PROCEEDINGS OF SCIENCE

LTH 1150, DESY 18-010, Nikhef 2018-004 January 2018

# Four-loop results on anomalous dimensions and splitting functions in QCD

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We report on recent progress on the flavour non-singlet splitting functions in perturbative QCD. The exact four-loop (N$^3$LO) contribution to these functions has been obtained in the planar limit of a large number of colours. Phenomenologically sufficient approximate expressions have been obtained for the parts not exactly known so far. Both cases include results for the four-loop cusp and virtual anomalous dimensions which are relevant well beyond the evolution of non-singlet quark distributions, for which an accuracy of (well) below 1% has now been been reached.

*13th International Symposium on Radiative Corrections (Applications of Quantum Field Theory to Phenomenology)*
*25-29 September 2017, St. Gilgen, Austria*

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## 1. Introduction

Up to power corrections, observables in *ep* and *pp* hard scattering can be schematically expressed as

$$O^{ep} \;=\; f_i \otimes c_i^{\rm o} \;, \quad O^{pp} \;=\; f_i \otimes f_k \otimes c_{ik}^{\rm o} \tag{1.1}$$

in terms of the respective partonic cross sections (coefficient functions) $c^{\rm o}$ and the universal parton distribution functions (PDFs) $f_i(x,\mu^2)$ of the proton at a (renormalization and factorization) scale $\mu$ of the order of a physical hard scale, e.g., $M_H$ for the total cross section for the production of the Higgs boson. The dependence of the PDFs on the momentum fraction $x$ is not calculable in perturbative QCD; their scale dependence is governed by the renormalization-group evolution equations

$$\frac{\partial}{\partial \ln \mu^2}\, f_i(x,\mu^2) \;=\; \big[P_{ik}(\alpha_{\rm s}(\mu^2)) \otimes f_k(\mu^2)\big](x) \tag{1.2}$$

where $\otimes$ denotes the Mellin convolution. The splitting functions, which are closely related to the anomalous dimensions of twist-2 operators in the light-cone operator-product expansion (OPE), and the coefficient functions can be expanded in powers of the strong coupling $a_{\rm s} \equiv \alpha_{\rm s}(\mu^2)/(4\pi)$,

$$P \;=\; \quad a_{\rm s}\, P^{(0)} + a_{\rm s}^2\, P^{(1)} + a_{\rm s}^3\, P^{(2)} + a_{\rm s}^4\, P^{(3)} + \,\ldots \quad , \tag{1.3}$$

$$c_a^{\rm o} \;=\; a_{\rm s}^{n_{\rm o}} \big[\, c_{\rm o}^{(0)} \;+ a_{\rm s}\, c_{\rm o}^{(1)} \;+ a_{\rm s}^2\, c_{\rm o}^{(2)} \;+ a_{\rm s}^3\, c_{\rm o}^{(3)} \;+ \ldots \big] \,. \tag{1.4}$$

Together the first three terms in eqs. (1.3) and (1.4) provide the next-to-next-to-leading order ($\rm N^2LO$) of perturbative QCD for the observables (1.1). This is now the standard approximation for many hard processes; see refs. [1–4] for the corresponding splitting functions.

Corrections beyond $\rm N^2LO$ are of phenomenological interest where high precision is required, such as in determinations of $\alpha_{\rm s}$ from deep-inelastic scattering (DIS) (see refs. [5, 6] for the $\rm N^3LO$ corrections to the most important structure functions), and where the perturbation series shows a slow convergence, such as for Higgs production via gluon-gluon fusion calculated in ref. [7] at $\rm N^3LO$. The size and structure of the corrections beyond $\rm N^2LO$ are also of theoretical interest.

Here we briefly report about considerable recent progress on the three four-loop ($\rm N^3LO$) non-singlet splitting functions. We focus on the quantities $P_{\rm ns}^{\pm(3)}(x)$ for the evolution of flavour-differences $q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k)$ of quark and antiquark distributions; for more details see ref. [8].

## 2. Diagram calculations of fixed-$N$ moments

Two methods have been applied for obtaining Mellin moments of the quantities $P^{(3)}$ in eq. (1.3). Depending on the function, both can be used to determine the same even-$N$ *or* the odd-$N$ moments.

In the first one calculates, via the optical theorem and a dispersion relation in $x$, the unfactorized structure functions in DIS, as done at two and three loops in refs. [9–12]. The construction of the FORCER program [13] has facilitated the extension of those computations (which also provide moments of the coefficient functions) to four loops. For the hardest diagrams, the complexity of these computations rises quickly with $N$, hence only $N \le 6$ has been covered completely so far [14]. Much higher $N$ can be accessed for simpler cases, e.g., values up to $N > 40$ have been reached for high-$n_f$ parts. These were sufficient to determine the complete $n_f^2$ and $n_f^3$ parts of the non-singlet splitting functions $P_{\rm ns}^{(3)}(x)$ and the $n_f^3$ parts of the corresponding flavour-singlet quantities [15].

The increase of the complexity of the Feynman integrals with $N$ is more benign for the second method based on the OPE which was applied to the present non-singlet cases at NLO in ref. [16], see also ref. [17]. FORCER calculations in this framework have reached $N = 16$ for all contributions to the functions $P_{\rm ns}^{(3)}$, $N = 18$ for their $n_f$ parts and $N = 20$ for the complete limit of a large number of colours $n_c$ [8]. See refs. [18–21] for earlier calculations of $P_{\rm ns}^{\pm(3)}$ at $N \leq 4$.

## 3. Towards all-$N$ expressions

If the anomalous dimensions $\gamma_{\rm ns}(N) = -P_{\rm ns}(N)$ at N$^{n>2}$LO are analogous to the lower orders, then they can be expressed in terms of harmonic sums $S_{\vec{w}}$ [22, 23] and denominators $D_a^k \equiv (N+a)^{-k}$ as

$$\gamma_{\rm ns}^{(n)}(N) \;=\; \sum_{w=0}^{2n+1} c_{00\vec{w}}\, S_{\vec{w}}(N) + \sum_a \sum_{k=1}^{2n+1} \sum_{w=0}^{2n+1-k} c_{ak\vec{w}}\, D_a^k\, S_{\vec{w}}(N)\;. \tag{3.1}$$

The denominators at the calculated values of $N$ indicate $a = 0, 1$ for $\gamma_{\rm ns}^{\pm}$, with coefficients $c_{00\vec{w}}$, $c_{ak\vec{w}}$ that are integer modulo low powers of 1/2 and 1/3. Sums up to weight $w = 2n+1$ occur at N$^n$LO.

Based on a conformal symmetry of QCD at an unphysical number of space-time dimensions $D$, it has been conjectured that the $\overline{\rm MS}$ functions $\gamma_{\rm ns}(N)$ are constrained by 'self-tuning' [24, 25],

$$\gamma_{\rm ns}(N) \;=\; \gamma_{\rm u}\left(N + \sigma\,\gamma_{\rm ns}(N) - \beta(a_{\rm s})/a_{\rm s}\right) \tag{3.2}$$

where $\beta(a_{\rm s}) = -\beta_0\, a_{\rm s}^2 - \beta_1\, a_{\rm s}^3 - \ldots$ is the beta function, for its present status see refs. [26, 27]. The initial-state (PDF) and final-state (fragmentation-function) anomalous dimensions are obtained for $\sigma = -1$ and $\sigma = 1$, respectively, and the universal kernel $\gamma_{\rm u}$ is reciprocity respecting (RR), i.e., invariant under replacement $N \to (1-N)$. Eq. (3.2) implies that the non-RR parts and the spacelike/timelike difference are inherited from lower orders. Hence 'only' $\gamma_{\rm u}$, which includes $2^{w-1}$ RR (combinations of) harmonic sums of weight $w$, needs to be determined at four loops.

Present information, given by the even-$N$ (odd-$N$) values $N \leq 16$ (15) of $\gamma_{\rm ns}^{+(3)}(N)$ ($\gamma_{\rm ns}^{-(3)}(N)$) and endpoint constraints (see below), is insufficient to determine the $n = 3$ coefficients in eq. (3.1). However, $\gamma_{\rm ns}^{+} = \gamma_{\rm ns}^{-}$ in the large-$n_c$ limit, hence the known even-$N$ *and* odd-$N$ values can be used. Moreover, alternating sums do not contribute to $\gamma_{\rm ns}^{\pm}$ in this limit, leaving 1, 1, 2, 3, 5, 8, 13 = Fibonacci$(w)$ RR sums at weight $w = 1, \ldots, 7$ and a total of 87 basis functions for $n = 3$ in eq. (3.1).

Large-$N$ and small-$x$ limits provide more than 40 constraints on their coefficients. At large-$N$, the non-singlet anomalous dimensions have the form [33–35]

$$\gamma_{\rm ns}^{(n-1)}(N) \;=\; A_n \ln\widetilde{N} - B_n + N^{-1}\{C_n \ln\widetilde{N} - \widetilde{D}_n + \tfrac{1}{2}A_n\} + O(N^{-2}) \tag{3.3}$$

with $\ln\widetilde{N} \equiv \ln N + \gamma_{\rm e}$, where $\gamma_{\rm e}$ denotes the Euler-Mascheroni constant. $C_n$ and $\widetilde{D}_n$ are given by

$$C(a_{\rm s}) \;=\; (A(a_{\rm s}))^2 \quad , \quad \widetilde{D}(a_{\rm s}) \;=\; A(a_{\rm s})\cdot(B(a_{\rm s}) - \beta(a_{\rm s})/a_{\rm s})\;, \tag{3.4}$$

in terms of lower-order information on the cusp anomalous dimension $A(a_{\rm s}) = A_1 a_{\rm s} + A_2\, a_{\rm s}^2 + \ldots$ and the quantity $B(a_{\rm s}) = B_1 a_{\rm s} + B_2\, a_{\rm s}^2 + \ldots$ sometimes called the virtual anomalous dimension.

The resummation of small-$x$ double logarithms [28–31] provides the four-loop coefficients of $x^a \ln^b x$ at $4 \leq b \leq 6$ and all $a$ in the large-$n_c$ limit (in full QCD, this holds only at even $a$ for $P_{\rm ns}^{+}(x)$ and odd $a$ for $P_{\rm ns}^{-}(x)$). Moreover, a relation leading to a single-logarithmic resummation at $a = 0$,

$$\gamma_{\rm ns}^{+}(N)\cdot\left(\gamma_{\rm ns}^{+}(N) + N - \beta(a_{\rm s})/a_{\rm s}\right) \;=\; O(1)\;, \tag{3.5}$$

has been conjectured in ref. [32]. As far as it can be checked so far, this relation is found to be correct except for terms with $\zeta_2 = \pi^2/6$ that vanish in the large-$n_c$ limit.

Taking into account all the above information, it is possible to set up systems of Diophantine equations for the coefficients $c_{00\vec{w}}$, $c_{ak\vec{w}}$ of $\gamma_{\rm ns}^{\pm(3)}(N)$ in the large-$n_c$ limit that can be solved using the moments $1 \leq N \leq 18$, leaving the results of the diagram calculation at $N = 19, 20$ as checks.

## 4. All-$N$ anomalous dimension in the large-$n_c$ limit

The exact expressions for the new $n_f^0$ and $n_f^1$ parts cannot be shown here due to their length, they can be found in eq. (3.6) and (3.7) of ref. [8]. For the $n_f^2$ and $n_f^3$ terms see ref. [15]. The resulting large-$N$ coefficients $A_{L,4}$ and $B_{L,4}$ – the subscript $L$ indicates the large-$n_c$ limit – are found to be

$$\begin{aligned}
A_{L,4} = \; & C_F n_c^3 \Big( \frac{84278}{81} - \frac{88832}{81}\,\zeta_2 + \frac{20992}{27}\,\zeta_3 + 1804\,\zeta_4 - \frac{352}{3}\,\zeta_2\zeta_3 - 352\,\zeta_5 - 32\,\zeta_3^2 - 876\,\zeta_6 \Big) \\
& - C_F n_c^2 n_f \Big( \frac{39883}{81} - \frac{26692}{81}\,\zeta_2 + \frac{16252}{27}\,\zeta_3 + \frac{440}{3}\,\zeta_4 - \frac{256}{3}\,\zeta_2\zeta_3 - 224\,\zeta_5 \Big) \\
& + C_F n_c n_f^2 \Big( \frac{2119}{81} - \frac{608}{81}\,\zeta_2 + \frac{1280}{27}\,\zeta_3 - \frac{64}{3}\,\zeta_4 \Big) - C_F n_f^3 \Big( \frac{32}{81} - \frac{64}{27}\,\zeta_3 \Big)
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
B_{L,4} = \; & C_F n_c^3 \Big( -\frac{1379569}{5184} + \frac{24211}{27}\,\zeta_2 - \frac{9803}{162}\,\zeta_3 - \frac{9382}{9}\,\zeta_4 + \frac{838}{9}\,\zeta_2\zeta_3 + 1002\zeta_5 + \frac{16}{3}\,\zeta_3^2 \\
& \qquad + 135\zeta_6 - 80\zeta_2\zeta_5 + 32\zeta_3\zeta_4 - 560\zeta_7 \Big) \\
& + C_F n_c^2 n_f \Big( \frac{353}{3} - \frac{85175}{162}\,\zeta_2 - \frac{137}{9}\,\zeta_3 + \frac{16186}{27}\,\zeta_4 - \frac{584}{9}\,\zeta_2\zeta_3 - \frac{248}{3}\,\zeta_5 - \frac{16}{3}\,\zeta_3^2 - 144\zeta_6 \Big) \\
& - C_F n_c n_f^2 \Big( \frac{127}{18} - \frac{5036}{81}\,\zeta_2 + \frac{932}{27}\,\zeta_3 + \frac{1292}{27}\,\zeta_4 - \frac{160}{9}\,\zeta_2\zeta_3 - \frac{32}{3}\,\zeta_5 \Big) \\
& - C_F n_f^3 \Big( \frac{131}{81} - \frac{32}{81}\,\zeta_2 - \frac{304}{81}\,\zeta_3 + \frac{32}{27}\,\zeta_4 \Big) \; .
\end{aligned} \tag{4.2}$$

The agreement of the four-loop cusp anomalous dimension (4.1) with the result obtained from the large-$n_c$ photon-quark form factor [36,37] provides a further non-trivial check of the determination of the all-$N$ expressions from the moments at $N \leq 18$, and hence also of the relations (3.1) – (3.5).

The maximum-weight $\zeta_3^2$ and $\zeta_6$ parts of eq. (4.1) also agree with the result obtained in planar $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory (MSYM) obtained before in ref. [38]. There is no such direct connection between the four-loop virtual anomalous dimension (4.2) and its counterparts in planar $\mathcal{N} = 4$ MSYM; see ref. [39] where the maximum-weight part of eq. (4.2) has been employed to derive the four-loop collinear anomalous dimension in planar $\mathcal{N} = 4$ MSYM.

The all-$N$ large-$n_c$ limit of $\gamma_{\rm ns}^{\pm(3)}(N)$ is compared in fig. 1 with the integer-$N$ QCD results at $N \leq 16$. As illustrated in the left panel, the former are a decent approximation to the latter for the individual $n_f^k$ contributions. However, as shown in the right panel, there are considerable cancellations between the these contributions. These cancellations are most pronounced for the physically relevant number of $n_f = 5$ light quark flavours outside the large-$N$ / large-$x$ region. Hence the large-$n_c$ suppressed contributions – indicated by the subscript $N$ below – need to be taken into account in phenomenological N$^3$LO analyses.

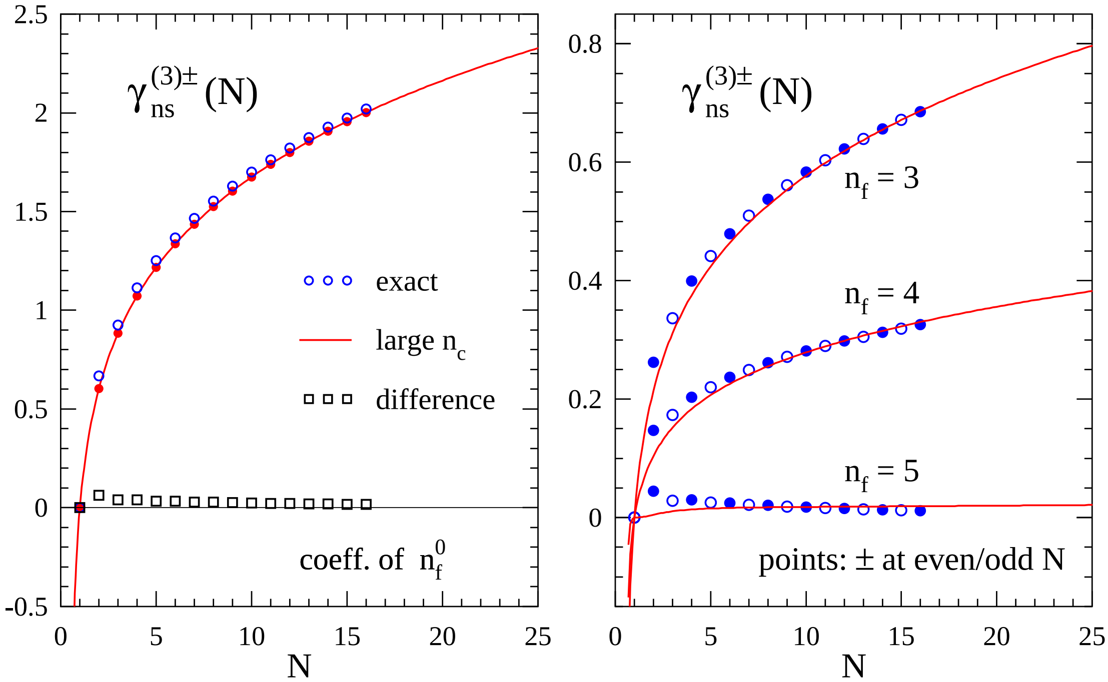


**Figure 1:** The large-$n_c$ limit of the four-loop anomalous dimensions $\gamma_{\rm ns}^{\pm(3)}(N)$ (lines) compared to the QCD results for $\gamma_{\rm ns}^{+(3)}(N)$ at even $N$ and $\gamma_{\rm ns}^{-(3)}(N)$ at odd $N$ (points). Left: the $n_f$-independent contributions. Right: the results for physically relevant values of $n_f$. The values have been converted to an expansion in $\alpha_s$.

## 5. $x$-space approximations of the large-$n_c$ suppressed parts

With eight integer-$N$ moments known for both $P_{\rm ns}^{+(3)}(x)$ and $P_{\rm ns}^{-(3)}(x)$ and the large-$x$ and small-$x$ knowledge discussed in section 2, it is possible to construct approximate $x$-space expressions which are analogous to (but more accurate than) those used before 2004 at N$^2$LO, see refs. [40–43]. For this purpose an ansatz consisting of

- the two large-$x$ parameters $A_4$ and $B_4$ in eq. (3.3),
- two of three suppressed large-$x$ logs $(1-x)\ln^k(1-x)$, $k = 1,2,3$,
- one of ten two-parameter polynomials in $x$ that vanish for $x \to 1$,
- two of the three unknown small-$x$ logarithms $\ln^k x$, $k = 1,2,3$

is built for the large-$n_c$ suppressed $n_f^0$ and $n_f^1$ parts $P_{\rm N,0/1}^{+(3)}$ of $P_{\rm ns}^{+(3)}(x)$. This results in 90 trial functions, the parameters of which can be fixed from the eight available moments. Of these functions, two representatives $A$ and $B$ are then chosen that indicate the remaining uncertainty, see fig. 2.

This non-rigorous procedure can be checked by comparing the same treatment for the large-$n_c$ parts to our exact results. Moreover, the trial functions lead to very similar values for the next moment, e.g., $N = 18$ for $P_{\rm ns}^{+(3)}$. The residual uncertainty at this $N$-value is a consequence of the width of the band at large $x$, which in turn is correlated with the uncertainties at smaller $x$. If the spread of the result $A$ and $B$ would underestimate the true remaining uncertainties, then a comparison with an additional analytic result at this next value of $N$ should reveal a discrepancy.

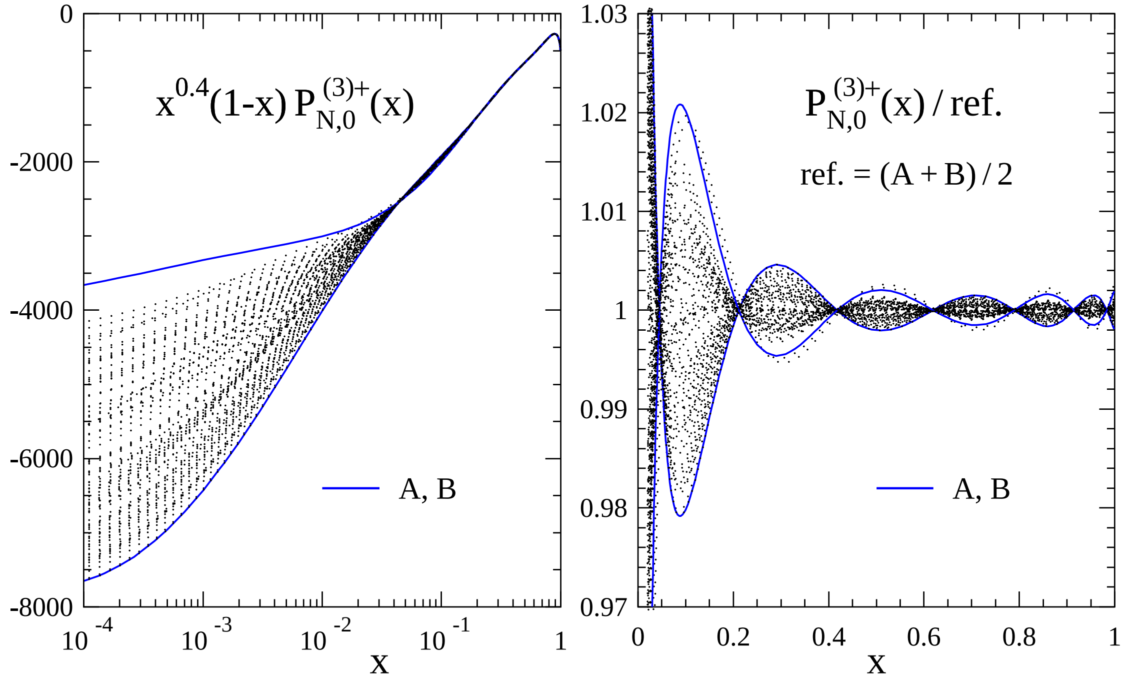


**Figure 2:** About 90 trial functions for the $n_f$-independent contribution to the large-$n_c$ suppressed part of splitting function $P_{\rm ns}^{+(3)}(x)$, multiplied by $x^{0.4}(1-x)$. The two functions chosen to represent the remaining uncertainty are denoted by $A$ and $B$ and shown by solid (blue) lines. Due to the factor $(1-x)$ the contribution $A_{N,4}$ to the four-loop cusp anomalous dimension can be read off at $x = 1$.

We were able to extend the diagram computations of the $n_f^1$ parts of $P_{\rm ns}^{(3)+}(x)$ to $N = 18$ and find

$$P_{\rm N,1}^{+(3)}(N=18) \;=\; 195.8888792_B \;<\; 195.8888857\ldots_{\rm exact} \;<\; 195.8888968_A \;. \tag{5.1}$$

A similar check for $P_{\rm N,0}^{+(3)}$ has been carried out by deriving a less accurate approximation using only seven moments and comparing the results to the now unused value at $N = 16$.

The case of $P_{\rm ns}^{-(3)}(x)$ has been treated in the same manner, but taking into account that only its leading small-$x$ logarithm is known up to now [29]. See ref. [8] for the (large-$N$ suppressed) additional $d^{abc}d_{abc}$ contribution $P_{\rm ns}^{s(3)}(x)$ to the splitting function for the total valence quark PDF.

## 6. Numerical results for the cusp and virtual anomalous dimensions

Combining the exact large-$n_c$ results, the approximations for the remaining $n_f^0$ and $n_f^1$ contributions and the complete high-$n_f$ contributions of ref. [15], the four-loop cusp anomalous dimension for QCD with $n_f$ quark flavours are given by

$$A_4 \;=\; 20702(2) - 5171.9(2)\, n_f + 195.5772\, n_f^2 + 3.272344\, n_f^3 \;, \tag{6.1}$$

where the numbers in brackets represent a conservative estimate of the remaining uncertainty. The conversion of this result to an expansion in powers of $\alpha_{\rm s}$ leads to

$$\begin{aligned} A_q(\alpha_{\rm s}, n_f\!=\!3) \;&=\; 0.42441\, \alpha_{\rm s}\, (1 + 0.72657\, \alpha_{\rm s} + 0.73405\, a_{\rm s}^2 + 0.6647(2)\, a_{\rm s}^3 + \ldots)\;, \\ A_q(\alpha_{\rm s}, n_f\!=\!4) \;&=\; 0.42441\, \alpha_{\rm s}\, (1 + 0.63815\, \alpha_{\rm s} + 0.50998\, a_{\rm s}^2 + 0.3168(2)\, a_{\rm s}^3 + \ldots)\;, \\ A_q(\alpha_{\rm s}, n_f\!=\!5) \;&=\; 0.42441\, \alpha_{\rm s}\, (1 + 0.54973\, \alpha_{\rm s} + 0.28403\, a_{\rm s}^2 + 0.0133(2)\, a_{\rm s}^3 + \ldots)\;. \end{aligned} \tag{6.2}$$

The corresponding results for the virtual anomalous dimension, i.e., the coefficient of $\delta(1-x)$ show a similarly benign expansion with

$$B_4 \;=\; 23393(10) - 5551(1)\, n_f + 193.8554\, n_f^2 + 3.014982\, n_f^3 \tag{6.3}$$

and

$$\begin{aligned}
B_q(\alpha_{\rm s}, n_f\!=\!3) \;&=\; 0.31831\, \alpha_{\rm s}\, (1 + 0.99712\, \alpha_{\rm s} + 1.24116\, a_{\rm s}^2 + 1.0791(13)\, a_{\rm s}^3 + \ldots)\,, \\
B_q(\alpha_{\rm s}, n_f\!=\!4) \;&=\; 0.31831\, \alpha_{\rm s}\, (1 + 0.87192\, \alpha_{\rm s} + 0.97833\, a_{\rm s}^2 + 0.5649(13)\, a_{\rm s}^3 + \ldots)\,, \\
B_q(\alpha_{\rm s}, n_f\!=\!5) \;&=\; 0.31831\, \alpha_{\rm s}\, (1 + 1.74672\, \alpha_{\rm s} + 0.71907\, a_{\rm s}^2 + 0.1085(13)\, a_{\rm s}^3 + \ldots)\,. 
\end{aligned} \tag{6.4}$$

Due to constraints by large-$N$ moments, the errors of $A_4$ and $B_4$ are fully correlated. The accuracy in eqs. (6.2) and (6.4) should be amply sufficient for phenomenological applications.

By repeating the approximation procedure in section 5 for individual colour factors, it is possible to obtain corresponding approximate coefficients for $A_4$ and $B_4$ which can be summarized as (for a table of the relevant group invariants see, e.g., appendix C of ref. [44])

| | $A_4$ | $B_4$ |
|---|---|---|
| $C_F^4$ | 0 | 197. ± 3. |
| $C_F^3 C_A$ | 0 | −687. ± 10. |
| $C_F^2 C_A^2$ | 0 | 1219. ± 12. |
| $C_F C_A^3$ | 610.3 ± 0.3 | 295.6 ± 2.4 |
| $d_R^{abcd} d_A^{abcd}/N_R$ | −507.5 ± 6.0 | −996. ± 45. |
| $n_f C_F^3$ | −31.00 ± 0.4 | 81.4 ± 2.2 |
| $n_f C_F^2 C_A$ | 38.75 ± 0.2 | −455.7 ± 1.1 |
| $n_f C_F C_A^2$ | −440.65 ± 0.2 | −274.4 ± 1.1 |
| $n_f\, d_R^{abcd} d_R^{abcd}/N_R$ | −123.90 ± 0.2 | −143.5 ± 1.2 |
| $n_f^2 C_F^2$ | −21.31439 | −5.775288 |
| $n_f^2 C_F C_A$ | 58.36737 | 51.03056 |
| $n_f^3 C_F$ | 2.454258 | 2.261237 |

where the exactly known $n_f^2$ and $n_f^3$ coefficients have been included for completeness. Due to the constraint provided by the exact large-$n_c$ limit, the errors in this table are highly correlated; for numerical applications in QCD eqs. (6.2) and (6.4) should be used instead. The above results show that both quartic group invariants definitely contribute to the four-loop cusp anomalous dimension, for this issue see also refs. [45–48] and references therein. This implies that the so-called Casimir scaling between the quark and gluon cases, $A_q = C_F/C_A\, A_g$, does not hold beyond three loops.

## 7. N$^3$LO corrections to the evolution of non-singlet PDFs

The effect of the fourth-order contributions on the evolution of the non-singlet PDFs can be illustrated by considering the logarithmic derivatives of the respective combinations of quark PDFs with respect to the factorization scale, $\dot{q}^{\,i}_{\rm ns} \equiv d \ln q^{\,i}_{\rm ns} / d \ln \mu_f^2$, at a suitably chosen reference point.

As in ref. [1], we choose the schematic, order-independent initial conditions

$$xq_{\rm ns}^{\pm,{\rm v}}(x,\mu_0^2) \;=\; x^{0.5}(1-x)^3 \quad \text{and} \quad \alpha_{\rm s}(\mu_0^2) \;=\; 0.2\,. \tag{7.1}$$

For $\alpha_{\rm s}(M_Z^2) = 0.114\ldots0.120$ this value for $\alpha_{\rm s}$ corresponds to $\mu_0^2 \simeq 25\ldots50$ GeV$^2$ beyond the leading order, a scale range typical for DIS at fixed-target experiments and at the *ep* collider HERA.

The new N$^3$LO corrections to $\dot{q}^{\,i}_{\rm ns}$ are generally small, hence they are illustrated in fig. 3 by comparing their relative effect to that of the N$^2$LO contributions for the standard identification $\mu_r = \mu_f \equiv \mu$ of the renormalization scale with the factorization scale. Except close to the sign change of the scaling violations at $x \simeq 0.07$, the relative N$^3$LO effects are (well) below 1% for the flavour-differences $q_{\rm ns}^+$ and $q_{\rm ns}^-$ (left and middle panel). The N$^2$LO and N$^3$LO corrections are larger for the valence distribution $q_{\rm ns}^{\rm v}$ at $x < 0.07$ due to the effect of the $d^{abc}d_{abc}$ 'sea' contribution $P_{\rm ns}^{\rm s}(x)$, note the different scale of the right panel in fig. 3. Also in this case the N$^3$LO evolution represents a clear improvement, and the relative four-loop corrections are below 2%.

The remaining uncertainty due to the approximate character of the four-loop splitting functions beyond the large-$n_c$ limit is indicated by the difference between the solid and dotted (red) curves in fig. 3 and fig. 4 below. Due to the small size of the four-loop contributions and the '$x$-averaging' effect of the Mellin convolution,

$$[P_{\rm ns} \otimes q_{\rm ns}](x) \;=\; \int_x^1 \frac{dy}{y}\, P_{\rm ns}(y)\, q_{\rm ns}\!\left(\frac{x}{y}\right)\,, \tag{7.2}$$

the results of section 4 are safely applicable to lower values of $x$ than one might expect from fig. 2.

The stability of the NLO, N$^2$LO and N$^3$LO results under variation of the renormalization scale over the range $\frac{1}{8}\mu_f^2 \le \mu_r^2 \le 8\mu_f^2$ is illustrated in fig. 4 at typical values of $x$. Except close to the sign change of $\dot{q}^{\,+}_{\rm ns}$, the variation is well below 1% for the conventional interval $\frac{1}{2}\mu_f \le \mu_r \le 2\mu_f$.

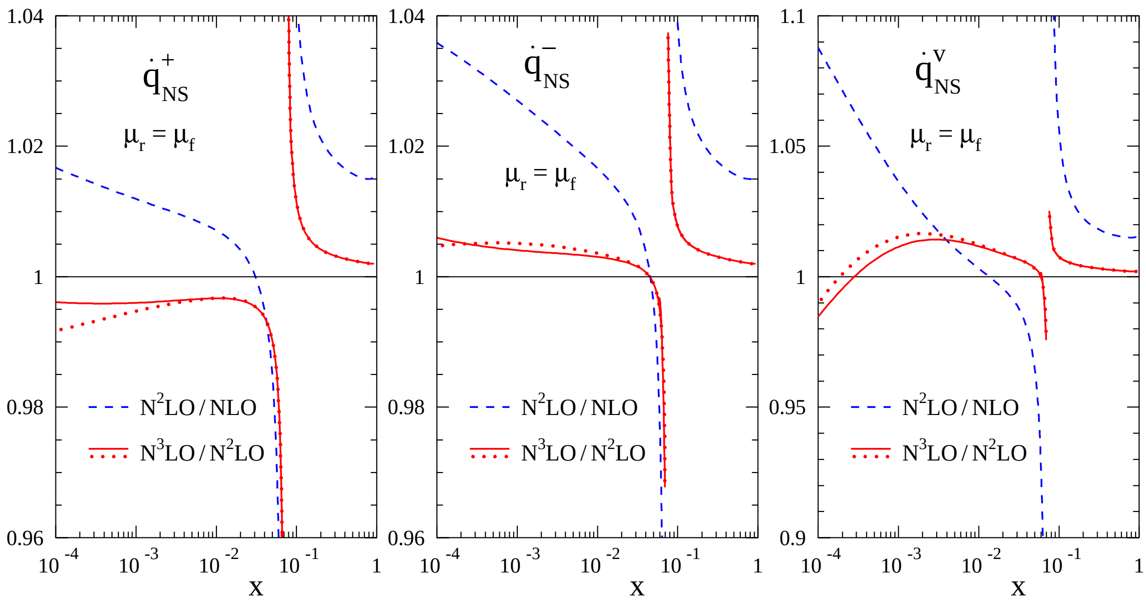


**Figure 3:** The relative N$^2$LO and N$^3$LO corrections to the logarithmic scale derivative of the non-singlet combinations $q_{\rm ns}^a$ of quark PDFs for the schematic order-independent input (7.1) for $n_f = 4$ at $\mu_r = \mu_f$.

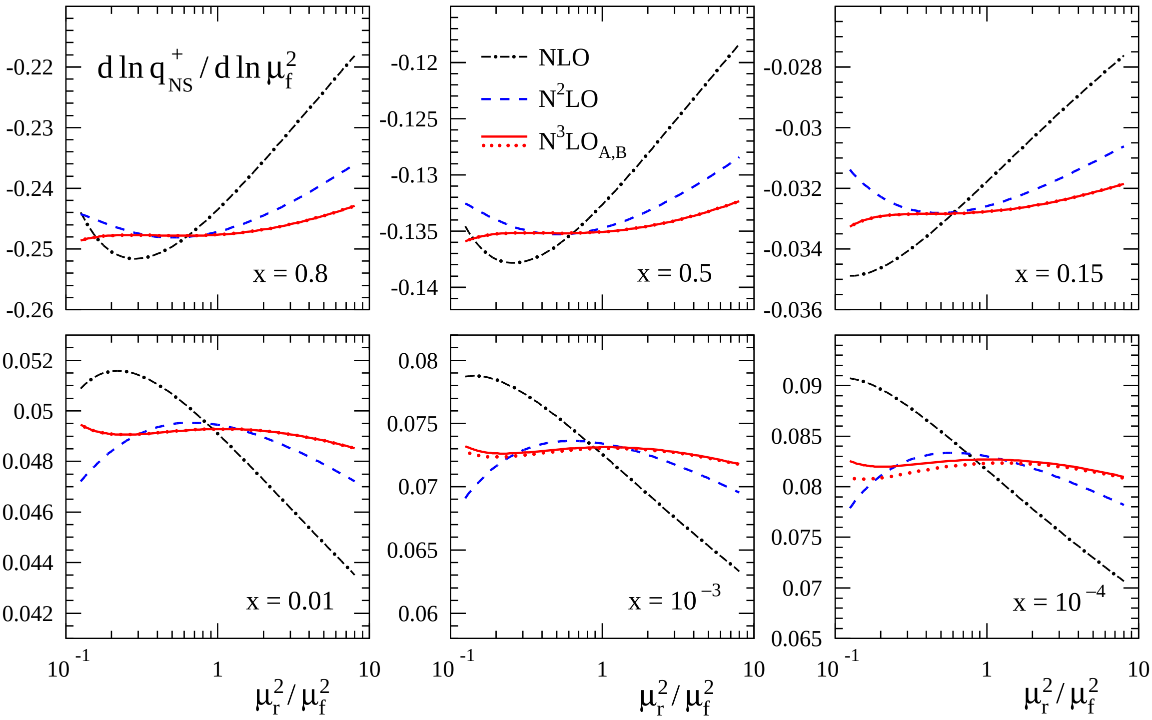


**Figure 4:** The dependence of the NLO, N$^2$LO and N$^3$LO results for $\dot{q}_{\rm ns}^{+} \equiv d\ln q_{\rm ns}^{+}/d\ln\mu_f^2$ on the renormalization scale $\mu_r$ at six typical values of $x$ for the initial conditions (7.1) and $n_f = 4$ flavours. The remaining uncertainty of the four-loop splitting function $P_{\rm ns}^{+(3)}(x)$ leads to the difference of the solid and dotted curves.

## 8. Summary and Outlook

The splitting functions for the non-singlet combinations of quark PDFs have been addressed at the fourth-order (N$^3$LO) of perturbative QCD. The quantities $P_{\rm ns}^{\pm(3)}$ are now known exactly in the limit of a large number of colours $n_c$. Present results for the large-$n_c$ suppressed contributions with $n_f^0$ and $n_f^1$ are still approximate, but sufficiently accurate for phenomenological applications in deep-inelastic scattering and collider physics. FORM and FORTRAN files of these results can be obtained by downloading the source of ref. [8] from `arXiv.org`.

It would be desirable, mostly for theoretical purposes, to obtain also the analytic forms $n_f^0$ and $n_f^1$ parts of $P_{\rm ns}^{\pm(3)}$. So far, only their contributions proportional to the values $\zeta_4$ and $\zeta_5$ of the Riemann $\zeta$-function have been completely determined, together with the (unpublished) $\zeta_3$ part of the $n_f^1$ contributions. The $\zeta_4$ parts are particularly simple; in fact, it turns out that they (and other $\pi^2$ terms) can be predicted via physical evolution kernels from lower-order quantities, see refs. [49,50].

The $\zeta_5$ part of $P_{\rm ns}^{\pm(3)}$, presented in appendix D of ref. [8], includes a (non large-$n_c$) contribution

$$-\,\frac{128}{3}\left\{3C_F^2C_A^2 - 2C_FC_A^3 + 12\,d_F^{abcd}d_A^{abcd}/N_R\right\}\,5\zeta_5\,[S_1(N)]^2\,. \qquad (8.1)$$

The resulting $\ln^2 N$ large-$N$ behaviour needs to be compensated by non-$\zeta_5$ terms. Eq. (8.1) looks exactly like the $\zeta_5$-'tail' of the so-called wrapping correction in the anomalous dimensions in $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory, see refs. [51,52].

Phenomenologically, of course, one rather needs corresponding results for the flavour-singlet splitting functions $P_{ij}^{(3)}(x)$, $i,j=q,g$. At present, it appears computationally too hard to obtain moments of all four functions beyond $N=6$ using the method of refs. [9–12]. Therefore one will need to resort to the OPE, which offers additional theoretical challenges in the massless flavour-singlet case, see refs. [53–55]. We hope to address this issue in a future publication.

**Acknowledgements**

The research reported here has been supported by the *European Research Council* (ERC) Advanced Grant 320651, *HEPGAME*, the grant ST/L000431/1 of the UK *Science & Technology Facilities Council* (STFC), and the *Deutsche Forschungsgemeinschaft* (DFG) grant MO 1801/1-2 and SFB 676 project A3. Part of our computations have been performed on a computer cluster in Liverpool funded by the STFC grant ST/H008837/1.

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