1,0}^{(2)} &= -\frac{1}{2}\mathcal{L}_{0,1} - \frac{1}{2}\mathcal{L}_{1,0} - 2\zeta_2, \end{aligned} \quad (5.46)$$

where we have left the SVHPL arguments $\{z, \bar{z}\} = \{1-x, 1-y\}$ implicit.

We wish to show that the dependence on x and y for arbitrary weight is captured by SVHPLs with arguments $1-x$ and $1-y$. Given the letters $\mathcal{S}_{\text{disc}}$ in eq. (5.7), the main issue is to show that the functions $\tilde{\mathcal{V}}_{i,k}^{(L)}(x, y, z)$ are single-valued at $0, 1, \infty$ in the complex plane for (x, y) . It is sufficient to look at two of the three limits, say where x and y both approach 1 or both approach ∞ . We can probe the first limit with the boundary conditions (5.41), which tell us that the monodromies around this point are dictated by the first and third columns of the matrix M in eq. (5.37). The first and third columns contain only $\ln z$, $\ln(xy)$, $\ln(x/y)$ and $\ln c = \ln[(x-1)(y-1)]$. The first three of these functions are smooth or vanish as $(x, y) \rightarrow (1, 1)$, and the fourth is real analytic (single valued). In other words, the potentially problematic entry $\ln[(x-1)/(y-1)]$ in the second and fourth columns is killed by the boundary condition (5.41). (This boundary condition is for $z = 1$, but the z dependence factorizes.) The single-valuedness at $x, y \rightarrow \infty$ can be seen by considering eqns. (5.28), (5.29), and (5.30), as well as the expansions (5.21) and (5.31), whereby the dependence on x and y in $\text{Disc}_c \mathcal{V}_i(s, g^2)$ takes the form $(xy)^{\pm j}$ times a regular expansion in powers of $1/x$ and $1/y$ in each term. Thus, the functions in $\text{Disc}_c \mathcal{V}_i(s, g^2)$ are single-valued in $1-x$ and $1-y$, making them SVHPLs.

In an ancillary file, `omegacdiscwt0-12.m`, we provide the SVHPL representation of all the c -discontinuity functions $\tilde{\mathcal{V}}_{i,k}^{(L)}(x, y, z)$ through weight 12.

Finally, to return to the ladder integrals themselves, we insert the series expansion of $\text{Disc}_c \mathcal{V}_i(s, g^2)$ in eq. (5.32) and the series expansion of the square root in terms of Catalan numbers,

$$s - \sqrt{s^2 - g^2} = \frac{g^2}{2s} \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n C_n \left(\frac{-g^2}{s^2}\right)^n, \quad (5.47)$$

into eq. (5.26). Performing the contour integral in s by residues at the origin, we obtain an expression for the c -discontinuity of all pentaladder integrals in terms of the $\tilde{\mathcal{V}}_{i,k}^{(L)}$:

$$\text{Disc}_c \{\mathcal{W}^{(L)}, \Omega^{(L)}, \tilde{\Omega}_e^{(L)}, \mathcal{O}^{(L)}\} = \sum_{n=0}^{L-1} \left(-\frac{1}{4}\right)^n C_n \tilde{\mathcal{V}}_{i,2n}^{(L-n)}. \quad (5.48)$$

The chief advantage of the enlarged set of $\tilde{\mathcal{V}}_{i,k}^{(L)}$, as opposed to looking only at the combinations in eq. (5.48), is that this set is closed under the coaction. This allows the $\tilde{\mathcal{V}}_{i,k}^{(L)}$ to be computed recursively in an efficient manner, and will be critical to “undoing” the discontinuity.

5.5 The Ω -functions in the coproduct formalism

Having now constructed a basis of functions describing the c -discontinuities of the Ω system and their derivatives, our next task is to “undo” the discontinuity to get the full functions. As in the box ladder example, the key is that Steinmann hexagon functions without c -discontinuity are extremely constrained.

A function with no c -discontinuity must have first entries in $\{a, b\}$. Using eqs. (B.15) and (B.16) one can check that such functions can only have symbol letters in the set $\{a, b, m_w\}$. Functions of this type were classified in ref. [69]; they are a subset of the functions called K functions there. These K functions can be expressed simply as products of logarithms $\ln^k(a/b)$ and HPLs in m_w . Thus, the c -discontinuity uniquely fixes the coproducts of our functions of interest, up to a few such K functions!

Since all the non-log, harmonic-polylogarithmic behavior of these K functions depends on w , they are naturally probed by values of the i loop integral $\Omega^{(i)}$ on the line where $u = v = 1$, namely $\Omega^{(i)}(1, 1, w)$ as given in eq. (4.38). In fact, with a bit of trial and error, we find that a special combination always occurs,

$$f_k(u, v, w) = - \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{1}{(k-2i)!} \ln^{k-2i}\left(\frac{u}{v}\right) \Omega^{(i)}(1, 1, w), \quad (5.49)$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x , and $f_0(u, v, w) = 1$. Note that $\ln(u/v) = \frac{1}{2} \ln(a/b)$. For the Ω space, we need only two additional transcendental K functions at each weight:

$$\kappa_k = f_k + (1-w)\partial_w f_{k+1}, \quad \tilde{\kappa}_k = -f_k + (1-w)\partial_w f_{k+1}. \quad (5.50)$$

These two functions can be constructed recursively from their nonvanishing coproducts:

$$\kappa_k^a = \kappa_{k-1}, \quad \tilde{\kappa}_k^b = -\tilde{\kappa}_{k-1}, \quad \kappa_k^{m_w} = -\tilde{\kappa}_k^{m_w} = -\frac{1}{2}(\kappa_{k-1} + \tilde{\kappa}_{k-1}), \quad (5.51)$$

together with the boundary condition at $(u, v, w) = (1, 1, 1)$:

$$f_k(1, 1, 1) = \kappa_k(1, 1, 1) = -\tilde{\kappa}_k(1, 1, 1) = \begin{cases} 0 & k \text{ odd,} \\ -\Omega^{(k/2)}(1, 1, 1) = (2k-2)\zeta_k & k \text{ even.} \end{cases} \quad (5.52)$$

For $k = 0$, $\kappa_0 = -\tilde{\kappa}_0 = 1$, since $f_0(1, 1, 1) = 1$ and the derivatives $(1-w)\partial_w f_{k+1}$ vanish uniformly at this point.

The functions $\mathcal{V}_{i,k}^{(L)} \equiv \{\mathcal{W}_k^{(L)}, \Omega_k^{(L)}, \tilde{\Omega}_{e,k}^{(L)}, \mathcal{O}_k^{(L)}\}$ are defined for $k \geq 0$ and $L \geq 1$ (if $k < 0$ or $L < 1$, they are set to zero, with the exception of $\Omega_0^{(0)} \equiv 1$). Given the formula (5.35) for the weight of these functions, the complete set of functions appearing at weight n is:

$$\begin{aligned} & \kappa_n, \quad \tilde{\kappa}_n, \\ & \mathcal{W}_{n+1-2L}^{(L)}, \quad L = 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor, \\ & \Omega_{n-2L}^{(L)}, \quad L = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor, \\ & \tilde{\Omega}_{e,n-2L}^{(L)}, \quad L = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor, \\ & \mathcal{O}_{n-1-2L}^{(L)}, \quad L = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor. \end{aligned} \quad (5.53)$$

The dimension of the space is

$$2 + \lfloor \frac{n+1}{2} \rfloor + 2 \times \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor = 2n + 1. \quad (5.54)$$

The coproducts of the $\mathcal{V}_{i,k}^{(L)}$ functions contain two types of terms. The first type involves other functions in \mathcal{V} ; they are determined by the matrix (5.37), which in the alphabet (5.5) reads

$$M(s, g^2) = \begin{pmatrix} s \ln \frac{m_v}{m_u} & -g^2 \ln c + 2s^2 \ln(m_u m_v) & g^2 \ln(m_u m_v) & 0 \\ \frac{1}{2} \ln(m_u m_v) & s \ln \frac{m_v}{m_u} & 0 & -\frac{g^2}{2} \ln(y_u y_v) \\ -\frac{1}{2} \ln c & 0 & s \ln \frac{m_v}{m_u} & -\frac{g^2}{2} \ln(y_u y_v y_w) + s^2 \ln(y_u y_v) \\ 0 & \ln(y_u y_v y_w) & \ln(y_u y_v) & s \ln \frac{m_v}{m_u} \end{pmatrix}. \quad (5.55)$$

The second type of terms involves κ and $\tilde{\kappa}$, and we have determined them by solving integrability conditions. (By ‘‘integrability’’ here we refer to the commutativity of partial derivatives, not to be confused with quantum integrability.)

Given the nonvanishing coproducts of these functions, one can define the Ω system recursively. The most complicated of these involve m_u and m_v for the even functions, and y_u

and y_v for the odd functions:

$$\mathcal{W}_k^{(L)m_u} = -\mathcal{W}_{k-1}^{(L)} + 2\Omega_{k-2}^{(L)} - \tilde{\Omega}_{e,k}^{(L-1)} + c_k^{(L-2)} \tilde{\kappa}_{2L+k-2}, \quad (5.56)$$

$$\Omega_k^{(L)m_u} = -\Omega_{k-1}^{(L)} + \frac{1}{2}\mathcal{W}_k^{(L)} + c_k^{(L-1)} \tilde{\kappa}_{2L+k-1}, \quad (5.57)$$

$$\tilde{\Omega}_{e,k}^{(L)m_u} = -\tilde{\Omega}_{e,k-1}^{(L)} + c_{k-1}^{(L)} \tilde{\kappa}_{2L+k-1}, \quad (5.58)$$

$$\mathcal{O}_k^{(L)y_u} = \Omega_k^{(L)} + \tilde{\Omega}_{e,k}^{(L)} + c_k^{(L)} \tilde{\kappa}_{2L+k}, \quad (5.59)$$

and

$$\mathcal{W}_k^{(L)m_v} = \mathcal{W}_{k-1}^{(L)} + 2\Omega_{k-2}^{(L)} - \tilde{\Omega}_{e,k}^{(L-1)} - c_k^{(L-2)} \kappa_{2L+k-2} \quad (5.60)$$

$$\Omega_k^{(L)m_v} = \Omega_{k-1}^{(L)} + \frac{1}{2}\mathcal{W}_k^{(L)} + c_k^{(L-1)} \kappa_{2L+k-1} \quad (5.61)$$

$$\tilde{\Omega}_{e,k}^{(L)m_v} = \tilde{\Omega}_{e,k-1}^{(L)} + c_{k-1}^{(L)} \kappa_{2L+k-1} \quad (5.62)$$

$$\mathcal{O}_k^{(L)y_v} = \Omega_k^{(L)} + \tilde{\Omega}_{e,k}^{(L)} - c_k^{(L)} \kappa_{2L+k}. \quad (5.63)$$

The first three terms on the first line of these, for example, come from the m_u and m_v terms in the first row of eq. (5.55). The remaining nonvanishing coproducts are then:

$$\mathcal{W}_k^{(L)c} = \Omega_k^{(L-1)}, \quad \Omega_k^{(L)y_u} = \Omega_k^{(L)y_v} = \frac{1}{2}\mathcal{O}_k^{(L-1)}, \quad (5.64)$$

$$\tilde{\Omega}_{e,k}^{(L)c} = -\frac{1}{2}\mathcal{W}_k^{(L)}, \quad \tilde{\Omega}_{e,k}^{(L)y_u} = \tilde{\Omega}_{e,k}^{(L)y_v} = \frac{1}{2}\mathcal{O}_k^{(L-1)} + \mathcal{O}_{k-2}^{(L)}, \quad (5.65)$$

$$\tilde{\Omega}_{e,k}^{(L)m_w} = \frac{1}{2}c_k^{(L-1)} (\kappa_{2L+k-1} + \tilde{\kappa}_{2L+k-1}), \quad \tilde{\Omega}_{e,k}^{(L)y_w} = \frac{1}{2}\mathcal{O}_k^{(L-1)}, \quad (5.66)$$

$$\mathcal{O}_k^{(L)m_u} = -\mathcal{O}_k^{(L)m_v} = -\mathcal{O}_{k-1}^{(L)}, \quad \mathcal{O}_k^{(L)y_w} = \Omega_k^{(L)} - c_k^{(L)} (\kappa_{2L+k} - \tilde{\kappa}_{2L+k}). \quad (5.67)$$

The binomial coefficients $c_k^{(L)}$, which intertwine the \mathcal{V} and κ systems, are:

$$c_k^{(L)} = 2^{k-1} \binom{k+L-1}{k}. \quad (5.68)$$

There are a few exceptional cases at low weights:

$$c_1^{(-1)} = -1, \quad c_0^{(0)} = \frac{1}{2}; \quad \text{otherwise } c_k^{(L)} = 0 \text{ for } k < 0 \text{ or } L < 1. \quad (5.69)$$

This concludes the complete recursive definition of the Ω space of functions to all weights.

Remarkably, when evaluated on the line $(1, 1, w)$, *every* function in the Ω space approaches an integer multiple of either $\Omega^{(m)}(1, 1, w)$ (for even weight $2m$) or $(1-w)d\Omega^{(m)}(1, 1, w)/dw$ (for odd weight $2m-1$). The integer multiples are given by the binomial coefficients $c_k^{(L)}$:

$$\begin{aligned} \kappa_k &\rightarrow -1, \quad \tilde{\kappa}_k \rightarrow (-1)^k \\ \{\mathcal{W}_k^{(L)}, \Omega_k^{(L)}, \tilde{\Omega}_{e,k}^{(L)}, \mathcal{O}_k^{(L)}\} &\rightarrow \{0, 0, 0, 0\}, \quad k \text{ odd,} \\ \{\mathcal{W}_k^{(L)}, \Omega_k^{(L)}, \tilde{\Omega}_{e,k}^{(L)}, \mathcal{O}_k^{(L)}\} &\rightarrow \{-c_{k+1}^{(L-2)}, c_{k+1}^{(L-1)}, c_k^{(L)}, c_{k+1}^{(L)}\}, \quad k \text{ even.} \end{aligned} \quad (5.70)$$

In other words, we can fix the boundary conditions for the coproduct description along the entire line $(1, 1, w)$, not just at the point $(1, 1, 1)$ given in eqs. (4.37), (4.39) and (4.48).

Finally, let us be explicit as to how the actual ladder integrals sit inside this basis, as Catalan-weighted sums along the same lines as eq. (5.48):

$$\{\mathcal{W}^{(L)}, \Omega^{(L)}, \tilde{\Omega}_e^{(L)}, \mathcal{O}^{(L)}\} = \sum_{n=0}^{L-1} \left(-\frac{1}{4}\right)^n C_n \{\mathcal{W}_{2n}^{(L-n)}, \Omega_{2n}^{(L-n)}, \tilde{\Omega}_{e,2n}^{(L-n)}, \mathcal{O}_{2n}^{(L-n)}\}, \quad (5.71)$$

and

$$\tilde{\Omega}_o^{(L)} = -\sum_{n=1}^{L-1} \left(-\frac{1}{4}\right)^n C_n \mathcal{O}_{2n-1}^{(L-n)}. \quad (5.72)$$

Note that $\tilde{\Omega}_e^{(1)}$ is not pure, so we should use eq. (3.30), not eq. (5.71) for that case. Also, the first two instances of $\mathcal{W}^{(L)}$ and the first instance of $\Omega^{(L)}$ are exceptional, needing additional κ contributions:

$$\mathcal{W}^{(1)} = \mathcal{W}_0^{(1)} + v \kappa_1 + u \tilde{\kappa}_1, \quad \mathcal{W}^{(2)} = \mathcal{W}_0^{(2)} - \frac{1}{4} \mathcal{W}_2^{(1)} + \frac{1}{2}(\kappa_3 + \tilde{\kappa}_3), \quad (5.73)$$

$$\Omega^{(1)} = \Omega_0^{(1)} + \frac{1}{2}(\tilde{\kappa}_2 - \kappa_2). \quad (5.74)$$

Apart from these exceptions, eqs. (5.71) and (5.72) locate the five integrals per loop perfectly inside the Ω space for all $L \geq 1$.

The space of Ψ functions for the pentabox ladders has an analogous description, which is not surprising since the Ψ integral is obtained from the Ω integral by letting $w \rightarrow 0$, $\Psi^{(L)}(u, v) = \Omega^{(L)}(u, v, 0)$. However, not all of the integrals are nonsingular in this limit. Instead of the $(2n+1)$ -dimensional space (5.53) at weight n , the following subspace survives,

$$\begin{aligned} \hat{\kappa}_n &\equiv \frac{1}{2}(\kappa_n - \tilde{\kappa}_n), \\ \hat{\mathcal{W}}_{n+1-2L}^{(L)} &\equiv \mathcal{W}_{n+1-2L}^{(L)} - \mathcal{O}_{n+1-2L}^{(L-1)} + c_{n+1-2L}^{(L-1)}(\kappa_n + \tilde{\kappa}_n), \quad L = 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor, \\ \hat{\Omega}_{n-2L}^{(L)} &\equiv \Omega_{n-2L}^{(L)}, \quad L = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor, \end{aligned} \quad (5.75)$$

with a total dimension of

$$1 + \lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor = n + 1. \quad (5.76)$$

Note that, while this dimension matches the size of the box ladder space, these spaces cannot be isomorphic because they involve a different number of symbol letters.

The 4×4 matrix M collapses to a 2×2 matrix \hat{M} acting on $(\hat{W}, \hat{\Omega})$:

$$\hat{M}(s, g^2) = \begin{pmatrix} s \ln z & -g^2 \ln(1-x) + s^2 \ln x \\ \frac{1}{2} \ln x & s \ln z \end{pmatrix}. \quad (5.77)$$

The reduction to a two-dimensional matrix occurs because there is only a single hypergeometric function of x in the finite-coupling formula (3.61) for Ψ , in contrast to the product of functions of x and y in the corresponding formula for Ω .

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$L = 2$	1	3	5	5	3	1	–	–	–	–	–	–	–	–
$L = 2, \text{P odd}$	0	0	0	1	1	1	–	–	–	–	–	–	–	–
$L = 3$	1	3	5	7	7	5	3	1	–	–	–	–	–	–
$L = 3, \text{P odd}$	0	0	0	1	1	2	1	1	–	–	–	–	–	–
$L = 4$	1	3	5	7	9	9	7	5	3	1	–	–	–	–
$L = 4, \text{P odd}$	0	0	0	1	1	2	2	2	1	1	–	–	–	–
$L = 5$	1	3	5	7	9	11	11	9	7	5	3	1	–	–
$L = 5, \text{P odd}$	0	0	0	1	1	2	2	3	2	2	1	1	–	–
$L = 6$	1	3	5	7	9	11	13	13	11	9	7	5	3	1
$L = 6, \text{P odd}$	0	0	0	1	1	2	2	3	3	3	2	2	1	1

Table 2. The dimensions of the spaces of $\{n, 1, \dots, 1\}$ coproducts of the odd ladder integral $\mathcal{O}^{(L)}$, and also that of the parity odd subspace. They are both palindromic sequences.

We remark that the perturbative Ω space is much smaller at each weight than what would be obtained solely by imposing proper branch cuts (at weight one) and the constraints of the Steinmann relations at weight two. As we describe in appendix C, there are additional constraints on pairs of adjacent letters in the Ω space, that are reminiscent of the Steinmann relations. In appendix B, we mention that there are similar “extended Steinmann relations” that apply to the more general space of hexagon functions [74], and are related to the cluster adjacency principle [115].

5.6 Other Ω space properties and embedding into hexagon function space

Although we have given a complete construction of the Ω space in the previous subsection, we can also ask how many functions can be obtained just as coproducts of a single function. This enumeration was useful for our initial understanding of the Ω space, before the above construction was discovered.

In particular, we can examine all the coproducts of the L loop odd ladder integral $\mathcal{O}^{(L)}$. Of the five ladder integrals at L loops, it has the highest weight, $2L + 1$. We iteratively construct all of the $\{n, 1, 1, \dots, 1\}$ coproducts of $\mathcal{O}^{(L)}$ at weight n . These coproducts are highly degenerate, so we only keep the linearly independent span of them at each weight. Then we differentiate each of those functions to go to the next lower weight, and again keep only the linearly independent ones, and so on. The results for the dimensions of these spaces, and for just the parity-odd subspaces, are tabulated in table 2 for each $L \leq 6$.

Table 2 shows a few interesting properties. First of all, the dimensions are “palindromic”: The number of independent functions increases by two with each successive differentiation,

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Ω space dimension	1	3	5	7	9	11	13	15	17	19	21	23	25	27
P even	1	3	5	6	8	9	11	12	14	15	17	18	20	21
P even, K	1	3	5	6	6	6	6	6	6	6	6	6	6	6
P odd	0	0	0	1	1	2	2	3	3	4	4	5	5	6

Table 3. Dimension of the full weight- n Ω space, and decomposed into even and odd subspaces under parity P. In the P-even sector we also list the number of K functions, which have no y_u, y_v, y_w letters in their symbol.

tracing the odd natural numbers, until it peaks and then declines again by two at each step, tracing out the same set of numbers.⁸ The same palindromic property holds for just the subspace that is odd under parity P, although the peak position is shifted up in weight.

Secondly, once the number of functions has reached its peak for a given L , the dimensions for weights below that peak equal the dimension of the full Ω space at that weight. We say that the space of coproducts becomes “saturated” below a given weight $n_s^{e,o}(L)$, which depends on whether the parity is even (e) or odd (o). Higher loop orders do not give additional functions for weight $n \leq n_s^{e,o}(L)$, and the dimension $n_\Omega(n)$ of the full Ω space can be read off. From table 2, we see that the dimensions saturate for even (or all) and odd functions at

$$n_s^e(L) = L, \quad n_s^o(L) = L + 2. \quad (5.78)$$

The total number of Ω functions at weight n is seen to be $n_\Omega(n) = 2n + 1$, matching the number in eq. (5.54).

In table 3 we list the dimensions of the even and odd subspaces for weight $n \leq 13$. Parity-odd functions necessarily contain the parity-odd letters $y_i \equiv \{y_u, y_v, y_w\}$ in their symbols. The P-even subspace has a further subspace of “ K ” functions [69] whose symbols contain *no* parity-odd letters. These functions are simply HPLs with arguments $1 - 1/u$, $1 - 1/v$, and $1 - 1/w$ in some cases combined with logarithms. We have already identified two of them, κ_n and $\tilde{\kappa}_n$, but there are four more “secret” K functions in the Ω space, for a total of six at each weight n (for $n > 3$):

$$\{\kappa_n, \tilde{\kappa}_n, \mathcal{W}_{n-1}^{(1)}, \mathcal{W}_{n-3}^{(2)}, \Omega_{n-2}^{(1)}, \Omega_{n-4}^{(2)} - \frac{1}{2}\tilde{\Omega}_{e,n-2}^{(1)}\}. \quad (5.79)$$

⁸The space of coproducts of the L -loop box ladder integral, which lives in the space enumerated in table 1, has the same palindromic property, except using all natural numbers instead of just the odd ones.

weight n	0	1	2	3	4	5	6	7	8
\mathcal{H} , P even, K	1	3	6	12	22	39	67	114	190
Ω_{cyc} , P even, K	1	3	6	12	18	16	15	15	15
\mathcal{H} , P even, non- K	0	0	0	0	3	9	25	56	123
Ω_{cyc} , P even, non- K	0	0	0	0	3	8	15	18	24
\mathcal{H} , P odd	0	0	0	1	2	6	13	30	59
Ω_{cyc} , P odd	0	0	0	1	2	6	6	9	9

Table 4. Dimension of the full weight- n hexagon function space \mathcal{H} , decomposed into even and odd subspaces under parity P, and compared with the corresponding dimensions for Ω_{cyc} . In the P-even sector we divide the space into non- K and K functions.

In terms of HPLs, the four secret K functions at weight n are

$$\begin{aligned}
& H_n\left(1 - \frac{1}{u}\right), \quad H_n\left(1 - \frac{1}{v}\right), \\
& \ln \frac{v}{w} H_{n-1}\left(1 - \frac{1}{u}\right) + \sum_{i=1}^{n-2} H_{i,n-i}\left(1 - \frac{1}{u}\right), \\
& \ln \frac{u}{w} H_{n-1}\left(1 - \frac{1}{v}\right) + \sum_{i=1}^{n-2} H_{i,n-i}\left(1 - \frac{1}{v}\right). \tag{5.80}
\end{aligned}$$

We have referred to the Ω space as a prototype or model for the full space of Steinmann hexagon functions \mathcal{H} . How many of the functions in \mathcal{H} does it capture or miss, as we go up in weight? The Ω space has a particular orientation, while \mathcal{H} is closed under all permutations of (u, v, w) . We define Ω_{cyc} to also include cyclic permutations of the Ω space functions under $(u, v, w) \rightarrow (v, w, u)$ and $(u, v, w) \rightarrow (w, u, v)$. For the most part, these permuted functions are independent. However, the top line of eq. (5.80) has two K functions, which after including cyclic permutations, become only three K functions in all, $H_n(1 - 1/u_i)$, $i = 1, 2, 3$. So we lose three K functions at each weight, and the number of K functions in Ω_{cyc} is $3 \times 6 - 3 = 15$ at weight 6 and above. There is a similar degeneracy under cyclic permutation for the few parity-odd functions at weights 3 and 4, and for the non- K parity-even functions at weights 4 and 5. (In the latter case, certain linear combinations of cyclic permutations of non- K functions are actually K functions.) Beyond weight 5 there are no non- K degeneracies, and so the dimension of the weight n part of Ω_{cyc} is $3(2n + 1) - 3 = 6n$ for $n \geq 6$. This dimension grows only linearly with n , whereas the dimension of the weight n part of \mathcal{N} grows much faster, roughly like 1.7^n .

In table 4 we compare the dimensions of the full hexagon function space \mathcal{H} , which has been trimmed to remove all inessential functions [74], with the dimensions of Ω_{cyc} through weight 8. We have split the functions into P even and P odd, and we have further split

the P-even functions into K functions and non- K functions. The space Ω_{cyc} spans the full hexagon function space through weight 3. At weight 4 it only misses 4 K functions, one of which is the constant ζ_4 . At weight 5 it misses numerous K functions, and a single P-even, non- K function, but it still captures all the P-odd functions at weight 5. The single missing P-even, non- K function evaluates to $5\zeta_5 - 2\zeta_2\zeta_3$ at $(u, v, w) = (1, 1, 1)$, while the weight 5 part of Ω_{cyc} vanishes at $(1, 1, 1)$. Presumably this weight 5 function is a \mathcal{W} -like seed for non-ladder DCI integral topologies beginning at three loops, in which three pentagons with appropriate numerators are joined together at a common vertex.

5.7 A nonperturbative coaction

That the dimensions of the Ω space saturate has an interesting implication: it allows us to define the coaction nonperturbatively. This can be illustrated by returning to the discontinuity functions $\text{Disc}_c \mathcal{V}_i(s, g^2)$, defined nonperturbatively in eq. (5.28). Using the differential equation these functions satisfy, we see they can also be defined by a path-ordered exponential, where the argument of the exponential is a 4×4 matrix:

$$\mathcal{U}_{ij}(s, g^2; x, y, z) = \left[\mathcal{P} \exp \left(\int_{(1,1,1)}^{(x,y,z)} dM^T(s, g^2; x, y, z) \right) \right]_{ij}. \quad (5.81)$$

Dotting \mathcal{U}_{ij} on the left with the initial condition in eq. (5.41) reproduces the vector $\text{Disc}_c \mathcal{V}_i$, but this matrix contains additional transcendental functions. Roughly speaking, we expect this space of functions to be necessary and sufficient to describe all the possible analytic continuations of the $\tilde{\mathcal{V}}_i$. A nonperturbative coaction can then be defined simply as a matrix product:

$$\Delta \mathcal{U}_{ij}(s, g^2; x, y, z) = \sum_k [\mathcal{U}_{ik}(s, g^2; x, y, z)] \otimes [\mathcal{U}_{kj}(s, g^2; x, y, z)]. \quad (5.82)$$

(See refs. [116, 117] for related constructions also involving hypergeometric functions.) The $\Delta_{\bullet,1}$ coproduct component from eq. (5.82) is easily seen to reproduce eq. (5.36), and perturbatively the $\Delta_{1,\dots,1}$ components reproduce the symbol of these functions. However, this equation defines Δ nonperturbatively, and in particular for all $\Delta_{\bullet,k}$ and $\Delta_{k,\bullet}$ for $k \geq 1$. (We haven't discussed boundary conditions, and in principle the definition in eq. (5.81) might need to be "twisted" by ζ -valued constants so as to match the ordinary coproduct of polylogarithms. We leave this exploration to future work.)

Note that, by construction, the coaction (5.82) satisfies a coaction principle [87–89]. That is, the first entry of the coaction is always contained within the original space of functions. A similar (perturbative) coaction principle has been observed in the full space of Steinmann hexagon functions, where it constrains the transcendental constants that can appear in the first entry in addition to restricting the symbols of these objects [69, 88]. The consequences of this coaction principle thus extend beyond what is currently understood in terms of physical principles (since only the symbol-level constraints are understood in terms of allowed branch cuts of Feynman integrals), as will be described in more detail in a forthcoming work [74].

The first entry of the coaction (5.82) manifestly realizes this same property. In particular, the first entry of the coaction maps to the same space of functions for any initial conditions one dots into \mathcal{U}_{ij} (after which the coaction principle more closely resembles those discussed in [88], since the second entry of the coaction will in general map to a larger space).

The nonperturbative coaction for the Ω system can be defined in an analogous fashion, writing in matrix form the general solution to the differential equations following from eqs. (5.56)–(5.69). The appearance of the κ and $\tilde{\kappa}$ functions in this space implies that \mathcal{U} will now appear as a 4×4 sub-block of a bigger matrix.

It is remarkable that the set of all coproducts of the Ω integrals (loosely speaking, the space of all their possible derivatives and analytic continuations) can be encoded nonperturbatively in a single matrix.

6 Conclusions

We have investigated a class of integrals, $\Omega^{(L)}$ and $\tilde{\Omega}^{(L)}$, that have representatives at each loop order. In doing so, we have found something remarkable: that their all-orders behavior can be expressed in terms of beautifully simple integral formulas, given in eqs. (3.26) and (3.32). Using these expressions, we can extract any desired loop coefficient, and have control over the full behavior of the functions via infinite sums.

We have also investigated the coproducts of these functions to all orders, allowing us to characterize the complete space of polylogarithms that envelops the $\Omega^{(L)}$ and $\tilde{\Omega}^{(L)}$ integrals and their derivatives. As a consequence, we can now efficiently construct a subspace of the Steinmann hexagon functions to arbitrarily high weight. This space is equipped with a coaction both perturbatively and at finite coupling, and obeys a coaction principle.

The inevitable next question is, can we characterize the full Steinmann hexagon space \mathcal{H} in a similar way? For example, can we find a systematic definition of the hexagon function coproducts, analogous to eqs. (5.56)–(5.64), which solves the integrability conditions to all orders? Or perhaps are there other subspaces of \mathcal{H} , larger than Ω , that we can describe to all orders, that capture a wider set of functions that are not in Ω_{cyc} ? Does the amplitude itself have a form like the $\Omega^{(L)}$ and $\tilde{\Omega}^{(L)}$ integrals, and could it be written as a finite-coupling expression involving (several) Mellin integrals? We suspect that this might be possible, although presumably it will have to include the full flux-tube dispersion relations [43, 49].

One reason for suspecting this is that the radius of convergence of the perturbative expansion of the Ω integrals appears to be much larger than that for amplitudes, as discussed in section 4.4. The radius of convergence for amplitudes is relatively close to that for the cusp anomalous dimension, which also controls the behavior of the flux-tube expansion at finite excitation number. Thus it seems likely that the large-order behavior of six-point amplitudes is controlled by other families of integrals that grow more quickly than the ladders.

In the past, several of the authors have observed differential equations linking the amplitude at different loop orders [69, 73]. While some of these relations do not hold at higher

orders [74] they still suggest that a larger piece of the Steinmann hexagon space has an iterative or recursive structure which awaits exploitation.

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A Hexagon Variables

The three cross ratios u, v, w used to describe hexagon functions are,

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad v = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2}, \quad w = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2}. \quad (\text{A.1})$$

The hexagon-function symbol alphabet is given by

$$\mathcal{S}_{\text{hex}} = \{u, v, w, 1 - u, 1 - v, 1 - w, y_u, y_v, y_w\} \quad (\text{A.2})$$

where [60]

$$y_u = \frac{u - z_+}{u - z_-}, \quad y_v = \frac{v - z_+}{v - z_-}, \quad y_w = \frac{w - z_+}{w - z_-}, \quad (\text{A.3})$$

and

$$z_{\pm} = \frac{1}{2} \left[-1 + u + v + w \pm \sqrt{\Delta} \right], \quad \Delta = (1 - u - v - w)^2 - 4uvw. \quad (\text{A.4})$$

The cross ratios u, v, w are rational in terms of y_u, y_v, y_w ,

$$u = \frac{y_u(1 - y_v)(1 - y_w)}{(1 - y_u y_v)(1 - y_u y_w)}, \quad v = \frac{y_v(1 - y_w)(1 - y_u)}{(1 - y_v y_w)(1 - y_v y_u)}, \quad w = \frac{y_w(1 - y_u)(1 - y_v)}{(1 - y_w y_u)(1 - y_w y_v)},$$

$$1 - u = \frac{(1 - y_u)(1 - y_u y_v y_w)}{(1 - y_u y_v)(1 - y_u y_w)}, \quad \text{etc.}, \quad \sqrt{\Delta} = \frac{(1 - y_u)(1 - y_v)(1 - y_w)(1 - y_u y_v y_w)}{(1 - y_u y_v)(1 - y_v y_w)(1 - y_w y_u)}. \quad (\text{A.5})$$

The corresponding momentum-twistor representations are

$$\begin{aligned} u &= \frac{\langle 6123 \rangle \langle 3456 \rangle}{\langle 6134 \rangle \langle 2356 \rangle}, & 1 - u &= \frac{\langle 6135 \rangle \langle 2346 \rangle}{\langle 6134 \rangle \langle 2356 \rangle}, \\ y_u &= \frac{\langle 1345 \rangle \langle 2456 \rangle \langle 1236 \rangle}{\langle 1235 \rangle \langle 3456 \rangle \langle 1246 \rangle}, & \sqrt{\Delta} &= \frac{\langle 1234 \rangle \langle 1256 \rangle \langle 3456 \rangle - \langle 2345 \rangle \langle 1236 \rangle \langle 1456 \rangle}{\langle 2356 \rangle \langle 1346 \rangle \langle 1245 \rangle}. \end{aligned} \quad (\text{A.6})$$

The representations for v , $1 - v$, y_v , and so on, can be obtained by cycling $Z_i \rightarrow Z_{i+1}$, remembering that under this transformation $u \rightarrow v \rightarrow w \rightarrow u$, while $y_u \rightarrow 1/y_v \rightarrow y_w \rightarrow 1/y_u$.

As discussed in section 3, for many purposes, a better set of variables for the Ω integrals is $\{x, y, z\}$, where

$$x = 1 + \frac{1 - u - v - w + \sqrt{\Delta}}{2uv} = \frac{1 - y_u y_v y_w}{y_u y_v (1 - y_w)}, \quad (\text{A.7})$$

$$y = 1 + \frac{1 - u - v - w - \sqrt{\Delta}}{2uv} = \frac{1 - y_u y_v y_w}{(1 - y_w)}, \quad (\text{A.8})$$

$$z = \frac{u(1 - v)}{v(1 - u)} = \frac{y_u(1 - y_v)^2}{y_v(1 - y_u)^2}. \quad (\text{A.9})$$

The inverse relations are,

$$\begin{aligned} u &= \frac{1}{1 + \sqrt{xy/z}}, & v &= \frac{1}{1 + \sqrt{xyz}}, & w &= \frac{(1 - x)(1 - y)}{(1 + \sqrt{xy/z})(1 + \sqrt{xyz})}, \\ y_u &= \frac{1 + \sqrt{y/(xz)}}{1 + \sqrt{x/(yz)}}, & y_v &= \frac{1 + \sqrt{yz/x}}{1 + \sqrt{xz/y}}, & y_w &= \frac{x(1 - y)}{y(1 - x)}. \end{aligned} \quad (\text{A.10})$$

Notice that x and y depend only on $y_u y_v$ and y_w . That is, the only dependence on y_u/y_v is through z .

Parity sends the dual coordinates $x_i \rightarrow x_{i+3} \pmod{6}$ and the momentum twistors $Z_i \rightarrow Z_{i+3} \pmod{6}$. Parity does not affect the cross ratios u, v, w but it exchanges $\sqrt{\Delta} \leftrightarrow -\sqrt{\Delta}$, so that $z_+ \leftrightarrow z_-$ and y_u, y_v, y_w are inverted: $y_i \leftrightarrow 1/y_i$. Eqs. (A.7)–(A.10) show that parity exchanges x and y , leaving z invariant.

The u derivative of a function F , holding v, w fixed, is given in terms of first coproducts by

$$\frac{\partial F}{\partial u} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1 - u - v - w}{u\sqrt{\Delta}} F^{y_u} + \frac{1 - u - v + w}{(1-u)\sqrt{\Delta}} F^{y_v} + \frac{1 - u + v - w}{(1-u)\sqrt{\Delta}} F^{y_w}, \quad (\text{A.11})$$

and derivatives with respect to v and w are obtained by cyclic permutations of this equation. Derivatives with respect to x, y, z are related to these derivatives by,

$$x\partial_x + y\partial_y = -u(1-u)\partial_u - v(1-v)\partial_v + (1-u-v)(1-w)\partial_w, \quad (\text{A.12})$$

$$x\partial_x - y\partial_y = -\sqrt{\Delta}\partial_w, \quad (\text{A.13})$$

$$z\partial_z = \frac{1}{2} \left[u(1-u)\partial_u - v(1-v)\partial_v - w(u-v)\partial_w \right]. \quad (\text{A.14})$$

In discussing the Ω space of functions in section 5, it is very useful to change the hexagon alphabet from \mathcal{S}_{hex} in eq. (A.2) to

$$\mathcal{S}'_{\text{hex}} = \{a, b, c, m_u, m_v, m_w, y_u, y_v, y_w\}, \quad (\text{A.15})$$

where

$$a = \frac{u}{vw}, \quad b = \frac{v}{wu}, \quad c = \frac{w}{uv}, \quad m_u = \frac{1-u}{u}, \quad m_v = \frac{1-v}{v}, \quad m_w = \frac{1-w}{w}. \quad (\text{A.16})$$

Given coproducts labelled using $\mathcal{S}'_{\text{hex}}$, we can convert them to those using \mathcal{S}_{hex} , by

$$F^u = F^a - F^b - F^c - F^{m_u}, \quad F^{1-u} = F^{m_u}, \quad (\text{A.17})$$

plus the relations obtained by cyclic permutations. To go in the opposite direction, we use,

$$F^a = -\frac{1}{2}(F^v + F^{1-v} + F^w + F^{1-w}), \quad F^{m_u} = F^{1-u}, \quad (\text{A.18})$$

plus the cyclic relations.

For example, the z derivative, using eq. (A.14) and expressed in terms of coproducts using the alphabet $\mathcal{S}'_{\text{hex}}$, is

$$z \frac{\partial F}{\partial z} = (1-v)F^a - (1-u)F^b - \frac{1}{2}(F^{m_u} - F^{m_v}) + \frac{u-v}{2(1-w)}F^{m_w} + \frac{\sqrt{\Delta}}{2(1-w)}(F^{y_u} - F^{y_v}). \quad (\text{A.19})$$

The x and y derivatives are more complicated, but are a bit more simply expressed in terms of the y_i variables and using the following combinations with opposite parity:

$$(x\partial_x - y\partial_y)F = \frac{(1-y_w)(1-y_u y_v y_w)}{y_w(1-y_u y_v)}(F^a + F^b - F^c) + \frac{(1-y_u y_w)(1-y_v y_w)}{y_w(1-y_u y_v)}F^{m_w} - F^{y_u} - F^{y_v} + \frac{y_u - y_v}{1-y_u y_v}(F^{y_u} - F^{y_v}) - \frac{(1-y_w)(1+y_u y_v y_w)}{y_w(1-y_u y_v)}F^{y_w}, \quad (\text{A.20})$$

$$(x\partial_x + y\partial_y)F = -\frac{(1+y_v y_w)(1-y_w)(1-y_u y_v y_w)}{y_w(1-y_u y_v)(1-y_v y_w)}F^a - \frac{(1+y_u y_w)(1-y_w)(1-y_u y_v y_w)}{y_w(1-y_u y_v)(1-y_u y_w)}F^b + \frac{(1+y_w)(1-y_u y_v y_w)}{y_w(1-y_u y_v)}F^c + F^{m_u} + F^{m_v} - \frac{1-y_u y_v y_w^2}{y_w(1-y_u y_v)}F^{m_w} + \frac{(1-y_w)(1-y_u y_v y_w)}{y_w(1-y_u y_v)}F^{y_w}. \quad (\text{A.21})$$

B Extended Steinmann Relations in the Full Hexagon Function Space

In this appendix we discuss properties of adjacent symbol entries in the full hexagon function space \mathcal{H} , as a prelude to a similar discussion for the Ω space and its c -discontinuity in the following appendix.

One advantage of the alphabet $\mathcal{S}'_{\text{hex}}$ is that the Steinmann relations are made transparent, insofar as each letter a, b, c contains a unique three-particle invariant [69]:

$$a = \frac{x_{13}^2 x_{46}^2}{x_{24}^2 x_{35}^2 x_{51}^2 x_{62}^2} (x_{25}^2)^2, \quad b = \frac{x_{24}^2 x_{51}^2}{x_{35}^2 x_{46}^2 x_{62}^2 x_{13}^2} (x_{36}^2)^2, \quad c = \frac{x_{35}^2 x_{62}^2}{x_{46}^2 x_{51}^2 x_{13}^2 x_{24}^2} (x_{41}^2)^2. \quad (\text{B.1})$$

A similar simplification occurs in the heptagon letters used in [63, 75], where each three-particle invariant only appears in a single letter a_{1j} . Thus the Steinmann constraints [69],

$$\text{Disc}_{x_{25}^2} (\text{Disc}_{x_{36}^2} A_6) = 0, \quad (\text{B.2})$$

and permutations thereof, are solved (at symbol level) simply by requiring that the first two entries of the symbol do *not* contain any of the six combinations,

$$\begin{aligned} a \otimes b \otimes \dots, \quad b \otimes c \otimes \dots, \quad c \otimes a \otimes \dots, \\ b \otimes a \otimes \dots, \quad c \otimes b \otimes \dots, \quad a \otimes c \otimes \dots \end{aligned} \quad (\text{B.3})$$

However, by examining the double coproducts of functions obtained by taking multiple coproducts of high loop six-point amplitudes, we have found that the same constraint also holds deeper into the symbol. That is, the combinations

$$\begin{aligned} \dots \otimes a \otimes b \otimes \dots, \quad \dots \otimes b \otimes c \otimes \dots, \quad \dots \otimes c \otimes a \otimes \dots, \\ \dots \otimes b \otimes a \otimes \dots, \quad \dots \otimes c \otimes b \otimes \dots, \quad \dots \otimes a \otimes c \otimes \dots \end{aligned} \quad (\text{B.4})$$

never appear [74]. We refer to this condition as the extended Steinmann constraints.

There are also 26 independent constraints on double coproducts from function-level integrability. Expressed in the alphabet \mathcal{S}_{hex} , they are contained in the following,

$$F^{[u_i, u_j]} = -F^{[y_i, y_j]}, \quad (\text{B.5})$$

$$F^{[1-u_i, 1-u_j]} = F^{[y_i, y_j]} + F^{[y_j, y_k]} + F^{[y_k, y_i]}, \quad (\text{B.6})$$

$$F^{[u_i, 1-u_j]} = -F^{[y_k, y_i]}, \quad (\text{B.7})$$

$$F^{[u_i, y_i]} = 0, \quad (\text{B.8})$$

$$F^{[u_i, y_j]} = F^{[u_j, y_i]}, \quad (\text{B.9})$$

$$F^{[1-u_i, y_i]} = F^{[1-u_j, y_j]} - F^{[u_j, y_k]} + F^{[u_k, y_i]}, \quad (\text{B.10})$$

$$F^{[1-u_i, y_j]} = -F^{[u_k, y_j]}, \quad (\text{B.11})$$

for all $i \neq j \neq k \in \{1, 2, 3\}$, where $F^{[x, y]} \equiv F^{x, y} - F^{y, x}$.

When we combine the constraint (B.4) with the integrability constraints, we find 52 independent pairs of double coproducts. However, when we construct the space of (extended) Steinmann hexagon functions iteratively in the weight, imposing correct branch cuts along with the additional constraint (B.4), we find only 40 independent pairs [74]. Of these pairs, 24 are parity even and 16 parity odd. Interestingly, these pairs also match those provided by the ‘‘cluster adjacency’’ principle described in ref. [115], once integrability is imposed.

In order to show linear combinations of symbol entry pairs more clearly, we denote a pair of allowed final entries using the following notation (not to be confused with a similar notation used in superscripts in eqs. (B.5)–(B.11)):

$$\dots \otimes x \otimes y \rightarrow [x, y], \quad (\text{B.12})$$

so that a sum of $[x, y]$ denotes symbols of the form

$$e_1 \otimes \dots \otimes e_j \otimes x \otimes y + e_1 \otimes \dots \otimes e_j \otimes z \otimes w \rightarrow [x, y] + [z, w]. \quad (\text{B.13})$$

We also use the multiplicative property of the symbol and make the following abbreviations,

$$[xy, z] \equiv [x, z] + [y, z], \quad [x/y, z] \equiv [x, z] - [y, z]. \quad (\text{B.14})$$

To denote cyclic classes, we write $a_i \in \{a, b, c\}$, $m_i \in \{m_u, m_v, m_w\}$, and $y_i \in \{y_u, y_v, y_w\}$. Again, $i \neq j \neq k$. In this notation, the 16 odd pairs are

$$\begin{aligned} & [a_i, y_i] + [y_i, a_i], \\ & [a_i, y_j y_k] + [y_j y_k, a_i], \\ & [m_j/m_k, y_i] + [y_i, m_j/m_k], \\ & [m_i, y_u y_v y_w] + [y_u y_v y_w, m_i], \\ & [a_i m_i, y_j y_k] - [m_j, y_j] - [m_k, y_k] - [y_j y_k, a_i m_i] + [y_j, m_j] + [y_k, m_k], \\ & [m_u, y_v y_w] + [m_v, y_u y_w] + [m_w, y_u y_v] - [y_v y_w, m_u] - [y_u y_w, m_v] - [y_u y_v, m_w], \end{aligned} \quad (\text{B.15})$$

while the 24 even pairs are

$$\begin{aligned} & [a_i, a_i], \\ & [m_i, m_i], \\ & [a_i, m_j] + [m_j, a_i], \quad [a_i a_j, m_k], \\ & [m_j, m_k] + [m_k, m_j] - [y_i, y_i], \\ & [a_i, m_j m_k] + [y_i, y_u y_v y_w], \\ & [y_u, y_u^2 y_v y_w] + [y_u^2 y_v y_w, y_u], \quad [y_v, y_u y_v^2 y_w] + [y_u y_v^2 y_w, y_v], \\ & [a, m_v] + [m_u, m_v] - [m_w, b] + [m_w, m_u] - [m_w, m_v] + [y_v, y_w]. \end{aligned} \quad (\text{B.16})$$

C Coproduct Relations in the Ω Space

C.1 Coproduct relations for general Ω functions

The spaces of single and double coproducts are much smaller in the Ω subspace than in the full space of Steinmann hexagon functions described in appendix B. At the single coproduct level, parity even functions in the Ω space are observed to have only 8, not 9 final entries: y_u and y_v do not appear separately, but only the combination $y_u y_v$. That is, $E^{y_u} - E^{y_v} = 0$

if $E \in \Omega$ and E is parity even. Notice from eq. (A.19) that the last, $\sqrt{\Delta}$ -containing term vanishes for the z derivatives of all even Ω functions.

Parity odd functions are found to be even more restricted; they have only the 4 final entries $\{m_u/m_v, y_u, y_v, y_w\}$. That is, $O^a = O^b = O^c = O^{m_w} = O^{m_u} + O^{m_v} = 0$ if $O \in \Omega$ and O is parity odd.

Parity-odd functions O in the Ω space have 8 allowed final entry pairs, 2 pairs of letters that are even under parity and 6 that are odd. They are

$$\begin{aligned} & [m_u/m_v, m_u/m_v] + [y_u y_v, y_u y_v], \\ & 2[y_u y_v, y_u y_v] + [y_u y_v, y_w] + [y_w, y_u y_v], \end{aligned} \quad (\text{C.1})$$

and

$$\begin{aligned} & [m_u/m_v, y_w] + [y_w, m_u/m_v], \\ & [m_u/m_v, y_u y_v] + [y_u y_v, m_u/m_v], \\ & 2[m_u m_v, y_w] + 2[m_w, y_u y_v] + [m_u m_v, y_u y_v] + [y_u/y_v, m_u/m_v], \\ & [a, y_v] + [a, y_w] + [m_u, y_w] - [m_w, y_w] - [y_v, m_u] + [y_v, m_v], \\ & [b, y_u] + [b, y_w] + [m_v, y_w] - [m_w, y_w] + [y_u, m_u] - [y_u, m_v], \\ & [c, y_u y_v] - [m_u m_v, y_u y_v] - [m_u m_v, y_w]. \end{aligned} \quad (\text{C.2})$$

Parity-even functions E in the Ω space have 20 allowed final entry pairs, 17 even and 3 odd. They are

$$\begin{aligned} & [m_u, m_u], \quad [m_v, m_v], \\ & [c, c] - [a, a], \quad [c, c] - [b, b], \\ & [m_w, a] - [b, m_w], \quad [a, m_w] - [m_w, b], \\ & [c, m_v] + [m_v, c], \quad [c, m_u] + [m_u, c], \\ & [a, m_v] - [m_v, c], \quad [b, m_u] - [m_u, c], \\ & [m_w, a] + [b, m_w] + [a, m_w] + [m_w, b] - 4[c, c], \\ & 2[m_u, c] + 2[m_v, c] + [y_u y_v, y_w] - [y_w, y_u y_v], \\ & 2[c, c] - [y_u y_v, y_w] - [y_w, y_u y_v] - 2[y_w, y_w], \\ & 2[m_u, m_v] + 2[m_v, m_u] + [y_u y_v, y_w] + [y_w, y_u y_v], \\ & -[a/b, m_w] + [m_u/m_v, c] - [y_u/y_v, y_u y_v] - [y_u/y_v, y_w], \\ & -[a/b, m_w] - [m_u/m_v, c] - [m_u m_v, m_u/m_v] + 2[m_w, m_u/m_v] - [y_u/y_v, y_w], \\ & 2[y_u y_v, y_u y_v] + [y_u y_v, y_w] + [y_w, y_u y_v], \end{aligned} \quad (\text{C.3})$$

and

$$\begin{aligned} & [m_u/m_v, y_w] + [y_w, m_u/m_v], \\ & [m_u/m_v, y_u y_v] + [y_u y_v, m_u/m_v], \\ & [y_u y_v, c] - [y_u y_v, m_u m_v] - [y_w, m_u m_v]. \end{aligned} \quad (\text{C.4})$$

C.2 Coproduct relations for the c Discontinuity

We define the space Ω_c to be the discontinuity in the c variable of all the functions in the Ω space. In Ω_c , the set of allowed single and double coproducts shrinks even further. Also, one of the three derivatives simplifies considerably. At the same time, we lose almost no information, because only the functions κ and $\tilde{\kappa}$ are set to zero, while the remaining functions are still linearly independent.

In particular, there are only five letters in the alphabet for Ω_c :

$$\{c, m_u, m_v, y_u y_v, y_w\}. \quad (\text{C.5})$$

Odd functions in Ω_c are further restricted to have only three final entries, $\{m_u/m_v, y_u y_v, y_w\}$.

Notice from eq. (A.19) that the z derivative of the c -discontinuity F_c of a function F simplifies greatly, to

$$z \frac{\partial F_c}{\partial z} = -F_c^{m_u}. \quad (\text{C.6})$$

In the finite-coupling solution (3.26), the z derivative is also simple, in that it does not touch the hypergeometric functions.

On the c -discontinuity, the 17 even pairs of final entries for parity-even functions E in eq. (C.3) reduce to 8 final entry pairs

$$\begin{aligned} & [m_u, m_u], \quad [m_v, m_v], \\ & [c, m_v] + [m_v, c], \quad [c, m_u] + [m_u, c], \\ & 2[m_u, c] + 2[m_v, c] + [y_u y_v, y_w] - [y_w, y_u y_v], \\ & 2[c, c] - [y_u y_v, y_w] - [y_w, y_u y_v] - 2[y_w, y_w], \\ & 2[m_u, m_v] + 2[m_v, m_u] + [y_u y_v, y_w] + [y_w, y_u y_v], \\ & 2[y_u y_v, y_u y_v] + [y_u y_v, y_w] + [y_w, y_u y_v], \end{aligned} \quad (\text{C.7})$$

while the 3 odd final entry pairs for parity-even functions in eq. (C.4) remain the same.

On the c -discontinuity, the 2 parity-even pairs of final entries for parity-odd functions O , in eq. (C.1), remain the same, while the 6 parity-odd final entry pairs in eq. (C.2) reduce to 3 final entry pairs,

$$\begin{aligned} & [m_u/m_v, y_w] + [y_w, m_u/m_v], \\ & [m_u/m_v, y_u y_v] + [y_u y_v, m_u/m_v], \\ & [c, y_u y_v] - [m_u m_v, y_u y_v] - [m_u m_v, y_w]. \end{aligned} \quad (\text{C.8})$$

C.3 Coproduct relations for Ω , $\tilde{\Omega}$, and \mathcal{O}

In this subsection we provide coproduct relations between the integrals $\Omega^{(L)}$, $\tilde{\Omega}^{(L)}$ and $\mathcal{O}^{(L)}$. While it is possible to read off all such relations from results in section 5.5, we can also derive many of them directly from the differential equations they satisfy. These relations were useful when constructing $\Omega^{(L)}$ at higher loops in an earlier stage of this work, and they

serve to illustrate the structure of the Ω functions in the coproduct formalism. The relations are all valid at sufficiently high loop order, using starting at either $L = 2$ or 3.

In appendix A, the x, y, z derivatives of any functions F are expressed in terms of coproducts for the alphabet $\mathcal{S}'_{\text{hex}}$. Consider, for example, the operator $z\partial_z$. Its action can be written as

$$z\frac{\partial F}{\partial z} = (1-v)F^a - (1-u)F^b - \frac{1}{2}(F^{m_u} - F^{m_v}) + \frac{u-v}{2(1-w)}F^{m_w} + \frac{\sqrt{\Delta}}{2(1-w)}(F^{y_u} - F^{y_v}). \quad (\text{C.9})$$

When applying a second-order differential operator, such as those appearing in the differential equations for the weight- $2L$ transcendental function $\Omega^{(L)}$, the weight can be reduced either by one or two. The former case occurs when the second derivative hits the rational factor in eq. (C.9) instead of the transcendental function. In the case of eq. (3.14), using the analogous expression for $x\partial_x$, we find that these weight- $(2L-1)$ terms combine to

$$\frac{1-x}{x} \left((x\partial_x)^2 - (z\partial_z)^2 \right) \Omega^{(L)} \Big|_{2L-1} = \frac{1}{1-x} \left(\Omega^a + \Omega^b - \Omega^c + \Omega^{m_w} + \Omega^{y_w} \right). \quad (\text{C.10})$$

Since the action of the operator should give $\Omega^{(L-1)}$, which has uniform weight $2L-2$, the right-hand side should vanish. This condition, together with the parity conjugate relation (3.15), implies that

$$\Omega^a + \Omega^b - \Omega^c + \Omega^{m_w} = 0, \quad \Omega^{y_w} = 0. \quad (\text{C.11})$$

Furthermore, combining eq. (3.35) with eqs. (3.43) and (3.44) gives second-order equations relating $\Omega^{(L)}$ to pure functions of weight $2L-2$, $\tilde{\Omega}_e^{(L-1)}$ and $\tilde{\Omega}_o^{(L-1)}$. By canceling the wrong-weight terms we find two more relations:

$$\Omega^{m_w} = 0, \quad \Omega^{y_u} = \Omega^{y_v}. \quad (\text{C.12})$$

Substituting the second of these equations into the derivative (3.35) that defines \mathcal{O} , we learn that \mathcal{O} is a pure function and that⁹

$$\Omega^{y_u} = \Omega^{y_v} = \frac{1}{2}\mathcal{O}^{(L-1)}. \quad (\text{C.13})$$

Now we look at derivatives of \mathcal{O} . We start with the relation $\tilde{\Omega}_o = -z\partial_z\mathcal{O}$ and set $F = \mathcal{O}$ in eq. (C.9). Because $\tilde{\Omega}_o$ is a pure transcendental function with no rational prefactor, all the terms containing non-constant rational prefactors must vanish. It is easy to see that no linear combinations of F^a , F^b , F^{m_w} and $F^{y_u} - F^{y_v}$ can produce a constant prefactor. Hence we obtain,

$$\mathcal{O}^a = \mathcal{O}^b = \mathcal{O}^{m_w} = 0, \quad \mathcal{O}^{y_u} = \mathcal{O}^{y_v}, \quad \mathcal{O}^{m_u} - \mathcal{O}^{m_v} = 2\tilde{\Omega}_o. \quad (\text{C.14})$$

Next we insert eq. (C.14) into eq. (A.21) for $(x\partial_x + y\partial_y)\mathcal{O}$, which appears in eq. (3.39) for Ω :

$$\Omega = \frac{1+y_w}{1-y_w}\mathcal{O}^c + 2\frac{y_w(1-y_u y_v)}{(1-y_w)(1-y_u y_v y_w)}(\mathcal{O}^{m_u} - \tilde{\Omega}_o) + \mathcal{O}^{y_w}. \quad (\text{C.15})$$

⁹ We have suppressed the (L) superscript for coproducts of L loop functions, but include a reminder in this equation that the odd ladder integral evaluated at one lower loop order, $\mathcal{O}^{(L-1)}$.

Purity of Ω in eq. (C.15) leads to the additional equations,

$$\mathcal{O}^c = 0, \quad \mathcal{O}^{m_u} = -\mathcal{O}^{m_v} = \tilde{\Omega}_o, \quad \mathcal{O}^{y_w} = \Omega. \quad (\text{C.16})$$

Substituting eqs. (C.14) and (C.16) into eq. (3.43) for $\tilde{\Omega}_e$, after evaluating it with the help of eqs. (A.12) and (A.13), yields

$$\tilde{\Omega}_e = \frac{xy}{x-y} \left[(1-x)\partial_x + (1-y)\partial_y \right] \mathcal{O} = \mathcal{O}^{y_u} - \Omega, \quad (\text{C.17})$$

so that

$$\mathcal{O}^{y_u} = \mathcal{O}^{y_v} = \Omega + \tilde{\Omega}_e. \quad (\text{C.18})$$

In summary, since we know all three derivatives of the odd ladder \mathcal{O} , we can determine all nine of its first coproducts,

$$\begin{aligned} \mathcal{O}^a = \mathcal{O}^b = \mathcal{O}^c = \mathcal{O}^{m_w} = 0, \quad \mathcal{O}^{m_u} = -\mathcal{O}^{m_v} = \tilde{\Omega}_o, \\ \mathcal{O}^{y_u} = \mathcal{O}^{y_v} = \Omega + \tilde{\Omega}_e, \quad \mathcal{O}^{y_w} = \Omega. \end{aligned} \quad (\text{C.19})$$

Returning to the derivatives of Ω , we find empirically that $z\partial_z\Omega$ is a pure function. This fact implies, via eq. (C.9), that

$$\Omega^a = \Omega^b = \Omega^c = 0. \quad (\text{C.20})$$

These additional relations imply that

$$x\partial_x\Omega = \frac{1}{2}(\Omega^{m_u} + \Omega^{m_v}) - \Omega^{y_u}, \quad y\partial_y\Omega = \frac{1}{2}(\Omega^{m_u} + \Omega^{m_v}) + \Omega^{y_u}, \quad (\text{C.21})$$

$$z\partial_z\Omega = \frac{1}{2}(-\Omega^{m_u} + \Omega^{m_v}). \quad (\text{C.22})$$

We have also found some first-order coproduct relations for $\tilde{\Omega}$:

$$0 = \tilde{\Omega}^a = \tilde{\Omega}^b = \tilde{\Omega}^{m_w} = \tilde{\Omega}^{m_u} + \tilde{\Omega}^{m_v} = \tilde{\Omega}^{y_u} - \tilde{\Omega}^{y_v} = -\tilde{\Omega}^{m_u} + \tilde{\Omega}^{y_u} - \tilde{\Omega}^{y_w}. \quad (\text{C.23})$$

These relations are equivalent to

$$x\partial_x\tilde{\Omega} = -\tilde{\Omega}^{m_u} - \frac{x}{1-x}(\tilde{\Omega}^c - \tilde{\Omega}^{y_w}), \quad (\text{C.24})$$

$$y\partial_y\tilde{\Omega} = \tilde{\Omega}^{m_u} - \frac{y}{1-y}(\tilde{\Omega}^c + \tilde{\Omega}^{y_w}), \quad (\text{C.25})$$

$$z\partial_z\tilde{\Omega} = -\tilde{\Omega}^{m_u}. \quad (\text{C.26})$$

There are also two relations that are specific to $\tilde{\Omega}_e$ and $\tilde{\Omega}_o$,

$$\tilde{\Omega}_e^{y_w} = \frac{1}{2}\mathcal{O}^{(L-1)}, \quad \tilde{\Omega}_o^c = 0. \quad (\text{C.27})$$

By taking derivatives of the all-orders representation (3.32) of $\tilde{\Omega}$, it is possible to show that the quantity $\tilde{\Omega}^{m_u}$ appearing in the x and y derivatives of $\tilde{\Omega}$ in eqs. (C.24) and (C.24)

is indeed the same as the one appearing in the z derivative (C.26). One can also show to all orders that

$$(x\partial_x + y\partial_y)\tilde{\Omega} = \frac{y}{1-y}(x\partial_x + z\partial_z)\Omega + \frac{x}{1-x}(y\partial_y - z\partial_z)\Omega. \quad (\text{C.28})$$

Inserting the coproduct representations of these derivatives, given above, we find relations between the first coproducts of $\tilde{\Omega}$ and Ω :

$$-\tilde{\Omega}^c + \tilde{\Omega}^{yw} = \Omega^{m_u} + \Omega^{y_u}, \quad \tilde{\Omega}^c + \tilde{\Omega}^{yw} = -\Omega^{m_v} + \Omega^{y_u}. \quad (\text{C.29})$$

By combining eqs. (C.13) and (C.19), we can write all three integrals, Ω , $\tilde{\Omega}_e$ and $\tilde{\Omega}_o$, as double coproducts of the Ω integral at one higher loop order:

$$\Omega^{(L-1)} = 2\Omega^{yw,y_u}, \quad \tilde{\Omega}_e^{(L-1)} = 2(\Omega^{y_u,y_u} - \Omega^{yw,y_u}), \quad \tilde{\Omega}_o^{(L-1)} = 2\Omega^{m_u,y_u}. \quad (\text{C.30})$$

We have found empirically that these integrals can also be written as double coproducts of $\tilde{\Omega}_e$ and $\tilde{\Omega}_o$:

$$\Omega^{(L-1)} = -2\tilde{\Omega}_e^{c,c}, \quad \tilde{\Omega}_e^{(L-1)} = 2(\tilde{\Omega}_e^{m_u,c} + \tilde{\Omega}_o^{m_u,y_w}), \quad \tilde{\Omega}_o^{(L-1)} = 2\tilde{\Omega}_e^{yw,m_u} = 2\tilde{\Omega}_o^{yw,y_u}, \quad (\text{C.31})$$

for $L > 2$. In both sets of equations, we suppress the (L) superscript on the right-hand side for clarity.

C.4 Improving the MHV-NMHV operator

Through five loops, the six-point MHV amplitude \mathcal{E} and NMHV amplitude E obey a curious relation that connects these amplitudes at different loop orders [69]. If we perform a cyclic permutation $u \rightarrow v \rightarrow w \rightarrow u$ on that relation, in order to give it the same $u \leftrightarrow v$ symmetry as the Ω integral, it becomes

$$X^{\text{old}}[\mathcal{E}(u, v, w)] = g^2(2E(v, w, u) - \mathcal{E}(u, v, w)), \quad (\text{C.32})$$

where

$$X^{\text{old}}[F] \equiv -F^{w,w} - F^{1-w,w} - 3F^{yw,y_w} + F^{y_u,y_u} + F^{y_v,y_v} + 2(F^{y_u,y_w} + F^{y_v,y_w}) - F^{y_u,y_v} - F^{y_v,y_u} \quad (\text{C.33})$$

is written in terms of the old alphabet \mathcal{S}_{hex} . (For an earlier version of this relation, see ref. [73].) In fact, this relation fails for amplitudes at six loops [74]. However, we will see that a version of it survives to arbitrary loop order in the pentagonal integrals.

In general, \mathcal{E} obeys many relations on its double coproducts, which allow the operator X to be rewritten without changing its action on \mathcal{E} . However, its action on the ladder integrals will generically change. It turns out that a better form for X , written in terms of the new alphabet $\mathcal{S}'_{\text{hex}}$, is

$$\begin{aligned} X[F] &= -3F^{yw,y_w} + F^{y_u,y_u} + F^{y_v,y_v} + 2(F^{y_u,y_u} + F^{y_w,y_v}) - F^{y_u,y_v} - F^{y_v,y_u} \\ &\quad + F^{a,a} + F^{b,b} + F^{c,c} + F^{a,m_w} + F^{b,m_w} - F^{m_w,a} - F^{m_w,b} \\ &\quad - 2(F^{c,m_u} + F^{c,m_v} - F^{m_u,c} - F^{m_v,c}). \end{aligned} \quad (\text{C.34})$$

This form is better because X now has a very simple action on the ladder integrals. We find that

$$X[\mathcal{W}(u, v, w)] = X[\mathcal{W}(v, w, u)] = X[\mathcal{W}(w, u, v)] = 0, \quad (\text{C.35})$$

$$X[\Omega(u, v, w)] = X[\Omega(v, w, u)] = X[\Omega(w, u, v)] = 0, \quad (\text{C.36})$$

$$X[\tilde{\Omega}(u, v, w)] = -2g^2 \tilde{\Omega}(u, v, w), \quad X[\tilde{\Omega}(v, w, u)] = X[\tilde{\Omega}(w, u, v)] = 0, \quad (\text{C.37})$$

$$X[\mathcal{O}(u, v, w)] = -2g^2 \mathcal{O}(u, v, w), \quad X[\mathcal{O}(v, w, u)] = X[\mathcal{O}(w, u, v)] = 0. \quad (\text{C.38})$$

There are anomalous terms in the even parts of $X[\tilde{\Omega}(v, w, u)]$, $X[\tilde{\Omega}(w, u, v)]$, $X[\Omega(v, w, u)]$ and $X[\Omega(w, u, v)]$ at $L = 2$, and in $X[\mathcal{W}(u, v, w)]$, $X[\mathcal{W}(v, w, u)]$, and $X[\mathcal{W}(w, u, v)]$ at both $L = 2$ and $L = 3$. But above three loops, there are no anomalies in the action on these integrals to any order. This can be contrasted with the operator's action on \mathcal{E} , which remains anomalous at six loops.

The operator X also has an interesting action on the full Ω space. In particular, note that for each ladder integral considered above (and ignoring low-weight anomalies), $X[F(v, w, u)] = X[F(w, u, v)] = 0$. While this is not quite true for the full space, we do find that for a general function $F(u, v, w) \in \Omega$,

$$X[F(v, w, u)], X[F(w, u, v)] \in \{\kappa, \tilde{\kappa}\}. \quad (\text{C.39})$$

That is, the action of the operator X on cyclic rotations of functions in the Ω space can be expressed entirely in terms of κ and $\tilde{\kappa}$ functions of the appropriate weight, which vanish on the c -discontinuity. In effect, the operator X annihilates the c -discontinuity of the cyclic and anti-cyclic rotations of the Ω functions. This is a surprising property, and one that suggests further investigation.

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