

Exclusive electroproduction of pion pairs

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Abstract

We investigate electroproduction of pion pairs on the nucleon in the framework of QCD factorization for hard exclusive processes. We extend previous analyses by taking the hard-scattering coefficients at next-to-leading order in α_s . The dynamics of the produced pion pair is described by two-pion distribution amplitudes, for which we perform a detailed theoretical and phenomenological analysis. In particular, we obtain constraints on these quantities by comparing our results with measurements of angular observables that are sensitive to the interference between two-pion production in the isoscalar and isovector channels.

1 Introduction

The theoretical treatment of hard exclusive processes has been a challenge in QCD for many years. With the advent of the generalized parton distribution (GPD) formalism a large class of such processes, all involving some hard scale Q^2 , can now be treated on a firm basis, with all non-perturbative physics described by suitable generalized parton distributions and similar quantities. GPDs allow one to relate the information from many different processes within an overall QCD description, including aspects that cannot be deduced directly from experiment, like the transverse spatial distribution of partons and their orbital angular momentum. Pioneering papers of this field are [1, 2, 3, 4], and extensive reviews are given in [5, 6, 7].

The large amount of information contained in GPDs implies that much and diverse data is needed to reliably determine their functional form with respect to the three variables x , ξ , and t (see below). One of the channels for which data is available is exclusive electroproduction of pion pairs [8]. This process has already been studied by some of us [9, 10] at leading order (LO) in α_s , and will be analyzed here at next-to-leading order (NLO). For this we can use various results [11, 12] obtained



earlier for similar processes, which greatly simplifies our task. Apart from GPDs, a second non-perturbative input in the description of pion pair production are two-pion distribution amplitudes (2π DAs), introduced in [1, 13]. Building on earlier work in [14, 15, 16] we will elaborate on the properties and phenomenological description of these quantities, which can be regarded as crossed-channel analogs of GPDs.

The paper is organized as follows. In the following two subsections we define the kinematics and the observables for the process which we will investigate, and recall its factorization property in Bjorken kinematics. In Section 2 we describe the model for GPDs used in our work, and in Section 3 we give a detailed discussion of two-pion distribution amplitudes and their representation in terms of dispersion integrals. The analytic form of the scattering amplitude at NLO in α_s is given in Section 4. In Section 5 we develop a number of model scenarios for 2π DAs, and in Section 6 we present our results for observables in two-pion production and their comparison with the HERMES data from [8]. We summarize our findings in Section 7.

1.1 Kinematics and observables

We describe exclusive two-pion electroproduction on a nucleon using the following momentum variables:

$$e(l) + N(p) \rightarrow e(l') + \pi^+(k) + \pi^-(k') + N(p'). \quad (1)$$

In the one-photon exchange approximation we can reduce our analysis to the hadronic subprocess

$$\gamma^*(q) + N(p) \rightarrow \pi^+(k) + \pi^-(k') + N(p'). \quad (2)$$

We specialize here to the case where the baryon is the same in the initial and final state, but the theory description can be easily extended to the case of a different outgoing baryon. In addition, results for the production of two neutral pions can readily be obtained from (1) using isospin symmetry. We use the conventional variables

$$q = l - l', \quad q^2 = -Q^2, \quad y = \frac{q \cdot p}{l \cdot p}, \quad W^2 = (q + p)^2, \quad x_B = \frac{Q^2}{2p \cdot q} \quad (3)$$

for deep inelastic scattering, and in addition define the momentum transfer

$$\Delta = p' - p, \quad \Delta^2 = t \quad (4)$$

to the nucleon. We denote the nucleon and pion mass by m_N and m_π , respectively, introduce the invariant mass of the two-pion system,

$$(k + k')^2 = m_{\pi\pi}^2 = s_\pi, \quad (5)$$

and neglect the lepton mass throughout. We also write $q' = k + k'$ for the total momentum of the pion pair. Finally, we define the polar and azimuthal angles θ and φ of the π^+ in the rest frame of the two pions, as shown in Fig. 1.

An important set of observables for analyzing the two-pion system are the Legendre moments

$$\langle P_l(\cos\theta) \rangle = \frac{\int_{-1}^1 d\cos\theta P_l(\cos\theta) \frac{d\sigma}{d\cos\theta}}{\int_{-1}^1 d\cos\theta \frac{d\sigma}{d\cos\theta}}, \quad (6)$$

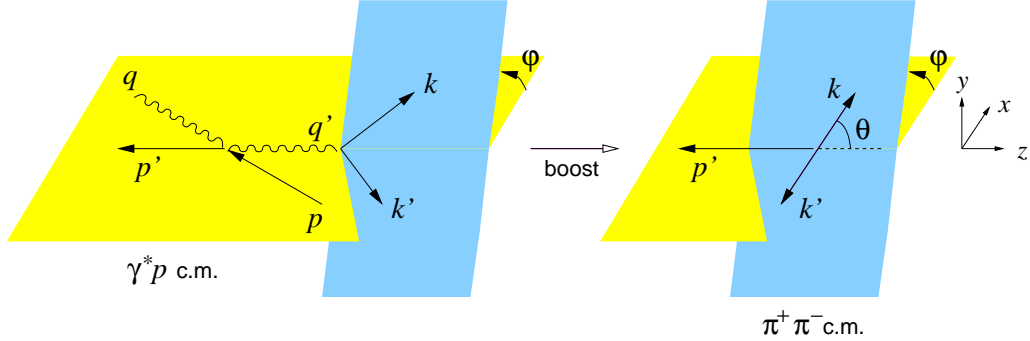


Figure 1: Definition of the angles θ and φ in the two-pion system.

where P_l is a Legendre polynomial. To make their content explicit, we decompose the cross section into partial waves of the produced two-pion system,

$$\frac{1}{\sigma} \frac{d\sigma}{d\cos\theta d\varphi} = \sum_{JJ'\lambda\lambda'} \rho_{\lambda\lambda'}^{JJ'} Y_{J\lambda}(\theta, \varphi) Y_{J'\lambda'}^*(\theta, \varphi), \quad \sum_{J\lambda} \rho_{\lambda\lambda}^{JJ} = 1, \quad (7)$$

where ρ is the spin density matrix of the pion pair. Its diagonal elements $\rho_{\lambda\lambda}^{JJ}$ give the probability that the two pions are in the state with total angular momentum J and angular momentum component λ along the z -axis in Fig. 1, whereas the off-diagonal terms describe the corresponding interference terms. Neglecting $J > 2$ contributions we have in particular [17, 6]

$$\begin{aligned} \langle P_1 \rangle &= \frac{1}{\sqrt{15}} \operatorname{Re} \left[4\sqrt{3} \rho_{11}^{21} + 4\rho_{00}^{21} + 2\sqrt{5} \rho_{00}^{10} \right], \\ \langle P_3 \rangle &= \frac{6}{7\sqrt{5}} \operatorname{Re} \left[-2\rho_{11}^{21} + \sqrt{3} \rho_{00}^{21} \right]. \end{aligned} \quad (8)$$

Note that the combination

$$\langle P_1(\cos\theta) + \frac{7}{3} P_3(\cos\theta) \rangle = \frac{2}{\sqrt{3}} \operatorname{Re} \left[\sqrt{5} \rho_{00}^{21} + \rho_{00}^{10} \right] \quad (9)$$

projects out the state with zero total helicity of the final two pions, whereas

$$\langle P_1(\cos\theta) - \frac{14}{9} P_3(\cos\theta) \rangle = \frac{2}{3} \operatorname{Re} \left[2\sqrt{5} \rho_{11}^{21} + \sqrt{3} \rho_{00}^{10} \right] \quad (10)$$

involves the helicity-one but not the helicity-zero state of the two pions in the $J = 2$ partial wave.

1.2 Factorization of the hadronic process

We consider the hadronic process (2) in the Bjorken limit, where the scattering energy and the virtuality of the exchanged photon are much larger than $\sqrt{-t}$, the nucleon mass, and the invariant mass of the produced two-pion system,

$$W^2, Q^2 \gg -t, m_N^2, m_{\pi\pi}^2. \quad (11)$$

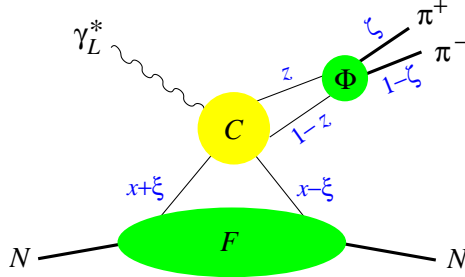


Figure 2: Factorization of hard exclusive pion pair production by a virtual photon. x , ξ and z , ζ are momentum fractions as explained in Sections 2 and 3.

According to the factorization theorem from [18, 19] the scattering amplitude in this limit can be written as the convolution of coefficient functions C^{ij} with nucleon matrix elements F^i parameterized by GPDs and with two-pion distribution amplitudes Φ^j ,

$$T = \frac{1}{Q} \sum_{ij} \int_{-1}^1 dx \int_0^1 dz F^i(x, \xi, t; \mu_F) C^{ij}(x, \xi, z; Q, \mu_F, \mu_R) \Phi^j(z, \zeta, s_\pi; \mu_F) + \mathcal{O}\left(\frac{1}{Q^2}\right), \quad (12)$$

where i and j stand for quarks or gluons, and μ_F and μ_R denote the factorization and renormalization scales. More details on F^i and Φ^j and on their arguments ξ and ζ will be given in the following two sections. The amplitude (12) refers to longitudinal polarization of the virtual photon and of the pion pair, i.e. to $\lambda = 0$ in (7). The amplitudes involving a transverse photon or nonzero λ decrease at least like $1/Q^2$ and are hence power suppressed in Bjorken kinematics. The factorization formula (12) is illustrated in Fig. 2.

We caution that the power corrections in (12) need not be numerically small for Q^2 of a few GeV^2 . Phenomenological estimates based on transverse parton momentum effects in the hard scattering [20, 21] or on renormalon calculations [22, 23] have indeed found substantial power corrections to the amplitude. On the other hand, the fair agreement of the leading-order calculation for $\pi^+\pi^-$ production in [10] with the experimental results [8] for the Legendre moments (8) gives hope that power corrections to these observables may be not too large in the kinematics of the HERMES measurement.

2 Modeling the GPDs

Generalized parton distributions parameterize matrix elements of light-cone separated quark or gluon operators. The distributions relevant for our process (2) are given by

$$\begin{aligned} F^q(x, \xi, t) &= \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{ix\lambda(P \cdot n)} \langle p' | \bar{q}(-\frac{1}{2}\lambda n) \not{n} q(\frac{1}{2}\lambda n) | p \rangle \\ &= \frac{1}{2P \cdot n} \left[H^q(x, \xi, t) \bar{u}(p') \not{n} u(p) + E^q(x, \xi, t) \bar{u}(p') \frac{i\sigma^{\alpha\beta} n_\alpha \Delta_\beta}{2m_N} u(p) \right], \\ F^g(x, \xi, t) &= \frac{1}{P \cdot n} \int \frac{d\lambda}{2\pi} e^{ix\lambda(P \cdot n)} n_\alpha n_\beta \langle p' | G^{\alpha\mu}(-\frac{1}{2}\lambda n) G_\mu{}^\beta(\frac{1}{2}\lambda n) | p \rangle \\ &= \frac{1}{2P \cdot n} \left[H^g(x, \xi, t) \bar{u}(p') \not{n} u(p) + E^g(x, \xi, t) \bar{u}(p') \frac{i\sigma^{\alpha\beta} n_\alpha \Delta_\beta}{2m_N} u(p) \right], \end{aligned} \quad (13)$$

where n is a lightlike auxiliary vector and we have omitted the dependence on μ_F for simplicity. For definiteness we consider proton distributions in this section, the corresponding expressions for a neutron target are readily obtained from isospin symmetry. The insertion of a Wilson line between the field operators is implied in (13). The light-cone momentum fractions of the partons with respect to the average nucleon momentum $P = \frac{1}{2}(p + p')$ are parameterized by x and ξ as shown in Fig. 2. The skewness $\xi = -2(\Delta \cdot n)/(P \cdot n)$ is related to the Bjorken variable in our process (2) by

$$\xi = \frac{x_B}{2 - x_B}. \quad (14)$$

In the forward limit, $p' = p$, the distributions $E^q(x, \xi, t)$ and $E^g(x, \xi, t)$ decouple in the matrix elements (13), whereas $H^q(x, \xi, t)$ and $H^g(x, \xi, t)$ reduce to the ordinary parton densities

$$\begin{aligned} H^q(x, 0, 0) &= q(x) && \text{for } x > 0, \\ H^q(x, 0, 0) &= -\bar{q}(-x) && \text{for } x < 0, \\ H^g(x, 0, 0) &= xg(x) && \text{for } x > 0. \end{aligned} \quad (15)$$

Following the notation in [6] we also use combinations $F^{q(C)}$ of quark GPDs corresponding to t -channel exchange with definite charge conjugation parity C ,

$$\begin{aligned} F^{q(+)}(x, \xi, t) &= F^q(x, \xi, t) - F^q(-x, \xi, t), \\ F^{q(-)}(x, \xi, t) &= F^q(x, \xi, t) + F^q(-x, \xi, t), \end{aligned} \quad (16)$$

with analogous decompositions for the distributions H and E . The gluon GPDs are even functions of x , i.e. $F^g(x, \xi, t) = F^g(-x, \xi, t)$, and of course correspond to $C = +1$ exchange.

In our calculations we use the same model for the functions H^q and H^g as in [10]. It is based on Radyushkin's double distribution ansatz [24, 25], supplemented by the Polyakov-Weiss term [26]. For the quark GPD at $t = 0$ we write

$$H^q(x, \xi, 0) = H_{\text{DD}}^q(x, \xi, 0) + \frac{1}{3}\theta(\xi - |x|)D\left(\frac{x}{\xi}\right) \quad (17)$$

and use for the Polyakov-Weiss term the estimate [27] obtained in the chiral quark-soliton model,

$$D(x) = -4.0(1 - x^2) \left[C_1^{3/2}(x) + 0.3 C_3^{3/2}(x) + 0.1 C_5^{3/2}(x) \right], \quad (18)$$

where $C_1^{3/2}(x)$ are Gegenbauer polynomials. An analogous representation holds for $H^g(x, \xi, 0)$, where we set the Polyakov-Weiss term to zero since there is no phenomenological estimate available for it in the literature. The double distribution parts are written as

$$\begin{aligned} H_{\text{DD}}^q(x, \xi, 0) &= \int_{-1}^1 d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \delta(x - \beta - \xi\alpha) \left[\theta(\beta) q(\beta) - \theta(-\beta) \bar{q}(-\beta) \right] h(\beta, \alpha), \\ H_{\text{DD}}^g(x, \xi, 0) &= \int_{-1}^1 d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \delta(x - \beta - \xi\alpha) \beta \left[\theta(\beta) g(\beta) - \theta(-\beta) g(-\beta) \right] h(\beta, \alpha) \end{aligned} \quad (19)$$

with a so-called profile function

$$h(\beta, \alpha) = \frac{\Gamma(2b + 2)}{2^{2b+1} [\Gamma(b + 1)]^2} \frac{[(1 - |\beta|)^2 - \alpha^2]^b}{(1 - |\beta|)^{2b+1}}, \quad (20)$$

where we take $b = 1$ for quarks and $b = 2$ for gluons. For the ordinary parton densities we will use the MRST 2004 NLO parameterization [28] at scale $Q^2 = 3.2 \text{ GeV}^2$ or $Q^2 = 7 \text{ GeV}^2$, depending on the value of Q^2 in the experimental observable to be described.

For the t -dependence of the GPDs we use a factorized ansatz [29, 30]. This is known to be oversimplified, both from general considerations [31] and from lattice calculations [32, 33]. In the present paper we will however concentrate on observables such as the Legendre moments (8), where the bulk of the t -dependence in the GPDs drops out and only details of this dependence matter that are currently not well constrained by phenomenology. For our purpose, a factorized ansatz should hence be sufficient, and for the $C = -1$ quark combination we write

$$H^{q(-)}(x, \xi, t) = H^{q(-)}(x, \xi, 0) \frac{F_1^q(t)}{F_1^q(0)}, \quad (21)$$

where $F_1^q(t)$ is the contribution from quark flavor q to the proton Dirac form factor. This ansatz ensures the sum rule

$$\int_{-1}^1 dx H^q(x, \xi, t) = F_1^q(t). \quad (22)$$

The input needed in (21) can be obtained from the measured electromagnetic nucleon form factors $F_1^p = \frac{2}{3}F_1^u - \frac{1}{3}F_1^d - \frac{1}{3}F_1^s$ and $F_1^n = \frac{2}{3}F_1^d - \frac{1}{3}F_1^u - \frac{1}{3}F_1^s$ combined with lattice calculations of the strangeness form factors. We have taken the parameterizations from [34] and [35], which should be sufficiently reliable given the limitations of the factorized form (21). For the gluon GPD we make an analogous ansatz

$$H^g(x, \xi, t) = H^g(x, \xi, 0) \frac{F_\theta(t)}{F_\theta(0)}, \quad (23)$$

where $F_\theta(t)$ is a form factor of the gluon part of the energy-momentum tensor, for which we use the model from [36],

$$F_\theta(t) = F_\theta(0) [1 - t/(3M_\theta^2)]^{-3} \quad (24)$$

with $M_\theta^2 = 2.6 \text{ GeV}^2$. The $C = +1$ quark combination (which is not constrained by the Dirac form factors) is modeled by a corresponding ansatz with the same form factor $F_\theta(t)$, motivated by the fact that $H^{q(+)}(x, \xi, t)$ and $H^g(x, \xi, t)$ mix under evolution.

We do not attempt to model E^q and E^g here and will neglect their contribution when calculating observables. This should be good enough for our purposes, because for an unpolarized target they enter with prefactors $t/(4m_N^2)$ or ξ^2 , which are small in the kinematics we will consider.

3 Two-pion distribution amplitudes

A central ingredient for the description of the process (1) are the two-pion distribution amplitudes. For the $\pi^+\pi^-$ system they are defined as [6]

$$\begin{aligned} \Phi^q(z, \zeta, s_\pi) &= \int \frac{d\lambda}{2\pi} e^{-iz\lambda(q'\cdot n)} \langle \pi^+(k) \pi^-(k') | \bar{q}(\lambda n) \not{n} q(0) | 0 \rangle, \\ \Phi^g(z, \zeta, s_\pi) &= \frac{1}{q' \cdot n} \int \frac{d\lambda}{2\pi} e^{-iz\lambda(q'\cdot n)} n_\alpha n_\beta \langle \pi^+(k) \pi^-(k') | G^{\alpha\mu}(\lambda n) G_\mu^\beta(0) | 0 \rangle, \end{aligned} \quad (25)$$

where n is a lightlike auxiliary vector and we have suppressed the dependence on the factorization scale μ_F as before. As in the case of GPDs, the insertion of an appropriate Wilson line between the

field operators is implied. The 2π DAs describe the exclusive fragmentation of a pair of quarks or gluons into the final pion pair [13]. The variable z is the light-cone momentum fraction of one of the two partons with respect to the total momentum q' of the pion pair. The variable ζ characterizes the distribution of the total momentum q' among the two pions,

$$\zeta = \frac{k \cdot n}{q' \cdot n}, \quad (26)$$

and is related to the polar angle θ and the relativistic velocity of the π^+ in the c.m. of the pair by

$$\beta \cos \theta = 2\zeta - 1, \quad \beta = \sqrt{1 - \frac{4m_\pi^2}{s_\pi}}. \quad (27)$$

It is useful to project out the combinations

$$\Phi^{q(\pm)}(z, \zeta, s_\pi) = \frac{1}{2} [\Phi^q(z, \zeta, s_\pi) \pm \Phi^q(z, 1 - \zeta, s_\pi)] \quad (28)$$

describing a two-pion system with definite charge conjugation parity $C = \pm 1$. Charge conjugation and isospin symmetry imply that

$$\Phi^{I=0} = \Phi^{u(+)} = \Phi^{d(+)}, \quad \Phi^{I=1} = \Phi^{u(-)} = -\Phi^{d(-)}, \quad (29)$$

where the combinations Φ^I associated with definite isospin I of the pion pair have been introduced in [14]. The 2π DAs for gluons and for strange quarks are of course pure isosinglet. We also remark that the distribution amplitude for u or for d quarks in a $\pi^0\pi^0$ pair is equal to $\Phi^{I=0}$ by isospin invariance.

Following [14, 15] we expand the distribution amplitudes in Gegenbauer polynomials $C_n^m(2z - 1)$ and Legendre polynomials $P_l(2\zeta - 1)$,

$$\begin{aligned} \Phi^{q(-)}(z, \zeta, s_\pi) &= 6z(1-z) \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \sum_{\substack{l=1 \\ \text{odd}}}^{n+1} B_{nl}^{q(-)}(s_\pi) C_n^{3/2}(2z-1) P_l(2\zeta-1), \\ \Phi^{q(+)}(z, \zeta, s_\pi) &= 6z(1-z) \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \sum_{\substack{l=0 \\ \text{even}}}^{n+1} B_{nl}^{q(+)}(s_\pi) C_n^{3/2}(2z-1) P_l(2\zeta-1), \\ \Phi^g(z, \zeta, s_\pi) &= 9z^2(1-z)^2 \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \sum_{\substack{l=0 \\ \text{even}}}^{n+1} B_{nl}^g(s_\pi) C_{n-1}^{5/2}(2z-1) P_l(2\zeta-1), \end{aligned} \quad (30)$$

where the restrictions to even or odd n and l follow from charge conjugation invariance.¹ We will also use the notation

$$B^{I=0} = B^{u(+)} = B^{d(+)}, \quad B^{I=1} = B^{u(-)} = -B^{d(-)} \quad (31)$$

corresponding to (29). The expansion of the z -dependence in Gegenbauer polynomials is chosen such that to leading order in α_s the coefficients B_{nl} evolve multiplicatively in the factorization scale μ_F , with mixing occurring only between B_{nl}^g and the quark singlet combination $\sum_q B_{nl}^{q(+)}$, see e.g. [16].

¹The coefficients B_{nl}^g follow the convention of [6] and are related to the coefficients A_{nl}^G in [15] by $3B_{nl}^g = 10A_{n-1,l}^G$.

The expansion of the ζ -dependence in Legendre polynomials is rather directly related to the partial wave expansion of the two-pion system, as we shall see shortly.

The coefficients $B_{nl}(s_\pi)$ parameterize matrix elements of local operators between a $\pi^+\pi^-$ state and the vacuum, i.e. they are form factors in the time-like region. By analytic continuation they are related to the spacelike form factors $A_{nk}(t_\pi)$ defined by

$$\begin{aligned} \langle \pi^+(p') | \bar{q}(0) \mathbf{S} \gamma_{\mu_1} i \overleftrightarrow{D}_{\mu_2} \dots i \overleftrightarrow{D}_{\mu_n} q(0) | \pi^+(p) \rangle &= 2 \sum_{\substack{k=0 \\ \text{even}}}^n A_{nk}^q(t_\pi) \mathbf{S} \Delta_{\mu_1} \dots \Delta_{\mu_k} P_{\mu_{k+1}} \dots P_{\mu_n}, \\ \langle \pi^+(p') | \mathbf{S} G_{\mu_1\nu}(0) i \overleftrightarrow{D}_{\mu_2} \dots i \overleftrightarrow{D}_{\mu_{n-1}} G^\nu_{\mu_n}(0) | \pi^+(p) \rangle &= 2 \sum_{\substack{k=0 \\ \text{even}}}^n A_{nk}^g(t_\pi) \mathbf{S} \Delta_{\mu_1} \dots \Delta_{\mu_k} P_{\mu_{k+1}} \dots P_{\mu_n}, \end{aligned} \quad (32)$$

where $\overleftrightarrow{D} = \frac{1}{2}(\overrightarrow{D} - \overleftarrow{D})$, $P = \frac{1}{2}(p + p')$, $\Delta = p' - p$, $t_\pi = \Delta^2$, and \mathbf{S} denotes symmetrization in all uncontracted Lorentz indices and subtraction of trace terms. These form factors are related to the Mellin moments of pion GPDs. For $t_\pi = 0$ they reduce to the moments of the usual quark and gluon densities in the pion, and one finds in particular

$$\begin{aligned} B_{n-1,n}^q(0) &= \frac{2}{3} \frac{2n+1}{n+1} A_{n0}^q(0) = \frac{2}{3} \frac{2n+1}{n+1} \int_0^1 dx x^{n-1} [q_\pi(x) + (-1)^n \bar{q}_\pi(x)], \\ B_{n-1,n}^g(0) &= \frac{8}{3} \frac{2n+1}{(n+1)(n+2)} A_{n0}^g(0) = \frac{8}{3} \frac{2n+1}{(n+1)(n+2)} \int_0^1 dx x^{n-1} g_\pi(x). \end{aligned} \quad (33)$$

Phenomenological experience with the distribution amplitudes of single mesons suggests that the coefficients of the expansion in Gegenbauer polynomials decrease reasonably fast with n , see e.g. [37, 38, 39]. This trend is enhanced for larger factorization scales since the Gegenbauer coefficients decrease faster with μ_F for increasing moment index n . Only the coefficient for $n = 0$ and a linear combination of the $n = 1$ quark and gluon coefficients are independent of μ_F and hence remain nonzero at asymptotically large μ_F . In our phenomenological application we will only retain the $n = 0$ and $n = 1$ terms in (30), keeping in mind that at moderately large scales this may not be a very accurate approximation. We then have

$$\Phi^{I=1}(z, \zeta, s_\pi) = 6z(1-z)(2\zeta-1)F_\pi(s_\pi), \quad (34)$$

where we have identified $B_{01}^{I=1}(s_\pi)$ with the electromagnetic pion form factor $F_\pi(s_\pi)$, and

$$\begin{aligned} \Phi^{I=0}(z, \zeta, s_\pi) &= 18z(1-z)(2z-1) \left[B_{10}^{I=0}(s_\pi) + B_{12}^{I=0}(s_\pi) P_2(2\zeta-1) \right], \\ \Phi^g(z, \zeta, s_\pi) &= 9z^2(1-z)^2 \left[B_{10}^g(s_\pi) + B_{12}^g(s_\pi) P_2(2\zeta-1) \right]. \end{aligned} \quad (35)$$

Notice that $F_\pi(s_\pi)$ is independent of the factorization scale since it is associated with the conserved quark vector current. The coefficients B_{1l}^q and B_{1l}^g depend on μ_F in the same way as the quark and gluon momentum fractions $\int_0^1 dx x [q_\pi(x) + \bar{q}_\pi(x)]$ and $\int_0^1 dx x g_\pi(x)$, in accordance with (33). The sum $B_{1l}^g + \sum_q B_{1l}^q$ is again μ_F independent since it is associated with the *total* energy-momentum tensor. The coefficients in (35) are related to the form factors in (32) by²

$$B_{10}(s_\pi) = \frac{5}{9} A_{20}(s_\pi) + \frac{20}{3} A_{22}(s_\pi), \quad B_{12}(s_\pi) = \frac{10}{9} A_{20}(s_\pi) \quad (36)$$

²We remark that the coefficients in eq. (90) of [6] should read $\frac{10}{9}$ and not $\frac{9}{10}$.

for both quarks and gluons. Chiral dynamics constrains these form factors for $|s_\pi| \ll \Lambda_\chi$, where $\Lambda_\chi = 4\pi f_\pi \approx 1.16$ GeV is the characteristic scale of chiral symmetry breaking [14, 40, 10]. From the one-loop calculation [41] in chiral perturbation theory we obtain

$$\begin{aligned} B_{10}(s_\pi) &= -B_{12}^{(0)} \left\{ 1 + c_{10}^{(m)} m_\pi^2 + c_{10}^{(s)} s_\pi + \frac{m_\pi^2 - 2s_\pi}{2\Lambda_\chi^2} \left[\ln \frac{m_\pi^2}{\mu_\chi^2} + \frac{4}{3} - \frac{s_\pi + 2m_\pi^2}{s_\pi} J(\beta) \right] \right\} + O(\Lambda_\chi^{-4}), \\ B_{12}(s_\pi) &= B_{12}^{(0)} \left\{ 1 + c_{12}^{(m)} m_\pi^2 + c_{12}^{(s)} s_\pi \right\} + O(\Lambda_\chi^{-4}), \end{aligned} \quad (37)$$

where $J(\beta) = 2 + \beta \ln[(\beta - 1)/(\beta + 1)]$ with β defined in (27), and $c_{1l}^{(m)}$, $c_{1l}^{(s)}$ are unknown low-energy constants, whose natural size is Λ_χ^{-2} . We have not displayed the dependence of $c_{10}^{(m)}$ and $c_{10}^{(s)}$ on the renormalization scale μ_χ , which cancels against the explicit logarithm in (37). We recover the soft-pion theorem $B_{10}(0) \approx -B_{12}(0)$ from [14] and obtain the leading chiral correction to it:

$$B_{10}(0) = -B_{12}(0) \left\{ 1 + \frac{m_\pi^2}{2\Lambda_\chi^2} \left[\ln \frac{m_\pi^2}{\mu_\chi^2} + 1 \right] + m_\pi^2 \left[c_{10}^{(m)} - c_{12}^{(m)} \right] \right\} + O(\Lambda_\chi^{-4}). \quad (38)$$

3.1 Partial wave decomposition and Omnès representation

The expansion (30) of the ζ dependence in Legendre polynomials resembles a partial wave decomposition of the two-pion system, which expands in $P_l(\cos \theta)$. Indeed one can readily rewrite the polynomials $P_l(2\zeta - 1) = P_l(\beta \cos \theta)$ in terms of $P_k(\cos \theta)$ with $k \leq l$. For $n = 2$ one obtains [16]

$$B_{10}(s_\pi) + B_{12}(s_\pi) P_2(2\zeta - 1) = \tilde{B}_{10}(s_\pi) + \tilde{B}_{12}(s_\pi) P_2(\cos \theta), \quad (39)$$

where the new coefficients

$$\tilde{B}_{10}(s_\pi) = B_{10}(s_\pi) - \frac{1 - \beta^2}{2} B_{12}(s_\pi) = B_{10}(s_\pi) - \frac{2m_\pi^2}{s_\pi} B_{12}(s_\pi), \quad \tilde{B}_{12}(s_\pi) = \beta^2 B_{12}(s_\pi) \quad (40)$$

describe the two pions in an S and a D wave, respectively. This holds for both quark and gluon coefficients, and we drop the corresponding superscript in the present subsection. The phase of $\tilde{B}_{nl}(s_\pi)$ reflects the interaction of two pions in the partial wave l . For values of s_π where $\pi\pi$ scattering is elastic, one can apply Watson's theorem and finds [14]

$$\tilde{B}_{nl}^*(s_\pi) = \tilde{B}_{nl}(s_\pi) \exp[-2i\delta_l(s_\pi)], \quad (41)$$

where $\delta_l(s_\pi)$ is the phase shift for elastic $\pi\pi$ scattering in the appropriate isospin channel ($I = 0$ for even l and $I = 1$ for odd l). This relation determines the phase of \tilde{B}_{nl} up to a multiple of π .

The form factors $B_{nl}(s_\pi)$ satisfy the usual analyticity properties in s_π , i.e. they have a branch cut on the real axis above threshold ($s_\pi \geq 4m_\pi^2$) and are real-valued for real s_π below threshold. Together with the phase information from Watson's theorem, one can write down an Omnès representation for the form factors, as was first pointed out in [14]. We need to review this issue here and start with a derivation of the Omnès representation in a form adapted to our purpose. Let $F(s_\pi)$ be a form factor which is nonzero at $s_\pi = 0$ and at $s_\pi = 4m_\pi^2$ and has the following properties:

1. $F(s_\pi)$ is analytic in the s_π plane except for a cut along the real axis for $s_\pi \geq 4m_\pi^2$, and it is real-valued for real $s_\pi < 4m_\pi^2$.

2. The complex phase of $F(s_\pi)/F(4m_\pi^2)$ is $\delta_F(s_\pi)$. With Watson's theorem we will have $\delta_F(s_\pi) = \delta_l(s_\pi)$ for s_π in the region above threshold where $\pi\pi$ scattering is elastic.
3. $\delta_F(s_\pi)$ tends to a constant for $|s_\pi| \rightarrow \infty$.
4. $F(s_\pi)$ has a finite number of simple zeroes at $s_\pi = s_1, s_\pi = s_2, \dots, s_\pi = s_n$, where n may also be zero. Because of property 1 the s_i are either real-valued or come in complex conjugate pairs $s_{i+1}^* = s_i$.

We now consider

$$G(s_\pi) = \ln \frac{F(s_\pi)}{F(0) (1 - s_\pi/s_1)(1 - s_\pi/s_2) \dots (1 - s_\pi/s_n)}, \quad (42)$$

where the Riemann sheet of the complex logarithm is chosen such that $G(s_\pi)$ is continuous and that $G(0) = 0$. Then $G(s_\pi)$ has the same analyticity properties as $F(s_\pi)$. Note that for this it was necessary to divide out possible zeroes of $F(s_\pi)$ before taking the logarithm. One can now write down a dispersion relation with $N \geq 1$ subtractions,

$$G(s_\pi) = \sum_{k=1}^{N-1} \frac{s_\pi^k}{k!} \frac{d^k}{ds_\pi^k} G(0) + \frac{s_\pi^N}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\delta_F(s)}{s^N (s - s_\pi - i\varepsilon)}, \quad (43)$$

where we have used that $\text{Im } G(s_\pi) = \delta_F(s_\pi)$ for real $s_\pi \geq 4m_\pi^2$. A term with $k = 0$ does not appear in the sum because $G(0) = 0$. The term $i\varepsilon$ implements the usual prescription for handling the singularity at $s = s_\pi$. We thus have

$$\begin{aligned} F(s_\pi) &= F(0) (1 - s_\pi/s_1)(1 - s_\pi/s_2) \dots (1 - s_\pi/s_n) \\ &\times \exp \left[\sum_{k=1}^{N-1} \frac{s_\pi^k}{k!} \frac{d^k}{ds_\pi^k} G(0) + \frac{s_\pi^N}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\delta_F(s)}{s^N (s - s_\pi - i\varepsilon)} \right]. \end{aligned} \quad (44)$$

This representation can readily be used for $F(s_\pi) = B_{12}(s_\pi)$. Assuming that $B_{12}(s_\pi)$ has no zero, we recover the representation of this form factor already given in [14],

$$\tilde{B}_{12}(s_\pi) = \beta^2 B_{12}(0) f_2(s_\pi) \quad (45)$$

with the Omnès function

$$\begin{aligned} f_2(s_\pi) &= \exp \left[\frac{s_\pi}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\tilde{\delta}_2(s)}{s(s - s_\pi - i\varepsilon)} \right] \\ &= \exp \left[s_\pi \frac{d}{ds_\pi} \ln B_{12}(0) + \frac{s_\pi^2}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\tilde{\delta}_2(s)}{s^2(s - s_\pi - i\varepsilon)} \right], \end{aligned} \quad (46)$$

where $\tilde{\delta}_2(s_\pi)$ is the phase of $B_{12}(s_\pi)/B_{12}(4m_\pi^2)$ and the forms with $N = 1$ and $N = 2$ subtractions are simultaneously valid. At small enough s_π one has $\tilde{\delta}_2(s_\pi) = \delta_2(s_\pi)$.

For the S wave the situation is more involved. To make use of the phase information from Watson's theorem one has to consider $\tilde{B}_{10}(s_\pi)$, which has a pole at $s_\pi = 0$ according to (40). We can however use the representation (44) for

$$F(s_\pi) = s_\pi \tilde{B}_{10}(s_\pi) = s_\pi B_{10}(s_\pi) - 2m_\pi^2 B_{12}(s_\pi). \quad (47)$$

According to the result (37) from chiral perturbation theory, this form factor has a zero for $s_1 \approx -2m_\pi^2$. For the following it is convenient to write

$$s_1 = \frac{2m_\pi^2}{1+\epsilon} \frac{B_{12}(0)}{B_{10}(0)}, \quad (48)$$

so that $F(0)(1-s_\pi/s_1) = -2m_\pi^2 B_{12}(0) + (1+\epsilon)s_\pi B_{10}(0)$. If we assume that $s_\pi \tilde{B}_{10}(s_\pi)$ only has the zero just discussed, we have

$$\frac{d}{ds_\pi} G(0) = \frac{d}{ds_\pi} \ln B_{12}(0) + \frac{\epsilon}{2m_\pi^2} \frac{B_{10}(0)}{B_{12}(0)}, \quad (49)$$

where both terms are of order Λ_χ^{-2} . The Omnès representation (44) for $F(s_\pi) = s_\pi \tilde{B}_{10}(s_\pi)$ now gives

$$\tilde{B}_{10}(s_\pi) = -B_{12}(0) \frac{3C - \beta^2}{2} f_0(s_\pi) \quad (50)$$

with

$$C = \frac{1}{3} - \frac{2(1+\epsilon)}{3} \frac{B_{10}(0)}{B_{12}(0)} \quad (51)$$

and

$$\begin{aligned} f_0(s_\pi) &= \exp \left[\frac{s_\pi}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\tilde{\delta}_0(s)}{s(s-s_\pi-i\epsilon)} \right] \\ &= \exp \left[s_\pi \left\{ \frac{d}{ds_\pi} \ln B_{12}(0) + \frac{\epsilon}{2m_\pi^2} \frac{B_{10}(0)}{B_{12}(0)} \right\} + \frac{s_\pi^2}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\tilde{\delta}_0(s)}{s^2(s-s_\pi-i\epsilon)} \right], \end{aligned} \quad (52)$$

where $\tilde{\delta}_0(s_\pi)$ is the phase of $\tilde{B}_{10}(s_\pi)/\tilde{B}_{10}(4m_\pi^2)$. With (37) the constants appearing in the Omnès representation can be expressed as

$$\frac{d}{ds_\pi} G(0) = -\frac{1}{\Lambda_\chi^2} \left[\ln \frac{m_\pi^2}{\mu_\chi^2} + \frac{17}{12} \right] + c_{10}^{(s)} + O(\Lambda_\chi^{-4}), \quad (53)$$

$$C = 1 + \frac{5m_\pi^2}{3\Lambda_\chi^2} \left[\ln \frac{m_\pi^2}{\mu_\chi^2} + \frac{4}{3} \right] + \frac{2m_\pi^2}{3} \left[c_{10}^{(m)} - c_{12}^{(m)} - 2c_{10}^{(s)} + 2c_{12}^{(s)} \right] + O(\Lambda_\chi^{-4}). \quad (54)$$

If we take $\mu_\chi = m_\rho$ then the first term in (53) gives -1.5 GeV^{-2} and the first two terms in (54) give 0.95. Without further dynamical input we can of course not estimate the values of the low-energy constants.

Inserting (45) and (50) into (39), we obtain

$$B_{10}(s_\pi) + B_{12}(s_\pi) P_2(2\zeta - 1) = -B_{12}(0) \left[\frac{3C - \beta^2}{2} f_0(s_\pi) - \beta^2 f_2(s_\pi) P_2(\cos \theta) \right] \quad (55)$$

with the Omnès functions f_0 and f_2 given in (52) and (46) and C specified by (51) and (48). This coincides with the representation given without derivation in [15] and used in [9, 10]. The paper [15] did not explicitly define the coefficient C , but for the quark isosinglet case it quoted an estimate

$C = 1 - bm_\pi^2 + O(m_\pi^4)$ with $b \approx -1.7 \text{ GeV}^{-2}$ from an instanton model calculation, which results in $C \approx 0.97$. The Omnès functions used in [15, 9, 10] were for $N = 1$ subtraction and coincide with the first lines of (52) and (46). For $N > 1$ subtractions our result differs from the dispersion relation for $B_{10}(s_\pi)$ given in these papers, where the transformation from $B_{10}(s_\pi)$ to $\tilde{B}_{10}(s_\pi)$ and the zero of $\tilde{B}_{10}(s_\pi)$ at $s_\pi \approx -2m_\pi^2$ is not discussed. In the Omnès function $f_0(s_\pi)$ this would result in a term $s_\pi \frac{d}{ds_\pi} \ln B_{10}(0)$ instead of $s_\pi \frac{d}{ds_\pi} G(0)$ specified by (49). Using the result (37) from chiral perturbation theory we find that

$$\frac{d}{ds_\pi} \ln B_{10}(0) = \frac{d}{ds_\pi} G(0) + \frac{19}{60} \frac{1}{\Lambda_\chi^2} + O(\Lambda_\chi^{-4}) \quad (56)$$

with $\frac{d}{ds} G(0)$ given in (53). Numerically, the second term on the r.h.s. is 0.23 GeV^{-2} .

The results of this subsection can be applied to both the isosinglet quark and to the gluon form factors, where the forward limits

$$B_{12}^{I=0}(0) = \frac{10}{9} A_{20}^u(0) = \frac{10}{9} A_{20}^d(0) \quad B_{12}^g(0) = \frac{10}{9} A_{20}^g(0) \quad (57)$$

with

$$A_{20}^q(0) = \int_0^1 dx x [q_\pi(x) + \bar{q}_\pi(x)], \quad A_{20}^g(0) = \int_0^1 dx x g_\pi(x) \quad (58)$$

are of order one. In contrast, $B_{12}^s(0)$ is $10/9$ times the momentum fraction carried by strange quarks and antiquarks in a pion and therefore quite small. This calls for a careful analysis of the size of different terms in the chiral expansion of $B_{12}^s(s_\pi)$. We shall not pursue this issue here since we will not include the 2π DA for strangeness in our phenomenological application.

4 The scattering amplitude at NLO

In this section we give the expression of the scattering amplitude for two-pion production to leading power in $1/Q$ and to NLO in α_s . We decompose the amplitude for $\gamma^* N \rightarrow \pi^+ \pi^- N$ into terms $T^{(C)}$ describing a two-pion state with definite charge conjugation parity $C = -1$ or $C = +1$,

$$T = T^{(-)} + T^{(+)} \quad (59)$$

We note that the corresponding amplitude for $\gamma^* N \rightarrow \pi^0 \pi^0 N$ is simply given by $T^{(+)}$. To leading power in $1/Q$ we have

$$T^{(-)} = \frac{2\pi\sqrt{4\pi\alpha}}{N_c \xi Q} \int_{-1}^1 dx \int_0^1 dz \sum_{q=u,d} e_q \Phi^{q(-)}(z, \zeta, s_\pi) \left[Q^{(+)}(z, x/\xi) F^{q(+)}(x, \xi, t) \right. \\ \left. + G^{(+)}(z, x/\xi) \frac{1}{2\xi} F^g(x, \xi, t) + R^{(+)}(z, x/\xi) \sum_{q'=u,d,s} F^{q'(+)}(x, \xi, t) \right], \quad (60)$$

and

$$T^{(+)} = \frac{2\pi\sqrt{4\pi\alpha}}{N_c \xi Q} \int_{-1}^1 dx \int_0^1 dz \sum_{q=u,d,s} e_q F^{q(-)}(x, \xi, t) \left[Q^{(-)}(z, x/\xi) \Phi^{q(+)}(z, \zeta, s_\pi) \right. \\ \left. + G^{(-)}(z, x/\xi) \Phi^g(z, \zeta, s_\pi) + R^{(-)}(z, x/\xi) \sum_{q'=u,d,s} \Phi^{q'(+)}(z, \zeta, s_\pi) \right], \quad (61)$$

where $N_c = 3$ denotes the number of colors, α the fine structure constant, and e_q the quark charge in units of the positron charge. As discussed in Section 1.2, these leading amplitudes in $1/Q$ are for longitudinal photon polarization.

The coefficient functions $R^{(\pm)}$, $G^{(\pm)}$ and $Q^{(\pm)}$ represent the amplitudes for the scattering of collinear partons, with the appropriate subtraction of ultraviolet and collinear singularities performed in the $\overline{\text{MS}}$ scheme. $R^{(+)}$ and $G^{(+)}$ were calculated in [12], where electroproduction of light vector mesons was studied at NLO. The coefficient function $Q^{(+)}$ for singlet quark exchange in the t -channel (a typical diagram is shown in Fig. 3(c)) can be obtained from the known result for the pion electromagnetic form factor as [11]

$$Q^{(+)}(z, x/\xi) = \left\{ \mathcal{Q}\left(z, \frac{\xi+x}{2\xi}\right) - \mathcal{Q}\left(\bar{z}, \frac{\xi-x}{2\xi}\right) \right\} + \{z \rightarrow \bar{z}\}, \quad (62)$$

where here and in the following we use the notation $\bar{z} = 1 - z$. We have

$$\begin{aligned} \mathcal{Q}(v, u) &= \frac{\alpha_s(\mu_R) C_F}{4vu} \left(1 + \frac{\alpha_s(\mu_R)}{4\pi} \mathcal{Q}^{(1)}(v, u) \right), \\ \mathcal{Q}^{(1)}(v, u) &= c_1 \left[2[3 + \ln(vu)] \ln\left(\frac{Q^2}{\mu_F^2}\right) + \ln^2(vu) + 6 \ln(vu) - \frac{\ln(v)}{\bar{v}} - \frac{\ln(u)}{\bar{u}} - \frac{28}{3} \right] \\ &+ c_2 \left[2 \frac{\bar{v}v^2 + \bar{u}u^2}{(v-u)^3} [\text{Li}_2(\bar{u}) - \text{Li}_2(\bar{v}) - \text{Li}_2(u) + \text{Li}_2(v) + \ln(\bar{v}) \ln(u) - \ln(\bar{u}) \ln(v)] \right. \\ &+ 2[\text{Li}_2(\bar{u}) + \text{Li}_2(\bar{v}) - \text{Li}_2(u) - \text{Li}_2(v) + \ln(\bar{v}) \ln(u) + \ln(\bar{u}) \ln(v)] \\ &+ 2 \frac{(v+u-2vu) \ln(\bar{v}\bar{u})}{(v-u)^2} + 4 \frac{vu \ln(vu)}{(v-u)^2} - 4 \ln(\bar{v}) \ln(\bar{u}) - \frac{20}{3} \left. \right] \\ &+ \beta_0 \left[\frac{5}{3} - \ln(vu) - \ln\left(\frac{Q^2}{\mu_R^2}\right) \right], \end{aligned} \quad (63)$$

where μ_R and μ_F denote the renormalization and factorization scales and

$$\begin{aligned} \beta_0 &= \frac{11}{3} N_c - \frac{2}{3} n_f, & \text{Li}_2(z) &= - \int_0^z \frac{dt}{t} \ln(1-t), \\ c_1 &= C_F = \frac{N_c^2 - 1}{2N_c}, & c_2 &= C_F - \frac{C_A}{2} = -\frac{1}{2N_c}. \end{aligned} \quad (64)$$

Starting at NLO there is a contribution from diagrams with the topology shown in Fig. 3(e),

$$R^{(+)}(z, x/\xi) = \frac{\alpha_s^2(\mu_R) C_F}{8\pi z \bar{z}} \mathcal{R}\left(z, \frac{x-\xi}{2\xi}\right), \quad (65)$$

where

$$\begin{aligned} \mathcal{R}(z, y) &= \left\{ \frac{2y+1}{y(y+1)} \left[\frac{y}{2} \ln^2(-y) - \frac{y+1}{2} \ln^2(y+1) \right. \right. \\ &+ [y \ln(-y) - (y+1) \ln(y+1)] \left. \left(\ln\left(\frac{Q^2 z}{\mu_F^2}\right) - 1 \right) \right] \\ &+ \frac{y \ln(-y) + (y+1) \ln(y+1)}{y(y+1)} - \frac{V(z, y)}{y+z} + \frac{y(y+1) + (y+z)^2}{(y+z)^2} W(z, y) \left. \right\} + \{z \rightarrow \bar{z}\} \end{aligned} \quad (66)$$

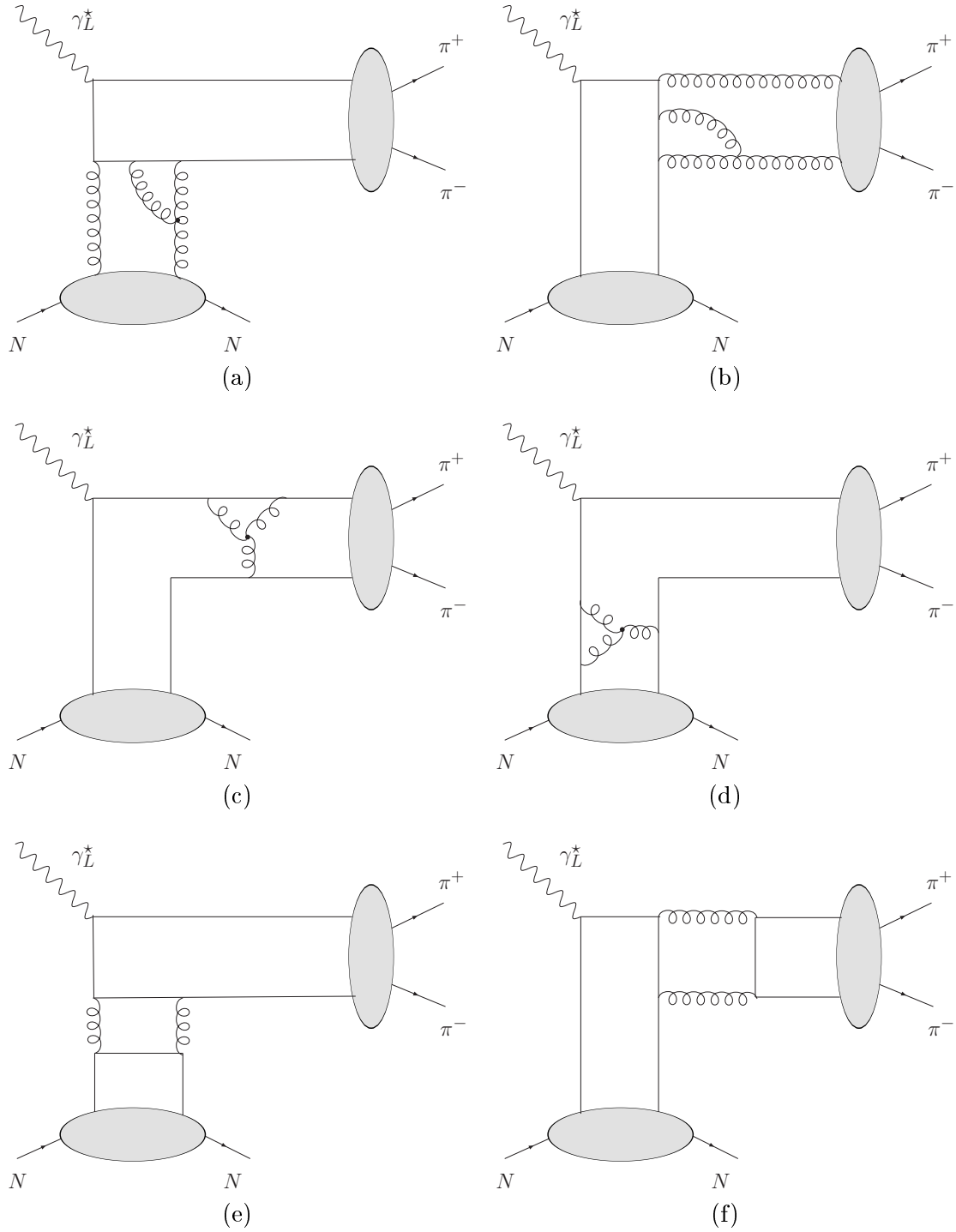


Figure 3: Typical NLO diagrams for the production of two pions in a state with $C = -1$ (left side) or $C = +1$ (right side). Diagrams related to each other by crossing are displayed side by side.

with the abbreviations

$$\begin{aligned} V(z, y) &= z \ln(-y) + \bar{z} \ln(y+1) + z \ln(z) + \bar{z} \ln(\bar{z}), \\ W(z, y) &= \text{Li}_2(y+1) - \text{Li}_2(-y) + \text{Li}_2(z) - \text{Li}_2(\bar{z}) + \ln(-y) \ln(\bar{z}) - \ln(y+1) \ln(z). \end{aligned} \quad (67)$$

For two-gluon exchange in the t -channel (Fig. 3(a)) we can use directly the NLO results obtained in Ref.[12],

$$G^{(+)}(z, x/\xi) = \mathcal{G}\left(z, \frac{x-\xi}{2\xi}\right) \quad (68)$$

with

$$\begin{aligned} \mathcal{G}(z, y) &= \frac{\alpha_s(\mu_R)}{2z\bar{z}y(y+1)} \left(1 + \frac{\alpha_s(\mu_R)}{4\pi} \mathcal{G}^{(1)}(z, y)\right), \\ \mathcal{G}^{(1)}(z, y) &= \left\{ \left(\ln\left(\frac{Q^2}{\mu_F^2}\right) - 1\right) \left[\frac{\beta_0}{2} - \frac{2(c_1 - c_2)[y^2 + (y+1)^2][(y+1)\ln(y+1) - y\ln(-y)]}{y(y+1)} \right. \right. \\ &\quad \left. \left. + \frac{c_1}{2} \left(\frac{y\ln(-y)}{y+1} + \frac{(y+1)\ln(y+1)}{y}\right) + c_1 \left(\frac{3}{2} + 2z\ln(\bar{z})\right) \right] \right. \\ &\quad - \frac{\beta_0}{2} \left(\ln\left(\frac{Q^2}{\mu_R^2}\right) - 1\right) - \frac{c_1(2y+1)V(z, y)}{2(y+z)} - \frac{3c_1 - 4c_2}{4} \left[\frac{y\ln^2(-y)}{y+1} + \frac{(y+1)\ln^2(y+1)}{y}\right] \\ &\quad + [\ln(-y) + \ln(y+1)] \left[c_1 \left(\bar{z}\ln(z) - \frac{1}{4}\right) + 2c_2 \right] + c_1 [z\ln^2(\bar{z}) + (1+3z)\ln(\bar{z}) - 2] \\ &\quad - (c_1 - c_2)[\ln(z\bar{z}) - 2] \left[\frac{y\ln(-y)}{y+1} + \frac{(y+1)\ln(y+1)}{y}\right] \\ &\quad + (c_1 - c_2)(2y+1)\ln\left(\frac{-y}{y+1}\right) \left[\frac{3}{2} + \ln(z\bar{z}) + \ln(-y) + \ln(y+1)\right] \\ &\quad \left. + \left(c_1[y(y+1) + (y+z)^2] - c_2(2y+1)(y+z)\right) \left[\frac{\ln(-y) - \ln(y+1) + \ln(z) - \ln(\bar{z})}{2(y+z)} \right. \right. \\ &\quad \left. \left. - \frac{V(z, y)}{(y+z)^2} + \frac{y(y+1) + (y+z)^2}{(y+z)^3} W(z, y) \right] \right\} + \{z \rightarrow \bar{z}\}. \end{aligned} \quad (69)$$

Note that in our NLO calculation we do not consider three-gluon exchange in the t -channel, which in collinear factorization only appears at NNLO in α_s and corresponds to odderon exchange. Such a contribution is relevant only at very high energies, see [42].

Since the photon has negative charge conjugation parity, the amplitude for a pion pair produced in the C -even channel involves the C -odd GPD combinations $F^{q(-)}$ and vice versa. The coefficient functions appearing in $T^{(+)}$ and $T^{(-)}$ are thus related by crossing symmetry. They coincide after the interchange of the t -channel and the s -channel parton pairs in the Feynman graphs and the corresponding interchange of the relative parton momentum fractions. One can easily convince oneself of this relationship by comparing the typical NLO diagrams on the left and right side in the Fig. 3. The prescription for the interchange of the momentum fractions in terms of variables reads

$$z \leftrightarrow \frac{\xi + x}{2\xi}, \quad \bar{z} \leftrightarrow \frac{\xi - x}{2\xi}, \quad (70)$$

so that we have

$$\begin{aligned}
Q^{(-)}(z, x/\xi) &= Q^{(+)}\left(\frac{\xi+x}{2\xi}, 2z-1\right), \\
R^{(-)}(z, x/\xi) &= R^{(+)}\left(\frac{\xi+x}{2\xi}, 2z-1\right), \\
G^{(-)}(z, x/\xi) &= G^{(+)}\left(\frac{\xi+x}{2\xi}, 2z-1\right).
\end{aligned} \tag{71}$$

In (62) to (71) we have omitted the $i\varepsilon$ prescription, which reads $(\xi+x)/(2\xi) - i\varepsilon$ and $(\xi-x)/(2\xi) - i\varepsilon$. Before giving more explicit expressions for the amplitude we make a number of simplifications:

1. We restrict ourselves to the asymptotic forms (34) and (35) for the z -dependence of the 2π DAs.
2. We take the Omnès functions $f_l(s_\pi)$ and the constant C in the representations (45), (50) to be equal for the quark isosinglet and for the gluon coefficients $\tilde{B}_{1l}^{I=0}(s_\pi)$ and $\tilde{B}_{1l}^g(s_\pi)$. For the Omnès functions this approximation should be good at least for low enough s_π , where the integrands in the first lines of (46) and (52) are determined by the $\pi\pi$ phase shifts in the dominant integration region. According to (54) we have $C \approx 1$ from chiral perturbation theory for both quarks and gluons.
3. We neglect the C -odd combination $F^{s(-)}(x, \xi, t)$ of nucleon GPDs. Its forward limit $s(x) - \bar{s}(x)$ is known to be very small, and we assume that the same holds for finite ξ and t .
4. We neglect the 2π DA for strangeness, $\Phi^{s(+)}$. Note that compared with $\Phi^{u(+)} = \Phi^{d(+)}$ this quantity appears in the amplitude (61) with a suppression factor of either $F^{s(-)}(x, \xi, t)$ or α_s . As discussed in Section 3, the coefficients $B_{12}^s(s_\pi)$ and $B_{10}^s(s_\pi)$ are small at the unphysical point $s_\pi = 0$, and we expect that at larger s_π they are at least not significantly larger than the corresponding coefficients for u and d quarks (and hence cannot compensate the suppression just mentioned).

As a compact notation we introduce

$$\begin{aligned}
I_Q^{(+)}(\xi, t) &= \int_{-1}^1 dx \int_0^1 dz \varphi_0(z) Q^{(+)}(z, x/\xi) \left[\frac{2}{3} F^{u(+)}(x, \xi, t) + \frac{1}{3} F^{d(+)}(x, \xi, t) \right], \\
I_G^{(+)}(\xi, t) &= \int_{-1}^1 dx \int_0^1 dz \varphi_0(z) G^{(+)}(z, x/\xi) \frac{1}{2\xi} F^g(x, \xi, t), \\
I_R^{(+)}(\xi, t) &= \int_{-1}^1 dx \int_0^1 dz \varphi_0(z) R^{(+)}(z, x/\xi) \left[F^{u(+)}(x, \xi, t) + F^{d(+)}(x, \xi, t) + F^{s(+)}(x, \xi, t) \right]
\end{aligned} \tag{72}$$

and

$$\begin{aligned}
I_Q^{(-)}(\xi, t) &= \int_{-1}^1 dx \int_0^1 dz \varphi_1(z) Q^{(-)}(z, x/\xi) \left[\frac{2}{3} F^{u(-)}(x, \xi, t) - \frac{1}{3} F^{d(-)}(x, \xi, t) \right], \\
I_G^{(-)}(\xi, t) &= \int_{-1}^1 dx \int_0^1 dz \varphi_G(z) G^{(-)}(z, x/\xi) \left[\frac{2}{3} F^{u(-)}(x, \xi, t) - \frac{1}{3} F^{d(-)}(x, \xi, t) \right], \\
I_R^{(-)}(\xi, t) &= \int_{-1}^1 dx \int_0^1 dz \varphi_1(z) R^{(-)}(z, x/\xi) \left[\frac{4}{3} F^{u(-)}(x, \xi, t) - \frac{2}{3} F^{d(-)}(x, \xi, t) \right]
\end{aligned} \tag{73}$$

with

$$\varphi_0(z) = z(1-z), \quad \varphi_1(z) = z(1-z)(2z-1), \quad \varphi_G(z) = z^2(1-z)^2. \quad (74)$$

The integrals over z can be performed analytically, whereas the x integral was evaluated numerically. The amplitudes for $\gamma^*p \rightarrow \pi^+\pi^-p$ then take the simple form

$$\begin{aligned} T^{(-)} &= \frac{2\pi\sqrt{4\pi\alpha}}{3\xi Q} 6\beta F_\pi(s_\pi) P_1(\cos\theta) I^{(+)}(\xi, t), \\ T^{(+)} &= \frac{2\pi\sqrt{4\pi\alpha}}{3\xi Q} \left[\frac{3C - \beta^2}{2} f_0(s_\pi) P_0(\cos\theta) - \beta^2 f_2(s_\pi) P_2(\cos\theta) \right] I^{(-)}(\xi, t) \end{aligned} \quad (75)$$

with

$$\begin{aligned} I^{(+)}(\xi, t) &= I_Q^{(+)}(\xi, t) + I_G^{(+)}(\xi, t) + I_R^{(+)}(\xi, t), \\ I^{(-)}(\xi, t) &= -10 \left[A_{20}^u(0) + A_{20}^d(0) \right] \left(I_Q^{(-)}(\xi, t) + I_R^{(-)}(\xi, t) \right) - 10 A_{20}^g(0) I_G^{(-)}(\xi, t) \end{aligned} \quad (76)$$

The corresponding expressions for $\gamma^*n \rightarrow \pi^+\pi^-n$ are readily obtained by interchanging $F^{u(\pm)}$ and $F^{d(\pm)}$ in (72) and (73), where the quark flavor label in the GPDs always refers to a proton target. Explicitly we have

$$\begin{aligned} I^{(-)}(\xi, t) &= \frac{40}{27} \left[A_{20}^u(0) + A_{20}^d(0) + \frac{3}{4} A_{20}^g(0) \right] \\ &\times \alpha_s(\mu_R) \int_{-1}^1 dx \frac{\xi^2}{(\xi+x)(\xi-x)} \left[2F^{u(-)}(x, \xi, t) - F^{d(-)}(x, \xi, t) \right] + O(\alpha_s^2). \end{aligned} \quad (77)$$

The momentum fraction integrals in the pion fulfill $A_{20}^u(0) + A_{20}^d(0) + \frac{3}{4} A_{20}^g(0) = 1 - A_{20}^s(0) - \frac{1}{4} A_{20}^g(0)$, so that $I^{(-)}(\xi, t)$ depends rather weakly on the precise values of these integrals, as was already reported in [10].

In terms of the amplitude $T = T^{(-)} + T^{(+)}$, the Legendre moments (6) take the form

$$\langle P_l(\cos\theta) \rangle = \frac{\sum_{\text{pol}} \int dy dx_B dt ds_\pi \frac{(1-y)\beta}{y^3 x_B} \int_{-1}^1 d\cos\theta P_l(\cos\theta) |T|^2}{\sum_{\text{pol}} \int dy dx_B dt ds_\pi \frac{(1-y)\beta}{y^3 x_B} \int_{-1}^1 d\cos\theta |T|^2}, \quad (78)$$

where \sum_{pol} denotes summation over the polarizations of the incoming and outgoing nucleon. With (75) and (76) we have

$$\begin{aligned} \int_{-1}^1 d\cos\theta |T|^2 &= \left(\frac{2\pi\sqrt{4\pi\alpha}}{3\xi Q} \right)^2 \left\{ \left(\frac{(3C - \beta^2)^2}{2} |f_0(s_\pi)|^2 + \frac{2\beta^4}{5} |f_2(s_\pi)|^2 \right) |I^{(-)}(\xi, t)|^2 \right. \\ &\quad \left. + 24\beta^2 |F_\pi(s_\pi)|^2 |I^{(+)}(\xi, t)|^2 \right\} \end{aligned} \quad (79)$$

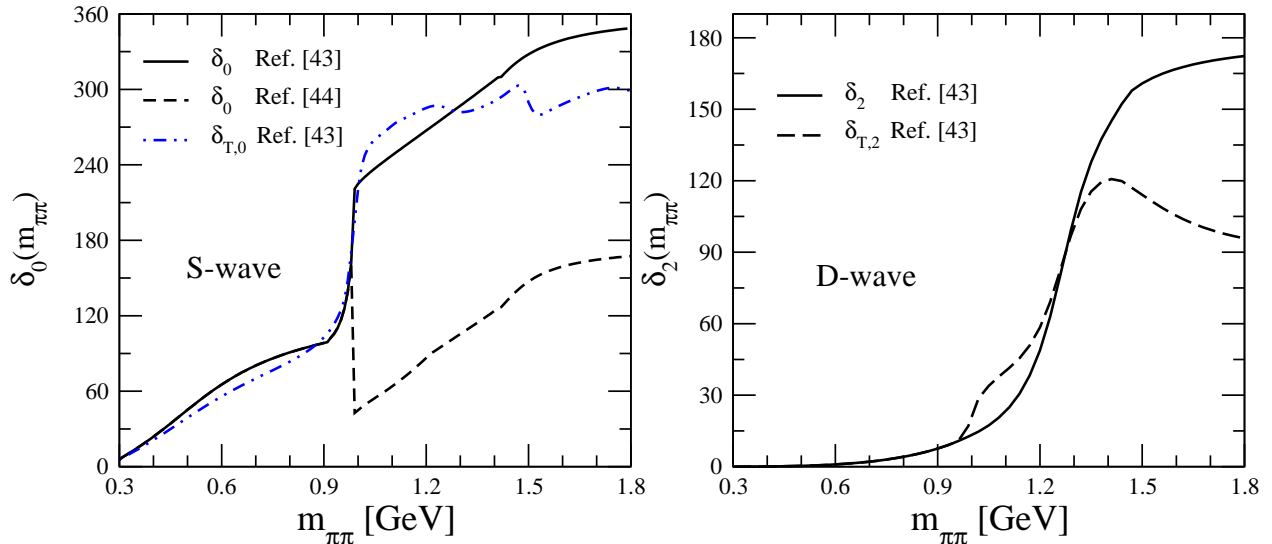


Figure 4: Phase shifts δ_l of $\pi\pi$ scattering in the isoscalar channel obtained by Kamiński et al. [43] and by Bugg [44]. For the parameterization [43] we also show the phase $\delta_{T,l}$ of the \mathcal{T} -matrix (81).

for the denominator and

$$\int_{-1}^1 d\cos\theta P_1(\cos\theta) |T|^2 = \left(\frac{2\pi\sqrt{4\pi\alpha}}{3\xi Q} \right)^2 12\beta \operatorname{Re} \left\{ [F_\pi(s_\pi)]^* \left[\frac{3C - \beta^2}{3} f_0(s_\pi) - \frac{4}{15} \beta^2 f_2(s_\pi) \right] \right. \\ \left. \times [I^{(+)}(\xi, t)]^* I^{(-)}(\xi, t) \right\},$$

$$\int_{-1}^1 d\cos\theta P_3(\cos\theta) |T|^2 = - \left(\frac{2\pi\sqrt{4\pi\alpha}}{3\xi Q} \right)^2 \frac{12\beta^3}{35} \operatorname{Re} \left\{ [F_\pi(s_\pi)]^* f_2(s_\pi) [I^{(+)}(\xi, t)]^* I^{(-)}(\xi, t) \right\} \quad (80)$$

for the numerator of (78).

5 Modeling the two-pion distribution amplitudes

In Section 3.1 we have represented the coefficients $\tilde{B}_{10}(s_\pi)$ and $\tilde{B}_{12}(s_\pi)$ of the 2π DAs as integrals involving the phases $\tilde{\delta}_l(s_\pi)$. With Watson's theorem (41) these phases are equal to the isoscalar phase shifts $\delta_l(s_\pi)$ from $s_\pi = 4m_\pi^2$ up to the value where $\pi\pi$ scattering in the appropriate partial wave becomes inelastic. Phenomenological analyses find that for the S and D waves $\pi\pi$ scattering is approximately elastic up to the two-kaon threshold at $s_\pi \sim 1 \text{ GeV}^2$. The S wave then becomes inelastic rather abruptly, whereas in the D wave inelasticity sets in more smoothly. The phases of the form factors we are interested in then differ from the corresponding phase shifts. In Fig. 4 we show the S and D wave phase shifts from the recent parameterization of Kamiński et al. [43], and for comparison also the result of the analysis of the S wave by Bugg [44].

As long as a small number of channels are relevant, one can attempt an explicit multi-channel analysis, say for the $\pi\pi$ and $K\bar{K}$ channels. For a local operator \mathcal{O} with appropriate symmetry

properties, time reversal relates the matrix element $\text{out}\langle\pi\pi|\mathcal{O}|0\rangle$ with $\text{in}\langle\pi\pi|\mathcal{O}|0\rangle^*$ and $\text{out}\langle K\bar{K}|\mathcal{O}|0\rangle$ with $\text{in}\langle K\bar{K}|\mathcal{O}|0\rangle^*$. In the region of s_π where the scattering matrix provides a closed relation between the states $|\pi\pi\rangle_{\text{out}}$, $|K\bar{K}\rangle_{\text{out}}$ and $|\pi\pi\rangle_{\text{in}}$, $|K\bar{K}\rangle_{\text{in}}$, one can then relate the phases of the operator matrix elements with those in the \mathcal{S} -matrix. Combining this information with a dispersion relation leads from the Omnès representation discussed in Section 3.1 to the Omnès-Muskhelishvili problem, whose solution is considerably more involved. Such analyses have for instance been performed for the trace of the energy-momentum tensor in [45] and for the scalar quark current in [45, 46, 47, 48]. We shall not attempt to do the same here for the twist-two part of the energy-momentum tensor, which is associated with the form factors $B_{10}(s_\pi)$ and $B_{12}(s_\pi)$. Instead we wish to point out salient results of the analysis in [48]. There it was found that just above the $K\bar{K}$ threshold the phase of the form factor $\Gamma(s_\pi) = \langle\pi\pi|m_u\bar{u}u + m_d\bar{d}d|0\rangle$ starts to deviate very strongly from the S wave phase shift, and instead is rather close to the phase of the corresponding \mathcal{T} -matrix element in the $\pi\pi$ channel. With the \mathcal{S} -matrix element in the two-pion channel parameterized as $\eta_l \exp[2i\delta_l]$, where η_l is the elasticity parameter, the corresponding \mathcal{T} -matrix element is

$$\frac{\eta_l \exp[2i\delta_l] - 1}{2i} \quad (81)$$

up to a normalization factor. Clearly, the phase $\delta_{T,l}$ of the \mathcal{T} -matrix element differs from the phase shift δ_l as soon as the elasticity deviates from $\eta_l = 1$. This is seen in Fig. 4, which shows both phases as obtained from the parameterization [43].

While the phase of $\Gamma(s_\pi)$ found in [48] is well approximated by $\delta_{T,0}(s_\pi)$, the phase of $\Delta(s_\pi) = \langle\pi\pi|m_s\bar{s}s|0\rangle$ was found to be closer to $\delta_0(s_\pi)$. This difference is perhaps not too surprising since the solution of the Omnès-Muskhelishvili problem depends not only on the \mathcal{S} -matrix in the pion and kaon channels but also on the relevant form factors for pions and for kaons at the subtraction point of the dispersion relation (typically $s_\pi = 0$). In the present analysis we will investigate the assumptions that the phases $\tilde{\delta}_l(s_\pi)$ are equal to either $\delta_l(s_\pi)$ or $\delta_{T,l}(s_\pi)$ for s_π above the $K\bar{K}$ threshold. We do this in the sense of exploring two rather extreme cases, keeping in mind that the true phases could be far from either of them. The phases $\tilde{\delta}_l$ can of course be different for $\tilde{B}_{1l}^{I=0}$ and for \tilde{B}_{1l}^g , but given our simple model ansatz we take them to be the same.

The Omnès representations (45) and (50) depend on the phases $\tilde{\delta}_l(s_\pi)$ at values of s_π above the point where the Omnès functions are evaluated. Clearly, the dependence on large s_π under the integrals is reduced for $N = 2$ subtractions as given in the second lines of (46) and (52), where the uncertainty due to the unknown behavior of the phases at large s_π is reduced at the expense of introducing an additional subtraction constant. In the following we shall work with the $N = 2$ Omnès functions, which permits a convenient estimate of uncertainties by varying these constants. The dynamical content of the representations with $N = 1$ and $N = 2$ is of course the same, and the simultaneous validity of the first and second lines in (46) and (52) implies sum rules

$$\begin{aligned} I_0 &= \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\tilde{\delta}_0(s)}{s^2} = \frac{d}{ds_\pi} \ln B_{12}(0) + \frac{\epsilon}{2m_\pi^2} \frac{B_{10}(0)}{B_{12}(0)}, \\ I_2 &= \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\tilde{\delta}_2(s)}{s^2} = \frac{d}{ds_\pi} \ln B_{12}(0). \end{aligned} \quad (82)$$

Using the parameterization of [43] we have evaluated the corresponding integrals in the range $4m_\pi^2 \leq s_\pi \leq 4m_K^2$, where the integrands are determined by the phase shifts. Under the rather weak assumption that above the $K\bar{K}$ threshold the phases $\tilde{\delta}_l(s_\pi)$ remain positive (even when dropping below

Table 1: The integrals I_0 (left) and I_2 (right) defined in (82), evaluated with different upper cutoffs s_{\max} on s_π . The phases $\tilde{\delta}_l(s_\pi)$ and $\delta_{T,l}(s_\pi)$ are taken from the parameterization in [43]. All integrals are given in units of GeV^{-2} .

	s_{\max}			s_{\max}			
	$4m_K^2$	$(1.8 \text{ GeV})^2$	∞	$4m_K^2$	$(1.8 \text{ GeV})^2$	∞	
$\tilde{\delta}_0 = \delta_0$	2.02	3.13	3.74	$\tilde{\delta}_2 = \delta_2$	0.04	0.37	0.67
$\tilde{\delta}_0 = \delta_{T,0}$	2.02	2.34	2.55	$\tilde{\delta}_2 = \delta_{T,2}$	0.04	0.33	0.48

$\delta_l(s_\pi)$), this gives a lower bound on the quantities in (82). We have further evaluated the integrals with an upper cutoff at $s_\pi = (1.8 \text{ GeV})^2$ and up to $s_\pi = \infty$, assuming either $\tilde{\delta}_l = \delta_l$ or $\tilde{\delta}_l = \delta_{T,l}$. The results are collected in Table 1 and provide an estimate of the possible contribution to the integrals (82) from the region $s_\pi \geq 4m_K^2$. For the S wave the contribution from $s_\pi \leq 4m_K^2$ gives an important part of the total result. In contrast, the D wave is strongly suppressed in that region, and practically the entire contribution to the integral I_2 comes from s_π above the two-kaon threshold. We note that the instanton model calculation reported in [14] obtained

$$\frac{d}{ds_\pi} \ln B_{12}^{I=0}(0) = \frac{d}{ds_\pi} \ln B_{10}^{I=0}(0) = \frac{N_c}{3} \frac{1}{(4\pi f_\pi)^2} \approx 0.73 \text{ GeV}^{-2}, \quad (83)$$

which together with the relation (56) from chiral perturbation theory gives

$$\frac{d}{ds_\pi} \ln B_{12}^{I=0}(0) + \frac{\epsilon}{2m_\pi^2} \frac{B_{10}^{I=0}(0)}{B_{12}^{I=0}(0)} \approx \frac{d}{ds_\pi} \ln B_{10}^{I=0}(0) - \frac{19}{60} \frac{1}{(4\pi f_\pi)^2} \approx 0.50 \text{ GeV}^{-2}. \quad (84)$$

For $\frac{d}{ds_\pi} \ln B_{12}^{I=0}(0)$ the value in (83) is not very different from our estimates for I_2 in Table 1, but for I_0 even our result from the region $s_\pi \leq 4m_K^2$ is significantly higher than (84). Since in that region the pion phase shifts are reasonably well known and equal to $\tilde{\delta}_0$, we must conclude that the result for $\frac{d}{ds_\pi} \ln B_{10}^{I=0}(0)$ in [14] is implausibly small—it would only be consistent with the sum rule (82) if $\tilde{\delta}_0$ became significantly negative for $s_\pi \geq 4m_K^2$ or if the relation (56) from one-loop chiral perturbation theory were invalidated by huge corrections from higher orders.

In Fig. 5 we show the absolute values of the Omnès functions evaluated with $N = 2$ subtractions, obtained with either $\tilde{\delta}_l = \delta_l$ or $\tilde{\delta}_l = \delta_{T,l}$ taken from the parameterization in [43]. To explore the dependence on the subtraction constants, we take as central values I_l^{cen} those obtained with $s_{\max} = \infty$ in Table 1. For the S wave we take as a lowest value I_0^{low} the one obtained with $s_{\max} = 4m_K^2$ and as highest value $I_0^{\text{hi}} = I_0^{\text{cen}} + (I_0^{\text{cen}} - I_0^{\text{low}})$. For the D wave we take instead $I_2^{\text{low}} = 0.5 I_2^{\text{cen}}$ and $I_2^{\text{hi}} = 1.5 I_2^{\text{cen}}$. For the S wave and the assumption $\tilde{\delta}_0 = \delta_0$ we also show the result from the parameterization in [44]. Here the integrals in both the Omnès function (52) and in the corresponding subtraction constant (82) are taken with an upper cutoff $s_{\max} = (1.8 \text{ GeV})^2$ since we do not have an analytic parameterization up to $s_\pi = \infty$ in this case. A corresponding truncation of the integrals with the parameterization of [43] has only a moderate effect, and we do not show the corresponding curve.

We see that the values of the subtraction constants have a visible effect on the Omnès functions, as well as the choice of parameterization for the $\pi\pi$ phase shifts and elasticity parameters. The largest

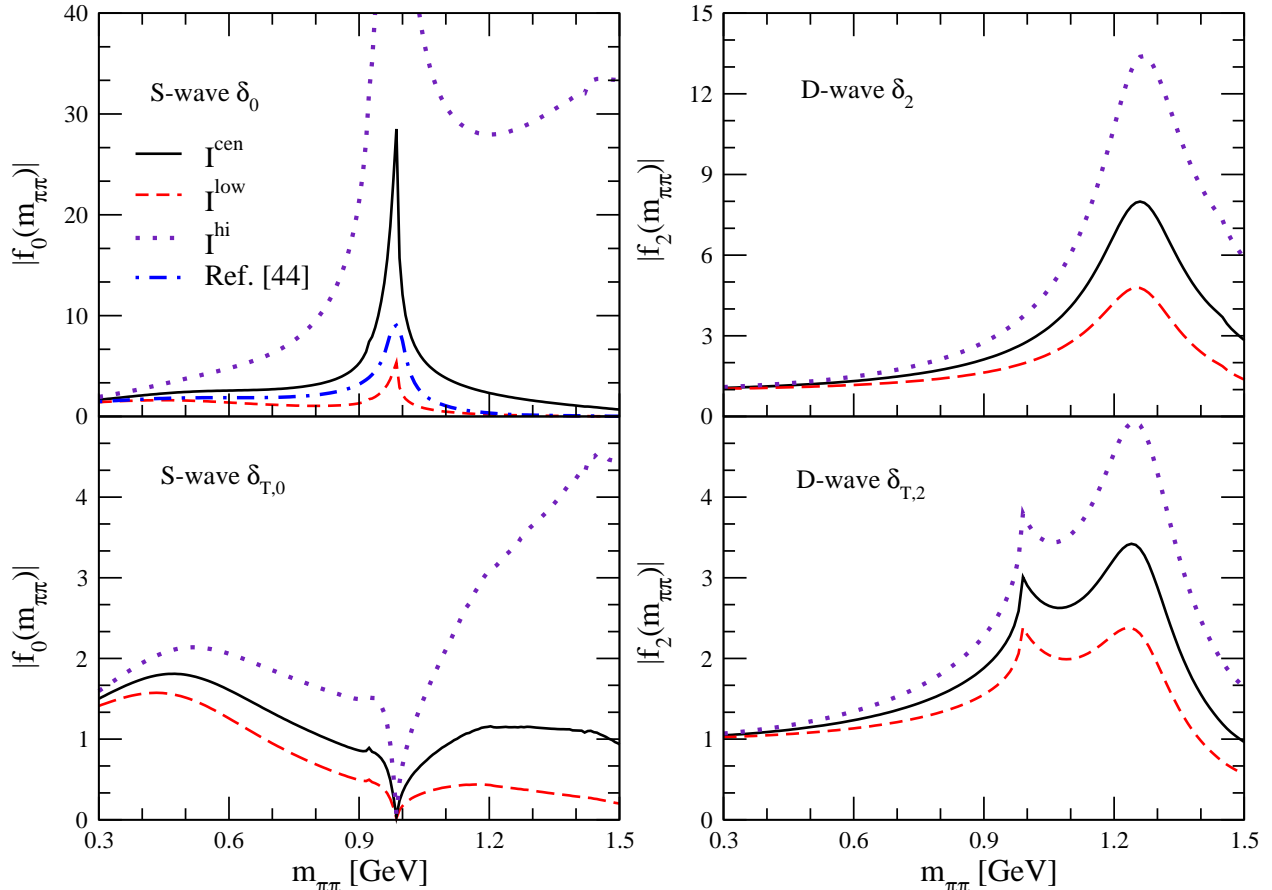


Figure 5: Absolute values of the Omnès functions for $N = 2$ subtractions, assuming $\tilde{\delta}_l = \delta_l$ (top) or $\tilde{\delta}_l = \delta_{T,l}$ (bottom). The different values taken for the subtraction constants are explained in the text.

uncertainty in the Omnès functions is however due to the values of $\tilde{\delta}_l(s_\pi)$ for s_π above the kaon threshold, as exemplified by the two assumptions $\tilde{\delta}_l = \delta_l$ and $\tilde{\delta}_l = \delta_{T,l}$. Whereas the former produces a clear peak of $|f_0(s_\pi)|$ around $s_\pi = 1 \text{ GeV}^2$, whose height depends on further details, the latter gives a dip at the same position. For $|f_2(s_\pi)|$ the differences are less extreme but still significant. In the next section we will compare the consequences of these Omnès functions on the two-pion mass spectrum and angular distribution observed at HERMES.

6 Results for two-pion electroproduction

We have now all ingredients necessary to evaluate the invariant mass spectrum and angular distribution of the two pions produced in $\gamma^* + N \rightarrow \pi^+ \pi^- + N$. We take the factorization and renormalization scales as $\mu_F = \mu_R = Q$ in the hard-scattering formulae. For the nucleon GPDs we use the model described in Section 2. The 2π DAs are calculated within the model specified above (72), based on the asymptotic z -dependence and the Omnès representations developed in Section 3.1. This leads to the expressions (75) and (76) for the scattering amplitude. We take $C = 1$ for the constant in (50)

and expect that the chiral corrections of order m_π^2/Λ_χ^2 to this quantity have a negligible effect on our results, given the other uncertainties we have to deal with. The Omnès functions are calculated with $N = 2$ subtractions, using the phases and subtraction constants presented in the previous section. For the quark and gluon momentum fractions appearing in (76) we take $A_{20}^u(0) + A_{20}^d(0) = A_{20}^g(0) = 0.5$, in line with the parton densities of the pion at moderate factorization scales [49]. A change of these values has only little effect on our results, as explained after (77).

For the timelike pion form factor $F_\pi(m_{\pi\pi}^2)$ we take the parameterization given in [50], which is in good agreement with data from $e^+e^- \rightarrow \pi^+\pi^-$ and also with the pion phase shift δ_1 in the P wave. We note that according to the analysis in [51], inelasticity has only a small effect on the difference between $\delta_1(m_{\pi\pi}^2)$ and the phase of $F_\pi(m_{\pi\pi}^2)$ for $m_{\pi\pi} \lesssim 1.3$ GeV. Our results are almost unchanged if we use the parameterizations for $F_\pi(m_{\pi\pi}^2)$ from [52] or [53] instead of [50].

When giving results for a deuteron target we will assume that the production process is incoherent, $\gamma^* + d \rightarrow \pi^+\pi^- + p + n$, with scattering either on the proton or the neutron. For the average $t = -0.29$ GeV² of the HERMES measurement [8] this should be a good approximation since elastic scattering on the deuteron is strongly suppressed at that value of t . In addition we neglect nuclear effects and treat proton and neutron as quasi-free. We then simply have

$$d\sigma(\gamma^*d) = d\sigma(\gamma^*p) + d\sigma(\gamma^*n), \quad \langle P_l \rangle_d = \frac{d\sigma_p}{d\sigma_p + d\sigma_n} \langle P_l \rangle_p + \frac{d\sigma_n}{d\sigma_p + d\sigma_n} \langle P_l \rangle_n, \quad (85)$$

where subscripts d, p, n refer to the different targets. In the following we give results for two kinematic points, which correspond to the average kinematics of the HERMES measurement [8]:

$$t = -0.43 \text{ GeV}^2, \quad Q^2 = 3.2 \text{ GeV}^2, \quad x_B = 0.16 \quad \text{for a proton target}, \quad (86)$$

$$t = -0.29 \text{ GeV}^2, \quad Q^2 = 3.3 \text{ GeV}^2, \quad x_B = 0.16 \quad \text{for a deuteron target}. \quad (87)$$

Our results for the invariant mass spectrum of two pions produced from a hydrogen target are given in Fig. 6. We show them for the Omnès functions calculated from the parameterizations of Kamiński et al. [43] and of Bugg [44] with the hypothesis $\tilde{\delta}_l = \delta_l$, and in the case of [43] also for $\tilde{\delta}_l = \delta_{T,l}$. In all cases, the central values I_l^{cen} for the subtraction constants have been used. As expected from our discussion of the Omnès functions, the ansatz $\tilde{\delta}_0 = \delta_0$ produces a clearly visible peak in the mass spectrum around $m_{\pi\pi} = 1$ GeV, although its height strongly depends on the phase shifts used. The HERMES measurement [8] did not find any indication of such a pronounced peak, and we conclude that these data strongly disfavor the hypothesis that the phase of \tilde{B}_{10} is given by the S wave phase shift above the two-kaon threshold. In the following we will therefore restrict the discussion to our alternative hypothesis $\tilde{\delta}_0 = \delta_{T,0}$. In the case of the D wave the assumption $\tilde{\delta}_2 = \delta_2$ produces a peak around the mass of the $f_2(1270)$. It is however much less pronounced than the one in the S wave, and we find that the invariant mass spectrum shown in [8] does not allow a strong conclusion on the phase of \tilde{B}_{12} . The same discussion applies for a deuterium target, i.e., the assumption $\tilde{\delta}_0 = \delta_0$ produces a clear mass peak around $m_{\pi\pi} = 1$ GeV, which is not seen in the data, whereas the mild peak around $m_{\pi\pi} = 1.27$ GeV produced by $\tilde{\delta}_2 = \delta_2$ cannot be ruled out by the data.

We also show in Fig. 6 the result of taking the LO approximation for the hard-scattering subprocess (for one choice of Omnès functions). The effect of the NLO corrections on this observable is clearly visible, but not of a size which would make us worry about the stability of the perturbative expansion. Note that we are not giving the absolute size of the cross section here: on one hand there is no experimental measurement to compare with, and on the other hand we expect important power corrections to our leading-twist calculation of this observable, as discussed in Section 1.2. The lowest

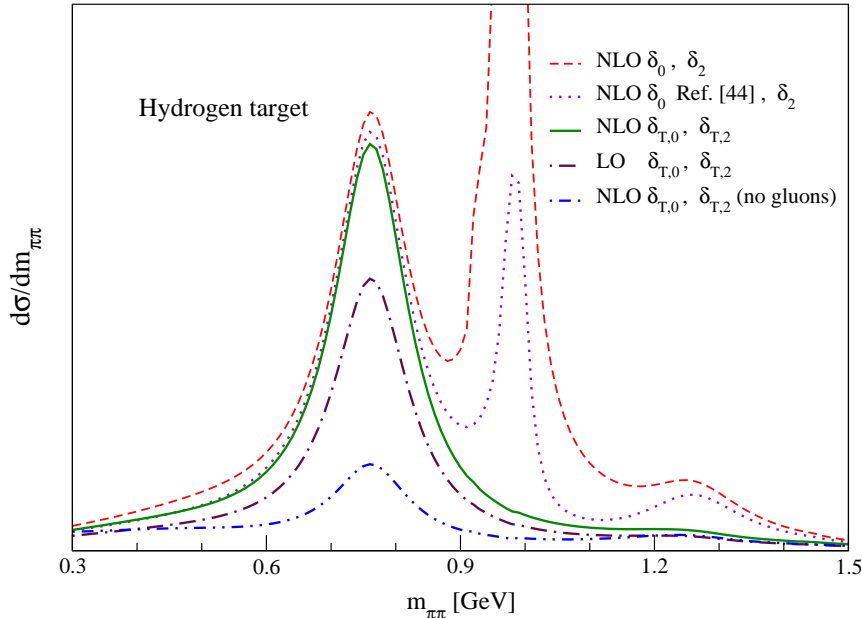


Figure 6: Two-pion invariant mass spectrum (in arbitrary units) for $\gamma^* + p \rightarrow \pi^+ \pi^- + p$, calculated with different assumptions for the Omnès functions as explained in the text. Phases are obtained from Ref. [43] unless explicitly indicated. The plot is for the average kinematics (86) of the HERMES measurement [8].

curve in Fig. 6 shows the result we obtain when setting the gluon GPD to zero. We see that even in HERMES kinematics there is a substantial contribution from gluon exchange to the P wave production amplitude, confirming similar findings in [54, 21].

In Fig. 7 we show our results for the Legendre moments (6) as a function of $m_{\pi\pi}$. Here and in the following figures we always use the phases from [43]. Given the experimental errors we find the overall agreement between data and theory fair, although clearly not perfect. The two curves correspond to the hypotheses $\tilde{\delta}_2 = \delta_2$ or $\tilde{\delta}_2 = \delta_{T,2}$. Whereas at face value the former hypothesis seems to be preferred by the data on $\langle P_1(\cos\theta) \rangle$, the opposite holds for $\langle P_3(\cos\theta) \rangle$. In Fig. 8 we compare the same two scenarios for the linear combinations (9) and (10) of Legendre moments at higher $m_{\pi\pi}$ values. The combination $\langle P_1 + \frac{7}{3}P_3 \rangle$ is only sensitive to amplitudes with total helicity $\lambda = 0$ of the two pions, which are those we can calculate using the factorization theorem. For this observable, the curve obtained with $\tilde{\delta}_2 = \delta_2$ is rather disfavored by the hydrogen data. The combination $\langle P_1 - \frac{14}{9}P_3 \rangle$ is sensitive to a $\lambda = 0$ contribution from the S wave, which comes out rather small in this $m_{\pi\pi}$ range, and to $\lambda = \pm 1$ in the D wave, which is absent in our leading-twist calculation (so that there is no difference between the two model curves). We also show in Fig. 7 the result obtained at LO in α_s for one choice of the Omnès functions. The effect of the NLO corrections is again found to be moderate but not negligible.

Figures 9 and 10 show our results obtained with $\tilde{\delta}_l = \delta_{T,l}$ in both S and D waves as a function of the subtraction constants I_0 and I_2 discussed in Section 5, where we have taken the high, central, or low values for both constants at a time. We see that the impact of these values on the observables is not negligible but beyond the accuracy of the presently available data.

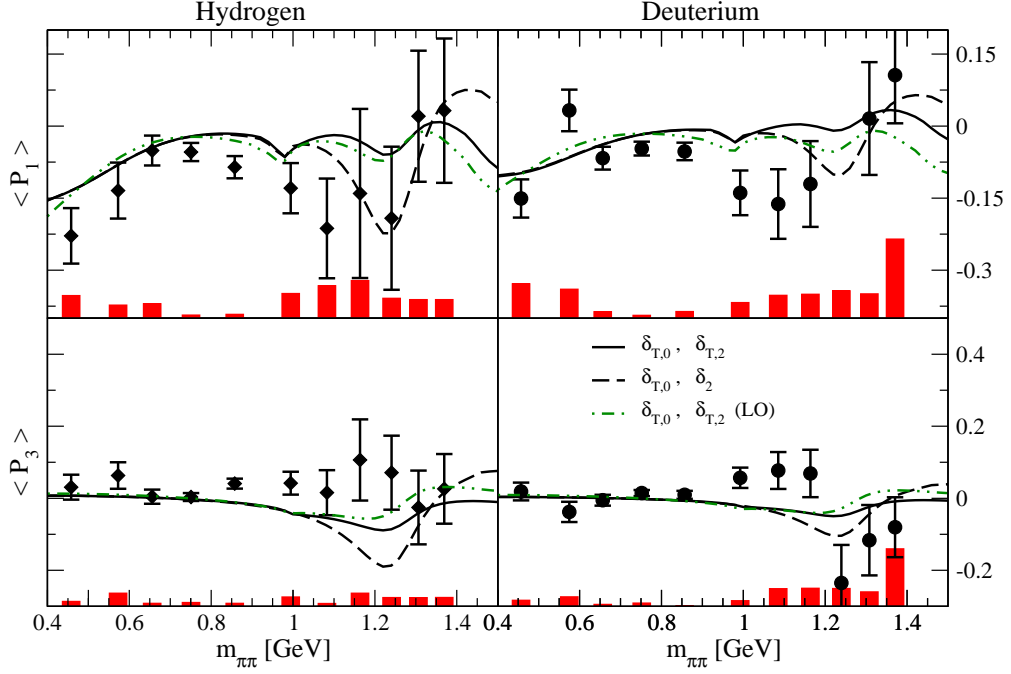


Figure 7: Legendre moments $\langle P_1 \rangle$ and $\langle P_3 \rangle$ for a hydrogen and deuterium target. The curves are calculated with the models specified in the text for the average kinematics (86) and (87) of the HERMES data [8]. The systematic uncertainty of the measurement is represented by the histograms.

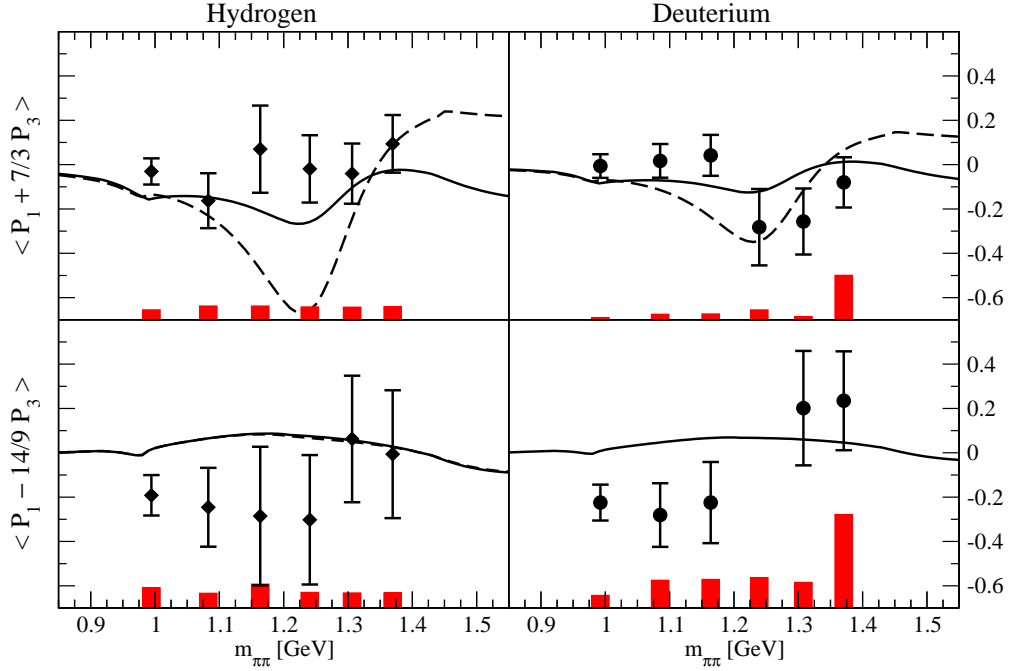


Figure 8: Data and theory for the linear combinations (9) and (10) of Legendre moments, obtained with the same Omnès functions as in Fig. 7. Curves here and in the following figures are for NLO.

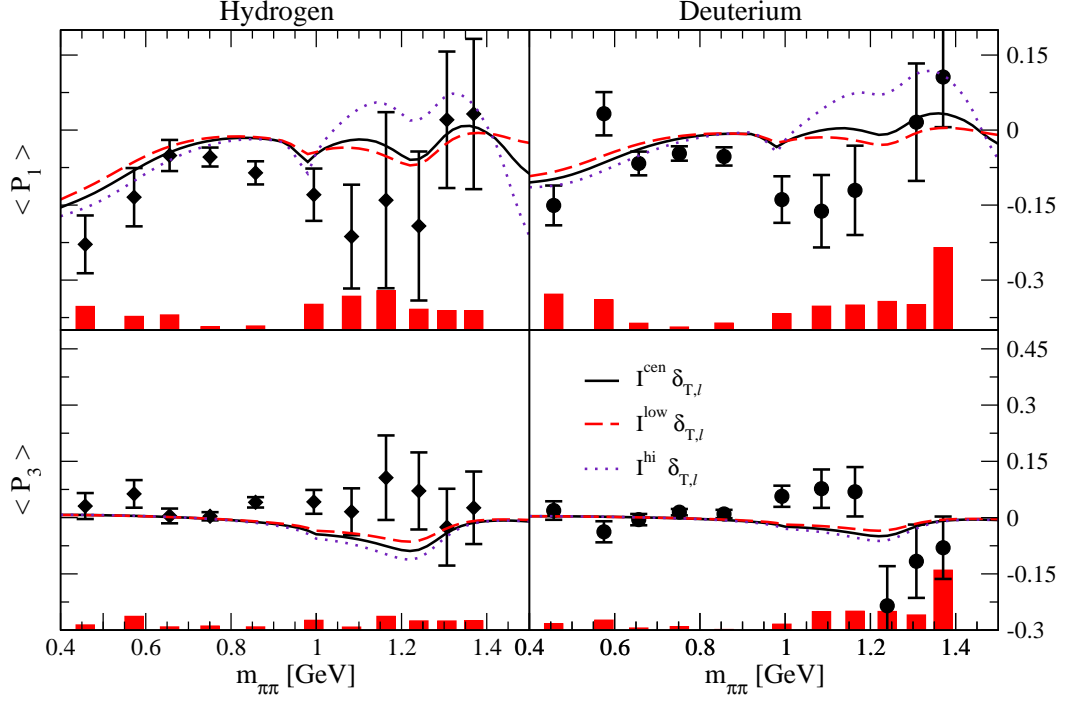


Figure 9: As Fig. 7 but for Omnès functions calculated with $\tilde{\delta}_l = \delta_{T,l}$ and our high, central, or low estimates for the subtraction constants I_l specified in Sect. 5.

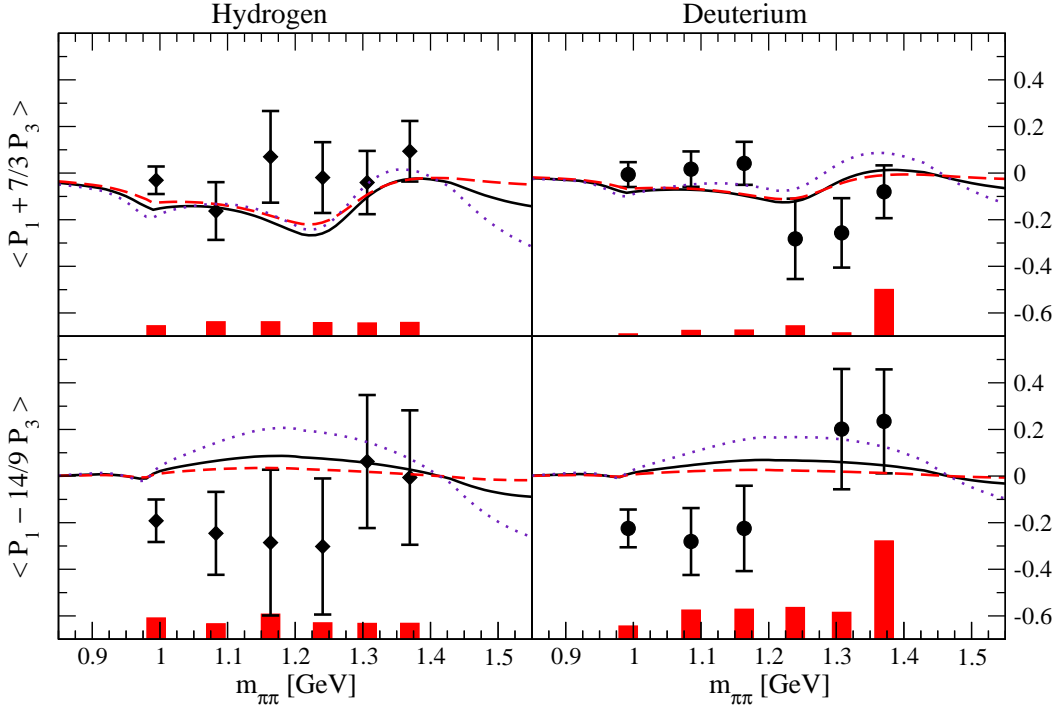


Figure 10: As Fig. 9 but for the linear combinations (9) and (10) of Legendre moments.

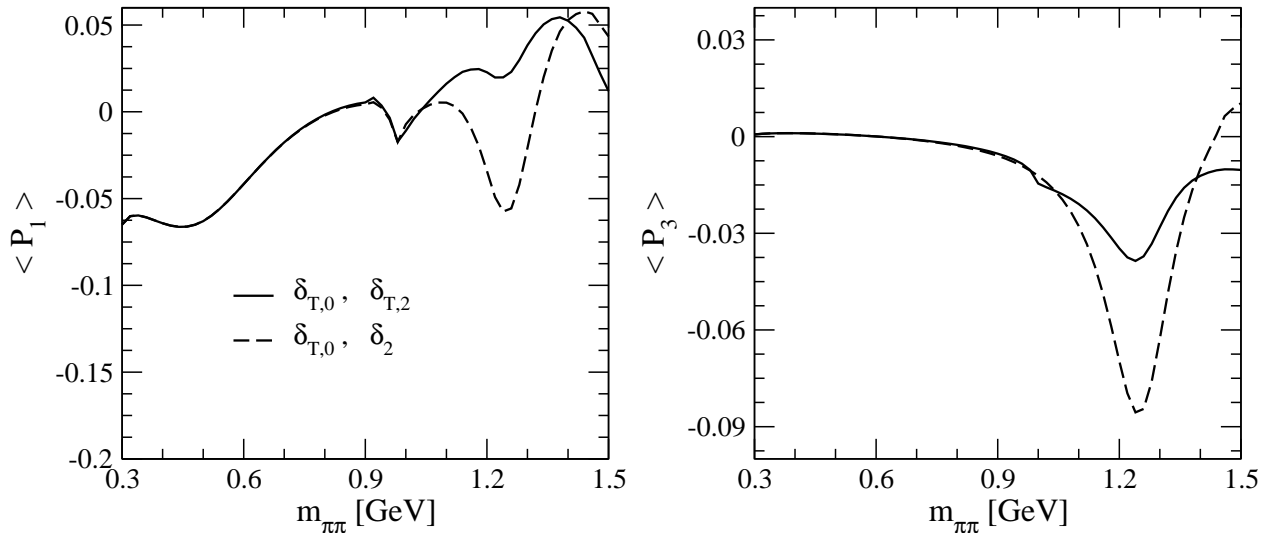


Figure 11: Predictions for the Legendre moments for a deuterium target in typical kinematics of the COMPASS experiment, $x_B = 0.08$, $t = -0.27 \text{ GeV}^2$ and $Q^2 = 7 \text{ GeV}^2$. The Omnès functions used are the same as in Fig. 7.

We finally show in Fig. 11 predictions for the Legendre moments in kinematics typical of the COMPASS experiment. In this case one can afford higher values of Q^2 , which increases the reliability of the leading-twist approach we use here. We see that compared with the HERMES kinematics the overall size of the Legendre moments is decreased. This is not too surprising, since the production of two pions in the P wave is enhanced by the growth of the gluon distribution with decreasing x_B , and the Legendre moments reflect the interference of the P wave with the S or D waves (which are insensitive to the gluon distribution). Comparing with the LO results, we find that the quantitative effect of NLO corrections on the invariant mass spectrum and on the Legendre moments in the kinematics of Fig. 7 is of similar size as in the HERMES case shown above.

7 Summary

We have calculated exclusive electroproduction of $\pi^+\pi^-$ pairs on the nucleon at NLO in α_s , focusing on the kinematics of the existing measurement at HERMES and of a possible analysis at COMPASS. We find that the effects of NLO corrections on the invariant mass spectrum are moderate, although not negligible. The same holds for NLO effects on the angular distribution of the produced pions, quantified by the Legendre moments (6). This indicates that the perturbative expansion is well behaved for the process in the kinematics we have studied.

In addition to generalized parton distributions, which appear in a number of hard exclusive processes, a crucial ingredient for the description of our reaction are two-pion distribution amplitudes, which describe the exclusive hadronization of a parton pair into $\pi^+\pi^-$. The lowest moments of these distribution amplitudes are form factors of the energy-momentum tensor. We have examined in detail their representation as dispersion integrals, which requires special care because of the presence of both S and D wave components. The behavior of these form factors directly influences the invariant mass and angular distribution of the pion pair in electroproduction. A salient feature of the HERMES

measurement [8] is the absence of a clear peak in the S wave for $m_{\pi\pi}$ around the mass of the $f_0(980)$. We take this as a strong indication that, once the two-kaon channel is opened, the phase of the relevant form factors differs strongly from the two-pion phase shift δ_0 . More detailed investigation of the angular distribution suggests that a similar statement may hold in the D wave for $m_{\pi\pi}$ above the mass of the $f_2(1270)$.

Considering two simple hypotheses for the phase of the two-pion distribution amplitudes we have seen that the range of model predictions is far greater than explored so far in the literature, and that the existing data can distinguish between different model assumptions. A more sophisticated treatment would be a two-channel analysis, similar to what has been done for the form factors of the scalar quark current [45, 46, 47, 48]. Such an analysis would involve further low-energy constants, and in our opinion would greatly benefit from further data with smaller statistical errors and, preferably, at higher values of Q^2 .

In conclusion we find that pion electroproduction is a case for which on one hand higher-order QCD corrections are under control and on the other hand chiral perturbation theory is rather advanced and well understood. This process may therefore be seen as a show case for the interplay of both approaches. Further progress can come from the improved knowledge of GPDs and GDAs, from extended calculations in chiral perturbation theory, and from experimental analysis of the two-pion system above the kaon threshold.

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