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Non–Hermitian spin chains with inhomogeneous coupling

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Abstract

An open $U_q(sl_2)$–invariant spin chain of spin S and length N with inhomogeneous coupling is investigated as an example of a non–Hermitian (quasi–Hermitian) model. For several particular cases of such a chain, the ranges of the deformation parameter γ are determined for which the spectrum of the model is real. For a certain range of γ, a universal metric operator is constructed and thus the quasi–Hermiticity of the model is established. The constructed metric operator is non–dynamical, its structure is determined only by the symmetry of the model. The results apply, in particular, to all known homogeneous $U_q(sl_2)$–invariant integrable spin chains with nearest–neighbour interaction. In addition, the most general form of a metric operator for a quasi–Hermitian operator in finite dimensional space is discussed.

Introduction

A bounded linear operator H in a complex Hilbert space $\mathfrak{H}$ equipped with the inner product $\langle x, y\rangle$ is said to be *symmetrizable* if there exists a Hermitian operator η such that $\eta \neq 0$ and

$$\eta\, \mathsf{H} = \mathsf{H}^* \eta\,. \tag{1}$$

Symmetrizable operators have been studied in mathematical literature since long ago [Za, Re, He, Di, S1, S2]. Following Dieudonne [Di], we will say that a symmetrizable operator H is *quasi–Hermitian* if the symmetrizing operator η is positive definite.

If η is invertible then a quasi–Hermitian operator H is similar to a Hermitian one and hence it has a real spectrum (the spectrum of H can be not entirely real if η is positive definite but not invertible, see [Di, S2]). This enables an interpretation [SGH] of an irreducible set of quasi–Hermitian operators as quantum mechanical observables if they share a common symmetrizing operator η. In this context η is called a *metric operator* since the observables become Hermitian operators with respect to the modified inner product $\langle x, y\rangle_\eta \equiv \langle x, \eta y\rangle$. Interesting motivating examples of non–Hermitian operators with a real spectrum are the Hamiltonian of the lattice Reggeon field theory [CS], the Hamiltonian of the Ising quantum spin chain in an imaginary magnetic field [Ge], the Hamiltonians of affine Toda field theories with an imaginary coupling constant [Ho], and the Schrödinger operator with an imaginary cubic potential [BZ]. The latter example was generalized [BB2] to a large class of symmetrizable Hamiltonians possessing the PT (parity and time–reversal) symmetry and having, according to Wiegner's theory [Wi] of anti–unitary operators, (partially) real spectra. Since then a lot of research in physical literature has been devoted to symmetrizable and, in particular, quasi–Hermitian Hamiltonians, leading to the construction of numerous interesting examples and the (re)discovery of many mathematical aspects; see [Be, M2] for reviews.

The Hamiltonian H of a physical model is often given by the sum or, more generally, a linear combination of local Hamiltonians H_n, $n=1,\ldots,N$ with real coefficients (coupling constants)

$$\mathsf{H} = \sum_{n=1}^{N} a_n \, \mathsf{H}_n \,, \qquad a_n \in \mathbb{R} \,. \tag{2}$$

Here we face an immediate difficulty not present in the theory of Hermitian operators: no general criterion is known that would determine whether H is a quasi–Hermitian operator given that all H_n are quasi–Hermitian operators (it is not assumed that they share a common symmetrizing operator). This problem naturally arises for Hamiltonians of various spin chains where the interaction between adjacent sites is described by quasi–Hermitian operators. For instance, the reality of spectra and the existence of metric operators for such compound chains have been investigated for the Ising chain in an imaginary magnetic field [Ge, CF], the Jordanian twist of the Heisenberg chain [KS], and the homogeneous XXZ model of spin $\frac{1}{2}$ [KW]. In the present paper we will address the problem of quasi–Hermiticity for an open spin chain of spin S with nearest–neighbour Hamiltonians H_n having most general form respecting $U_q(sl_2)$ symmetry.

The paper is organized as follows. In Section 1.1, we provide the necessary facts about quasi–Hermitian operators, and in Section 1.2, discuss the most general form of a metric operator. In Section 2.1, we recall the basic notions related to the quantum algebra $U_q(sl_2)$, discuss the phenomenon of non–Hermiticity for the tensor product of its representations in the case of $q=e^{i\gamma}$, $\gamma\in\mathbb{R}$, and introduce an open $U_q(sl_2)$–invariant spin chain of length N with inhomogeneous coupling. In Sections 2.2 and 2.3, we investigate the reality of spectra of particular cases of such a chain for $N=3,4,5$ by considering the minimal polynomials of the corresponding Hamiltonians. Extrapolating our results, we formulate two conjectures on the range of γ in which the spectrum is real. In Section 2.4, we construct a multi–parametric family of universal, i.e. independent of coupling constants, symmetrizing operators for the most general $U_q(sl_2)$–invariant open spin chain with a nearest–neighbour interaction. The construction exploits solely the quantum algebraic symmetry of the model and is formulated in terms of related algebraic objects such as the R–matrix and the comultiplication. For a one–parametric subfamily of symmetrizing operators, we determine the range of γ in which it contains positive definite operators and thus the Hamiltonian of the model is quasi–Hermitian. In Conclusion we summarize and briefly discuss our results. Appendix contains proofs of the statements given in the main text and some technical details on R–matrices and projectors on irreducible subspaces in tensor products.

1 Quasi–Hermitian operators and metric operators

1.1 Preliminaries

Consider the eigenvalue problem for a quasi–Hermitian operator H,

$$\mathsf{H}\,\omega_j = \lambda_j\,\omega_j \,, \qquad \langle \omega_j, \omega_j \rangle = 1 \,. \tag{3}$$

Let $\{\omega_j\}$ be the set of normalized eigenvectors of H and $\mathrm{Spec}(\mathsf{H}) \equiv \{\lambda_j\}$ be the set of the corresponding eigenvalues. Here and below we will restrict our consideration to the case of finite dimensional Hilbert space, $d \equiv \dim \mathfrak{H} < \infty$. In this case, the metric operator η is invertible and the quasi–Hermitian operator H is similar to a Hermitian operator $\eta^{\frac{1}{2}}\mathsf{H}\eta^{-\frac{1}{2}}$. Whence it is immediate that $\mathrm{Spec}(\mathsf{H}) \subset \mathbb{R}$, and the set $\{\omega_j\}$ is a complete set of vectors in $\mathfrak{H}$.

Remark 1. The converse is also true, see [S1, Thm. 3.3]: if a linear operator H in a finite dimensional complex Hilbert space $\mathfrak{H}$ has a real spectrum and the set $\{\omega_j\}$ of its eigenvectors is complete, then H is quasi–Hermitian. A metric operator for a given H can be constructed as follows (see e.g. [M1]): take an arbitrary orthonormal basis $\{e_j\}$ in $\mathfrak{H}$ and define a linear operator Ω such that $\Omega\omega_j = e_j$. Then Ω is invertible and $\mathsf{H}_0 = \Omega\mathsf{H}\Omega^{-1}$ is Hermitian. Whence it follows that $\eta_0 = \Omega^*\Omega$ is a metric operator for H. Note that η_0 does not actually depend on the choice of the basis $\{e_j\}$.

Remark 2. In physical literature on PT–symmetric models [BBJ, Be, M2, AF], one considers also *pseudo–Hermitian* operators, i.e. symmetrizable operators for which η is invertible but not positive definite. Pseudo–Hermiticity of H implies only that, if $\lambda \in \mathrm{Spec}(\mathsf{H})$, then $\bar{\lambda} \in \mathrm{Spec}(\mathsf{H})$, as for instance in the case of $\mathsf{H} = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $\eta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Furthermore, the set of the eigenvectors of a pseudo–Hermitian operator is not necessarily a complete set of vectors in $\mathfrak{H}$, as another simple example demonstrates: $\mathsf{H} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\eta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

The eigenvectors $\{\omega_j\}$ of a quasi–Hermitian operator H provide a *non–orthogonal* basis in $\mathfrak{H}$. Consider the corresponding *Gram matrix* G with entries $G_{kn} = \langle \omega_k, \omega_n \rangle$. The matrix G is invertible, Hermitian (with respect to the conjugate transpose operation), and positive definite. The set of vectors $\{\tilde{\omega}_j\}$, where $\tilde{\omega}_j = \sum_{n=1}^{d} (G^{-1})_{nj}\omega_n$, provides another non–orthogonal basis in $\mathfrak{H}$. Its Gram matrix is G^{-1}. The bases $\{\omega_j\}$ and $\{\tilde{\omega}_j\}$ form a *bi–orthogonal* system:

$$\langle \omega_k, \omega_j \rangle = \delta_{kj}\,, \qquad \langle \omega_k, \tilde{\omega}_j \rangle = \delta_{kj}\,, \qquad \langle \tilde{\omega}_k, \tilde{\omega}_j \rangle = (G^{-1})_{kj}\,. \tag{4}$$

Remark 3. Note that $\tilde{\omega}_j$ are, in general, *not normalized*. Indeed, positive definiteness of G^{-1} implies only that $(G^{-1})_{jj} > 0$ for all j.

Any vector $x \in \mathfrak{H}$ defines a linear functional $x^\dagger : \mathfrak{H} \mapsto \mathbb{C}$ such that $x^\dagger(y) = \langle x, y \rangle$. Since $\{\omega_j\}$ and $\{\tilde{\omega}_j\}$ are bases in $\mathfrak{H}$, any linear operator A acting in $\mathfrak{H}$ can be written in the form

$$\mathsf{A} = \sum_{k,n=1}^{d} O(\mathsf{A})_{kn}\omega_k\, \omega_n^\dagger = \sum_{k,n=1}^{d} \tilde{O}(\mathsf{A})_{kn}\tilde{\omega}_k\, \tilde{\omega}_n^\dagger\,, \tag{5}$$

where $O(\mathsf{A})$ and $\tilde{O}(\mathsf{A})$ are complex matrices (we will call them *symbols* of A). It is useful to observe that $O(\mathsf{A}^*) = \big(O(\mathsf{A})\big)^*$, $\tilde{O}(\mathsf{A}^*) = \big(\tilde{O}(\mathsf{A})\big)^*$, and

$$O(\mathsf{A}\,\mathsf{B}) = O(\mathsf{A})\, G\, O(\mathsf{B})\,, \qquad \tilde{O}(\mathsf{A}\,\mathsf{B}) = \tilde{O}(\mathsf{A})\, G^{-1}\tilde{O}(\mathsf{B})\,, \tag{6}$$

$$\tilde{O}(\mathsf{A}) = G O(\mathsf{A}) G\,, \qquad \tilde{O}(\mathsf{A})\, O(\mathsf{A}^{-1}) = E\,, \tag{7}$$

where E is the identity matrix, and the last relation makes sense if A is invertible.

Let P_j and $\tilde{\mathsf{P}}_j$ denote projectors in $\mathfrak{H}$ on ω_j and $\tilde{\omega}_j$, respectively, i.e. $\mathsf{P}_j\, \omega_k = \delta_{jk}\, \omega_j$ and $\tilde{\mathsf{P}}_j\, \tilde{\omega}_k = \delta_{jk}\, \tilde{\omega}_j$. Relations (4) imply that these projectors are given by

$$\mathsf{P}_j = \omega_j\, \tilde{\omega}_j^\dagger = \sum_{n=1}^{d} (G^{-1})_{jn}\, \omega_j\, \omega_n^\dagger = \sum_{n=1}^{d} G_{nj}\, \tilde{\omega}_n\, \tilde{\omega}_j^\dagger\,, \qquad \tilde{\mathsf{P}}_j = \mathsf{P}_j^* = \tilde{\omega}_j\, \omega_j^\dagger\,. \tag{8}$$

The resolutions of the unity, $\sum\limits_{j=1}^{d} \mathsf{P}_j = 1 = \sum\limits_{j=1}^{d} \mathsf{P}_j^*$, are due to the completeness of the sets $\{\omega_j\}$ and $\{\tilde{\omega}_j\}$.

1.2 General form of metric operator

Consider a quasi–Hermitian operator H which has $d' \le d$ distinct eigenvalues $\{\lambda_j\}$ with multiplicities $\mu_j \ge 1$, so that we have $\sum_{j=1}^{d'} \mu_j = d$. The eigenvectors corresponding to a given eigenvalue λ_j span the subspace $\mathfrak{H}_j \subset \mathfrak{H}$. Let $\{\omega_{j,k}\}$, $k = 1,\ldots,\mu_j$ be a basis of $\mathfrak{H}_j$ (it is not unique if $\mu_j > 1$) and let $\mathsf{P}_{j,k}$ denote the projector on $\omega_{j,k}$.

***Proposition* 1.** *a) For a quasi–Hermitian operator* H *which has the spectrum* $\{\lambda_j\}$ *with multiplicities* μ_j*, fix some basis* $\{\omega_{j,k}\}$ *in each subspace* $\mathfrak{H}_j$*. Then, for this* H*, the most general form of a metric operator and its inverse is the following*

$$\eta = \sum_{j=1}^{d'} \sum_{k,n=1}^{\mu_j} \left(\Phi_j\right)_{kn} \tilde{\omega}_{j,k}\, \tilde{\omega}_{j,n}^{\dagger}\,, \qquad \eta^{-1} = \sum_{j=1}^{d'} \sum_{k,n=1}^{\mu_j} \left(\Phi_j^{-1}\right)_{kn} \omega_{j,k}\, \omega_{j,n}^{\dagger}\,, \tag{9}$$

where Φ_j *are arbitrary Hermitian positive definite matrices of size* $\mu_j \times \mu_j$*.*
b) For a quasi–Hermitian operator H *which has the spectrum* $\{\lambda_j\}$ *with multiplicities* μ_j*, take some metric operator* η*. Then there exists a choice of bases* $\{\omega_{j,k}\}$ *of subspaces* $\mathfrak{H}_j$ *such that the given operator* η *and its inverse are given by*

$$\eta = \sum_{j=1}^{d'} \sum_{k=1}^{\mu_j} \Phi_{j,k}\, \mathsf{P}_{j,k}^{*}\, \mathsf{P}_{j,k}\,, \qquad \eta^{-1} = \sum_{j=1}^{d'} \sum_{k=1}^{\mu_j} \widetilde{\Phi}_{j,k}\, \mathsf{P}_{j,k}\, \mathsf{P}_{j,k}^{*}\,, \tag{10}$$

where $\Phi_{j,k}$ *are arbitrary positive numbers and* $\widetilde{\Phi}_{j,k} = \big((G^{-1})_{\{j,k\},\{j,k\}}\, \Phi_{j,k}\big)^{-1}$.

Remark 4. It is natural to regard metric operators differing only by a positive constant scalar factor as equivalent. Thus, formulae (10) describe $(d-1)$–parametric families of operators. If the spectrum of a quasi–Hermitian operator H is simple, then these formulae give the most general form of the corresponding metric operator and its inverse.

Remark 5. As noted in the previous Remark, the parts *a)* and *b)* of Proposition 1 are just different forms of the same statement if the spectrum of H is simple. The difference appears if the spectrum of H is degenerate. Indeed, although any given metric operator can be brought to the form (10) which involves only the projectors on the eigenvectors of H, this requires a change of the basis in the Hilbert space *after* we have chosen the metric operator. But if we work with a *fixed* basis, then the most general form of a metric operator (9) cannot in general be re–expressed only in terms of the projectors on the eigenvectors of H if it has a degenerate spectrum. This is so because $\mathsf{P}_{j,k}^{*}\mathsf{P}_{j,n} = G_{\{j,k\},\{j,n\}} \tilde{\omega}_{j,k} \tilde{\omega}_{j,n}^{\dagger}$, and the corresponding entry of the Gram matrix can be zero. (In fact, it is zero, if we choose an orthonormal basis in the subspace $\mathfrak{H}_j$.)

Remark 6. If all Φ_j are identity matrices, then (9) yields the operator η_0 considered in Remark 1. Indeed, it easy to see that $\Omega^{-1} = \sum_{j=1}^{d} \omega_j\, e_j^{\dagger}$, whence $\eta_0^{-1} = \Omega^{-1}(\Omega^{*})^{-1} = \sum_{j=1}^{d} \omega_j\, \omega_j^{\dagger}$.

Remark 7. If H has a simple spectrum, we can rewrite formulae (10) using Eqs. (54) into a form that does not use eigenvectors explicitly:

$$\eta = \sum_{j=1}^{d} \Theta_j \left(\prod_{n\neq j}^{d} (\mathsf{H}^{*} - \lambda_n\, 1)\right) \left(\prod_{m\neq j}^{d} (\mathsf{H} - \lambda_m\, 1)\right), \tag{11}$$

$$\eta^{-1} = \sum_{j=1}^{d} \widetilde{\Theta}_j \left(\prod_{m\neq j}^{d} (\mathsf{H} - \lambda_m\, 1)\right) \left(\prod_{n\neq j}^{d} (\mathsf{H}^{*} - \lambda_n\, 1)\right), \tag{12}$$

where Θ_j are arbitrary positive numbers and $\widetilde{\Theta}_j = \big((G^{-1})_{jj}\,\Theta_j\big)^{-1}$.

As an example, consider the following operator acting in $\mathbb{C}^2$ (it is related to the Hamiltonian (93) in [Be] by a change of variables which ensures reality of the spectrum):

$$\mathsf{H} = \begin{pmatrix} e^{i\theta}\,\sinh z & \sin\theta\,\cosh z \\ \sin\theta\,\cosh z & e^{-i\theta}\,\sinh z \end{pmatrix} = (\sinh z)\,e^{i\theta\sigma_3} + (\sin\theta\,\cosh z)\,\sigma_1\,, \qquad \theta, z \in \mathbb{R}. \quad (13)$$

Here and below we use the standard notations for the Pauli matrices: $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Operator (13) is not Hermitian but has real eigenvalues $\lambda_\pm = \cos\theta\,\sinh z \pm \sin\theta$. Observe that its spectral resolution can be written in the following form

$$\mathsf{H} = \lambda_+ \mathsf{P}_+ + \lambda_- \mathsf{P}_-\,, \qquad \mathsf{P}_\pm = e^{-\frac{z}{2}\sigma_2}\,\frac{(1 \pm \sigma_1)}{2}\,e^{\frac{z}{2}\sigma_2}\,, \quad (14)$$

which makes it obvious that $\mathsf{H} = \Omega^{-1}\mathsf{H}_0\Omega$, where $\Omega = e^{\frac{z}{2}\sigma_2}$ and H_0 is Hermitian. Whence, by Remark 1, we have $\eta_0 = \Omega^*\Omega = e^{z\sigma_2}$, whereas (10) yields a one parametric family of metric operators. Namely, taking $\Phi_\pm = e^{\pm\varphi}/\cosh z$, where $\varphi \in \mathbb{R}$, we obtain

$$\eta_\varphi = e^{\frac{z}{2}\sigma_2}\,e^{\varphi\sigma_1}\,e^{\frac{z}{2}\sigma_2}\,. \quad (15)$$

In this form, positive definiteness of η_φ is self–evident, and we recover η_0 for $\varphi = 0$.

2 Spin chains with inhomogeneous coupling

2.1 Spin chains with $U_q(sl_2)$ symmetry

We will consider one dimensional lattice models (open chains with free boundary conditions) which have $U_q(sl_2)$ symmetry. Recall that the algebra $U_q(sl_2)$ has the following defining relations

$$[E,F] = \tfrac{K^2-K^{-2}}{q-q^{-1}}, \qquad KE = qEK, \qquad KF = q^{-1}FK. \quad (16)$$

A comultiplication consistent with these relations can be chosen as follows:

$$\Delta(E) = E\otimes K^{-1} + K\otimes E\,, \quad \Delta(F) = F\otimes K^{-1} + K\otimes F\,, \quad \Delta(K) = K\otimes K\,. \quad (17)$$

Let S be a positive integer or semi–integer number, and let $q = e^{i\gamma}$, where $\gamma \in \mathbb{R}$ and $2S|\gamma| < \pi$. Let $V^S \simeq \mathbb{C}^{2S+1}$ be an irreducible highest weight $U_q(sl_2)$ module and $\{\omega_k\}_{k=-S}^{S}$ be its canonical orthonormal basis in which K is diagonalized. We will consider the standard representation π_S of $U_q(sl_2)$ on V^S:

$$\begin{aligned} \pi_S(E)\,\omega_k &= \sqrt{[S-k][S+k+1]}\,\omega_{k+1}, \\ \pi_S(F)\,\omega_k &= \sqrt{[S+k][S-k+1]}\,\omega_{k-1}, \end{aligned} \qquad \pi_S(K)\,\omega_k = q^k\,\omega_k, \quad (18)$$

where $[t] \equiv \frac{\sin\gamma t}{\sin\gamma}$. In particular, $\pi_{\frac{1}{2}}(E) = \sigma^+ \equiv \frac{1}{2}(\sigma_1 + i\sigma_2)$, $\pi_{\frac{1}{2}}(F) = \sigma^- \equiv \frac{1}{2}(\sigma_1 - i\sigma_2)$, $\pi_{\frac{1}{2}}(K) = e^{i\frac{\gamma}{2}\sigma_3}$. For $2S|\gamma| < \pi$, the non–zero matrix entries of $\pi_S(E)$ and $\pi_S(F)$ are positive, and these matrices are conjugate transposed to each other. Therefore, Eqs. (18) can be regarded as a representation of the algebra $U_q(sl_2)$ with the involution

$$E^* = F\,, \qquad F^* = E\,, \qquad K^* = K^{-1}\,. \tag{19}$$

However, the algebra $U_q(sl_2)$ with such an involution is not a Hopf $*$–algebra, i.e., $\big(\Delta(X)\big)^* \neq \Delta(X^*)$ in general. Instead we have $\big(\Delta(X)\big)^* = \mathbb{P}\Delta(X^*)\mathbb{P}$, where $\mathbb{P}$ is the operator of permutation of the tensor factors in $U_q(sl_2)^{\otimes 2}$. This is the origin of non–Hermiticity of models that will be considered below.

The comultiplication (17) determines the decomposition $V^S \otimes V^S = \oplus_{s=0}^{2S} V^s$, where each V^s is an irreducible $U_q(sl_2)$–submodule. The inner product on V^S gives rise to an inner product on $V^S \otimes V^S$: $\langle \omega_k \otimes \omega_m, \omega_{k'} \otimes \omega_{m'} \rangle = \delta_{kk'}\delta_{mm'}$. A basis for $V^S \otimes V^S$ can be taken to be $\{\omega_{s,k}\}$, where $s = 0, \ldots, 2S$, and, for given s, vectors $\omega_{s,k}$, $k = -S, \ldots, S$ comprise the canonical basis of V^s.

An important difference between the cases $q \in \mathbb{R}$ and $|q| = 1$ is that in the latter case vectors from different submodules can be non–orthogonal. For instance, the basis for $V^{\frac12} \simeq \mathbb{C}^2$ is $\omega_{\frac12} = \binom{1}{0}$, $\omega_{-\frac12} = \binom{0}{1}$, and the basis for $V^{\frac12} \otimes V^{\frac12} = V^0 \oplus V^1$ is

$$\omega_{0,0} = \tfrac{1}{\sqrt{\varkappa}} \begin{pmatrix} 0 \\ q^{-\frac12} \\ -q^{\frac12} \\ 0 \end{pmatrix}, \quad \omega_{1,1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \omega_{1,0} = \tfrac{1}{\sqrt{\varkappa}} \begin{pmatrix} 0 \\ q^{\frac12} \\ q^{-\frac12} \\ 0 \end{pmatrix}, \qquad \omega_{1,-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{20}$$

For $q \in \mathbb{R}$, these vectors are orthogonal, and normalization requires to set $\varkappa = [2]$. For $|q| = 1$, the vectors are normalized if $\varkappa = 2$, and we have $\langle \omega_{0,0}, \omega_{1,0} \rangle = i \sin\gamma$.

Remark 8. Only those basis vectors from different submodules can be non–orthogonal that have equal eigenvalues under the action of $\mathsf{K}_{12} = (\pi_S \otimes \pi_S)\Delta(K)$. Indeed, it follows from (17) and (19) that K_{12} is unitary, $\mathsf{K}^*_{12} = \mathsf{K}^{-1}_{12}$. Therefore, if $\mathsf{K}_{12}\omega = q^k \omega$ and $\mathsf{K}_{12}\omega' = q^{k'}\omega'$, then $\langle \omega', \mathsf{K}_{12}\omega \rangle = q^k \langle \omega', \omega \rangle$ and hence $q^{-k}\langle \omega, \omega' \rangle = \langle \omega, \mathsf{K}^*_{12}\omega' \rangle = \langle \omega, \mathsf{K}^{-1}_{12}\omega' \rangle = q^{-k'}\langle \omega, \omega' \rangle$, which implies that $q^k = q^{k'}$ if $\langle \omega, \omega' \rangle \neq 0$.

Let $\mathsf{P}^{S,s}$ denote the projector onto the irreducible submodule V^s in $V^S \otimes V^S$. Some details on the structure of these projectors are given in Appendix A.2. In particular, the projectors $\mathsf{P}^{S,s}$ are not Hermitian but they are symmetrizable operators:

$$\big(\mathsf{P}^{S,s}\big)^* = \mathsf{P}^{S,s}\,\big|_{q \to \overline{q}} = \mathbb{P}\,\mathsf{P}^{S,s}\,\mathbb{P}\,. \tag{21}$$

In fact, by Remark 1, it is evident that these projectors are quasi–Hermitian operators.

Consider a one dimensional lattice which contains N nodes, each node carries an irreducible module V^S as a local Hilbert space. For an operator A in V^S or in $(V^S)^{\otimes 2}$, we will use the standard notations A_n and A_{nm} for its embedding in operators in $\mathfrak{H} = \big(V^S\big)^{\otimes N}$ that act non–trivially only in the n–th or in the n–th and m–th tensor components, respectively. The following operator

$$\mathsf{H}^{S,s}_{\{a_1,\ldots,a_{N-1}\}} = \sum_{n=1}^{N-1} a_n\, \mathsf{P}^{S,s}_{n,n+1}\,, \qquad a_n \in \mathbb{R}, \tag{22}$$

can be regarded as the Hamiltonian of an open spin chain with *inhomogeneous* coupling. This Hamiltonian commutes with the global action of $U_q(sl_2)$ in $\mathfrak{H}$, i.e. we have (see Appendix A.2)

$$\big[\,\mathsf{H}^{S,s}_{\{a_1,\ldots,a_{N-1}\}}, \pi_S^{\otimes N}\big(\Delta^{(N-1)}(X)\big)\,\big] = 0\,, \qquad \text{for any } \; X \in U_q(sl_2)\,. \tag{23}$$

Here and in the rest of the text we use the abbreviation $\pi_S^{\otimes N} \equiv (\pi_S \otimes \ldots \otimes \pi_S)$.

Recall that the positive integer power of the comultiplication used in (23) is defined recursively: $\Delta^{(1)} \equiv \Delta$ and $\Delta^{(N)} = \Delta_{N,n} \circ \Delta^{(N-1)}$. Here and below we denote $\Delta_{N,n} \equiv id_{n-1} \otimes \Delta \otimes id_{N-n}$, where n can be taken any from 1 to N thanks to the coassociativity of Δ, i.e. $\Delta_{2,1} \circ \Delta = \Delta_{2,2} \circ \Delta$.

Remark 9. The Hamiltonian (22) is pseudo–Hermitian in the *homogeneous* case ($a_1 = \ldots = a_{N-1}$) for any N and in the *two–periodic* case ($a_{2n+1} = a_1$, $a_{2n} = a_2$) for even N. The symmetrizing operator for these cases is given by $\eta = \mathbb{P}_{1,N}\mathbb{P}_{2,N-1}\ldots$.

In general, a lattice model with Hamiltonian (22) is not integrable. However, its homogeneous case is integrable for $s=0$. The corresponding R–matrix is constructed by a Baxterization of the Temperley–Lieb algebra (see, e.g. [Ku]). In particular, for $S=\frac{1}{2}$ and $s=0$, setting $a_1 = a_2 = \ldots = -\cos\gamma$, we recover the Hamiltonian of the well known XXZ model of spin $\frac{1}{2}$ (which is an integrable deformation of the Heisenberg chain),

$$\mathsf{H}^{\frac{1}{2},0}_{\{-\cos\gamma,\ldots\}} = \sum_{n=1}^{N-1} \Big(\tfrac{1}{2}(\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+) + \tfrac{\cos\gamma}{4}\big(\sigma_n^3\sigma_{n+1}^3 - 1\big) + \tfrac{i\sin\gamma}{4}(\sigma_n^3 - \sigma_{n+1}^3)\Big). \quad (24)$$

2.2 $N=2$ and $N=3$

We commence by studying spectra of short chains. Since $\mathfrak{H}$ is finite dimensional, we have $\operatorname{Spec}\mathsf{H} = \big\{\lambda : \mathcal{P}_{\mathsf{H}}(\lambda) = 0\big\}$, where $\mathcal{P}_{\mathsf{H}}(\lambda)$ is the minimal polynomial for H, i.e. the least degree non–zero polynomial such that $\mathcal{P}_{\mathsf{H}}(\mathsf{H}) = 0$. In the simplest case, $N=2$, we have $\mathsf{H}^{S,s}_{\{a_1\}} = a_1\mathsf{P}^{S,s}_{12}$. The corresponding minimal polynomial is $\mathcal{P}^{S,s}_{a_1}(\lambda) = \lambda^2 - a_1\lambda$, which shows that the spectrum consists of points 0 and a_1 and thus is real.

For $N=3$, we have $\mathsf{H}^{S,s}_{\{a_1,a_2\}} = a_1\mathsf{P}^{S,s}_{12} + a_2\mathsf{P}^{S,s}_{23}$. Let us consider first the case $s=0$. In this case the projectors satisfy the relations of the Temperley–Lieb algebra [BB1, B2]:

$$\mathsf{P}^{S,0}_{n-1,n}\,\mathsf{P}^{S,0}_{n,n+1}\,\mathsf{P}^{S,0}_{n-1,n} = \mu_S\,\mathsf{P}^{S,0}_{n-1,n}\,, \qquad \mu_S = \tfrac{1}{[2S+1]^2}\,. \quad (25)$$

Using these relations (see Appendix A.3), we find the minimal polynomial for $\mathsf{H}^{S,0}_{\{a_1,a_2\}}$:

$$\mathcal{P}^{S,0}_{a_1,a_2}(\lambda) = \lambda\left(\lambda^2 - (a_1+a_2)\,\lambda + a_1a_2\,(1-\mu_S)\right). \quad (26)$$

Hence it follows that all eigenvalues of $\mathsf{H}^{S,0}_{\{a_1,a_2\}}$ are real iff $\mathcal{D}^{S,0} \equiv (a_1-a_2)^2 + 4a_1a_2\mu_S$ is non–negative, that is iff

$$\Big(\frac{\sin(2S+1)\gamma}{\sin\gamma}\Big)^2 \geq -\frac{4a_1a_2}{(a_1-a_2)^2}\,. \quad (27)$$

Clearly, this condition holds always if a_1 and a_2 are both positive (or both negative). If $a_1a_2<0$, then the spectrum of $\mathsf{H}^{S,0}_{\{a_1,a_2\}}$ is not real for those values of γ where (27) does not hold. Note that the r.h.s. of (27) attains the maximal value equal to 1 when $a_2 = -a_1$. Hence we infer that, even for $a_1a_2<0$, the spectrum of $\mathsf{H}^{S,0}_{\{a_1,a_2\}}$ is guaranteed to be real for sufficiently small values of γ, namely for $|\gamma| < \gamma_{S,0}$, where

$$\gamma_{S,0} = \frac{\pi}{2(S+1)} \quad (28)$$

is the minimal positive solution of the equation $\sin(2S+1)\gamma = \sin\gamma$.

For $s \neq 0$, the projectors $\mathsf{P}^{S,s}$ do not satisfy relations of the type (25). However, by evaluating (57) and (60) in the representation (18), one can find an explicit matrix form of these projectors and then search for the coefficients of the minimal polynomial for $\mathsf{H}^{S,s}_{\{a_1,a_2\}}$. The author performed these steps for $S=1,\frac{3}{2}$ and $s \leq 2S$ using *Mathematica*™. The polynomials obtained are:

$$\mathcal{P}^{S,s}_{a_1,a_2}(\lambda) = \lambda^{\epsilon_{S,s}} \prod_k \big(\lambda^2 - (a_1+a_2)\,\lambda + a_1 a_2\,(1-d^{S,s}_k)\big)\,, \tag{29}$$

where the coefficients $d^{S,s}_k$ are listed in Appendix A.4. In (29) we have $\epsilon_{S,s}=0$ if there is $d^{S,s}_k=1$ in the list for given S and s (which occurs for $s=2S$) and $\epsilon_{S,s}=1$ otherwise.

From (29) we infer that all eigenvalues of $\mathsf{H}^{S,s}_{\{a_1,a_2\}}$ are real iff all $\mathcal{D}^{S,s}_k \equiv (a_1-a_2)^2 + 4a_1a_2 d^{S,s}_k$ are non–negative, that is iff

$$\big(d^{S,s}_k\big)^{-1} \geq -\frac{4a_1a_2}{(a_1-a_2)^2}\,. \tag{30}$$

Thus, we see that, for the considered values of S, the spectrum of $\mathsf{H}^{S,s}_{\{a_1,a_2\}}$ is real if $a_1a_2>0$ and is not real for some values of γ if $a_1a_2<0$. In the latter case, the spectrum of $\mathsf{H}^{S,s}_{\{a_1,a_2\}}$ is guaranteed to be real for $|\gamma| < \gamma_{S,s} = \min_k \gamma^{\{k\}}_{S,s}$, where $\gamma^{\{k\}}_{S,s}$ is the minimal positive solution of the equation $d^{S,s}_k = 1$. In Appendix A.4, the coefficients $d^{S,s}_k$ are listed in such a way that $k=1$ corresponds to the minimal value among $\gamma^{\{k\}}_{S,s}$. The list (65) of resulting values $\gamma_{S,s}$ together with formula (28) allows us to conjecture the following.

***Conjecture* 1.** *For $a_1a_2<0$, the spectrum of $\mathsf{H}^{S,s}_{\{a_1,a_2\}}$ is real for $|\gamma| < \gamma_{S,s}$, where*

$$\gamma_{S,s} = \frac{\pi}{2(s+S+1-\delta_{s,2S})}\,. \tag{31}$$

Remark 10. Appearance of the correction for $s=2S$ in (31) seems to be related to the fact that $\mathsf{P}^{S,2S} = 1 - \sum_{s\neq 2S} \mathsf{P}^{S,s}$. In particular, (31) yields $\gamma_{\frac{1}{2},1} = \gamma_{\frac{1}{2},0}$, as should be anticipated because $\mathsf{H}^{\frac{1}{2},1}_{\{a_1,a_2\}}$ and $\mathsf{H}^{\frac{1}{2},0}_{\{a_1,a_2\}}$ differ only by a sign and a shift by a real multiple of the identity operator.

2.3 $N=4$ and $N=5$ for $s=0$

For $N=4$ and $s=0$, a computation analogous to that in Appendix A.3 yields the following minimal polynomial

$$\begin{aligned}\mathcal{P}^{S,0}_{a_1,a_2,a_3}(\lambda) = \lambda\,&\big(\lambda^2 - (a_1+a_2+a_3)\,\lambda + (a_1+a_3)\,a_2\,(1-\mu_S)\big)\\ &\times\big(\lambda^3 - (a_1+a_2+a_3)\lambda^2 + \big(a_1a_3 + a_2(a_1+a_3)(1-\mu_S)\big)\lambda - a_1a_2a_3\,(1-2\mu_S)\big)\,.\end{aligned} \tag{32}$$

Analysis of the reality of the roots of the cubic factor is fairly complicated. Therefore, we restrict our consideration to the case $a_3=a_1$ (which, in particular, includes the homogeneous case). In this case, (32) simplifies and acquires the following form:

$$\begin{aligned}\mathcal{P}^{S,0}_{a_1,a_2,a_1}(\lambda) = \lambda\,(\lambda-a_1)&\big(\lambda^2 - (a_1+a_2)\,\lambda + a_1a_2\,(1-2\mu_S)\big)\\ &\times\big(\lambda^2 - (2a_1+a_2)\,\lambda + 2a_1\,a_2\,(1-\mu_S)\big)\,.\end{aligned} \tag{33}$$

It follows from (33) that all eigenvalues of $\mathsf{H}^{S,0}_{\{a_1,a_2,a_1\}}$ are real iff both $\tilde{\mathcal{D}}^{S,0}_1 \equiv (2a_1-a_2)^2 + 8a_1a_2\mu_S$ and $\tilde{\mathcal{D}}^{S,0}_2 \equiv (a_1-a_2)^2 + 8a_1a_2\mu_S$ are non–negative. Thus, we conclude that the spectrum of $\mathsf{H}^{S,0}_{\{a_1,a_2,a_1\}}$ is real if $a_1a_2>0$ and is not real for some values of γ if $a_1a_2<0$. In the latter case, we note that $\tilde{\mathcal{D}}^{S,0}_1 - \tilde{\mathcal{D}}^{S,0}_2 = a_1(3a_1-2a_2)>0$. Therefore, for $a_1a_2<0$, the spectrum of $\mathsf{H}^{S,0}_{\{a_1,a_2,a_1\}}$ is real iff $\tilde{\mathcal{D}}^{S,0}_2>0$, that is iff

$$\Big(\frac{\sin{(2S{+}1)\gamma}}{\sin\gamma}\Big)^2 \geq -\frac{8a_1a_2}{(a_1-a_2)^2}\,. \tag{34}$$

The r.h.s. of (34) attains the maximal value equal to 2 when $a_2=-a_1$. Thus, for $a_1a_2<0$, the spectrum of $\mathsf{H}^{S,0}_{\{a_1,a_2,a_1\}}$ is guaranteed to be real for $|\gamma| < \tilde{\gamma}_{S,0}$, where $\tilde{\gamma}_{S,0}$ is the minimal positive solution of the equation $\sin^2{(2S{+}1)\gamma} = 2\sin^2\gamma$. Taking into account that, for $S\geq\frac{1}{2}$, we have $\sin{(2S{+}1)\gamma}/\sin\gamma>\sqrt{2}$ on some interval that contains the point $\gamma=0$, the value $\tilde{\gamma}_{S,0}$ can be equivalently determined as the minimal positive solution of the equation

$$U_{2S}(\cos\gamma) = \sqrt{2}\,, \tag{35}$$

where $U_n(t)$ is the Chebyshev polynomial of the second kind ($U_1(t)=2t$, $U_2(t)=4t^2-1$, etc.) In particular, we have

$$\tilde{\gamma}_{\frac{1}{2},0} = \frac{\pi}{4}\,, \qquad \tilde{\gamma}_{1,0} = \arccos\frac{\sqrt{1{+}\sqrt{2}}}{2} \approx 0.217\,\pi\,. \tag{36}$$

For $N = 5$ and $s=0$, even in the reduced case $a_3=a_1$, $a_4=a_2$, the minimal polynomial $\mathcal{P}^{S,0}_{a_1,a_2,a_1,a_2}(\lambda)$ contains factors which are fourth and fifth degree polynomials in λ. However, for $a_1=a_3=a$, $a_2=a_4=-a$, it simplifies and acquires the following form

$$\begin{aligned}\mathcal{P}^{S,0}_{a,-a,a,-a}(\lambda) = \lambda\,\big(\lambda^4 + a^2(3\mu_S-2)\,\lambda^2 + a^4\,(\mu_S^2-3\mu_S+1)\big)\\ \times\,\big(\lambda^4 + a^2(6\mu_S-5)\,\lambda^2 + a^4\,(5\mu_S^2-10\mu_S+4)\big)\,.\end{aligned} \tag{37}$$

The first bi–quadratic factor here has only real roots iff $\mu_S\leq\frac{3-\sqrt{5}}{2}$. For this range of μ_S, the second bi–quadratic factor has also only real roots. Thus, the spectrum of $\mathsf{H}^{S,0}_{\{a,-a,a,-a\}}$ is guaranteed to be real for $|\gamma| < \tilde{\gamma}_{S,0}$, where $\tilde{\gamma}_{S,0}$ is the minimal positive solution of the equation $\sin{(2S{+}1)\gamma} = \big(\frac{3+\sqrt{5}}{2}\big)^{1/2}\sin\gamma$, or, equivalently, of the equation

$$U_{2S}(\cos\gamma) = \frac{1+\sqrt{5}}{2}\,. \tag{38}$$

In particular, we have

$$\tilde{\gamma}_{\frac{1}{2},0} = \tilde{\gamma}_{1,0} = \frac{\pi}{5}\,, \qquad \tilde{\gamma}_{\frac{3}{2},0} \approx 0.172\,\pi\,. \tag{39}$$

Equations (28), (35), and (38) allow us to make the following conjecture about a chain with *alternating* coupling ($a_1=-a_2=a_3=-a_4=\ldots$).

***Conjecture* 2.** *For an alternating chain with $N\geq 3$ nodes, the spectrum of $\mathsf{H}^{S,0}_{\{a,-a,a,-a,\ldots\}}$ is real for $|\gamma| < \tilde{\gamma}_{S,0}$, where $\tilde{\gamma}_{S,0}$ is the minimal positive solution of the equation*

$$U_{2S}(\cos\gamma) = 2\cos\frac{\pi}{N}\,. \tag{40}$$

Remark 11. For the alternating chain of spin $S=\frac{1}{2}$ and length N, Eq. (40) yields

$$\tilde{\gamma}_{\frac{1}{2},0} = \frac{\pi}{N}\,, \tag{41}$$

which is the most natural extrapolation of the values $\tilde{\gamma}_{\frac{1}{2},0}$ given by Eqs. (28), (36), and (39).

2.4 A universal metric operator

The most general form of a $U_q(sl_2)$)–invariant open spin chain Hamiltonian with a nearest–neighbour interaction and an inhomogeneous coupling is the following

$$\mathsf{H}_N^S = \sum_{n=1}^{N-1} \sum_{s=0}^{2S} b_{n,s}\, \mathsf{P}^{S,s}_{n,n+1}\,, \qquad b_{n,s} \in \mathbb{R}\,. \tag{42}$$

The previously considered Hamiltonian (22) is a particular case of (42) corresponding to the choice $b_{n,s'} = a_n \delta_{ss'}$. A particular homogeneous case of (42) corresponding to the choice $b_{n,s} = (\sin\gamma) \sum_{k=1}^{s} \cot(\gamma k)$ recovers the Hamiltonian of the integrable XXZ model of spin S (see e.g. [B1]). For spin $S=1$, another integrable model recovered as a homogeneous case of (42) is the spin chain generated by the Izergin–Korepin R–matrix [IK].

Now our aim is to construct a *universal* metric operator η_N for the Hamiltonian (42), i.e. such that relation (1) holds irrespective of the choice of the coupling coefficients $b_{n,s}$. As seen from Eq. (21), it suffices to find such η_N that the relation

$$\eta_N\, \mathsf{P}^{S,s}_{n,n+1} = \big(\mathsf{P}^{S,s}_{n,n+1}\big)^*\, \eta_N = \mathsf{P}^{S,s}_{n+1,n}\, \eta_N \tag{43}$$

holds for all $n = 1,\ldots,N-1$.

Recall that the Hopf algebra $U_q(sl_2)$ is quasi–triangular [D1], i.e. it possesses a universal R–matrix which is an invertible element of (a completion of) $U_q(sl_2)^{\otimes 2}$ with the following properties

$$R\,\Delta(X) = \Delta'(X)\,R\,, \qquad \text{for any } \; X \in U_q(sl_2)\,, \tag{44}$$

$$(\Delta \otimes id)\,R = R_{13}\,R_{23}\,, \qquad (id \otimes \Delta)\,R = R_{13}\,R_{12}\,, \tag{45}$$

where $\Delta'(X) \equiv \mathbb{P}\,\Delta(X)\,\mathbb{P}$. In fact, there exist two universal R–matrices because, if $R^+ \equiv R$ satisfies (44)–(45), then so does $R^- = \mathbb{P}\big(R^+\big)^{-1}\mathbb{P}$. The explicit form of the universal R–matrices $R^\pm$ consistent with the comultiplication (17) is given in Appendix A.5.

Let us denote $\mathsf{R}^\pm \equiv (\pi_S \otimes \pi_S) R^\pm$. Eq. (44) along with the fact that $\mathsf{P}^{S,s}$ is a function of $(\pi_S \otimes \pi_S)\Delta(C)$ (see Eq. (60)) implies that the projectors $\mathsf{P}^{S,s}$ are symmetrizable by $\mathsf{R}^\pm$, i.e.

$$\mathsf{R}^\pm_{n,n+1}\, \mathsf{P}^{S,s}_{n,n+1} = \mathsf{P}^{S,s}_{n+1,n}\, \mathsf{R}^\pm_{n,n+1}\,. \tag{46}$$

Eq. (68) implies that $\eta_2^S(\alpha) = e^{i\alpha}\,\mathsf{R}^+ + e^{-i\alpha}\,\mathsf{R}^-$ is a Hermitian operator if $\alpha \in \mathbb{R}$. This, along with (46), means that $\eta_2^S(\alpha)$ is a one–parametric family of symmetrizing operators for a chain of length $N=2$. We will extend this observation to a chain of arbitrary length as follows (a proof is given in Appendix A.6).

***Proposition* 2.** *a) For a chain of length N, the following operators satisfy relations (43)*

$$\eta_N^{\pm} = \overleftarrow{\mathsf{R}}_N \ldots \overleftarrow{\mathsf{R}}_2, \qquad \textit{where} \quad \overleftarrow{\mathsf{R}}_n = \mathsf{R}_{n-1,n} \ldots \mathsf{R}_{1,n}\,. \tag{47}$$

b) These operators can also be represented as follows

$$\eta_N^{\pm} = \overrightarrow{\mathsf{R}}_1 \ldots \overrightarrow{\mathsf{R}}_{N-1}, \qquad \textit{where} \quad \overrightarrow{\mathsf{R}}_n = \mathsf{R}_{n,n+1} \ldots \mathsf{R}_{n,N}\,. \tag{48}$$

c) These operators are conjugate to each other,

$$\big(\eta_N^{+}\big)^* = \eta_N^{-}\,. \tag{49}$$

Remark 12. The proof of Proposition 2 is facilitated by an observation that the operation $\Delta^{\pm} \equiv R^{\pm}\Delta$ is coassociative (but note that it is not an algebra homomorphism) and that the operators (47) can be expressed in terms of its power: $\eta_N^{\pm} = \pi_S^{\otimes N}\big(\Delta_{\pm}^{(N-1)}(1)\big)$, see Lemma 2.

As seen from (49), the symmetrizing operators $\eta_N^{\pm}$ are not Hermitian. However, we can utilize them to build a multi–parametric family of Hermitian symmetrizing operators as follows:

$$\eta_N^S(\alpha_1,\ldots|\beta_1,\ldots) = \sum_{n\geq 1} \beta_n \Big(e^{i\alpha_n}\, \eta_N^{+} \big((\eta_N^{-})^{-1}\, \eta_N^{+}\big)^{n-1} + e^{-i\alpha_n}\, \eta_N^{-} \big((\eta_N^{+})^{-1}\, \eta_N^{-}\big)^{n-1}\Big), \tag{50}$$

where all α_n and β_n are real. Here we used a simple fact: if η, η', and η'' are symmetrizing operators for an operator H, then so is $\eta(\eta')^{-1}\eta''$ if η' is invertible. In our case, $\eta_N^{\pm}$ are invertible because so are the universal R–matrices.

Note that, for $\gamma = 0$, we have $\mathsf{R}^{\pm} = 1 \otimes 1$ and $\eta_N^{\pm} = 1_N$. Therefore, for sufficiently small values of γ and appropriately chosen coefficients $\{\alpha_n\}$, $\{\beta_n\}$, operator (50) is positive definite and, thus, is a metric operator for the Hamiltonian (42).

For $\gamma \neq 0$, it is not straightforward to determine the values of $\{\alpha_n\}$ and $\{\beta_n\}$ for which (50) is positive definite. In the present article, we restrict our consideration to a one–parametric family,

$$\eta_N^S(\alpha) = e^{i\alpha}\, \eta_N^{+} + e^{-i\alpha}\, \eta_N^{-}\,, \qquad \alpha \in \mathbb{R}\,. \tag{51}$$

Let $\gamma(\alpha)$ denote the maximal positive value of γ for which (51) is positive definite for given α, and let $\hat{\gamma}_S \equiv \sup_\alpha \gamma(\alpha)$. At least one of the eigenvalues of $\eta_N^S(\alpha)$ vanishes at $\gamma = \hat{\gamma}_S$. Therefore, $\hat{\gamma}_S$ can be determined from the condition $\det\big(\eta_N^S(\alpha)\big) = 0$.

***Lemma* 1.** *The following relation holds*

$$\det\big(\eta_N^S(\alpha)\big) = \prod_{s=s_o}^{SN} \Big(e^{i\alpha} q^{s(s+1)-NS(S+1)} + e^{-i\alpha} q^{NS(S+1)-s(s+1)}\Big)^{(2s+1)\nu_s}\,, \tag{52}$$

where ν_s are the multiplicities of the irreducible submodules in the decomposition $\big(V^S\big)^{\otimes N} = \bigoplus_{s=s_o}^{NS} \nu_s V^s$. Here $s_o = 0$ if NS is integer and $s_o = \frac{1}{2}$ if NS is half–integer.

The range of γ that includes the point $\gamma = 0$ and in which (52) does not vanish is maximal if we set $\alpha = \alpha_0 \equiv \frac{\gamma}{2}\big(NS(2S+1-NS) - s_o(s_o+1)\big)$. Then we have $\det\big(\eta_N^S(\alpha_0)\big) > 0$ for $|\gamma| < \hat{\gamma}_S$, where

$$\hat{\gamma}_S = \frac{\pi}{(NS - s_o)(NS + s_o + 1)}\,. \tag{53}$$

Since $\frac{1}{2}\eta_N^S(0) = 1$ for $\gamma = 0$, we conclude that $\eta_N^S(\alpha_0)$ is positive definite for $|\gamma| < \hat{\gamma}_S$. Thus, we have established the following.

***Proposition* 3.** *The Hamiltonian H_N^S given by (42) is quasi–Hermitian for any choice of the coupling constants $b_{n,s}$ provided that $|\gamma| < \hat{\gamma}_S$, where $\hat{\gamma}_S$ is given by (53).*

Conclusion

It is well known that for a given quasi–Hermitian operator H there are many metric operators [SGH, Be, M2]. In the physical literature on non–Hermitian Hamiltonians, the one most frequently discussed is the operator η_0 considered in Remark 1. For the case of H having a simple spectrum, a generalization of η_0 to an operator of the type (9) was given in [ZG]. In the present article, we have given the most general form of a metric operator for a finite dimensional quasi–Hermitian operator H not assuming its spectrum to be simple.

As an example of a compound operator (2) given by the sum of quasi–Hermitian operators, we studied the Hamiltonians (22) and (42) of an open $U_q(sl_2)$–invariant spin chain of spin S and length N. For these Hamiltonians, we constructed two symmetrizing operators $\eta_N^{\pm}$ in terms of products of local R–matrices (let us note that similar products appeared in a different context in [TV]). From the operators $\eta_N^{\pm}$ we built a multi–parametric family of metric operators. These metric operators are universal, i.e. independent of the coupling constants, and thus non–dynamical, i.e. their construction does not require the knowledge of the eigenvectors of a Hamiltonian.

By optimizing the value of the free parameter in a one–parametric subfamily of universal metric operators, we obtained an estimate (53) on the range of the deformation parameter γ in which the considered Hamiltonians are quasi–Hermitian. Note that this range is in general narrower than the ranges of γ for which the short chains considered in Section 2.2 and 2.3 have real spectra. We expect that better estimates of the quasi–Hermiticity range can be obtained by using the multi–parametric family (50).

It is worth mention that the most general family (42) of Hamiltonians includes, in particular, all known (see, e.g. [B2]) integrable $U_q(sl_2)$–invariant spin chains with nearest–neighbour interaction: the XXZ model of spin S, the Temperley–Lieb spin chain of spin S, and, for spin 1, the spin chain generated by the Izergin–Korepin R–matrix. So our construction of the metric operators applies also to these cases.

Let us conclude with several remarks on the "experimental" data obtained in Section 2.2 and 2.3 for the ranges of γ in which the Hamiltonian (22) has a real spectrum. First, it is very interesting to note that the value of $\tilde{\gamma}_{\frac{1}{2},0}$ in (41) for an alternating XXZ chain of spin $\frac{1}{2}$ is exactly the same as the boundary of the quasi–Hermiticity range for a homogeneous XXZ chain of spin $\frac{1}{2}$ found in [KW] by means of the path basis technique. Actually, the results for short chains seem to indicate that, for given S and N, the alternating chain ($a_1 = -a_2 = a_3 = -a_4 \ldots$) is the most non–Hermitian one, at least in the subclass of chains with a two–periodic coupling ($a_{2n+1} = a_1$, $a_{2n} = a_2$). Thus, we have a reason to expect that Conjecture 2 may hold not only for alternating but also for two–periodic chains and, possibly, even for arbitrary ones.

Finally, let us remind that in the general $N = 3$ case and the two–periodic $N = 4$ case the spectra are always real if all coupling constants are positive. This observation is supported by numerical checks in a number of other cases. It is thus tempting to suggest the following.

***Conjecture* 3.** *For $|\gamma| < \frac{\pi}{2S}$, the Hamiltonian (22) of a spin chain with inhomogeneous coupling has a real spectrum if all $a_n > 0$.*

A Appendix

A.1 Proof of Proposition 1

The spectral resolutions of a quasi–Hermitian operator H and its adjoint are $\mathsf{H} = \sum_{j=1}^{d'} \lambda_j \mathfrak{P}_j$, $\mathsf{H}^* = \sum_{j=1}^{d'} \lambda_j \mathfrak{P}_j^*$, where $\mathfrak{P}_j = \sum_{k=1}^{\mu_j} \mathsf{P}_{j,k}$ are the projectors onto the subspaces $\mathfrak{H}_j$. Hence

$$\mathfrak{P}_j = \prod_{n \neq j}^{d'} \frac{\mathsf{H} - \lambda_n 1}{\lambda_j - \lambda_n}, \qquad \mathfrak{P}_j^* = \prod_{n \neq j}^{d'} \frac{\mathsf{H}^* - \lambda_n 1}{\lambda_j - \lambda_n}. \tag{54}$$

It follows from relation (1) that $\eta\, \mathsf{H}^n = (\mathsf{H}^*)^n \eta$ for all $n \in \mathbb{N}$. Therefore $\eta f(\mathsf{H}) = \big(f(\mathsf{H})\big)^* \eta$, where $f(t)$ is an arbitrary polynomial with real coefficients. Along with (54) it implies that a positive definite operator η is a metric operator for H iff

$$\eta\, \mathfrak{P}_j = \mathfrak{P}_j^*\, \eta, \qquad j = 1, \dots, d'. \tag{55}$$

As the basis of $\mathfrak{H}$ we take a naturally ordered set $\{\omega_{1,1}, \dots, \omega_{1,\mu_1}, \omega_{2,1}, \dots, \omega_{d',\mu_{d'}}\}$. Then, according to (8), we have $\tilde{O}(\mathfrak{P}_j) = G\, E_j$ and $\tilde{O}(\mathfrak{P}_j^*) = E_j G$, where E_j is a diagonal matrix with μ_j consecutive entries equal to 1 and others being 0; the identity matrix has the resolution $E = \sum_{j=1}^{d'} E_j$. Using (6), we find that $\tilde{O}(\eta\, \mathfrak{P}_j) = \tilde{O}(\eta) E_j$ and $\tilde{O}(\mathfrak{P}_j^* \eta) = E_j \tilde{O}(\eta)$. Therefore, (55) holds iff $\tilde{O}(\eta)$ commutes with E_j for all j, that is iff $\tilde{O}(\eta)$ is a block diagonal matrix. The second relation in (7) implies that $O(\eta^{-1})$ is inverse to $\tilde{O}(\eta)$ and so it is also a block diagonal matrix. Whence Eqs. (9) follow. The Hermiticity of η is equivalent to $(\tilde{O}(\eta))^* = (\tilde{O}(\eta))$ which implies that blocks Φ_j in (9) must be Hermitian. Since η is invertible, it is positive definite whenever η^{-1} is so. The latter condition requires, in particular, that $\langle x_j, \eta^{-1} x_j \rangle > 0$, for any non–zero vector $x_j \in \mathfrak{H}_j$. Which is equivalent to $\sum_{k,n=1}^{\mu_j} \big(\Phi_j^{-1}\big)_{kn} \overline{\beta_k} \beta_n > 0$, where $\beta_k \equiv \langle \omega_{j,k}, x_j \rangle$ can be arbitrary (but not all zero). Thus, Φ_j^{-1} must be positive definite, and hence so does Φ_j.

To prove the part b), we fix some bases $\{\omega^\circ_{j,k}\}$ of subspaces $\mathfrak{H}_j$. Consider η and η^{-1} given by (9) with some matrices Φ_j°. Let U_j be such unitary matrices that $\Phi_j = U_j \Phi_j^\circ U_j^{-1}$ are diagonal. Then, introducing new basis vectors, $\omega_{j,k} = \sum_n (U_j^{-1})_{kn} \omega^\circ_{j,n}$, we achieve that, in the new basis, the symbol $O(\eta^{-1})$ becomes a diagonal matrix. The second relation in (7) implies that $\tilde{O}(\eta)$ also becomes a diagonal matrix. It remains to use formulae (8) to obtain Eqs. (10).

A.2 Projectors $\mathsf{P}^{S,s}$

Let $q = e^{i\gamma}$. The algebra (16) has the following Casimir element:

$$C = \tfrac{1}{2}\big(E\, F + F\, E\big) - \tfrac{\cos\gamma}{4 \sin^2\gamma}\big(K - K^{-1}\big)^2. \tag{56}$$

Its value in an irreducible representation V^S is $\pi_S(C) = [S][S+1]$, where the q–numbers are defined as $[t] \equiv \frac{\sin\gamma t}{\sin\gamma}$. The tensor Casimir element is an operator in $V^S \otimes V^S$ given by

$$\mathsf{C}^{S,S} = (\pi_S \otimes \pi_S)\Delta(C) = (\pi_S \otimes \pi_S)\Big((K\, E) \otimes (F\, K^{-1}) + (F\, K^{-1}) \otimes (K\, E) \tag{57}$$
$$+ \tfrac{1}{2\sin^2\gamma}\big((1 \otimes 1 + K^2 \otimes K^{-2})\ \cos\gamma - (1 \otimes K^{-2} + K^2 \otimes 1)\ \cos\big(\gamma(2S+1)\big)\big)\Big).$$

Obviously, we have $[\mathsf{C}^{S,S},(\pi_S\otimes\pi_S)\big(\Delta(X)\big)]=0$ for any $X\in U_q(sl_2)$. Furthermore, we have

$$[\mathsf{C}^{S,S}_{n,n+1},\pi_S^{\otimes N}\big(\Delta^{(N-1)}(X)\big)]=0\,, \tag{58}$$

for any X and $n=1,\ldots,N-1$. This can be verified by evaluating $\pi_S^{\otimes N}\big(\Delta_{N-1,n}(Y)\big)$, where $Y=[C_n,\big(\Delta^{(N-2)}(X)\big)]=0$.

With respect to the involution (19), the tensor Casimir element is not Hermitian but is a symmetrizable operator,

$$\big(\mathsf{C}^{S,S}\big)^*=\mathsf{C}^{S,S}_{q^{-1}}=\mathbb{P}\,\mathsf{C}^{S,S}\,\mathbb{P}\,. \tag{59}$$

Here $\mathsf{C}^{S,S}_{q^{-1}}$ is the tensor Casimir element of the algebra $U_{q^{-1}}(sl_2)$ (which is obtained by the mapping $E\to E$, $F\to F$, $K\to K^{-1}$, $q\to q^{-1}$).

The projectors $\mathsf{P}^{S,s}$ can be constructed as follows (see e.g. [B1])

$$\mathsf{P}^{S,s}=\prod_{\substack{l=0\\ l\neq s}}^{2S}\frac{\mathsf{C}^{S,S}-[l][l+1]}{[s-l][s+l+1]}\,. \tag{60}$$

In particular, for $S=\frac{1}{2}$ we have

$$\mathsf{P}^{\frac{1}{2},0}=\frac{1}{\kappa}\begin{pmatrix}0&&&\\&q^{-1}&-1&\\&-1&q&\\&&&0\end{pmatrix},\qquad \mathsf{P}^{\frac{1}{2},1}=\frac{1}{\kappa}\begin{pmatrix}\kappa&&&\\&q&1&\\&1&q^{-1}&\\&&&\kappa\end{pmatrix},\qquad \kappa=q+q^{-1}.$$

Note that matrix entries of $\mathsf{P}^{S,s}$ can have singularities at some values of γ. This means that at these points the Gram matrix of the basis of $V^S\otimes V^S$ is not invertible (cf. Eq. (8)) and some basis vectors become linear dependent. We shall exclude such values of γ from consideration.

Since $\mathsf{P}^{S,s}$ are polynomials (with real coefficients) in $\mathsf{C}^{S,S}$, they satisfy the same relations (58) and (59), i.e.,

$$[\mathsf{P}^{S,s}_{n,n+1},\pi_S^{\otimes N}\big(\Delta^{(N-1)}(X)\big)]=0\,,\qquad \big(\mathsf{P}^{S,s}\big)^*=\mathsf{P}^{S,s}_{q^{-1}}=\mathbb{P}\,\mathsf{P}^{S,s}\,\mathbb{P}\,. \tag{61}$$

The first equality in the second relation implies, in particular, that $\tilde\omega_{s,k}\simeq\omega_{s,k}|_{q\to\bar q}=\overline{\omega}_{s,k}$, where $\simeq$ means equality up to a normalization (recall that $\tilde\omega$ are, in general, not normalized, cf. Remark 3). Using this relation and formulae (8), we can write down a more explicit expression for $\mathsf{P}^{S,s}$,

$$\mathsf{P}^{S,s}=\sum_{k=-s}^{s}\mathsf{P}_{s,k}=\sum_{k=-s}^{s}\tfrac{1}{\kappa_{s,k}}\,\omega_{s,k}\,\overline{\omega}^{\dagger}_{s,k}\,, \tag{62}$$

where $\kappa_{s,k}=\langle\overline{\omega}_{s,k},\omega_{s,k}\rangle=||\omega_{s,k}||^2_{q\in\mathbb{R}}$, which is the norm of $\omega_{s,k}$ for $q\in\mathbb{R}$. Consider, for instance, the case of $s=0$. The corresponding submodule V^0 is one dimensional and it is easy to find its basis vector $\omega_{0,0}$ (which is annihilated by both $(\pi_S\otimes\pi_S)\Delta(E)$ and $(\pi_S\otimes\pi_S)\Delta(F)$),

$$\omega_{0,0}=\sum_{k=-S}^{S}\frac{(-1)^{S-k}\,q^{-k}}{\sqrt{2S+1}}\,\omega_k\otimes\omega_{-k}\,, \tag{63}$$

so that $\kappa_{0,0}=\frac{[2S+1]}{2S+1}$. Substituting $\omega_{0,0}$ in (62) and identifying $\omega_k\simeq e_{S+1-k}$, where e_k is a vector in $\mathbb{C}^{2S+1}$ such that $(e_k)_r=\delta_{kr}$, we obtain the following matrix form of $\mathsf{P}^{S,0}$,

$$\mathsf{P}^{S,0}=\sum_{m,n=1}^{2S+1}\frac{(-1)^{m+n}\,q^{m+n-2S-2}}{[2S+1]}\,E_{m,n}\otimes E_{2S+2-m,2S+2-n}\,, \tag{64}$$

where $E_{m,n}$ are matrices of size $2S{+}1$ such that $\big(E_{m,n}\big)_{kl}=\delta_{mk}\delta_{nl}$.

A.3 Minimal polynomial $\mathcal{P}^{S,0}_{a_1,a_2}$

For $\mathsf{H} = a_1 \mathsf{P}^{S,0}_{12} + a_2 \mathsf{P}^{S,0}_{23}$ we have

$$\mathsf{H}^2 = a_1^2 \mathsf{P}^{S,0}_{12} + a_2^2 \mathsf{P}^{S,0}_{23} + a_1 a_2 (\mathsf{P}^{S,0}_{12}\mathsf{P}^{S,0}_{23} + \mathsf{P}^{S,0}_{23}\mathsf{P}^{S,0}_{12})\,.$$

Multiplying this expression by H and using (25) we find

$$\mathsf{H}^3 = a_1^3 \mathsf{P}^{S,0}_{12} + a_2^3 \mathsf{P}^{S,0}_{23} + a_1 a_2 (a_1 + a_2)(\mathsf{P}^{S,0}_{12}\mathsf{P}^{S,0}_{23} + \mathsf{P}^{S,0}_{23}\mathsf{P}^{S,0}_{12}) + \mu_S\, a_1 a_2 \mathsf{H}\,.$$

Whence $\mathsf{H}^3 - (a_1+a_2)\,\mathsf{H}^2 = (\mu_S - 1)\, a_1 a_2 \mathsf{H}$. Thus, the minimal polynomial for H is (26).

A.4 Coefficients $d^{S,s}_k$ for minimal polynomials $\mathcal{P}^{S,s}_{a_1,a_2}$

Let us denote $[t] \equiv \frac{\sin\gamma t}{\sin\gamma}$ and $\{t\} \equiv 2\cos\gamma t$. The coefficients $d^{S,s}_k$ in (29) are given by

$$S{=}1,\ s{=}1\ :\quad d^{1,1}_1 = \frac{1}{\{2\}^2}\,,\quad d^{1,1}_2 = \Big(\frac{\{3\}}{\{1\}\{2\}}\Big)^2;$$

$$S{=}1,\ s{=}2\ :\quad d^{1,2}_1 = \frac{1}{\{2\}^2}\,,\quad d^{1,2}_2 = \Big(\frac{1}{\{2\}[3]}\Big)^2,\quad d^{1,2}_3 = 1;$$

$$S{=}\tfrac{3}{2},\ s{=}1\ :\quad d^{\frac{3}{2},1}_1 = \Big(\frac{[3]}{\{2\}[5]}\Big)^2,\quad d^{\frac{3}{2},1}_2 = \frac{1}{\{2\}^2}\,,\quad d^{\frac{3}{2},1}_3 = \Big(\frac{[2][6]-1}{[4][5]}\Big)^2;$$

$$S{=}\tfrac{3}{2},\ s{=}2\ :\quad d^{\frac{3}{2},2}_1 = \frac{1}{\{3\}^2}\,,\quad d^{\frac{3}{2},2}_2 = \frac{1}{\{2\}^2}\,,\quad d^{\frac{3}{2},2}_3 = \Big(\frac{\{5\}}{\{2\}\{3\}}\Big)^2,\quad d^{\frac{3}{2},2}_4 = \Big(\frac{[5]-2}{\{2\}\{3\}}\Big)^2;$$

$$S{=}\tfrac{3}{2},\ s{=}3\ :\quad d^{\frac{3}{2},3}_1 = \frac{1}{\{3\}^2}\,,\quad d^{\frac{3}{2},3}_2 = \Big(\frac{\{1\}}{\{3\}[5]}\Big)^2,\quad d^{\frac{3}{2},3}_3 = \Big(\frac{1}{\{2\}\{3\}[5]}\Big)^2,\quad d^{\frac{3}{2},3}_4 = 1.$$

The minimal positive solutions $\gamma_{S,s}$ of the equation $d^{S,s}_1 = 1$ are the following:

$$\gamma_{1,1} = \gamma_{1,2} = \frac{\pi}{6}\,,\qquad \gamma_{\frac{3}{2},1} = \frac{\pi}{7}\,,\qquad \gamma_{\frac{3}{2},2} = \gamma_{\frac{3}{2},3} = \frac{\pi}{9}\,. \tag{65}$$

Let us mention in passing an interesting pattern in the minimal positive solutions of the equation $d^{S,s}_k = 1$ for $s = 2S$: we have $\gamma^{\{1\}}_{1,2} = \frac{\pi}{6}$, $\gamma^{\{2\}}_{1,2} = \frac{\pi}{5}$, and $\gamma^{\{1\}}_{\frac{3}{2},3} = \frac{\pi}{9}$, $\gamma^{\{2\}}_{\frac{3}{2},3} = \frac{\pi}{8}$, $\gamma^{\{3\}}_{\frac{3}{2},3} = \frac{\pi}{7}$.

A.5 Universal R–matrix

Drinfeld has shown [D1] that relations (44) and (45) are satisfied for R^+ and $R^- \equiv \mathbb{P}(R^+)^{-1}\mathbb{P}$, where R^+ is given by

$$R^+ = q^{H\otimes H} \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}(n^2-n)}}{\prod_{k=1}^{n}[k]_q} \big((q-q^{-1})F\otimes E\big)^n\, q^{H\otimes H}\,. \tag{66}$$

Here H is related to K via $K = q^H$. Relations (44)–(45) imply the Yang–Baxter equation,

$$R^{\pm}_{12}\, R^{\pm}_{13}\, R^{\pm}_{23} = R^{\pm}_{23}\, R^{\pm}_{13}\, R^{\pm}_{12}\,. \tag{67}$$

Note that $R^+\big|_{q\to q^{-1}} = \big(R^+\big)^{-1}$. Therefore, for $|q| = 1$ we have

$$\big(R^+\big)^* = R^-\,. \tag{68}$$

A.6 Proof of Proposition 2

Let us introduce an operation $\Delta^{\pm} \equiv R^{\pm}\Delta$ and define its action on $X \in U_q(sl_2)^{\otimes N}$ by the following formula: $\Delta^{\pm}_{N,n}(X) \equiv R^{\pm}_{n,n+1}\Delta_{N,n}(X)$ (recall that $\Delta_{N,n}$ was defined after Eq. (23)).

***Lemma* 2.** *a) $\Delta^{\pm}$ is coassociative, i.e.*

$$\Delta^{\pm}_{2,1} \circ \Delta^{\pm} = \Delta^{\pm}_{2,2} \circ \Delta^{\pm} . \tag{69}$$

Therefore, a positive integer power of $\Delta^{\pm}$ can be defined in the same way as it is done for Δ, i.e.

$$\big(\Delta^{\pm}\big)^{(N)} = \Delta^{\pm}_{N,n} \circ \big(\Delta^{\pm}\big)^{(N-1)} . \tag{70}$$

The operations Δ^{+} and Δ^{-} are conjugate to each other in the following sense:

$$\big(\Delta^{+}(X)\big)^{*} = \Delta^{-}(X^{*}) , \tag{71}$$

for any $X \in U_q(sl_2)$.
b) The symmetrizing operators (47) can be equivalently represented as follows

$$\eta^{\pm}_{N+1} = \pi_S^{\otimes N+1}\Big(\Delta^{\pm}_{N,n}\left(\tilde{\eta}^{\pm}_{N}\right)\Big) = \pi_S^{\otimes(N+1)}\Big(\big(\Delta^{\pm}\big)^{(N)}(1)\Big) , \tag{72}$$

where $\tilde{\eta}^{\pm}_N$ are given by (47) with $R^{\pm}_{nm}$ instead of $\mathsf{R}^{\pm}_{nm}$, and $\tilde{\eta}^{\pm}_1 \equiv 1$.
In (70) and (72), n can be taken any from 1 to N.

Proof. a) The coassociativity of $\Delta^{\pm}$ follows from the coassociativity of Δ along with the Yang–Baxter equation:

$$\begin{aligned}
&\Delta^{\pm}_{2,1} \circ \Delta^{\pm}(X) = \Delta^{\pm}_{2,1}\big(R^{\pm}\Delta(X)\big) \overset{(45)}{=} R^{\pm}_{12}\, R^{\pm}_{13}\, R^{\pm}_{23}\, \Delta_{2,1}(X) \\
&\overset{(67)}{=} R^{\pm}_{23}\, R^{\pm}_{13}\, R^{\pm}_{12}\, \Delta_{2,2}(X) \overset{(44)}{=} \Delta^{\pm}_{2,2}\big(R^{\pm}\Delta(X)\big) = \Delta^{\pm}_{2,2} \circ \Delta^{\pm}(X) .
\end{aligned}$$

The property (71) is easily checked:

$$\big(\Delta^{+}(X)\big)^{*} = \big(R^{+}\Delta(X)\big)^{*} \overset{(68)}{=} \Delta'(X^{*})\, R^{-} \overset{(44)}{=} R^{-}\, \Delta(X^{*}) = \Delta^{-}(X^{*}) .$$

b) First, we will prove the first equality in (72) by an induction in the case of $n = N-1$. The base of the induction, for $N = 2$, holds by the definition of $\Delta^{\pm}$ and the relation $\Delta(1) = 1 \otimes 1$. The inductive step (which can be regarded as an extension of the lattice by an additional node) is checked as follows

$$\begin{aligned}
\eta^{\pm}_{N+1} &\overset{(47)}{=} \overleftarrow{\mathsf{R}}^{\pm}_{N+1}\, \overleftarrow{\mathsf{R}}^{\pm}_{N}\, \eta^{\pm}_{N-1} = \mathsf{R}^{\pm}_{N,N+1}\mathsf{R}^{\pm}_{N-1,N+1} \ldots \mathsf{R}^{\pm}_{1,N+1}\mathsf{R}^{\pm}_{N-1,N} \ldots \mathsf{R}^{\pm}_{1N}\, \eta^{\pm}_{N-1} \\
&= \mathsf{R}^{\pm}_{N,N+1}(\mathsf{R}^{\pm}_{N-1,N+1}\mathsf{R}^{\pm}_{N-1,N} \ldots \mathsf{R}^{\pm}_{n,N+1}\mathsf{R}^{\pm}_{n,N} \ldots \mathsf{R}^{\pm}_{1,N+1}\mathsf{R}^{\pm}_{1,N})\, \eta^{\pm}_{N-1} \\
&\overset{(45)}{=} \pi_S^{\otimes(N+1)}\big(R^{\pm}_{N,N+1}\Delta_{N,N}(R^{\pm}_{N-1,N} \ldots R^{\pm}_{1N})\, \tilde{\eta}^{\pm}_{N-1}\big) \\
&= \pi_S^{\otimes(N+1)}\big(R^{\pm}_{N,N+1}\Delta_{N,N}(\overleftarrow{R}^{\pm}_{N}\, \tilde{\eta}^{\pm}_{N-1})\big) \overset{(47)}{=} \pi_S^{\otimes(N+1)}\big(\Delta^{\pm}_{N,N}(\tilde{\eta}^{\pm}_{N})\big).
\end{aligned}$$

Whence $\eta^{\pm}_{N+1} = \pi_S^{\otimes(N+1)}\big(\Delta^{\pm}_{N,N} \circ \Delta^{\pm}_{N-1,N-1} \circ \cdots \circ \Delta^{\pm}_{1,1}(\tilde{\eta}^{\pm}_1)\big) \overset{(70)}{=} \pi_S^{\otimes(N+1)}\big((\Delta^{\pm})^{(N)}(1)\big)$. That is, we have proved the equality of $\eta^{\pm}_{N+1}$ to the last expression in (72). The latter in turn is equal to the middle expression in (72), because n in the definition (70) can be any from 1 to N. This completes the proof of the Lemma 1.

Proof of Proposition 2.

We commence by proving the part *b)*. Choosing $n=1$ in (72), we can write $\eta^{\pm}_{N+1}$ as follows: $\eta^{\pm}_{N+1}=\pi_S^{\otimes N+1}\big((\Delta^{\pm})^{(N)}(1)\big) = \pi_S^{\otimes N+1}\big(\Delta^{\pm}_{N,1}\circ\Delta^{\pm}_{N-1,1}\circ\cdots\circ\Delta^{\pm}_{1,1}(1)\big)$. Then expressions (48) can be obtained by an induction analogous to that was performed in the proof of Lemma 1 but this time one should use the first relation in (45).

Relation (49) in the part *c)* of Proposition 2 is an immediate consequence of applying relation (71) to formula (72).

To prove the part *a)* of Proposition 2, we show first that $\eta^{\pm}_N$ are symmetrizing operators for the tensor Casimir element:

$$\eta^{\pm}_N\,\mathsf{C}_{n,n+1} \overset{(72)}{=} \pi_S^{\otimes N}\big(R^{\pm}_{n,n+1}\Delta_{N-1,n}(\tilde\eta^{\pm}_{N-1}\,C_n)\big) = \pi_S^{\otimes N}\big(R^{\pm}_{n,n+1}\Delta_{N-1,n}(C_n\,\tilde\eta^{\pm}_{N-1})\big)$$
$$\overset{(72)}{=} \mathsf{R}^{\pm}_{n,n+1}\,\mathsf{C}_{n,n+1}\,\big(\mathsf{R}^{\pm}_{n,n+1}\big)^{-1}\,\eta^{\pm}_N \overset{(44)}{=} \mathsf{C}_{n+1,n}\,\eta^{\pm}_N\,.$$

Therefore $\eta^{\pm}_N$ are symmetrizing operators also for an arbitrary polynomial in $\mathsf{C}_{n,n+1}$ with real coefficients. Whence, taking formula (60) into account, we conclude that relation (43) holds. Thus, Proposition 2 is proven.

A.7 Proof of Lemma 1

The bialgebra defined by relations (16)–(17) turns into a Hopf algebra if the antipode $\mathcal{S}$ (an anti-homomorphism) is defined as follows: $\mathcal{S}(E)=-q^{-1}E$, $\mathcal{S}(F)=-qF$, $\mathcal{S}(K)=K^{-1}$.

The R–matrix (66) has the following form: $R^{+}=\sum_a r_a^{(1)}\otimes r_a^{(2)}$. Consider the element $\chi=K^2\left(\sum_a\mathcal{S}(r_a^{(2)})r_a^{(1)}\right)$. From the results of [D2], it follows that χ is a central element, which acquires the value $q^{-2S(S+1)}$ on an irreducible module V^S, and that χ satisfies the following relation:

$$\chi_1\,\chi_2\,\Delta(\chi^{-1}) = \big(R^{-}\big)^{-1}R^{+}\,. \tag{73}$$

Let us prove that

$$\chi_1\ldots\chi_N\,\Delta^{(N-1)}(\chi^{-1}) = \big(\tilde\eta^{-}_N\big)^{-1}\,\tilde\eta^{+}_N\,. \tag{74}$$

For $N=2$, this relation coincides with (73). For $N\geq 3$, it is verified by induction:

$$\chi_1\ldots\chi_{N+1}\,\Delta^{(N)}(\chi^{-1}) \overset{(73)}{=} \big(R^{-}_{12}\big)^{-1}R^{+}_{12}\,\Delta_{N,1}\big(\chi_1\ldots\chi_N\,\Delta^{(N-1)}(\chi^{-1})\big)$$
$$\overset{(74)}{=} \big(R^{-}_{12}\big)^{-1}R^{+}_{12}\,\Delta_{N,1}\big((\tilde\eta^{-}_N)^{-1}\big)\,\Delta_{N,1}\big(\tilde\eta^{+}_N\big) = \big(\Delta^{-}_{N,1}(\tilde\eta^{-}_N)\big)^{-1}\,\Delta^{+}_{N,1}(\tilde\eta^{+}_N) \overset{(72)}{=} \big(\tilde\eta^{-}_{N+1}\big)^{-1}\,\tilde\eta^{+}_{N+1}.$$

If q is not a root of unity, the center of the algebra $U_q(sl_2)$ is generated by the Casimir element (56). Therefore, there exists a function φ_q such that $\chi=\varphi_q(C)$. Consequently, the operator $\Delta^{(N-1)}(\chi)=\varphi(\Delta^{(N-1)}(C))$ acts in each irreducible submodule $V^s\subset(V^S)^{\otimes N}$ as multiplication by $q^{-2s(s+1)}$. This, along with formula (74), implies that

$$\big(\eta^{-}_N\big)^{-1}\eta^{+}_N = \sum_{s=s_\circ}^{NS} q^{2s(s+1)-2NS(S+1)}\,\mathcal{P}_s\,, \tag{75}$$

where $\mathcal{P}_s$ denotes the projector of rank $\nu_s(2s+1)$ onto the reducible invariant subspace $\oplus^{\nu_s} V^s \subset (V^S)^{\otimes N}$.

Using (75), we derive formula (52):

$$\det\big(e^{i\alpha}\eta_N^+ + e^{-i\alpha}\eta_N^-\big) = \det(\eta_N^-)\ \det\big(e^{i\alpha}(\eta_N^-)^{-1}\eta_N^+ + e^{-i\alpha}\,1\big)$$
$$\stackrel{(75)}{=} \det\Big(\sum_{s=s_\circ}^{NS}(e^{i\alpha}q^{2s(s+1)-2NS(S+1)} + e^{-i\alpha})\,\mathcal{P}_s\Big)$$
$$= \rho_{N,S}\prod_{s=s_\circ}^{SN}\Big(e^{i\alpha}q^{s(s+1)-NS(S+1)} + e^{-i\alpha}q^{NS(S+1)-s(s+1)}\Big)^{\nu_s(2s+1)},$$

where $\rho_{N,S} \equiv \prod_{s=s_\circ}^{SN} q^{\nu_s(2s+1)(s(s+1)-NS(S+1))} = 1$, which follows from (75) and the relation $\det\eta_N^\pm = 1$ (note that $\det \mathsf{R}^\pm = 1$).

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