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On the difference between the pole and the $\overline{\text{MS}}$ masses of the top quark at the electroweak scale

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Abstract

We argue that for a Higgs boson mass $M_H \sim 125$ GeV, as suggested by recent Higgs searches at the LHC, the inclusion of electroweak radiative corrections in the relationship between the pole and $\overline{\text{MS}}$ masses of the top quark reduces the difference to about 1 GeV. This is relevant for the scheme dependence of electroweak observables, such as the ρ parameter, as well as for the extraction of the top quark mass from experimental data. In fact, the value currently extracted by reconstructing the invariant mass of the top quark decay products is expected to be close to the pole mass, while the analysis of the total cross section of top quark pair production yields a clean determination of the $\overline{\text{MS}}$ mass.

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1 Introduction

For the precise understanding of the relationship between running and pole masses of particles, within the framework of the Standard Model (SM) of electroweak (EW) and strong interactions, it is mandatory to use the full SM renormalization group (RG) equations. In this paper, we focus in particular on the top quark mass. In published results, commonly only the QCD corrections are applied, but also the corresponding EW corrections are important. Here we discuss the EW contributions to the SM RG equations and the related matching conditions and their numerical significance for the pole mass. The relevant corrections have been derived for the top quark in Refs. [1, 2, 3]. Assuming the particle recently discovered at the CERN LHC [4] to be the SM Higgs boson, it is possible to specify the corrections numerically. As we know the top quark, like no other quark, is accessible to perturbative predictions by virtue of its very large mass and small width, which let the top quark decay before it can form hadrons.

Since free quarks are not observable in nature, their masses primarily are Lagrangian parameters which parametrize the chiral symmetry breaking in terms of masses as required by observation, mainly by the observed mass spectrum of the hadronic states, which consist of permanently confined quarks and gluons. In any case, quark masses are needed as input parameters for calculations of SM predictions [5] and must be tuned to account for corresponding mass effects in hadronic reactions. The most frequently used definitions of mass are the pole and $\overline{\text{MS}}$ ones, which for quarks both are formal definitions. They both are popular because of their simple access in perturbation theory. One should note that the $\overline{\text{MS}}$ scheme is intrinsically only defined in the perturbative approach.

Applying dimensional regularization [6] and the $\varepsilon = (4 - d)/2$ expansion, the RG functions are uniquely defined, order by order in perturbation theory, by the ultraviolet (UV) properties of the model, represented by the $1/\varepsilon$ counterterms [7]. In order to determine the value of a running mass at some scale, the matching condition between the running mass and some observable has to be evaluated (see e.g. Ref. [8]). Since the SM includes both EW and QCD type UV singularities, the corresponding RG equations have to take into account both, too.

The pole mass is a well defined quantity within perturbation theory. It is related to the pole of the renormalized propagator in the complex energy plane. The position of the pole is a gauge invariant and infrared finite quantity [9, 10]. A shortcoming is the fact that the pole mass suffers from renormalon contributions, which worsen the convergence properties of the perturbative expansion. The corresponding uncertainty is of the order of $\Lambda_{\text{QCD}} \sim 200$ MeV [11, 12], which is not too large for a particle as heavy as the top quark, but leads to an intrinsic limitation of the possible precision. The top quark being a colored object, the pole of its propagator is not an observable per se, although it seems that the color singlet recombination via gluonic strong-interaction effects does not affect the location of the top quark propagator pole very much. These problems and deficiencies have triggered many discussions about the accuracy of the top quark mass and its extraction from experimental data, and actually other mass definitions which look to be closer to observable quantities have been worked out [13, 14]. Usually, alternative masses are nevertheless converted into pole and/or $\overline{\text{MS}}$ masses, which thus both remain useful concepts, and their relationship remains of primary

interest. However, up to now, mostly QCD corrections have been included in the conversion between pole and $\overline{\text{MS}}$ masses of the top quark. In this note, we shall discuss how to account for the EW contributions and evaluate their size. We shall denote a pole mass by capital M and a $\overline{\text{MS}}$ mass by lowercase m in the following.

2 Running masses in the SM

The first systematic inclusion of the EW corrections in the definition of the running mass of a fermion has been achieved in Ref. [1]. By including all self-energy diagrams (including tadpoles), one obtains a gauge invariant relation between pole and bare masses [15]. By applying minimal subtraction to the UV counterterms of this relation, the one-loop relation between a $\overline{\text{MS}}$ mass m_f and the corresponding pole mass M_f , as well as the threshold relation $\delta_{f,\alpha}$ between the corresponding Yukawa coupling $y_f(\mu^2)$ and M_f , have been derived. In this approach, care has to be exercised, especially at the multiloop level, to include all the contributing diagrams including tadpoles, while it is not sufficient to select gauge invariant subsets. As an illustrative example, we mention the $O(\alpha_s)$ mixing contributions to the pole masses of quarks. The definition, via a “gauge invariant set of diagrams including tadpole contributions”, was complemented in Ref. [16] by a theorem about the interrelation between the RG functions for the massive parameters (masses of particles, as well as the Fermi constant) calculated in the broken phase of the SM with RG functions of parameters of the unbroken phase of the SM, in accord with the expectation that spontaneous symmetry breaking does not affect the UV structure of the SM. In other words, the EW UV counterterms in the broken phase of the SM can be obtained in terms of the UV counterterms in the unbroken phase.¹ The above-mentioned theorem has been verified explicitly by a two-loop analysis of the UV counterterms evaluated in the broken phase of SM [2, 3, 16]. This approach gives rise to the same set of quark self-energy Feynman diagrams [20] as well as to an equivalent definition [21] of the threshold relations [1, 22].

Before we proceed, let us remind the reader of some basic definitions needed for the following discussion. Applying dimensional regularization [6] in the broken phase, the SM UV counterterms for the quark masses in the $\overline{\text{MS}}$ scheme have the following form:

$$m_{q,\text{bare}} = m_q(\mu^2) \left[1 + \alpha_s \sum_{i=0} \alpha_s^i \sum_{k=1}^{i+1} \frac{\delta Z_{\alpha_s}^{(i,k)}}{\varepsilon^k} + \alpha \sum_{i,j=0} \alpha^i \alpha_s^j \sum_{k=1}^{i+j+1} \frac{\delta Z_{\alpha}^{(i,j,k)}}{\varepsilon^k} \right]. \quad (1)$$

The first series in this relation corresponds to the QCD corrections, the second one to the EW contribution mixed in higher orders with QCD. In accordance with 't Hooft's

¹A different theorem states that tadpole terms, which are absent in the symmetric phase, drop out from observable quantities. However, if one omits tadpole terms in relations between bare and renormalized quantities, as frequently done in SM calculations [8, 17, 18, 19], one not only loses a manifestly gauge invariant relationship between the bare and the renormalized theory, also the UV structure is not preserved and one does not get the same RG equations. In fact, tadpoles are related to quadratic divergences which show up in the renormalization of the mass parameter m^2 of the Higgs potential in the symmetric phase.

prescription [7], the quark mass anomalous dimension, defined by

$$\mu^2 \frac{d}{d\mu^2} \ln m_q^2 = \gamma_q(\alpha_s, \alpha) = \left[\alpha_s \frac{\partial}{\partial \alpha_s} + \alpha \frac{\partial}{\partial \alpha} \right] \left[\alpha_s \sum_{i=0} \alpha_s^i \delta Z_{\alpha_s}^{(i,1)} + \alpha \sum_{i,j=0} \alpha^i \alpha_s^j \delta Z_{\alpha}^{(i,j,1)} \right], \quad (2)$$

can be split into two parts: the QCD and EW contributions $\gamma_q(\alpha_s, \alpha) = \gamma_q^{\text{QCD}} + \gamma_q^{\text{EW}}$, where γ_q^{QCD} includes all terms which are proportional to powers of α_s only and γ_q^{EW} includes all other terms proportional to at least one power of α , and beyond one loop multiplied by further powers of α and/or α_s . We call γ_t^{QCD} the QCD anomalous dimension and γ_t^{EW} the EW one. As has been shown in Ref. [16], the EW contribution to the fermion anomalous dimension γ_t^{EW} in the $\overline{\text{MS}}$ scheme may be written in terms of RG functions of parameters in the unbroken phase of the SM as

$$\gamma_t^{\text{EW}} = \gamma_{y_t} + \frac{1}{2} \gamma_{m^2} - \frac{1}{2} \frac{\beta_\lambda}{\lambda}, \quad (3)$$

where $\gamma_{m^2} = \mu^2 \frac{d}{d\mu^2} \ln m^2$, $\beta_\lambda = \mu^2 \frac{d}{d\mu^2} \lambda$, $\gamma_{y_q} = \mu^2 \frac{d}{d\mu^2} \ln y_q$, y_q is the Yukawa coupling of quark q , and m^2 and λ are the parameters of the scalar potential $V = \frac{m^2}{2} \phi^2 + \frac{\lambda}{24} \phi^4$.

It has also been shown [23] that the coefficients of the higher poles in ε in the mass counterterms (1) in the broken phase are uniquely determined by the lower-order coefficients and the RG functions defined by Eq. (3).

The RG equation for the square of the Higgs vacuum expectation value (VEV) $v(\mu^2)$ follows from the RG equations for masses and massless coupling constants and reads

$$\begin{aligned} \mu^2 \frac{d}{d\mu^2} v^2(\mu^2) &= 4 \mu^2 \frac{d}{d\mu^2} \left[\frac{m_W^2(\mu^2)}{g^2(\mu^2)} \right] = 4 \mu^2 \frac{d}{d\mu^2} \left[\frac{m_Z^2(\mu^2) - m_W^2(\mu^2)}{g'^2(\mu^2)} \right] \\ &= 3 \mu^2 \frac{d}{d\mu^2} \left[\frac{m_H^2(\mu^2)}{\lambda(\mu^2)} \right] = 2 \mu^2 \frac{d}{d\mu^2} \left[\frac{m_f^2(\mu^2)}{y_f^2(\mu^2)} \right] = v^2(\mu^2) \left[\gamma_{m^2} - \frac{\beta_\lambda}{\lambda} \right], \end{aligned} \quad (4)$$

where g' and g are the $U(1)_Y$ and $SU(2)_L$ gauge couplings, respectively, and we assume the running of g and g' as well as of y_t and λ to be the same in the broken and the unbroken phases [21, 24, 25, 26, 27, 28, 29, 30]. Since the relation² $G_F = \frac{1}{\sqrt{2} v^2}$ is valid for bare as well as for on-shell parameters, the RG equation for the $\overline{\text{MS}}$ version of the running Fermi constant follows from $G_F^{\overline{\text{MS}}}(\mu^2) = \frac{1}{\sqrt{2} v^2(\mu^2)}$. The corresponding anomalous dimension γ_{G_F} of $G_F^{\overline{\text{MS}}}$ is then given by

$$\gamma_{G_F} = \mu^2 \frac{d}{d\mu^2} \ln G_F^{\overline{\text{MS}}}(\mu^2) = -\mu^2 \frac{d}{d\mu^2} \ln v^2(\mu^2) = - \left[\gamma_{m^2} - \frac{\beta_\lambda}{\lambda} \right], \quad (5)$$

i.e., by minus the anomalous dimension of v^2 .

We note that the anomalous dimension of $v^2(\mu^2)$ defined by Eq. (4) via diagrammatic calculations differs from the anomalous dimension of the scalar field as obtained in the effective-potential approach [31].

²By G_F we denote a generic Fermi constant, by G_μ the physical on-shell one, and by $G_F^{\overline{\text{MS}}}$ the $\overline{\text{MS}}$ variant.

The RG equations (2) have to be complemented by matching conditions between pole and running masses, which we may write in the form

$$M_t - m_t(\mu^2) = m_t(\mu^2) \sum_{j=1} \left(\frac{\alpha_s(\mu^2)}{\pi} \right)^j \rho_j + m_t(\mu^2) \sum_{i=1; j=0} \left(\frac{\alpha(\mu^2)}{\pi} \right)^i \left(\frac{\alpha_s(\mu^2)}{\pi} \right)^j r_{ij}. \quad (6)$$

The QCD corrections ρ_j were calculated in Refs. [32, 33, 34] up to $j = 3$, while the $O(\alpha)$ and $O(\alpha\alpha_s)$ corrections r_{10} and r_{11} , respectively, are available in analytic form from Refs. [1, 2, 3]. The $O(\alpha\alpha_s)$ result for r_{11} with tadpoles dropped was also evaluated using asymptotic expansions in Ref. [35] and numerical agreement with Refs. [2, 3] was found after subtraction of the tadpoles. The leading part of the $O(G_\mu M_t^2 \alpha_s)$ contribution to r_{11} was confirmed in Ref. [36] after including the tadpole contribution in the result of Ref. [17]. The correction r_{12} has been evaluated in Ref. [19] in the gaugeless-limit approximation of the SM.

3 Behavior of the RG equations at low and high energies

Let us analyze the behavior of the full SM RG equation for a quark mass in the $\overline{\text{MS}}$ scheme

$$\mu^2 \frac{d}{d\mu^2} \ln m_f(\mu^2) = \gamma_q^{\text{QCD}} + \gamma_{y_f} - \frac{1}{2} \gamma_{G_F}, \quad (7)$$

in which the EW part follows from Eqs. (3) and (4). Let us consider the low-energy limit first. In the weak sector of the SM, there is no decoupling because masses and couplings are interrelated by the Higgs mechanism. So “decoupling by hand” as usually applied in QCD by considering an effective ‘ n_f active flavors’ QCD to be matched at successive flavor thresholds, and which can be applied to QED as well, does not make sense in the weak sector. Note that there is no decoupling for the W and Z bosons: the limit $M_W \rightarrow \infty$ can be achieved by letting $g \rightarrow \infty$ or $v \rightarrow \infty$ or both. In nature, only the limit $g \rightarrow \infty$ leads to the observed low-energy limit of the effective four-fermion theory with $\sqrt{2} G_\mu = 1/v^2$ fixed, by nuclear β decay etc. This obviously is a non-decoupling effect. In contrast to QED or QCD, the low-energy effective theory (obtained after elimination of the heavy state) is a non-renormalizable one exhibiting a completely wrong high-energy behavior. So, in general, “decoupling by hand,” as it is commonly utilized in $\overline{\text{MS}}$ -parametrized QCD, is not very sensible when the Appelquist-Carazzone theorem [37] does not apply.

Nevertheless, in calculations of EW radiative corrections for LEP processes, covering scales up to 200 GeV, the standard on-shell parametrization in terms of the most precise parameters α , G_μ , and M_Z (besides the fermion and Higgs-boson masses) reveals that, while α is running strongly, keeping G_μ as scale independent³ provides an

³This assertion has been checked experimentally by comparing the standard low-energy quantity G_μ determined via the muon lifetime $\tau_\mu = 1/\Gamma_\mu(\mu \rightarrow e\bar{\nu}_e\nu_\mu)$ versus the corresponding effective coupling extracted from the leptonic W -boson decay rate $\hat{G}_\mu = 12\pi\Gamma_{W\ell\nu}/(\sqrt{2}M_W^3)$, which involves W -boson mass scale observables only. The fact that $\hat{G}_\mu \approx G_\mu$ with good accuracy is not surprising because the tadpole corrections,

excellent parametrization in terms of $\alpha(M_Z^2)$, G_μ and M_Z for LEP observables. The latter parametrization incorporates the leading-logarithmic resummation as effectuated by the RG. Usually the scale insensitivity of an effective G_F is “explained” by a “decoupling by hand” argument via inspection of the one-loop RG equation

$$\mu^2 \frac{dG_F^{\overline{\text{MS}}}}{d\mu^2} = \frac{G_F^{\overline{\text{MS}}}}{8\pi^2\sqrt{2}} \left\{ \sum_f \left(m_f^2 - 4 \frac{m_f^4}{m_H^2} \right) - 3 M_W^2 + 6 \frac{M_W^4}{m_H^2} - \frac{3}{2} M_Z^2 + 3 \frac{M_Z^4}{m_H^2} + \frac{3}{2} m_H^2 \right\}, \quad (8)$$

which follows from the counterterm given first in Ref. [15]. If we only sum terms with $m_f < \mu$, there is effectively no running (because of the smallness of the light-fermion masses) before M_W , M_Z , M_H , and M_t come into play.

As mentioned earlier, ambiguities enter if we are to represent predictions in terms of the not-so-physical $\overline{\text{MS}}$ parameters.⁴ On phenomenological grounds, as G_F has been measured to agree at the M_Z scale with its low-energy version G_μ and because Yukawa couplings run as they do in the symmetric phase, below of the EW scale, one may define effective light-fermion masses to run via their Yukawa couplings only:

$$\hat{m}_f(\mu^2) = 2^{-3/4} G_\mu^{-1/2} y_f(\mu^2). \quad (9)$$

As the Yukawa couplings $y_f(\mu^2)$ are not affected by the Higgs mechanism, the EW corrections to the Yukawa couplings are free of tadpoles [1, 20] and/or quadratic divergences. Since real physical observables are also free of tadpole contributions, this property is an additional argument why Eq. (9) is a good candidate for the evaluation of the EW contributions to the ratio between pole and $\overline{\text{MS}}$ masses of lighter quarks, such as the bottom and charm quarks (see also the discussion in Ref. [38]). In short, fermion masses and Yukawa couplings have equivalent RG evolutions as long as G_F or, equivalently, v can be taken not to be running, so that one may identify $G_F^{\overline{\text{MS}}}(\mu^2) = G_\mu$. Alternatively, and more consequently concerning the decoupling issue, the proper $\overline{\text{MS}}$ definition of a running fermion mass is

$$m_f(\mu^2) = 2^{-3/4} \left(G_F^{\overline{\text{MS}}}(\mu^2) \right)^{-1/2} y_f(\mu^2), \quad (10)$$

where $G_F^{\overline{\text{MS}}}(\mu^2)$ and $m_f(\mu^2)$ are solutions of Eq. (5) and (7), respectively. For the $\overline{\text{MS}}$ top quark mass, we consequently advocate to utilize Eq. (10), which among others includes the tadpole contributions. Note that the difference between Eqs. (9) and (10) is particularly significant for the top quark. As both versions are gauge invariant by definition, the difference is not just dropping the tadpole terms or not.

The running of $G_F^{\overline{\text{MS}}}$ definitely starts at about $\mu \sim 2M_W$,⁵ when the scale of a process exceeds the masses of the weak gauge bosons. Since the top quark is the

which potentially lead to substantial corrections, are absent in relations between observable quantities as we know.

⁴The $\overline{\text{MS}}$ parameters other than $v(\mu^2)$ (i.e. $\overline{\text{MS}}$ gauge, Yukawa, and Higgs self couplings) are likely the most natural parameters in the unbroken phase of the SM, where an S matrix does not exist due to infrared problems. Other renormalization schemes that can be applied in this case include the MOM-type schemes, which are, however, gauge dependent.

⁵As the on-shell version of G_F at the Z -boson mass scale can be identified with G_μ , it is justified to match $G_F^{\overline{\text{MS}}}$ with G_μ at the scale M_Z .

heaviest particle in the SM, at least here the “decoupling by hand” prescription becomes obsolete, and we have to take full SM parameter relations as they are.

One of the most well-known non-decoupling effects related to the top quark is the EW ρ parameter $\rho_{\text{eff}}(0) = G_{\text{NC}}/G_{\text{CC}}(0)$, where $G_{\text{CC}}(0)$ is the Fermi coupling $G_F = G_\mu$ and G_{NC} the low-energy effective axial-vector Z -boson coupling to fermions. As is well known, in the SM we have

$$\rho = 1 + \frac{N_c G_\mu}{8\pi^2 \sqrt{2}} \left(m_t^2 + m_b^2 - \frac{2m_t^2 m_b^2}{m_t^2 - m_b^2} \ln \frac{m_t^2}{m_b^2} \right) \approx 1 + \frac{N_c y_t^2}{32\pi^2}, \quad (11)$$

which measures the weak-isospin breaking by the Yukawa couplings of the heavy fermions at zero momentum. Within the SM, this quantity is strongly constrained by LEP data, and, in spite of the fact that the top quark was by far too heavy to be produced at LEP, the top quark contribution and indirectly the top quark mass have been constrained by LEP data. Actually, a first strong indication of a heavy top quark had been found earlier by the ARGUS experiment, which discovered, unexpectedly, a substantial $B\bar{B}$ oscillation (in the SM enhanced by a contribution $\propto y_t^2$), which turned out to be much larger than anticipated before. So recipes like “decoupling by hand” make no sense to be applied to the weak sector of the SM, as heavy-particle effects definitely cannot be renormalized away.

For large values of μ^2 , the behavior of the running Fermi constant $G_F(\mu^2)$ is defined by the Higgs self-coupling and the sign of its beta-function β_λ :

$$\mu^2 \frac{d}{d\mu^2} \ln G_F(\mu^2) \sim \frac{\beta_\lambda(\mu^2)}{\lambda(\mu^2)}. \quad (12)$$

The detailed perturbative analysis of the r.h.s. of this equation was performed recently (see Refs. [21, 24, 25, 26, 28, 29]) and reveals that the beta function β_λ is negative up to a scale of about 10^{17} GeV, where it changes sign. Above the zero of β_λ , the effective coupling starts to increase again, and the key question is whether at the zero of the beta function the effective coupling is still positive. In the latter case, it will remain positive although small up to the Planck scale. In any case, at moderately high scales where $\beta_\lambda < 0$, and provided that λ is still positive, the following behavior is valid for the Fermi constant:

$$G_F(\mu^2) \Big|_{\mu^2 \rightarrow \infty} \sim (\mu^2)^{\frac{\beta_\lambda(\mu^2)}{\lambda(\mu^2)}} \rightarrow 0, \quad (13)$$

being decreasing, which means that $v^2(\mu^2)$ is increasing at these scales (where $\beta_\lambda < 0$ and $\lambda > 0$). The analysis of Ref. [24] finds that λ turns negative (unstable or meta-stable Higgs potential) before the beta function reaches its zero. This may happen at rather low scales, at around 10^{10} GeV. In this case, we would get an infinite Higgs vacuum expectation value far below the Planck scale as an essential singularity. Given the present uncertainty in the value of M_t , there is a good chance that λ remains positive up to the zero of the beta function and as a consequence up to the Planck scale [21, 25]. Then $G_F(\mu^2)$ would start to increase again, and $v(\mu^2)$ would start to decrease but remain finite (about 685 GeV) at the Planck scale, implying that all effective masses stay bounded. The effectively massless symmetric phase of the SM would then be obtained at high energies by the fact that mass effects are suppressed

for dimensional reasons: according to the RG, for a vertex function under a dilatation of all momenta, $\{p_i\} \rightarrow \{\kappa p_i\}$, up to the overall dynamical dimension and wave-function renormalizations, the result is given by replacing $g_i \rightarrow g_i(\kappa)$ and $m_i \rightarrow m_i(\kappa)/\kappa$ at fixed $\{p_i\}$ and renormalization scale μ . I.e. provided that $m(\kappa)/\kappa \rightarrow 0$ as $\kappa \rightarrow \infty$, the high-energy asymptotic effective theory is effectively massless as expected in the symmetric phase.

4 Numerical result for $m_t - M_t$

In the previous section, we have presented the arguments, why decoupling does not apply in the EW sector, in particular not to the top quark mass effects. In this section, we will check how significant the EW contribution to matching and running of the top quark mass is. For that purpose, the inverse of the relation (6), $m_t(\mu^2)$ as a function of the pole mass M_t , is required (see Eq. (5.54) in Ref. [2]). For the numerical analysis, we adopt the following values for the input parameters [39]:

$$\begin{aligned} M_Z &= 91.1876(21) \text{ GeV}, & M_W &= 80.385(15) \text{ GeV}, & M_t &= 173.5(1.0) \text{ GeV},^6 \\ G_F &= 1.16637 \times 10^{-5} \text{ GeV}^{-2}, & \alpha^{-1} &= 137.035999, & \alpha_s^{(5)}(M_Z^2) &= 0.1184(7). \end{aligned} \quad (14)$$

Furthermore, we take the effective fine-structure constant at the Z boson mass scale to be $\alpha^{-1}(M_Z^2) = 127.944$. All light-fermion masses M_f ($f \neq t$) give negligible effects and do not play any role in our consideration. Up to the three-loop order, the QCD relation between the running and pole masses is given by (see Eq. (12) in Ref. [34])

$$\begin{aligned} \left\{ m_t(M_t^2) - M_t \right\}_{\text{QCD}} &= M_t \left[-\frac{4}{3} \frac{\alpha_s^{(6)}(M_t^2)}{\pi} - 9.125 \left(\frac{\alpha_s^{(6)}(M_t^2)}{\pi} \right)^2 \right. \\ &\quad \left. - 80.405 \left(\frac{\alpha_s^{(6)}(M_t^2)}{\pi} \right)^3 \right]. \end{aligned} \quad (15)$$

Using $\alpha_s^{(6)}(M_t^2) = 0.1079(6)$ [41], which follows from the value of $\alpha_s^{(5)}(M_Z^2)$ in Eq. (14) via four-loop evolution and three-loop matching [42], we obtain the numerical result

$$\left\{ m_t(M_t^2) - M_t \right\}_{\text{QCD}} = -7.95 \text{ GeV} - 1.87 \text{ GeV} - 0.57 \text{ GeV} = -10.38 \text{ GeV}. \quad (16)$$

A numerical estimation of the $O(\alpha_s^4)$ term, given in Ref. [43], is ~ -0.02 GeV, which is not included in Eq. (16). The analytic result for the EW corrections at the one-loop order has a more complicated form and may be found in Refs. [1, 15]. The two-loop corrections of order $O(\alpha^2)$ are not yet known. Exploring the results of Ref. [19], we estimate it to be of order $O(1 \text{ GeV})$. Another way to estimate the two-loop contribution

⁶The values of the top quark mass quoted by the experimental collaborations correspond to parameters in Monte Carlo event generators in which, apart from parton showering, the partonic subprocesses are calculated at the tree level, so that a rigorous theoretical definition of the top quark mass is lacking [39, 40]. For definiteness, we take the value from Ref. [39] to be the pole mass M_t .

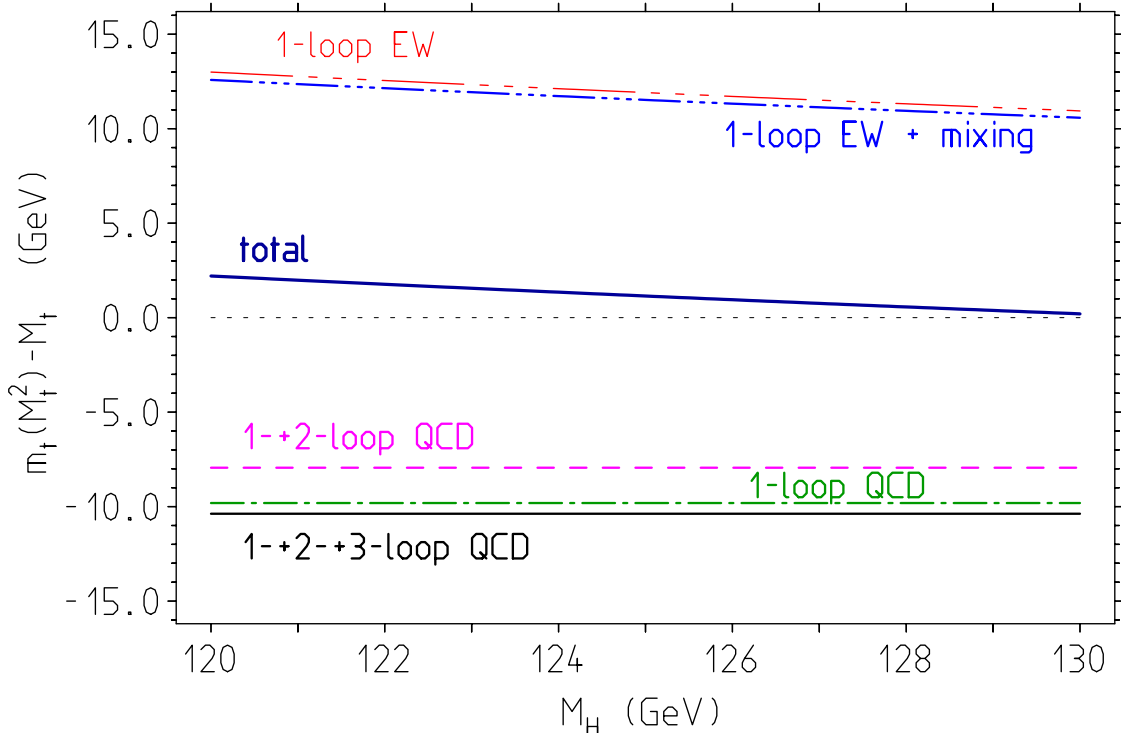


Figure 1: Numerical results for the difference $m_t(M_t^2) - M_t$. The red line represents the $O(\alpha)$ correction, the blue line the $O(\alpha) + O(\alpha\alpha_s)$ correction, the green line the $O(\alpha_s)$ correction, the magenta line the $O(\alpha_s) + O(\alpha_s^2)$ correction, the black line the $O(\alpha_s) + O(\alpha_s^2) + O(\alpha_s^3)$ correction, and the dark blue line the sum of all these corrections. The input parameters are specified in Eq. (14).

follows from results of Ref. [16] and the observation that the largest contribution is coming from tadpole diagrams:

$$\left\{ \frac{m_t(\mu^2)}{M_t} \right\} \sim \left\{ \frac{m_t(\mu^2)}{M_t} \right\}_{\text{tadpole}} \sim \sqrt{\frac{m_W^2(\mu^2)}{M_W^2}} = 1 + \frac{1}{2}\delta_{W,\alpha} + \frac{1}{2}\delta_{W,\alpha\alpha_s} + \frac{1}{2}\delta_{W,\alpha^2} - \frac{1}{8}\delta_{W,\alpha}^2, \quad (17)$$

where $\delta_{W,\alpha}$, δ_{W,α^2} , and $\delta_{W,\alpha_s\alpha}$ are the corrections in the relation $\frac{m_W^2(\mu^2)}{M_W^2} = 1 + \delta_{W,\alpha} + \delta_{W,\alpha_s\alpha} + \delta_{W,\alpha^2}$ and may be found in Ref. [16]. This also allows us to estimate the error due to the unknown higher-order corrections, which is about 1 GeV.

A detailed comparison of the individual contributions is presented in Fig. 1. For a set of experimentally most probable values of M_H [4], $M_H = \{124, 125, 126\}$ GeV, the numerical values of the EW and QCD contributions to the difference $m_t(M_t^2) - M_t$ and their sum are collected in Table 1. As a result, we observe a large EW correction, which for the assumed M_H range almost perfectly cancels the QCD correction. The relationship between $m_t(M_t^2)$ and M_t can be parametrized in the range displayed in

Table 1: The various contributions to $m_t(M_t^2) - M_t$ in GeV.

M_H [GeV]	$O(\alpha)$	$O(\alpha\alpha_s)$	$O(\alpha) + O(\alpha\alpha_s)$	$O(\alpha_s) + O(\alpha_s^2) + O(\alpha_s^3)$	total
124	12.11	-0.39	11.72	-10.38	1.34
125	11.91	-0.39	11.52	-10.38	1.14
126	11.71	-0.38	11.32	-10.38	0.94

Fig. 1 as

$$\left\{ m_t(M_t^2) - M_t \right\}_{\text{SM}} = \left\{ m_t(M_t^2) - M_t \right\}_{\text{QCD}} + \left[0.0664 - 0.00115 \times \left(\frac{M_H}{1 \text{ GeV}} - 125 \right) \right] M_t. \quad (18)$$

The almost perfect cancellation between the QCD and EW effects for the given Higgs boson mass is certainly accidental, but must be taken into account in comparisons with experimental data. Our calculation shows that the large leading correction, of $\mathcal{O}(\alpha)$, to the shift $m_t(M_t^2) - M_t$ is not substantially modified by the next-to-leading term, of $\mathcal{O}(\alpha\alpha_s)$. Radiative corrections beyond the presently known ones are likely to be small and not to change the observed quenching qualitatively.

5 Conclusions

We calculated the shift $m_t(M_t^2) - M_t$ of the top-quark mass in the SM by strictly taking into account all diagrams generated by the Feynman rules, including the tadpole ones, as is required to manifestly respect the Slavnov-Taylor and Ward-Takahashi identities.

SM transition matrix elements of physical processes renormalized according to the EW on-shell scheme are manifestly devoid of tadpoles to all orders of perturbation theory [44]. This has led to the quite common practice to set tadpole contributions to zero. On the other hand, the tadpoles are gauge dependent, and the mass counterterms are only gauge independent if the tadpole contributions are included, as may be observed already at one loop [1, 15]. Also, if tadpoles drop out from physical quantities, or relations between them, it does not mean that carrying them along in a calculation would not lead to a correct result. In contrast, tadpole cancellation may serve as a useful check of a calculation.

Upon on-shell renormalization, the SM transition matrix elements of physical processes are gauge-independent functions of the pole masses and the other renormalized parameters, i.e. the couplings and mixing angles. By finite reparametrizations, these transition matrix elements may be converted to any other renormalization scheme, in our case to the $\overline{\text{MS}}$ scheme. The relationships between the on-shell parameters and the $\overline{\text{MS}}$ parameters are gauge independent only if tadpole contributions are retained. Tadpoles are artifacts of spontaneous symmetry breaking, where they show up in the Higgs vacuum expectation value, which induces the masses. Correspondingly, tadpoles affect all mass counterterms, and only these. The dimensionless gauge and Yukawa couplings as well as the Higgs self-coupling and their counterterms are not affected.

We advocate to keep the relationships between the on-shell and bare parameters in the dimensionally regularized theory and their relationships to the closely related $\overline{\text{MS}}$ parameters gauge invariant. Otherwise, the expressions of the transition matrix elements in terms of the renormalized parameters acquire artificial gauge dependence, and the choice of gauge must always be specified, too, whenever the value of a renormalized parameter extracted from experimental data is communicated.

This leads us to include the tadpole contribution in the relationship between the pole mass M_t and the $\overline{\text{MS}}$ mass $m_t(\mu^2)$ of the top quark, on which we focus our attention in this paper. Assuming the recently-discovered Higgs-like boson to be the missing link of the SM, then the smallness of its mass M_H renders the positive tadpole contribution to the difference $m_t(M_t^2) - M_t$ so sizable that it almost perfectly cancels the familiar -10 GeV shift induced by pure QCD corrections, and it is one of the major purposes of this work to publicize this intriguing coincidence. As a welcome consequence of this near-quenching, the theoretical uncertainty due to scheme dependence in any physical observable that depends on the top quark mass at leading order is greatly reduced. In fact, this uncertainty is proportional to the shift $|m_t(M_t^2) - M_t|$ itself, and is thus reduced by an order of magnitude if $|m_t(M_t^2) - M_t|$ is. This may easily be understood as follows. Let $O = f(M)(1 + \delta)$ be an M -dependent observable with radiative correction δ in the on-shell scheme. In the $\overline{\text{MS}}$ scheme, this observable is then given by $\overline{O} = f(m)(1 + \overline{\delta})$ with $\overline{\delta} = \delta + (M - m)\partial \ln f(m)/\partial m$, and the leading scheme dependence corresponds to the magnitude of $O/\overline{O} - 1 = \delta(M - m)\partial \ln f(m)/\partial m$.

The shift $m_t(M_t^2) - M_t$ is of paramount phenomenological importance for the combination of different determinations of the top quark mass in ongoing experiments. In fact, the value currently extracted by reconstructing the invariant mass of the top quark decay products is expected to be close to M_t [13, 39, 40], while the analysis of the total cross section of top quark pair production yields a clean determination of $m_t(M_t^2)$ [25, 39, 40, 45]. The EW $\mathcal{O}(\alpha)$ correction to the $t\bar{t}$ production cross section is available in the on-shell scheme [18, 46]. In order to consistently incorporate it in the QCD analysis of Refs. [25, 45], it needs to be converted to the $\overline{\text{MS}}$ scheme as described above. This will generate an explicit tadpole contribution in the radiative corrections to the cross section. In turn, the scheme dependence will be substantially reduced because $m_t(M_t^2)$ and M_t almost coincide.

We have analyzed the EW contributions to the running and scheme dependence of the top quark mass above the W boson threshold, when G_F can not be treated any longer as a low-energy constant in one-to-one correspondence with the muon lifetime, but turns into a running effective parameter. This effect is similar to the running electromagnetic coupling $\alpha(\mu^2)$, which, however, is strongly scale dependent right from zero momentum and is sensitive to non-perturbative hadronic vacuum polarization effects there. Like the running couplings g, g', y_f , and λ , also the running of G_F is scheme dependent. In the $\overline{\text{MS}}$ scheme, the scale at which G_F effectively starts to run, is not uniquely defined. SM non-decoupling effects have to be taken into account. In any case, light-fermion contributions including the one of the bottom quark are tiny. The quantitative analysis shows that the main contribution comes from the matching relation (6), which supplements the RG equation (7). At low energies, the running of the quark mass is equivalent to the running of the Yukawa coupling via Eq. (10) and by standard QCD corrections.

As the $\overline{\text{MS}}$ scheme is a renormalization scheme with mass-independent anomalous dimensions, mass effects drop out at high energies on account of their positive canonical mass dimension. This is in contrast to the on-shell renormalization scheme, where masses are utilized as renormalization scales, which leads to residual mass effects in the high-energy asymptotic regime via renormalization effects, with the Callan-Symanzik equation replacing the $\overline{\text{MS}}$ RG equations.

As our focus is on physics at the EW scale, a precise treatment of mass effects of the heavier SM states (t, H, Z, W) is mandatory for a precise interpretation of related experimental data. In particular, for the top quark, which as we know decays before it can form hadrons, it is not sufficient to take into account QCD corrections only, as our analysis shows.

In conclusion, for the current value of the Higgs mass, $122 < M_H < 128$ GeV [4], the one-loop EW corrections to $m_t(M_t^2) - M_t$ are large and have opposite sign relative to the QCD contributions, so that the total correction is actually small and approximately equal to $[1 \pm O(1)]$ GeV (see Table 1). As a result, taking into account EW radiative corrections, besides the QCD ones, reduces the scheme dependence for EW observables that depend on the top quark mass.

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