

Exact AdS/CFT spectrum: Konishi dimension at any coupling

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We compute the full dimension of Konishi operator in planar N=4 Super Yang-Mills (SYM) theory for a wide range of couplings, from weak to strong coupling regime, and predict the subleading terms in its strong coupling asymptotics. For this purpose we solve numerically the integral form of the AdS/CFT Y-system equations for the exact energies of excited states proposed by us and A. Kozak.

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INTRODUCTION

Four-dimensional Yang-Mills theories are at the heart of modern high energy physics, describing all fundamental interactions except gravity. Nevertheless, in spite of considerable efforts during almost 40 years, we still don't have a satisfactory quantitative description of the most interesting YM theories, such as QCD, in the region of intermediate and strong couplings. The low energy quantum dynamics of YM field is mostly known only from computer simulations of lattice YM theories. A few important exact results concerned the topological, BPS sectors of N=1,2 SYM were obtained.

Recently, when the hopes on complete exact 4D solutions in particular, for the quantities given by nontrivial 4D Feynman series seemed to start waning, N=4 supersymmetric Yang-Mills theory gave us serious hopes for a better understanding of the dynamics of strongly interacting 4D gauge theories. Due to the AdS/CFT correspondence [1], as well as to the quantum integrability discovered on both sides of this duality in the planar limit (when the number of colors $N \rightarrow \infty$ with the 't Hooft coupling $\lambda = g_{YM}^2 N$ fixed) [2–11], we acquire little by little tools for the study of the most important quantities in N=4 SYM, such as the spectrum of dimensions $\Delta(\lambda)$ of local operators as functions of λ - the scale independent coupling constant in this superconformal 4D theory. The weak coupling behaviour ($\lambda \rightarrow 0$) of $\Delta(\lambda)$ is given by Feynman perturbation theory. The dual string worldsheet σ -model with the coupling $g = \sqrt{\lambda}/4\pi$ allows to find the strong coupling asymptotics of various dimensions. Integrability allows us to connect the two regimes. In particular, the asymptotic Bethe ansatz (ABA) of [13] gives us the asymptotic spectrum of single trace operators containing an asymptotically large number of elementary fields.

However, for short operators, such as Konishi operator $\text{Tr}[D, Z]^2$ [12] [39], the calculation of anomalous dimensions is still an interesting and important challenge.

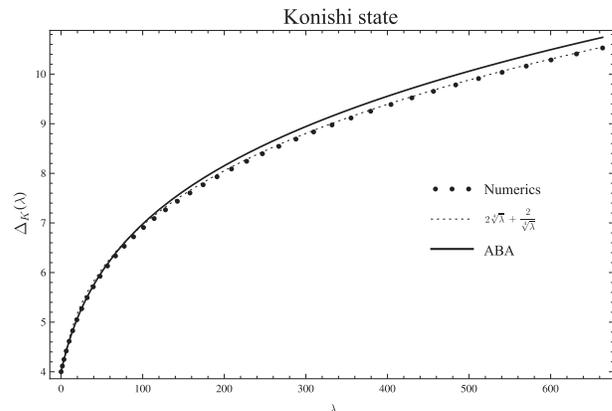


Figure 1: Numerical solution of exact finite size integral Y-system equations for the Konishi dimension $\Delta_K(\lambda)$ in a wide range of 't Hooft couplings λ , compared to the asymptotic Bethe ansatz curve and to the predicted large λ asymptotics $\Delta_K(\lambda) \simeq 2\lambda^{1/4} + 2/\lambda^{1/4}$ obtained by fit.

Recently we proposed the Y-system for the planar AdS/CFT [14], a set of functional equations defining the anomalous dimensions of *all* operators of planar N=4 SYM theory *at any coupling*. The integral form of the Y-system for excited states in $SL(2)$ sector, including the one corresponding to Konishi operator, was presented in [15]. The integral equations for the BPS vacuum energy were independently obtained in [16, 17] by the thermodynamical Bethe ansatz (TBA) technique based on the dynamics of bound states [18–24] (see also [9, 25]) of the mirror theory [26, 27]. The solutions of the integral Y-system are also solutions of the functional Y-system [15–17, 28]. The combination of functional and integral versions of the Y-system appears to be quite efficient to compute numerically the exact spectrum. In this work, we use the functional form of the Y-system to derive the large volume (L) [40] asymptotic solution and then, departing from it, we solve the integral form of the Y-system iteratively. As a demonstration of the power of

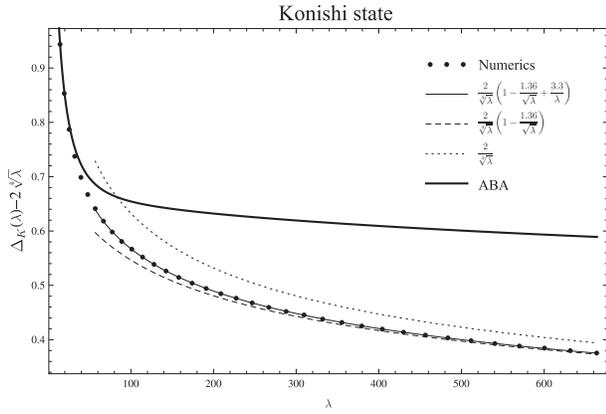


Figure 2: Plot of $\Delta_K(\lambda) - 2\lambda^{1/4}$ from the numerical data compared with the Bethe ansatz prediction and some fits. The fits in this plot are done assuming the asymptotics $\Delta_K(\lambda) = 2\lambda^{1/4} + 2/\lambda^{1/4} + \dots$

our method, we calculate numerically the dimension of Konishi operator in a wide range of the 't Hooft coupling covering both the weak and strong coupling regimes. The results appear to be quite satisfactory: we manage to compute the dimension of Konishi operator in the interval of couplings $0 \lesssim \lambda \lesssim 700$ and to confirm, within the precision of our numerics, all the existing data concerning this quantity: the perturbative results [41] up to 4 loops (up to λ^4 terms) [29–32] and the large λ asymptotics $2\lambda^{1/4}$ matches the prediction of [33] for the lowest level of the string spectrum. Fitting our numerical data with $\lambda > 60$ we find (with uncertainty in last digit)

$$\Delta_K = 2\lambda^{1/4} \left(1.0002 + \frac{0.994}{\lambda^{1/2}} - \frac{1.30}{\lambda} + \frac{3.1}{\lambda^{3/2}} + \dots \right) \quad (1)$$

The leading term reproduces indeed the expected large λ behavior within our numerical precision. It was also argued in [34] that the subleading coefficient ought to be integer (the next corrections could be transcendental [35]). Indeed, our numerics seems to indicate that $\Delta_K = 2\lambda^{1/4} + 2/\lambda^{1/4} + \dots$ thus predicting the value of this integer! [42] We also obtained predictions for two further subleading corrections (with a lower precision of course).

Our results are represented in Figs.1 and 2.

Y-SYSTEM FUNCTIONAL AND INTEGRAL EQUATIONS FOR ADS/CFT

The Y-system defining the spectrum of all local operators in planar AdS/CFT correspondence reads [14]

$$\frac{Y_{a,s}^+ Y_{a,s}^-}{Y_{a+1,s} Y_{a-1,s}} = \frac{(1 + Y_{a,s+1})(1 + Y_{a,s-1})}{(1 + Y_{a+1,s})(1 + Y_{a-1,s})} \quad (2)$$

where the functions $Y_{a,s}(u)$ correspond only to the nodes marked by $\circ, \oplus, \otimes, \Delta, \bullet$ on Fig.3 (we will use these nota-

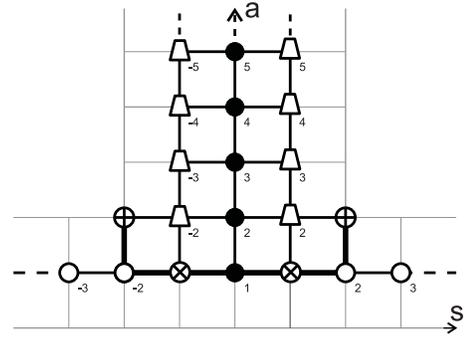


Figure 3: T-shaped domain (T-hook) [36]. It defines the interactions between Y's in the AdS/CFT Y-system equations.

tions for Y-functions in what follows). The one particle energy at infinite length $\epsilon^{(a)}(u) = a + \frac{2ig}{x^{[-a]}} - \frac{2ig}{x^{[+a]}}$ is defined through the Zhukovski map $x(u) + \frac{1}{x(u)} = \frac{u}{g}$ and $f^{[\pm a]} \equiv f(u \pm ia/2)$, $f^\pm \equiv f(u \pm i/2)$ for any function $f(u)$. A solution of Y-system with a given set of quantum numbers defines the energy of a state (or dimension of an operator in N=4 SYM) through the formula (3) where the Bethe roots u_j are fixed by the exact Bethe ansatz equations $Y_{\bullet_1}(u_j) = -1$. In this paper we restrict ourselves to the integral form of the Y-system for the $SL(2)$ -excited states obtained in [15]. Furthermore we focus ourself for simplicity on the Konishi operator where we have only two roots $u_1 = -u_2$ which we can encode into the ‘‘Baxter functions’’ $R^{(\pm)}(u) = (x(u) - x_1^\mp)(x(u) - x_2^\mp)$ and their complex conjugates $B(u) = \bar{R}(u)$ where $x_{1,2}^\mp = x(u_{1,2} \mp i/2)$ with the *physical* choice of branches, such that $|x(u)| > 1$, while for the free variable $x(u)$ we should use the *mirror* kinematics which corresponds to the branches where $\text{Im}(x(u)) > 0$ [27]. Unless it is explicitly said otherwise, we will be always using this latter choice in what follows. With the mirror choice, $x(u)$ has a semi-infinite cut for $u \in (-\infty, -2g) \cup (2g, +\infty)$. The energy of the Konishi state is computed from

$$\Delta_K = 2 + 2\epsilon^{(1)}(u_1) + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \partial_u \epsilon^{(a)} \log(1 + Y_{\bullet_a}) \quad (3)$$

where the integral equations defining Y_{\bullet_a} read [15]

$$\begin{aligned} \log Y_{\otimes} &= K_{m-1} * \log(1 + 1/Y_{\circ_m}) / (1 + Y_{\Delta_m}) \\ &\quad + \mathcal{R}^{(0m)} * \log(1 + Y_{\bullet_m}) + \log \frac{-R^{(+)}}{R^{(-)}} \\ \log Y_{\Delta_n} &= \mathcal{M}_{nm} * \log(1 + Y_{\bullet_m}) - K_{n-1} \otimes \log(1 + Y_{\otimes}) \\ &\quad - K_{n-1, m-1} * \log(1 + Y_{\Delta_m}) + \log \frac{R_n^{(+)} B_{n-2}^{(+)}}{R_n^{(-)} B_{n-2}^{(-)}} \\ \log Y_{\circ_n} &= K_{n-1, m-1} * \log(1 + 1/Y_{\circ_m}) + K_{n-1} \otimes \log(1 + Y_{\otimes}) \\ \log Y_{\bullet_n} &= \mathcal{T}_{nm} * \log(1 + Y_{\bullet_m}) + 2\mathcal{R}^{(n0)} \otimes \log(1 + Y_{\otimes}) \\ &\quad + 2\mathcal{N}_{nm} * \log(1 + Y_{\Delta_m}) + i\Phi_n \quad (4) \end{aligned}$$

We use here the kernels and sources defined in [15] and presented in the appendix for completeness. The integrals in convolutions $K * f = \int dv K(u, v) f(v)$ go along the real axis, but slightly below the poles in the terms involving Y_{Δ_2} (due to the last term in the corresponding integral equation, see [15]). The convolutions \otimes should be understood in the sense of a B-cycle (see [15]), e.g. $\mathcal{R}^{(n_0)} \otimes \log(1 + Y_{\otimes})$ stands for

$$\int_{-2g}^{2g} dv \left[\mathcal{R}^{(n_0)} \log(1 + Y_{\otimes}) - \mathcal{B}^{(n_0)} \log(1 + 1/Y_{\oplus}) \right]$$

where $\frac{1}{Y_{\oplus}}$ is the analytical continuation of Y_{\otimes} across the cut $u \in (-\infty, -2g) \cup (2g, +\infty)$. Summation over the repeated index m is assumed with $m \geq 2$ for $\Delta_{\pm m}$ and $\circ_{\pm m}$, and $m \geq 1$ for \bullet_m .

A remarkable feature of all these equations, crucial for the success of our numerics and noticed in [15], is the reality of all Y -functions in the integral equations.

EXACT BETHE EQUATIONS

The Y -system integral equations for the functions $Y_{a,s}$ need to be supplemented by the *exact* Bethe equations $Y_{\bullet_1}^{\text{ph}}(u_j) = -1$ which reproduce the asymptotic Bethe equations of Beisert and Staudacher in the large L limit [14]. To use this equation, we need to analytically continue the last of eq.(4) in the free variable u from mirror to physical plane and then evaluate it at $u = u_1$. After some manipulations with contours we find

$$\begin{aligned} \log Y_{\bullet_1}^{\text{ph}}(u_1) &= \tilde{\mathcal{T}}_{1m} * \log(1 + Y_{\bullet_m}) + \log Z_{\Delta_2}(u_1) \\ &+ i\Phi_{\text{ph}}(u_1) + 2(\mathcal{R}_{\text{ph,mir}}^{(10)} \otimes K_{m-1} + K_{m-1}^-) *_{p.v.} \log(Z_{\Delta_m}) \\ &+ 2\mathcal{R}_{\text{ph,mir}}^{(10)} \otimes \log(1 + Y_{\otimes}) - 2 \log \frac{u_1 - i/2}{i} - 2 \sum_{j=1}^2 \log \frac{\frac{1}{x_1^+} - x_j^+}{\frac{1}{x_1} - x_j^+} \end{aligned} \quad (5)$$

where $*_{p.v.}$ stands for principal value integration, $\tilde{\mathcal{T}}_{1m}$ is the dressing phase in the physical kinematics while $i\Phi_{\text{ph}}$ is the same as $i\Phi$ but evaluated in the physical region (see appendix). We used $1/Y_{\oplus}(u_j \pm i/2) = 0$ (following from (4)) and denoted $Z_{\Delta_m} = (1 + Y_{\Delta_m})(1 - 1/m^2)(u^2 - u_1^2)^{\delta_{m,2}}$ to isolate the poles in Y_{Δ_2} at $u = u_j$ and to ensure decreasing asymptotics at large u which is of course useful for the numerics. Finally, in contrast to $Y_{\bullet_1}^{\text{ph}}(u_1)$, the term $Z_{\Delta_2}(u_1)$ is evaluated in mirror kinematics.

NUMERICS AND ITS INTERPRETATION

We solve the integral equations (4) iteratively for the Konishi state. Namely, we find the Y -functions at a step n by plugging the Y -functions of step $n-1$ into the r.h.s. of these equations. As the first step of the iterations we

use the large L , ABA solutions of the Y -system [14]. At each step of iterations we also update the position of the Bethe roots by solving the exact Bethe equation of the previous section. It is important to note that the r.h.s. of (5) remains, within our precision, purely imaginary in the process of iterations.

We should also truncate the infinite set of Y -functions to some finite number. We explicitly iterate the first 25 Y_{Δ_n} 's and 25 Y_{\circ_n} 's at each step. Then we also extrapolate them to obtain extra 40 Y_{Δ_n} 's and 40 Y_{\circ_n} 's with higher numbers to replace the infinite sums in (4) on the next step of iteration. For Y_{\bullet_n} we can truncate the sums much earlier (typically $Y_{\bullet_2}/Y_{\bullet_1} \ll 0.1$): the first 5 of them are largely enough for our precision. Finally, the integrals along the real axis are computed along the stretch $(-X, X)$ with X being a large cut-off. With these approximations, our absolute precision for the energy is around ± 0.001 .

We solved the integral equations for a wide range of couplings $0 \lesssim \lambda \lesssim 700 \gg 1$ stretching from the perturbative region up to this value, which is already a deep strong coupling regime [43]. We found no sign of any singularity and the curve looks perfectly smooth. By this reason we believe that any new singularities, such as those related to the Lüscher μ -terms, will not appear. This seems to be the case perturbatively [37] and our numerics seems to indicate that the integral form of Y -system we are solving describes exactly the full spectrum of planar $N=4$ SYM theory in $SL(2)$ sector. Although we cannot discard a possibility that some new singularities could collide with the integration contours for even larger values of the 't Hooft coupling (such extra singularities could be easily incorporated into our code), our numerical results suggest that this possibility is very unlikely.

With a precision of 0.001 we can approximate the Konishi dimension in the range we considered by $\sqrt[4]{g^2 + 1} \frac{252.93h^4 + 384.74h^3 + 674.13h^2 + 128.17h + 4}{35.67h^4 + 51.43h^3 + 99.71h^2 + 29.29h + 1}$ where $h = g^2 / \sqrt{g^2 + 1}$ and $\lambda = 16\pi^2 g^2$ (this function is just a shorthand for the data in Table I).

CONCLUSIONS

We presented here a numerical method for solving the Y -system for the Konishi dimension in planar $N=4$ SYM. It opens the way to the systematic study of the spectrum of many interesting states at any values of the coupling. We also hope to simplify the Y -system in the future using the underlying Hirota integrable dynamics and reducing it to a finite system of integral equations.

APPENDIX

We use: $\mathcal{P}^{(n)}(v) \equiv -\frac{1}{2\pi} \frac{d}{dv} \log \frac{x_v^{[+n]}}{x_v^{[-n]}}$, $K_n \equiv \frac{2n/\pi}{n^2+4u^2}$ and

$$\mathcal{R}^{(nm)}(u, v) \equiv \frac{\partial_v}{2\pi i} \log \frac{x_u^{[+n]} - x_v^{[-m]}}{x_u^{[-n]} - x_v^{[+m]}} - \frac{1}{2i} \mathcal{P}^{(m)}(v),$$

$$\mathcal{B}^{(nm)}(u, v) \equiv \frac{\partial_v}{2\pi i} \log \frac{1/x_u^{[+n]} - x_v^{[-m]}}{1/x_u^{[-n]} - x_v^{[+m]}} - \frac{1}{2i} \mathcal{P}^{(m)}(v),$$

$$\mathcal{M}_{nm} \equiv K_{n-1} \otimes \mathcal{R}^{(0m)} + K_{n-1, m-1}^\neq,$$

$$\mathcal{N}_{nm} \equiv \mathcal{R}^{(n0)} \otimes K_{m-1} + K_{n-1, m-1}^\neq,$$

$$K_{nm} \equiv \mathcal{F}_n^u \circ \mathcal{F}_m^v \circ K_2(u-v), K_{nm}^\neq \equiv \mathcal{F}_n^u \circ \mathcal{F}_m^v \circ K_1(u-v),$$

where the fusion operation $\mathcal{F}_n^u \circ A \equiv \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} A(u+ik)$.

Finally, we also use a nice integral representation [15] of the kernel $\mathcal{T}_{n,m}$ given by

$$\begin{aligned} \mathcal{T}_{n,m}(u, v) &= -K_{n,m}(u-v) - \frac{in}{2} \mathcal{P}^{(m)}(v), \\ &-2 \sum_{a=1}^{\infty} \int \left[\mathcal{B}_{n1}^{(10)}(u, w + i\frac{a}{2}) \mathcal{B}_{1m}^{(01)}(w - i\frac{a}{2}, v) + c.c. \right] dw, \end{aligned} \quad (6)$$

where $\mathcal{B}_{nm}^{(10)} = \mathcal{F}_n^u \circ \mathcal{F}_m^v \circ \mathcal{B}^{(10)}$. For the exact Bethe equations we should use this kernel in the mixed representation,

$$\begin{aligned} \tilde{\mathcal{T}}_{1m} &= -\sum_{a=1}^{\infty} 2\mathcal{B}_{\text{ph,mir}}^{(10)}(u, w + i\frac{a}{2}) * \mathcal{B}_{\text{mir,mir}}^{(0m)}(w - i\frac{a}{2}, v) \\ &- \sum_{a=1}^{\infty} 2\mathcal{B}_{\text{ph,mir}}^{(10)}(u, w - i\frac{a}{2} - i0) * \mathcal{B}_{\text{mir,mir}}^{(0m)}(w + i\frac{a}{2}, v) - K_{1m}. \end{aligned}$$

Finally the source term $\Phi_n(u) = \mathcal{F}_n^u \circ \Phi(u)$, with

$$\Phi(u) = \frac{1}{i} \log \left[\left(\frac{x^-}{x^+} \right)^{L+M} S^2 \frac{B^{(+)+} R^{(-)-}}{B^{(-)-} R^{(+)+}} \right] \quad (7)$$

where the BES [13] dressing phase $S(u) = \prod_{j=1}^2 \sigma(x_j^{[\pm]}, x_j^{\pm})$ should be taken in the mixed, mirror-physical representation in the integral equations and in the physical-physical representation for Φ_{ph} appearing in the exact Bethe equations. We use the mirror-physical integral representation [15]

$$\begin{aligned} \log S &= \log \frac{B^{(-)+}}{B^{(+)+}} \\ &+ \left(\mathcal{B}^{(10)}(u, w + i0) * \mathcal{G} * \log \frac{R^{(+)}(u-i0)}{R^{(-)}(u-i0)} + c.c. \right) \end{aligned} \quad (8)$$

where $\mathcal{G}(u) \equiv \frac{\partial_u}{2\pi i} \log \frac{\Gamma(1-iu)}{\Gamma(1+iu)}$ while for the physical-physical representation we can use the DHM integral representation [38]. Finally $\mathcal{K}_{\text{ph,mr}}(u, v)$ represents a kernel where we use the physical (mirror) branches for u

(v). For Konishi $L = 2$ and it has $M = 2$ derivatives.

Acknowledgments

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Table I: Numerical values of the Konishi anomalous dimension. The numerical error should be within ± 0.001

$\frac{\sqrt{\lambda}}{4\pi}$	$\Delta_K(\lambda)$	$\frac{\sqrt{\lambda}}{4\pi}$	$\Delta_K(\lambda)$	$\frac{\sqrt{\lambda}}{4\pi}$	$\Delta_K(\lambda)$	$\frac{\sqrt{\lambda}}{4\pi}$	$\Delta_K(\lambda)$
0.00	4.0000	0.55	5.9251	1.05	7.7689	1.60	9.3874
0.05	4.0297	0.60	6.1330	1.10	7.9301	1.65	9.5207
0.10	4.1155	0.65	6.3342	1.15	8.0876	1.70	9.6524
0.15	4.2488	0.70	6.5300	1.20	8.2428	1.75	9.7823
0.20	4.4189	0.75	6.7207	1.25	8.3943	1.80	9.9101
0.25	4.6147	0.80	6.9079	1.30	8.5431	1.85	10.0361
0.30	4.8269	0.85	7.0885	1.35	8.6895	1.90	10.1609
0.35	5.0476	0.90	7.2646	1.40	8.8343	1.95	10.2847
0.40	5.2710	0.95	7.4366	1.45	8.9752	2.00	10.4063
0.45	5.4934	1.00	7.6044	1.50	9.1149	2.05	10.5265
0.50	5.7120			1.55	9.2519		

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- [39] Here Z is one of the complex scalars and D is a covariant derivative in a light cone direction.
- [40] L is the number of Z fields in an operator in $SL(2)$ sector.
- [41] meaningful until the convergency radius $g < g_c = 1/4$; in this region our numerical data are indistinguishable from the BAE for our accuracy. At weak coupling, an interesting numerical prediction could be the order g^{10} . However our numerical error ± 0.001 is larger than $(g_c)^{10}$ and therefore does not allow for such predictions.
- [42] Assuming the leading coefficient to be $2\lambda^{1/4}$ one gets 1.001 instead of 0.994 for the subleading term, even closer to 1 - the value predicted here.
- [43] The expected world-sheet perturbation theory expansion at strong coupling is $\lambda^{1/4}(a_1 + a_2/\sqrt{\lambda} + \dots)$. The first two coefficients a_i - are of order 1 and we assume that all $a_i \sim 1$. In our numerics $1/\sqrt{\lambda} \sim 0.04$. The appropriate extrapolation procedure can easily increase the precision by one order and we expect the absolute numerical error for our a_2 coefficient to be within 0.005.