DESY 12-189

Spin-$k/2$-spin-$k/2$ $SU(2)$ two-point functions on the torus

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Abstract

We discuss a class of two-point functions on the torus of primary operators in the $SU(2)$ Wess-Zumino-Witten model at integer level k. In particular, we construct an explicit expression for the current blocks of the spin-$\frac{k}{2}$-spin-$\frac{k}{2}$ torus two-point functions for all k. We first examine the factorization limits of the proposed current blocks and test their monodromy properties. We then prove that the current blocks solve the corresponding Knizhnik-Zamolodchikov-like differential equations using the method of Mathur, Mukhi and Sen.

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1 Introduction

The $SU(2)$ Wess-Zumino-Witten (WZW) model is an integral part of string theory on $AdS_3 \times S^3$. In this string theory, worldsheet correlators were computed on the sphere and agreement with the correlators in the dual conformal field theory was found [1]. The computation was done in a particular large N limit in which higher-genus contributions are suppressed. In order to leave the large N limit in AdS/CFT [2], it would be desirable to repeat such computations on higher-genus surfaces. Unfortunately, while correlators in the $SU(2)$ WZW model have been much studied on the sphere [3, 4], not much is known at higher genera (see however [5]-[10] for general progress). This makes it difficult to go beyond large N and motivates our renewed interest in the $SU(2)$ WZW model on Riemann surfaces with genus $g \geq 1$.

A few correlators of $SU(2)$ primary fields are known on the torus ($g = 1$). The zero-point functions, *i.e.* the characters, were determined in [11]. The one-point functions [12] vanish as they are not singlets under $SU(2)$. The two-point functions were determined at level $k = 1, 2$ by the identification of the $SU(2)$ WZW model with free field theories [13], by solving appropriate differential equations [14, 15, 16], and by pinching genus-2 characters [17, 18]. Not so much is known for levels $k \geq 3$ at which the $SU(2)$ WZW model becomes interacting. An integral representation of the two-point spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ current blocks on the torus was found in [19] (for all k), using a Coulomb gas approach, and in [20], using the free field representation developed by Bernard and Felder [7]. In principle, these methods can be used to find expressions for the other classes of torus two-point functions but may be cumbersome to apply. Other correlators on the torus and related work can be found in [21, 22].

In this paper we find a relatively simple expression for the spin-$\frac{k}{2}$-spin-$\frac{k}{2}$ torus two-point functions $\langle\Phi_{k/2}(z)\Phi_{k/2}(0)\rangle$, where $\Phi_{k/2}$ is a level-k primary with maximal spin $j=\frac{k}{2}$ and conformal weight $h=\frac{k}{4}$. A special feature of these two-point functions is that they only have the identity running in their intermediate channel. In the limit, when the two insertion points are close together, their current blocks therefore factorize into the two-point function on the sphere and a level-k character. This suggests to write the current blocks as

$$f_l \sim \frac{\chi_l(z,\tau)}{E(z,\tau)^{2h}}\,, \qquad (l=0,...,\tfrac{k}{2})\,, \tag{1.1}$$

where $E(z,\tau)$ is the torus prime form, and $\chi_l(z,\tau)$ is an extension of the character $\chi_l(\tau)=\lim_{z\to 0}\chi_l(z,\tau)$ to non-zero values of the distance z of the two insertions. The non-trivial task is to find a function $\chi_l(z,\tau)$ which has the correct monodromy properties with respect to shifts $z\to z+1$ and $z\to z+\tau$. Fortunately, in the spin-$\frac{k}{2}$ case, the expression $\chi_l(z,\tau)$ provided by Kac and Peterson [11] already has the correct monodromy properties, and the f_l's given by (1.1) become the current blocks of $\langle\Phi_{k/2}(z)\Phi_{k/2}(0)\rangle$. This will be proven using the differential equation technique developed by Mathur, Mukhi and Sen [14, 15] (see also [23]).

The paper is organized as follows. In section 2, we review some basic prerequisites and state our conjecture for the spin-$\frac{k}{2}$-spin-$\frac{k}{2}$ torus two-point functions. In section 3, we examine the factorization limits of the corresponding current blocks and study their monodromy properties. In section 4, we eventually prove our conjecture using the differential equation technique of [14, 15]. In section 5, we conclude with a few comments on other $SU(2)$ torus two-point functions.

2 The spin-$\frac{k}{2}$-spin-$\frac{k}{2}$ two-point function on the torus

Consider the $SU(2)$ WZW model at integer level k. It is invariant under the combined action of the Virasoro and affine $su(2)$ algebra. The primary fields Φ_j have the conformal weights

$$h_j = \frac{j(j+1)}{k+2}\,, \qquad j=0,...,\tfrac{k}{2}\,, \tag{2.1}$$

and satisfy

$$J_n^a|j\rangle = 0\,, \quad n>0\,, \qquad J_0^+|j\rangle = 0\,, \qquad J_0^0|j\rangle = j|j\rangle\,, \tag{2.2}$$

where J_n^a $(a=\pm,0)$ are the modes of the holomorphic $SU(2)$ currents. The central charge is $c=\frac{3k}{k+2}$.

The current blocks of the two-point function $\langle\Phi_j(z)\Phi_j(0)\rangle$ of a primary field Φ_j on the torus are shown in figure 1. The field running in the torus is denoted by $[\Phi_l]$, while the

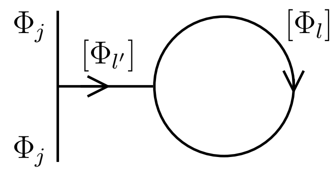

Figure 1: Current block for the torus two-point function in the $z \to e^{2\pi i}z$ eigenstate basis.

field $[\Phi_{l'}]$ connects the propagator to the loop.[1] For a given field Φ_j, the connecting fields allowed by the fusion rules are $[\Phi_{l'}]$ with $0 \le l' \le \min(2j, k-2j)$.[2] To each field $[\Phi_{l'}]$, the fusion rules allow $k+1-2l'$ fields $[\Phi_l]$ $(l = l'/2, ..., (k-l')/2)$ in the loop. Each two-point correlator therefore contains

$$n = \sum_{l'=0}^{\min(2j,k-2j)} (k+1-2l') = (2j+1)(k-2j+1) \tag{2.3}$$

current blocks.

A simple class of two-point functions is $\langle \Phi_j(z)\Phi_j(0)\rangle$ with spin $j = \frac{k}{2}$ (corresponding to $h = \frac{k}{4}$), which is the maximal spin at level k. In this case the only field in the intermediate channel allowed by the fusion rules is the identity ($l' = 0$), and there are $k+1$ fields $[\Phi_l]$ $(l = 0, \frac{1}{2}, ..., \frac{k}{2})$ in the loop. As argued in the introduction, this class of two-point functions is constructed from the characters of the $SU(2)$ WZW model and the torus prime form. The level-k characters of the $SU(2)$ WZW model on the torus were derived in [11] and are given by

$$\chi_l^{(k)}(\tau) \equiv \lim_{z\to 0} \chi_l^{(k)}(z,\tau) = \lim_{z\to 0} \frac{\Theta_{2l+1,k+2}(z,\tau) - \Theta_{-2l-1,k+2}(z,\tau)}{\Theta_{1,2}(z,\tau) - \Theta_{-1,2}(z,\tau)}\,, \qquad 0 \le l \le \tfrac{k}{2}\,, \tag{2.4}$$

where the Kac-Peterson theta functions $\Theta_{m,k}(z,\tau)$ are defined as

$$\Theta_{m,k}(z,\tau) \equiv \sum_{n\in\mathbb{Z}+\frac{m}{2k}} q^{kn^2} x^{kn}\,, \tag{2.5}$$

and $q = e^{2\pi i\tau}$, $x = e^{2\pi i z}$. The characters $\chi_l^{(k)}$ formally correspond to the torus one-point function of the identity $\langle\Phi_0\rangle_l$ with a primary field Φ_l running in the loop.

The prime form on the torus $E(z,\tau)$ is defined by

$$E(z,\tau) = \frac{\theta_1(z,\tau)}{\theta_1'(0,\tau)}\,, \tag{2.6}$$

where $\theta_1(z,\tau)$ is the first Jacobi theta function.

We summarize our claim for this class of two-point functions in the following theorem:

[1]Square brackets [...] denote all fields allowed by the fusion rules, *i.e.* the primary and all its current algebra descendants.

[2]The $SU(2)$ fusions rules are given in appendix B.

Theorem 2.1 *The current blocks of the level-k spin-$\frac{k}{2}$-spin-$\frac{k}{2}$ two-point function on the torus* ($k \in \mathbb{N}$),

$$\langle \Phi_{\frac{k}{2}}(z,\bar{z}) \Phi_{\frac{k}{2}}(0,0) \rangle = \sum_{l=0,\, l \in \mathbb{N}_0/2}^{k/2} |f_l(z,\tau)|^2 \,, \tag{2.7}$$

are

$$f_l(z,\tau) = \frac{\chi_l^{(k)}(z,\tau)}{E(z,\tau)^{2h}} \,, \qquad l = 0, \tfrac{1}{2}, ..., \tfrac{k}{2} \,, \tag{2.8}$$

where $\chi_l^{(k)}(z,\tau)$ *is given by (2.4), and* $h = \frac{k}{4}$ *is the conformal dimension of the primary* $\Phi_{\frac{k}{2}}$.

We remark here that the theorem is formulated for the A-type modular invariant theory, which exists for all $k \geq 1$. In this case the two-point function is simply the sum over the product of holomorphic and anti-holomorphic current blocks. For $k = 4\rho$ ($\rho \in \mathbb{N}$), there is also a D-type modular invariant theory, and the torus two-point function is

$$\langle \Phi_{\frac{k}{2}}(z,\bar{z}) \Phi_{\frac{k}{2}}(0,0) \rangle = \sum_{l=0,\, l \in \mathbb{N}_0/2}^{(k-2)/4} \left| f_l(z,\tau) + f_{\frac{k}{2}-l}(z,\tau) \right|^2 + 2 \left| f_{\frac{k}{4}}(z,\tau) \right|^2 \,, \tag{2.9}$$

where $\Phi_{\frac{k}{2}} \equiv \Phi_{\frac{k}{2},\frac{k}{2}}$. Likewise, we construct the corresponding two-point functions for the other D- and E-type modular invariant theories following Table 2 in [24].

3 Factorization limits and monodromy properties

Before we will give a proof of theorem 2.1 in the next section, we will first examine the factorization limits and monodromy properties of the current blocks (2.8).

3.1 Factorization in the limits $z \to 0$ and $q \to 0$

In the limit $z \to 0$, the prime form $E(z,\tau) \to z$ and the current blocks (2.8) trivially factorize as

$$f_l(z,\tau) \xrightarrow{z \to 0} \frac{1}{z^{2h}} \cdot \chi_l^{(k)}(\tau) \,, \tag{3.1}$$

i.e. into the two-point function on the sphere [3] and the character $\chi_l^{(k)}(\tau)$ [11], as expected by pinching the intermediate channel in figure 1.

Another interesting limit is $q \to 0$ (*i.e.* $\tau \to i\infty$), which corresponds to pinching the torus along the a-cycle. In this limit the appropriate representation of the current blocks f_l is given in figure 2. Here the Φ_j's are the external states, and $[\Phi_l]$ and $[\Phi_{l'}]$ denote the primaries (and their descendants) running between the two insertions. As argued in

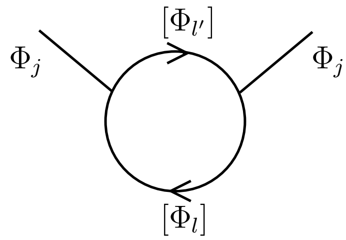

Figure 2: Current block for the torus two-point function in the $z \to z+1$ eigenstate basis.

[25, 23], in this limit a general current block f_l is expected to degenerate into a four-point function on the sphere with two $\Phi_{l'}$'s and two Φ_j's as external states and Φ_l in the intermediate channel,

$$f_l(z,\tau) \stackrel{q\to 0}{\longrightarrow} q^{h_l-\frac{c}{24}} x^h \langle l'|\Phi_j(1)\Phi_j(x)|l'\rangle_l \,. \tag{3.2}$$

Here $x = e^{2\pi i z}$ is the cross-ratio and c the central charge of $SU(2)$ at level k. Writing the four-point function as a product of three-point vertices and expanding around $x = 0$, we get the following behavior:

$$f_l \stackrel{q\to 0}{\longrightarrow} q^{h_l-c/24} x^{h_{l'}-h_l}\left(1 + K_{1,2l+1}x + K_{2,2l+1}x^2 + \ldots\right) , \tag{3.3}$$

where h_l and $h_{l'}$ are the conformal dimensions of Φ_l and $\Phi_{l'}$, respectively. The coefficients of the subleading terms $K_{i,2l+1}$ $(i = 1, 2, \ldots)$ can in principle be determined by elementary methods [26].

Let us now consider the $q \to 0$ limit of the current blocks (2.8). In the double limit $q \to 0$, $x \to 0$, the prime form and functions $\chi_l^{(k)}(z,\tau)$ given by (2.4) behave to leading order in q as

$$E(z,\tau) \sim x^{-1/2}(1 - x + \ldots)\,, \tag{3.4}$$

$$\chi_l^{(k)}(z,\tau) \sim q^{h_l-c/24} x^{-l}(1 + \mathcal{O}(x))\,, \tag{3.5}$$

such that the current blocks (2.8) behave as

$$f_l(z,\tau) \sim q^{h_l-c/24} x^{h-l}\left(1 + \tilde{K}_{1,2l+1}x + \tilde{K}_{2,2l+1}x^2 + \ldots\right) . \tag{3.6}$$

To leading order in x, we get agreement with (3.3), if

$$h - l = h_{l'} - h_l \,. \tag{3.7}$$

A general current block does not satisfy this condition. However, (3.7) is satisfied for the class of two-point functions of primary operators $\Phi_j(z)$ with $j = \frac{k}{2}$. In that case, the conformal dimension is $h = \frac{k}{4}$ and $l + l' = j$ such that both sides of (3.7) become equal to $\frac{k}{4} - l$, and we find that to leading order in x (2.8) has the correct asymptotics in the limit $q \to 0$. In appendix C we also find agreement to next-to-leading order in x by showing $K_1 = \tilde{K}_1$ for a few special cases.

3.2 Monodromy properties

The complete two-point function is a uniform, doubly periodic function on the torus. Its current blocks however have non-trivial monodromy properties, *i.e.* there exist monodromy matrices M_σ and M_τ such that under translations $z \to z+1$ and $z \to z+\tau$, $f_l \to (M_\sigma)_{lm} f_m$ and $f_l \to (M_\tau)_{lm} f_m$, respectively [15]. As shown in [14], there exists a basis of functions f_l in which M_σ is diagonal with eigenvalues $e^{2\pi i(h_l - h_{l'})}$ such that

$$f_l(z+1,\tau) = e^{2\pi i(h_l - h_{l'})} f_l(z,\tau)\,. \tag{3.8}$$

This basis corresponds to choosing the holomorphic blocks as in figure 2. Here h_l and $h_{l'}$ are the conformal weights of the primaries Φ_l and $\Phi_{l'}$ running between the two insertions. By modular invariance, the eigenvalues of M_τ are the same as those of M_σ, but M_σ and M_τ cannot be diagonalized simultaneously [15].

Let us consider the monodromy properties of the current blocks (2.8). The behavior under monodromy transformations can be deduced from that of the Kac-Peterson theta functions given by (A.4) in appendix A. Under the monodromy $z \to z+1$, the prime form and Kac-Peterson characters pick up some phases,

$$\begin{aligned} E(z+1,\tau) &= -E(z,\tau)\,, \\ \chi_l(z+1,\tau) &= (-1)^{2l} \chi_l(z,\tau)\,, \end{aligned} \tag{3.9}$$

such that the current blocks (2.8) transform as

$$f_l(z+1,\tau) = (e^{\pi i})^{2(l-h)} f_l(z,\tau)\,. \tag{3.10}$$

Comparing this with (3.8), we find again the requirement (3.7), which is satisfied for $j = \frac{k}{2}$, as argued above.

Under the monodromy $z \to z+\tau$, the prime form and Kac-Peterson characters transform as

$$\begin{aligned} E(z+\tau,\tau) &= e^{-\pi i} \frac{1}{x q^{\frac{1}{2}}} E(z,\tau)\,, \\ \chi_l(z+\tau,\tau) &= x^{-\frac{k}{2}} q^{-\frac{k}{4}} \chi_{\frac{k}{2}-l}(z,\tau)\,, \end{aligned} \tag{3.11}$$

and therefore

$$f_l(z+\tau,\tau) = (e^{\pi i})^{2h} x^{2h-\frac{k}{2}} q^{h-\frac{k}{4}} f_{\frac{k}{2}-l}(z,\tau)\,. \tag{3.12}$$

For the case $j = \frac{k}{2}$ ($h = \frac{k}{4}$), the transformation becomes independent of z and τ,

$$f_l(z+\tau,\tau) = (e^{\pi i})^{2h} f_{\frac{k}{2}-l}(z,\tau)\,, \tag{3.13}$$

a fact which will be of importance in the proof.

In summary, M_σ and M_τ are $k+1$-dimensional matrices of the type

$$M_\sigma = \begin{pmatrix} (e^{\pi i})^{-2h} & & \\ & \ddots & \\ & & (e^{\pi i})^{\frac{k}{2}-2h} \end{pmatrix}\,, \qquad M_\tau = (e^{\pi i})^{2h} \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}\,, \tag{3.14}$$

with $h = \frac{k}{4}$. For instance, for $k = 1$, $M_\sigma = i\sigma_3 = \mathrm{diag}(e^{i\pi/2}, e^{-i\pi/2})$ and $M_\tau = i\sigma_1$ in agreement with [14]. It can be shown that, for $h = \frac{k}{4}$, M_τ has the same eigenvalues as M_σ, as required. The current blocks (2.8) thus have the proper monodromy properties with respect to the transformations $z \to z + 1$ and $z \to z + \tau$.

4 Differential equations for the two-point conformal blocks on the torus

In this section we will prove theorem 2.1 by showing that the current blocks (2.8) satisfy an appropriate Knizhnik-Zamolodchikov-like differential equation. Such differential equations were first constructed in [14] using a combination of the current Ward identity and a current algebra null-vector relation. It was then noticed [15] (see also [23]) that the same equations can be derived exclusively from modular invariance and the Wronskians of the current blocks.

4.1 Brief review of the method

Let us consider the two-point correlator of a primary field Φ_j at level k. It is constructed from n holomorphic conformal blocks f_l ($l = 1, ..., n$), which are the solutions of a single nth-order differential equation of the form

$$W(z,q)\partial^n f_l + \sum_{k=0}^{n-1} (-1)^{n-k} W_k(z,q)\partial^k f_l = 0\,, \tag{4.1}$$

where $\partial \equiv \partial_z$, $W = W_n$ and W_k is defined by

$$W_k = \det \begin{bmatrix} f_1 & \cdots & f_n \\ \partial_z f_1 & \cdots & \partial_z f_n \\ \vdots & & \vdots \\ \partial_z^{k-1} f_1 & \cdots & \partial_z^{k-1} f_n \\ \partial_z^{k+1} f_1 & \cdots & \partial_z^{k+1} f_n \\ \vdots & & \vdots \\ \partial_z^{n} f_1 & \cdots & \partial_z^{n} f_n \end{bmatrix}, \tag{4.2}$$

see [15] for the derivation of (4.1).

The Wronskians W_k are meromorphic functions of z with a single pole at $z = 0$ and invariant under monodromy transformations [15]. Since the derivatives of the Weierstrass $\wp$-function form a basis for meromorphic functions on the torus with poles at $z = 0$, the W_k's can be expanded as

$$W_k = \sum_{r=0}^{p+n-k-2} A_r^{(k)}(\tau)\partial^{p+n-k-2-r}\wp(z,\tau) + A_{p+n-k}^{(k)}(\tau)\,, \tag{4.3}$$

where p is the order of the pole of $W = W_n$ at $z = 0$.[3] $A_r^{(k)}(\tau)$ are modular forms of weight r. Note that $A_r^{(k)}(\tau) = 0$ for $r = 2$ and odd r, since there is no modular form with weight 2 or odd weight. In the simplest case, when the space M_r of modular forms of weight r is one-dimensional (*e.g.* for $r = 4, 6, 8, 10, 14$), we may express $A_r^{(k)}(\tau)$ in terms of the Eisenstein series, $A_r^{(k)}(\tau) = a_r^{(k)} E_r$ with $a_r^{(k)}$ some $\mathbb{C}$-valued coefficient.

The constants $a_r^{(k)}$ can be found by studying the differential equation (4.1) both at $z = 0$ and $z = i\infty$. In particular, the coefficients $a_0^{(k)}$ are obtained by analyzing (4.1) around $z = 0$. Near $z = 0$, the Weierstrass-$\wp$ function is $\wp(z, \tau) \approx z^{-2}$. Substituting $f(z) \sim z^\mu$ into (4.1), we get the linear equation

$$\sum_{k=0}^{n} (-1)^{n-k} a_0^{(k)} \left(\prod_{j=0}^{k-1} (\mu - j) \right) \left(\prod_{l=0}^{p+n-k-3} (-2-l) \right) = 0 \,, \tag{4.4}$$

where the second product equals one if $p + n - k - 3 < 0$.

Let us now determine the behavior of the current blocks $f_l \sim z^\mu$ near $z = 0$. For this, we define $n_{l'} = k + 1 - 2l'$ as the number of conformal blocks f_l with the field $[\Phi_{l'}]$ in the intermediate channel, see figure 1. Their asymptotic behavior near $z = 0$ is

$$z^{-(2h - h_{l'} - l')}, z^{-(2h - h_{l'} - l' - 2)}, \ldots, z^{-(2h - h_{l'} - 2(n_{l'} - 1))} \,. \tag{4.5}$$

For $l' = 0$, the leading behavior is just z^{-2h}. For $l' > 0$, the one-point function of $\Phi_{l'}$ vanishes on the torus. In fact, all descendants of $\Phi_{l'}$ up to level $l' - 1$ have vanishing one-point functions [23]. The first non-vanishing one-point function corresponds to the state $J_{-1}^{a_1} \cdots J_{-1}^{a_{l'}} |\Phi_{l'}\rangle$, which has conformal dimension $h_{l'} + l'$. For W_k to be non-vanishing, we also keep the non-leading terms of the current blocks, corresponding to the descendants $J_{-1}^{a_1} \cdots J_{-1}^{a_{l'+2}} |\Phi_{l'}\rangle$, $J_{-1}^{a_1} \cdots J_{-1}^{a_{l'+4}} |\Phi_{l'}\rangle$ etc. Since $n = \sum_{l'} n_{l'}$, we find n values for μ and therefore n independent linear equations of the type (4.4), enough to determine the n coefficients $a_0^{(0)}, \ldots, a_0^{(n-1)}$ (Since the differential equation (4.1) can be multiplied by a constant, we may set the coefficient $a_0^{(n)} = 1$). The remaining coefficients $a_r^{(k)}$ $(r \neq 0)$ are obtained in a similar way by expanding around $z = i\infty$.

In appendix C, we list the differential equations (4.1) for the current blocks of the torus two-point function $\langle \Phi_{\frac{k}{2}}(z) \Phi_{\frac{k}{2}}(0) \rangle$ at level k for $k = 1, 2, \ldots, 5$. As a first test of our solution, we substitute (2.8) (for all $k \leq 5$ and all $l = 0, \frac{1}{2}, \ldots, \frac{k}{2}$) into (4.1) and expand the left-hand side around $q = 0, z = 0$. We find that (2.8) satisfies (4.1) up to the order we expanded in z and q.[4] We now give a more rigorous proof of theorem 2.1.

4.2 Proof of theorem 2.1

For the proof we will need a variation of Liouville's theorem for elliptic functions:

[3]The behavior $W \sim z^{-p}$ near $z = 0$ can be determined from the asymptotic behavior of the current blocks near $z = 0$, see (4.5) below.

[4]Using computer algebra (mathematica), this can be done to arbitrary orders in z and q and is only limited by computer power.

Theorem 4.1 *An entire elliptic (i.e. doubly periodic) function is constant.*

For a special class of quasi-periodic functions, this theorem can be extended to the

Proposition 4.2 Let $g(z)$ be an entire quasi-periodic function $g(z)$,

$$g(z+1) = e^{i\phi_1} g(z)\,, \qquad g(z+\tau) = e^{i\phi_2} g(z)\,, \tag{4.6}$$

whose multipliers $e^{i\phi_i}$ $(i = 1, 2)$ are just phases ($\phi_i = const.$). If $g(0) = 0$, then $g(z) \equiv 0$ for all z.[5]

Proof of proposition 4.2: In this case the absolute values of $g(z)$ are invariant under shifts $z \to z+1$ and $z \to z+\tau$, *i.e.* $|g(z)| = |g(z+1)| = |g(z+\tau)|$ as for elliptic functions. Therefore, if $g(z)$ is holomorphic and its absolute value is bounded inside the fundamental parallelogram, by the quasi-periodicity of $g(z)$, it is bounded for all z and thus constant. The requirement $g(0) = 0$ sets the constant to zero. □

To prove theorem 2.1, we substitute the $n = k+1$ current blocks of the spin-$\frac{k}{2}$-spin-$\frac{k}{2}$ two-point function,

$$f_l(z,\tau) = \frac{\chi_l^{(k)}(z,\tau)}{E(z,\tau)^{2h}}\,, \qquad l = 0, \tfrac{1}{2}, ..., \tfrac{k}{2}\,, \qquad h = \tfrac{k}{4}\,, \tag{4.7}$$

with $\chi_l^{(k)}(z,\tau)$ as in (2.4), into the differential equation (4.1) for a fixed but arbitrary level k. We then define the function $g_l = g_l(z,\tau)$ as the left-hand side of (4.1),

$$g_l \equiv \partial^n f_l + \sum_{r=0}^{n-1} (-1)^{n-r} \frac{W_r}{W} \partial^r f_l\,, \tag{4.8}$$

where the Wronskians are those of the spin-$\frac{k}{2}$-spin-$\frac{k}{2}$ two-point function at level k. The Wronskians are in principle known for all levels k and can be constructed as described in the previous subsection. For the proof we do not need to do this explicitly. It will turn out to be enough to know a few properties of the Wronskians. We need to show that $g_l(z,\tau) \equiv 0$ for all z and τ.

It follows from proposition 4.2 that $g_l(z,\tau) \equiv 0$ for all z and τ if $g_l(z,\tau)$

(i) is a quasi-periodic function of the type (4.6) with periods 1 and 2τ,

(ii) has no poles in the z-plane inside the parallelogram with sides 1 and 2τ,

(iii) vanishes at $z = 0$.

[5] We thank Yuri Aisaka for discussions which led to the formulation of this proposition.

(i) Quasi-periodicity of g_l:

The function g_l inherits the monodromies of f_l,

$$g_l(z+1,\tau) = e^{2\pi i(h_l - h_{l'})} g_l(z,\tau)\,, \qquad g_l(z+\tau,\tau) = (e^{\pi i})^{2h} g_{\frac{k}{2}-l}(z,\tau)\,, \tag{4.9}$$

as follows from the following observations:

- The derivatives of the holomorphic blocks f_l have the same monodromy properties as f_l. This follows from the fact that the monodromies of f_l are independent of z, see (3.10) and (3.13) for $h = \frac{k}{4}$. The derivatives $\partial^r f_l$ are therefore quasi-periodic with the same monodromies as f_l.

- The Wronskians W and W_r are monodromy invariant [15].

Therefore, $g_l(z+2\tau,\tau) = (e^{\pi i})^{4h} g_l(z,\tau)$, *i.e.* the periods of g_l are 1 and 2τ.

(ii) Poles of g_l:

The function g_l can have poles at most at $z = m + n\tau$ $(m,n \in \mathbb{Z})$. This follows from the fact that the individual terms in g_l, which are of the type $\frac{W_r}{W}\partial^r f_l$, have poles at $z = m + n\tau$:

- The only poles of $f_l(z,\tau)$ are at $z = m + n\tau$. Note that the theta functions are holomorphic in z and the denominator of $\chi_l^{(k)}(z,\tau)$,

$$\Theta_{1,2}(z,\tau) - \Theta_{-1,2}(z,\tau) = i\theta_1(z,\tau)\,, \tag{4.10}$$

  is zero at $z = m + n\tau$. A similar argument holds for the inverse of the torus prime form $E(z,\tau)$.

- The derivatives $\partial^r f_l$ have poles at the same locations as the functions f_l.

- In general, W_r/W is a meromorphic single-valued function of z with poles at $z = 0$ and at the locations of the zeros of W [15]. However, in the differential equations for spin-$\frac{k}{2}$-spin-$\frac{k}{2}$ two-point functions, W is a (non-vanishing) constant [15]. This leaves only $z = 0$ as a possible location for a pole of W_r/W.

Even though the individual terms have poles at $z = m + n\tau$, the total sum of the pole contributions vanishes in g_l, as follows from property (iii).

(iii) $g_l = 0$ at $z = 0$:

- Near $z = 0$, the current blocks f_l behave like $z^{-k/2}$ $(= z^{-2h})$. The solutions f_l therefore trivially satisfy (4.1) near $z = 0$, since the coefficients $a_0^{(k)}$ are constructed such that $\mu = -2h$ is a solution of (4.4). In other words, (4.1) is constructed such that $f \sim z^{-2h}$ is a solution of it near $z = 0$. Thus, $g_l = 0$ at $z = 0$ (This excludes poles of g_l at $z = m + n\tau$ $(m,n \in \mathbb{Z})$ such that g_l does not have any poles).

In conclusion, by (i), (ii) and (iii), g_l is a quasi-periodic function of the type (4.6), free of poles and $g_l(0) = 0$. By proposition 4.2, $g_l \equiv 0$ for all z.

A final remark on the overall normalization of the current blocks as a function of τ is in order. The differential equation (4.1) determines the current blocks only up to an overall τ-dependent factor and cannot be used to check the normalization of f_l. However, the current blocks f_l have the correct τ-dependence by construction. This follows from the fact that in the limit $z \to 0$ the total torus spin-$\frac{k}{2}$-spin-$\frac{k}{2}$ two-point function factorizes into the corresponding two-point function on the sphere and the partition function $Z = \sum_l |\chi_l|^2$. The transformation properties of the torus two-point function under the modular transformations $S : \tau \to -\frac{1}{\tau}$ and $T : \tau \to \tau + 1$ therefore follow directly from those of the characters. $\square$

5 Conclusions

The main result of this paper is the expression (2.8) for the current blocks of the $SU(2)$ spin-$\frac{k}{2}$-spin-$\frac{k}{2}$ torus two-point function at arbitrary level k. The theorem was proven using the differential equation technique developed in [14, 15].

One may wonder whether the expression (2.8) also describes the current blocks of other torus $SU(2)$ two-point functions which have the identity $[\Phi_0]$ in the intermediate channel. Consider for instance the current block $f_{l=1}$ of the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ two-point function at level $k = 2$. This block has $[\Phi_0]$ in the intermediate channel and $[\Phi_1]$ in the loop. For this case, the expression (2.8) with $\chi_l(z, \tau)$ given by (2.4) has the wrong factorization property in the limit $q \to 0$, even though its factorization in the limit $z \to 0$ is correct. Note that the important condition (3.7) is not satisfied since $h - l = \frac{3}{16} - 1 = -\frac{13}{16} \notin \{\pm\frac{3}{16}, \pm\frac{5}{16}\}$, which is the set of values $h_{l'} - h_l$ for the four conformal blocks of this correlator. It would therefore be interesting to study how the character $\chi_l(\tau)$ is properly extended to a function $\widetilde{\chi}_l(z, \tau) \neq \chi_l(z, \tau)$ for such current blocks. Eventually it would be interesting to find an expression for the current blocks of all $SU(2)$ torus two-point functions and higher n-point functions.

Acknowledgments

We thank Yuri Aisaka, Matthias Gaberdiel, Terry Gannon, Volker Schomerus and Cosimo Restuccia for helpful comments and discussions related to this work. I.K. is grateful to Samir Mathur for email correspondence. P.K. thanks the Deutsches Elektronen-Synchrotron (DESY) for its hospitality during the summer student programme 2012 and Frank Tackmann for initial support.

Appendix

A Theta functions

A.1 Theta functions with characteristics

The (first-order) theta function with characteristics is defined as [27]

$$\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i(\frac{1}{2}(n+\delta)^2\tau+(z+\varepsilon)(n+\delta))}\,. \tag{A.1}$$

Under translations $z \to z + a + b\tau$, it transforms as

$$\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(z+a+b\tau,\tau) = e^{2\pi i(-\frac{1}{2}b^2\tau - bz + a\delta - b\varepsilon)}\theta\begin{bmatrix}\delta+b\\ \varepsilon\end{bmatrix}(z,\tau)\,. \tag{A.2}$$

A.2 Kac-Peterson theta functions for $SU(2)$

The Kac-Peterson theta functions are

$$\Theta_{m,k}(z,\tau) \equiv \theta\begin{bmatrix}\frac{m}{2k}\\ 0\end{bmatrix}(kz,2k\tau) = \sum_{n\in\mathbb{Z}+\frac{m}{2k}} q^{kn^2}x^{kn}\,, \tag{A.3}$$

where $q = e^{2\pi i\tau}$, $x = e^{2\pi i z}$. Using (A.2), we find the monodromy transformations

$$\begin{aligned}\Theta_{m,k}(z+1,\tau) &= e^{\pi i m}\Theta_{m,k}(z,\tau)\,,\\ \Theta_{m,k}(z+\tau,\tau) &= e^{\pi i(-k\tau/2-kz)}\Theta_{m+k,k}(z,\tau)\,.\end{aligned} \tag{A.4}$$

Under modular transformations, they transform as

$$T: \quad \Theta_{m,k}(z,\tau+1) = e^{2\pi i\frac{m^2}{4k}}\Theta_{m,k}(z,\tau)\,, \tag{A.5}$$

$$S: \quad \Theta_{m,k}(z/\tau,-1/\tau) = e^{2\pi i(z^2/4\tau)}(-i\tau)^{1/2}\sum_{m'=-k+1}^{k} B^{(k)}_{mm'}\Theta_{m',k}(z,\tau)\,, \tag{A.6}$$

$$B^{(k)}_{mm'} = \frac{1}{\sqrt{2k}}e^{\pi\, imm'/k}\,.$$

Moreover,

$$\Theta_{m,k}(-z,\tau) = \Theta_{-m,k}(z,\tau)\,, \qquad \Theta_{m+2k,k}(z,\tau) = \Theta_{m,k}(z,\tau)\,. \tag{A.7}$$

The prime form $E(\tau,z)$ transforms as

$$E(z,\tau+1) = E(z,\tau)\,, \tag{A.8}$$

$$E(z/\tau,-1/\tau) = \frac{e^{\pi i(z^2/2\tau)}}{\tau}E(z,\tau)\,. \tag{A.9}$$

B Fusion rules

The fusion rules for the $SU(2)$ WZW model at level k can be expressed in terms of the fusion matrices $N_{j_1 j_2}^{(j_3)} = N_{j_1 j_2}^{j_3}$ as

$$[\phi_{j_1}] \times [\phi_{j_2}] = \sum_{j_3} N_{j_1 j_2}^{j_3} [\phi_{j_3}] \,, \tag{B.1}$$

where $N_{j_1 j_2}^{j_3} = 1$ if $|j_1 - j_2| \leq j_3 \leq \min(j_1 + j_2, k - j_1 - j_2)$, otherwise $N_{j_1 j_2}^{j_3} = 0$ [3].

C Differential equations

In this appendix we list the differential equations for the current blocks of the torus two-point function $\langle \phi_{k/2}(z) \phi_{k/2}(0) \rangle$ at level k for $k = 1, 2, ..., 5$. We also give the coefficients $K_{1,2l+1}$ in the asymptotic expansion of the current blocks,

$$f_l(z,\tau) \sim q^{h_l - c/24} x^{h_{l'} - h_l} \left(1 + K_{1,2l+1} x + ...\right) \,. \tag{C.1}$$

The coefficients $K_{1,1} = \frac{k}{2}$ and $K_{1,2} = \frac{k}{2} + 1$ follow from our procedure and agree with the corresponding coefficients $\tilde{K}_{1,1}$ and $\tilde{K}_{1,2}$ of the small q and x expansion of the solution (2.8), as given by (3.6). The remaining coefficients can be computed by elementary methods [26].

The differential equations for the spin-$\frac{k}{2}$-spin-$\frac{k}{2}$ current blocks are:

$k = 1, j = 1/2$:

$$f''(z,\tau) - \frac{3}{4} \wp(z,\tau) f(z,\tau) = 0 \,, \tag{C.2}$$

$$K_{1,1} = \tfrac{1}{2} \,, \quad K_{1,2} = \tfrac{3}{2} \,, \tag{C.3}$$

$k = 2, j = 1$:

$$f^{(3)}(z,\tau) - 3\wp(z,\tau) f'(z,\tau) - \frac{3}{2} \wp'(z,\tau) f(z,\tau) = 0 \,, \tag{C.4}$$

$$K_{1,1} = 1 \,, \quad K_{1,2} = 2 \,, \quad K_{1,3} = 2 \,, \tag{C.5}$$

$k = 3, j = 3/2$:

$$\begin{aligned} f^{(4)}(z,\tau) - \frac{15}{2} \wp(z,\tau) f''(z,\tau) - \frac{15}{2} \wp'(z,\tau) f'(z,\tau) \\ + \left(\frac{9\pi^4}{16} \mathrm{E}_4(\tau) - \frac{45}{32} \wp''(z,\tau) \right) f(z,\tau) = 0 \,, \end{aligned} \tag{C.6}$$

$$K_{1,1} = \tfrac{3}{2} \,, \quad K_{1,2} = \tfrac{5}{2} \,, \quad K_{1,3} = \tfrac{20}{8} \,, \quad K_{1,4} = \tfrac{20}{8} \,, \tag{C.7}$$

$k = 4, j = 2$:

$$f^{(5)}(z,\tau) - 15\wp(z,\tau)f^{(3)}(z,\tau) - \frac{45}{2}\wp'(z,\tau)f''(z,\tau) + \left(4\pi^4\mathrm{E}_4(\tau) - \frac{15}{2}\wp''(z,\tau)\right)f'(z,\tau) = 0\,, \tag{C.8}$$

$$K_{1,1} = 2\,,\quad K_{1,2} = 3\,,\quad K_{1,3} = 3\,,\quad K_{1,4} = 3\,.\quad K_{1,5} = 3\,, \tag{C.9}$$

$k = 5, j = 5/2$:

$$f^{(6)}(z,\tau) - \frac{105}{4}\wp(z,\tau)f^{(4)}(z,\tau) - \frac{105}{2}\wp'(z,\tau)f^{(3)}(z,\tau) + \left(\frac{259\pi^4}{16}\mathrm{E}_4(\tau) - \frac{735}{32}\wp''(z,\tau)\right)f''(z,\tau) + \frac{105}{32}\wp^{(3)}(z,\tau)f'(z,\tau) + \left(-\frac{735\pi^4}{64}\mathrm{E}_4(\tau)\wp(z,\tau) - \frac{5\pi^6}{16}\mathrm{E}_6(\tau) + \frac{1155}{512}\wp^{(4)}(z,\tau)\right)f(z,\tau) = 0\,, \tag{C.10}$$

$$K_{1,1} = \tfrac{5}{2}\,,\quad K_{1,2} = \tfrac{7}{2}\,,\quad K_{1,3} = \tfrac{7}{2}\,,\quad K_{1,4} = \tfrac{7}{2}\,,\quad K_{1,5} = \tfrac{7}{2}\,,\quad K_{1,6} = \tfrac{7}{2}\,, \tag{C.11}$$

where $f^{(n)}$ denotes the nth derivative in z (also $' = \partial_z$). $E_r(\tau)$ is the Eisenstein series of weight r and $\wp(z,\tau)$ is the Weierstrass $\wp$-function. The differential equations for $k = 1, 2, 3, 4$ are agree with those in [14, 15].

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