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Global Analysis of General $SU(2) \times SU(2) \times U(1)$ Models with Precision Data

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Abstract

We present the results of a global analysis of a class of models with an extended electroweak gauge group of the form $SU(2) \times SU(2) \times U(1)$, often denoted as $G(221)$ models, which include as examples the left-right, the lepto-phobic, the hadro-phobic, the fermio-phobic, the un-unified, and the non-universal models. Using an effective Lagrangian approach, we compute the shifts to the coefficients in the electroweak Lagrangian due to the new heavy gauge bosons, and obtain the lower bounds on the masses of the Z' and W' bosons. The analysis of the electroweak parameter bounds reveals a consistent pattern of several key observables that are especially sensitive to the effects of new physics and thus dominate the overall shape of the respective parameter contours.

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I. INTRODUCTION

Despite the tremendous success of the Standard Model, there are still open questions that are unanswered and motivate further model-building. One of the most common model-building tools is to extend the gauge structure of the Standard Model. The simplest extension involves an additional $U(1)_X$ gauge symmetry (and thus an extra gauge boson Z'). One of the next-simplest extensions involves an additional $SU(2)$, with the left-right model [1][2][3] being perhaps the most widely-studied case of such models. On the other hand, given the extended gauge group $SU(2)_1 \times SU(2)_2 \times U(1)_X$ in the electroweak sector, there are many other models besides the left-right model that can be constructed, and these models, despite having a common fundamental gauge group, may have very different low-energy phenomenology. In this paper we present a unified, systematic study of many such models, which are commonly called $G(221)$ models in the literature.

The most important feature of $G(221)$ models is the existence of new heavy gauge bosons, W' and Z'. The existence of the gauge boson Z' has influences on the low-energy neutral-current processes, the Z-pole data at LEP-I and high energy LEP-II data [4][5]. The existence of the W' boson has implications to the search of new physics beyond the Standard Model (SM) via studying charged-current processes. In low energy experiments, the most sensitive probes of charged currents come from flavor physics, such as the $K\bar{K}$, $b\bar{b}$ mixing processes and semileptonic decays of the b quark [6][7]. However, the low energy impact depends sensitively on the details of the flavor sectors, for which there is little experimental input [8]. There is thus a large uncertainty on the constraints on W' and its interactions.

In this paper, we classify the $G(221)$ models by the patterns of symmetry breaking summarized in Table II (see section II). Our main goals are to obtain the bounds on the masses of the W' and Z' bosons for these various models, and, through the results of the global-fit analysis, to identify the key observables that are most sensitive to the new physics in these models. Our key results are that, at the 95% confidence level, the lower bounds on the masses of new heavy gauge bosons can be very light for breaking pattern I, which includes left-right, lepto-phobic, hadro-phobic and fermio-phobic models, for example, $M_{Z'} \sim 1.6$ TeV and $M_{W'} \sim 0.3$ TeV in the left-right model and hadro-phobic model; $M_{Z'} \sim 1.7$ TeV and $M_{W'} \sim 0.7$ TeV in the lepto-phobic and fermio-phobic models. In breaking pattern II, which includes un-unified and non-universial models, because of the degeneracy of the masses of the W' and Z', the lower bounds on their masses are quite heavy, for example, $M_{Z'} = M_{W'} \sim 2.5$ TeV in the un-unified model.

We organize this paper as follows. In Section II, we lay out the various $G(221)$ models and discuss the results of the relevant literature. In Section III, we give the effective Lagrangians, both at the electroweak scale (obtained by integrating out W' and Z') and below the electroweak scale (by integrating out the W and Z). In Section IV, we discuss the global-fit procedure and present our results obtained using the code *Global Analysis for Particle Properties* (GAPP) [9], a software that utilizes the CERN library MINUIT [10] and was used for the Particle Data Group global analysis [28]. We also discuss which observables are especially sensitive to the new physics contributions in these various models. We conclude in Section VI with a summary and outlook of our key findings. The Appendix contains the explicit effective Lagrangians for the $G(221)$ models.

II. THE $G(221)$ MODELS

We focus on the so-called $G(221)$ models having a $SU(2)_1 \times SU(2)_2 \times U(1)_X$ gauge structure that ultimately breaks to $U(1)_{\rm em}$. Relative to the Standard Model, these models have three additional massive gauge bosons, and their phenomenology depends on the specific patterns of symmetry breaking as well as the charge assignments of the SM fermions. For our studies, we consider the following different $G(221)$ models: left-right (LR) [1][2][3], lepto-phobic (LP), hadro-phobic (HP), fermio-phobic (FP) [11][12][13], un-unified (UU) [14][15], and non-universal (NU) [16][17][18]. The charge assignments of the SM fermions in these models are given in Table I, and these models can be categorized by two patterns of symmetry breaking (summarized in Table II):

- Breaking pattern I (the LR, LP, HP, and FP models):
  We identify $SU(2)_1$ as $SU(2)_L$ of the SM. The first stage of symmetry breaking then is $SU(2)_2 \times U(1)_X \to U(1)_Y$, giving rise to three heavy gauge bosons $W'^{\pm}$ and Z' at the TeV-scale. The second stage is $SU(2)_L \times U(1)_Y \to U(1)_{\rm em}$ at the electroweak scale.

- Breaking pattern II (the UU and NU models):
  We identify $U(1)_X$ as $U(1)_Y$ of the SM. The first stage of symmetry breaking is $SU(2)_1 \times SU(2)_2 \to SU(2)_L$. The second stage is $SU(2)_L \times U(1)_Y \to U(1)_{\rm em}$ at the electroweak scale.

In addition to specifying the gauge group and the fermion charge assignments, a complete $G(221)$ model should also include the ingredients of the Higgs sectors and the Yukawa couplings. While the observed relationships between the masses of W and Z bosons leave little freedom in the Higgs representation used for electroweak symmetry breaking (EWSB), we have freedoms in the choices of the Higgs representation used to break the fundamental $G(221)$ gauge group to the SM gauge group. In breaking pattern I we assume the two simplest cases of symmetry breaking: via a doublet or a triplet Higgs. In the breaking pattern II we assume the simplest case of using a bi-doublet Higgs to achieve this symmetry breaking. The model-specific Higgs representations and vacuum expectation values (VEV's) are given in Table III. For heavy Higgs boson, Wang *et al.* [19] used a non-linear effective theory approach to obtain an electroweak chiral Lagrangian for W'. In our paper, by assuming a

TABLE I: The charge assignments of the SM fermions under the $G(221)$ gauge groups. Unless otherwise specified, the charge assignments apply to all three generations.

Model	$SU(2)_1$	$SU(2)_2$	$U(1)_X$
Left-right (LR)	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$\begin{pmatrix} u_R \\ d_R \end{pmatrix}, \begin{pmatrix} \nu_R \\ e_R \end{pmatrix}$	$\frac{1}{6}$ for quarks, $-\frac{1}{2}$ for leptons.
Lepto-phobic (LP)	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$\begin{pmatrix} u_R \\ d_R \end{pmatrix}$	$\frac{1}{6}$ for quarks, $Y_{\rm SM}$ for leptons.
Hadro-phobic (HP)	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$\begin{pmatrix} \nu_R \\ e_R \end{pmatrix}$	$Y_{\rm SM}$ for quarks, $-\frac{1}{2}$ for leptons.
Fermio-phobic (FP)	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$		$Y_{\rm SM}$ for all fermions.
Un-unified (UU)	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$Y_{\rm SM}$ for all fermions.
Non-universal (NU)	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}_{1^{\rm st},2^{\rm nd}}, \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}_{1^{\rm st},2^{\rm nd}}$	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}_{3^{\rm rd}}, \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}_{3^{\rm rd}}$	$Y_{\rm SM}$ for all fermions.

TABLE II: Summary of the two different breaking patterns and the two different stages of symmetry breaking in $G(221)$ models.

Pattern	Starting Point	First stage breaking	Second stage breaking
I	Identify $SU(2)_1$ as $SU(2)_L$	$SU(2)_2 \times U(1)_X \to U(1)_Y$	$SU(2)_L \times U(1)_Y \to U(1)_{em}$
II	Identify $U(1)_X$ as $U(1)_Y$	$SU(2)_1 \times SU(2)_2 \to SU(2)_L$	$SU(2)_L \times U(1)_Y \to U(1)_{em}$

light Higgs, we analyze the low-energy constraints by using a linearlized effective Lagrangian approach.

The lepto-phobic, hadro-phobic, and un-unified models are, with the current set-up, incomplete because of gauge anomalies. It is entirely possible that the additional matter content used to address the anomalies can alter the low-energy phenomenologies and the results of our studies. Nevertheless, for completeness, we include these models with the current set-up in our studies, in which we focus on effects originated from the interactions of W' and Z' bosons to the SM fields. In the cases of the lepto-phobic and hadro-phobic models, one can view them as transitions between the left-right models (where both right-handed leptons and quarks are charged under $SU(2)_2$) and the fermio-phobic model (where neither are charged).

There have already been many theoretical and phenomenological studies of various $G(221)$ models, and we focus our brief literature review here mainly to those works that perform a

First stage breaking		
	Rep.	Multiplet and VEV
LR-D, LP-D HP-D, FP-D	$\Phi \sim (1,2,\frac{1}{2})$	$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \ \langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \tilde{u}_{\rm D} \end{pmatrix}$
LR-T, LP-T HP-T, FP-T	$\Phi \sim (1,3,1)$	$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi^+ & \sqrt{2}\phi^{++} \\ \sqrt{2}\phi^0 & -\phi^+ \end{pmatrix}, \ \langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ \tilde{u}_{\rm T} & 0 \end{pmatrix}$
UU, NU	$\Phi \sim (2,\overline{2},0)$	$\Phi = \begin{pmatrix} \phi^0+\pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & \phi^0-\pi^0 \end{pmatrix}, \ \langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \tilde{u} & 0 \\ 0 & \tilde{u} \end{pmatrix}$

Second stage breaking		
	Rep.	Multiplet and VEV
LR-D, LP-D HP-D, FP-D	$H \sim (2,\overline{2},0)$	$H = \begin{pmatrix} h_1^0 & h_1^+ \\ h_2^- & h_2^0 \end{pmatrix}, \langle H\rangle = \frac{\tilde{v}}{\sqrt{2}}\begin{pmatrix} c_{\tilde{\beta}} & 0 \\ 0 & s_{\tilde{\beta}} \end{pmatrix}$
LR-T, LP-T HP-T, FP-T	$H \sim (2,\overline{2},0)$	$H = \begin{pmatrix} h_1^0 & h_1^+ \\ h_2^- & h_2^0 \end{pmatrix}, \langle H\rangle = \frac{\tilde{v}}{\sqrt{2}}\begin{pmatrix} c_{\tilde{\beta}} & 0 \\ 0 & s_{\tilde{\beta}} \end{pmatrix}$
UU, NU	$H \sim (1,2,\frac{1}{2})$	$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}, \langle H\rangle = \frac{\tilde{v}}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

TABLE III: These tables display the model-specific Higgs representations and VEVs that achieve the symmetry breaking of $G(221)$ models.

global fitting in the same spirit as our work. In the symmetric left-right model (where the couplings of the W' are of the same strength as those of the W), Polak and Zralek obtained the constraints on parameters from the Z-pole data [20] and low energy data [21], separately. While for the non-symmetric case, Chay, Lee and Nam [22] considered phenomenological constraints on three parameters: the mass of the Z', the mixing angles $\tilde{\phi}$ (the analog of the Weinberg angle in the breaking of $SU(2)_R \times U(1)_X \to U(1)_Y$) and the Z-Z' mixing angle ξ, by combining the precision electroweak data from LEP I (through $\epsilon_1, \epsilon_2, \epsilon_3$) and the low-energy neutral-current experimental data. For the non-symmetric case, the combined bounds at the 95% confidence level are $0.0028 < \xi < 0.0065$ and $M_{Z'} \geq 400$ GeV for all $\tilde{\phi}$, while for the symmetric case, a more severe bound $M_{Z'} \geq 1.6$ TeV is obtained.

In the fermio-phobic model, Donini *et al.* [23] used the Z-pole and low-energy data, and the flavor physics data from flavor-changing neutral-current (FCNC) processes and $b \to s\gamma$, to put constraints on the parameter space (W-W' mixing angle $\alpha_\pm$, and Z-Z' mixing angle α_0) by fixing several sets of representative values of $M_{W'}$ and x (strength of the coupling of the fermiophobic gauge group, relative to $SU(2)_L$ of the Standard Model). For the input parameters in the the ranges 100 GeV $< M_{W'} <$ 1000 GeV and $0.6 < x < 15$, and for a low Higgs mass of 100 GeV, the best-fit values of $|\alpha_0|$ and $|\alpha_\pm|$ increases with increasing x, when holding $M_{W'}$ fixed. On the other hand, when holding x fixed, increasing $M_{W'}$ leads to an increase in the best-fit values of $|\alpha_0|$ and a decrease in the best-fit values of $|\alpha_\pm|$. In the entire range of parameter space, the magnitude of the best-fit values of α_0 and $\alpha_\pm$ are at the percent level.

In the non-unified model, Malkawi and Yuan [16] performed a global fit of the parameter space (x, ϕ) using the Z-pole data, and found that the lower bound is $M_{Z'} = M_{W'} \geq 1.3$ TeV if no flavor physics is included. Chivukula et. al [24] used the data from precision electroweak measurements to put stringent bounds on the un-unified Standard Model [14] [15]. They found a lower bound on the masses of the heavy W' and Z' of approximately 2 TeV at the 95% confidence level.

III. THE EFFECTIVE LAGRANGIAN APPROACH

To analyze the low-energy constraints, we will take an effective Lagrangian approach, and follow the general procedures laid out by Burgess in Ref. [25] to extract the effects of new physics. Although the details of each of these models are different, we first perform a generic analysis that can be applied to any $G(221)$ model we consider in this work.

Per the convention in Burgess [25], we denote the gauge couplings as $\tilde{g}_1$, $\tilde{g}_2$, and $\tilde{g}_X$ respectively for the gauge groups $SU(2)_1$, $SU(2)_2$, and $U(1)_X$. The tilde (˜) on the couplings and VEVs emphasizes the fact that these are model parameters, as opposed to quantities that can be directly measured in experiments, such as the physical mass of the Z boson. As an extension to the convention in Burgess [25], we also denote with tilde (˜) any combination constructed from the model parameters. We also abbreviate the trigonometric functions

$$c_x \equiv \cos(x), \;\; s_x \equiv \sin(x), \;\; \text{and } t_x \equiv \tan(x). \tag{1}$$

A. Mixing Angles and Gauge Couplings

We define the mixing angle $\tilde{\phi}$ at the first breaking stage as

$$t_{\tilde{\phi}} \left(= \tan\tilde{\phi}\right) \equiv \begin{cases} \tilde{g}_X/\tilde{g}_2 & \text{(LR, LP, HP, FP models)} \\ \tilde{g}_2/\tilde{g}_1 & \text{(UU, NU models)}, \end{cases} \tag{2}$$

and define the couplings

$$\tilde{g}_L \equiv \begin{cases} \tilde{g}_1, & \text{(LR, LP, HP, FP models)} \\ \left(\dfrac{1}{\tilde{g}_1^2} + \dfrac{1}{\tilde{g}_2^2}\right)^{-1/2} & \text{(UU, NU models)}, \end{cases}$$
$$\tilde{g}_Y \equiv \begin{cases} \left(\dfrac{1}{\tilde{g}_2^2} + \dfrac{1}{\tilde{g}_X^2}\right)^{-1/2} & \text{(LR, LP, HP, FP models)} \\ \tilde{g}_X, & \text{(UU, NU models)}. \end{cases} \tag{3}$$

The couplings $\tilde{g}_L$ and $\tilde{g}_Y$ are respectively the gauge couplings of the unbroken $SU(2)_L \times U(1)_Y$ gauge groups after the first stage of symmetry breaking. Similarly to the Standard Model, for both breaking patterns we define the weak mixing angle ($\tilde{\theta}$) as

$$t_{\tilde{\theta}} \left(= \tan \tilde{\theta}\right) \equiv \frac{\tilde{g}_Y}{\tilde{g}_L}. \tag{4}$$

For both breaking patterns, the electric charge ($\tilde{e}$) is given by

$$\frac{1}{\tilde{e}^2} = \frac{1}{\tilde{g}_1^2} + \frac{1}{\tilde{g}_2^2} + \frac{1}{\tilde{g}_X^2}, \tag{5}$$

and we also define $\tilde{\alpha}_e \equiv \tilde{e}^2/4\pi$.

With the angles $\tilde{\theta}$ and $\tilde{\phi}$, the gauge couplings can be expressed as

$$\tilde{g}_1 = \begin{cases} \tilde{e}/(s_{\tilde{\theta}}), & \text{(LR, LP, HP, FP models)} \\ \tilde{e}/(s_{\tilde{\theta}} s_{\tilde{\phi}}), & \text{(UU, NU models)} \end{cases} \tag{6}$$

$$\tilde{g}_2 = \begin{cases} \tilde{e}/(c_{\tilde{\theta}} s_{\tilde{\phi}}), & \text{(LR, LP, HP, FP models)} \\ \tilde{e}/(s_{\tilde{\theta}} c_{\tilde{\phi}}), & \text{(UU, NU models)} \end{cases} \tag{7}$$

$$\tilde{g}_X = \begin{cases} \tilde{e}/(c_{\tilde{\theta}} c_{\tilde{\phi}}), & \text{(LR, LP, HP, FP models)} \\ \tilde{e}/(c_{\tilde{\theta}}). & \text{(UU, NU models)} \end{cases} \tag{8}$$

B. The Effective Lagrangian

1. *Gauge Interactions of Fermions*

In this sub-section we parameterize the gauge interactions of the fermions that is applicable to all the $G(221)$ models under considerations here. We will obtain both the SM-like effective theory applicable at the electroweak scale as well as the four-fermion effective theory below the electroweak scale. We do this by first building up the fundamental Lagrangian in stages, and then successively integrating out the massive gauge bosons. The Z-pole data measured at the electroweak scale, and measurements of the four-fermion neutral-current interactions are some of the most precise measurements to-date, and provide stringent bounds on new physics models.

As discussed earlier, we consider the symmetry breaking to take two stages:

$$SU(2)_1 \times SU(2)_2 \times U(1)_X \to SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}. \tag{9}$$

We denote the gauge bosons of the $G(221)$ models as:

$$\begin{aligned} SU(2)_1 &: W^{\pm}_{1,\mu}, W^3_{1,\mu}, \\ SU(2)_2 &: W^{\pm}_{2,\mu}, W^3_{2,\mu}, \\ U(1)_X &: X_\mu. \end{aligned} \tag{10}$$

After the first-stage breaking, the neutral gauge eigenstates mix as follows

$$\hat{B}_\mu \equiv \begin{cases} s_{\tilde{\phi}} W^3_{2,\mu} + c_{\tilde{\phi}} X_\mu & \text{(LR, LP, HP, FP models)} \\ X_\mu & \text{(UU, NU models)} \end{cases}$$

$$\hat{W}^3_\mu \equiv \begin{cases} W^3_{1,\mu} & \text{(LR, LP, HP, FP models)} \\ s_{\tilde{\phi}} W^3_{1,\mu} + c_{\tilde{\phi}} W^3_{2,\mu} & \text{(UU, NU models)} \end{cases}$$

$$\hat{Z}'_\mu \equiv \begin{cases} c_{\tilde{\phi}} W^3_{2,\mu} - s_{\tilde{\phi}} X_\mu & \text{(LR, LP, HP, FP models)} \\ c_{\tilde{\phi}} W^3_{1,\mu} - s_{\tilde{\phi}} W^3_{2,\mu}, & \text{(UU, NU models)} \end{cases} \quad (11)$$

and for the charged gauge bosons, we have

$$\hat{W}^\pm_\mu \equiv \begin{cases} W^\pm_{1,\mu} & \text{(LR, LP, HP, FP models)} \\ s_{\tilde{\phi}} W^\pm_{1,\mu} + c_{\tilde{\phi}} W^\pm_{2,\mu}, & \text{(UU, NU models)} \end{cases}$$

$$\hat{W}'^\pm_\mu \equiv \begin{cases} W^\pm_{2,\mu}, & \text{(LR, LP, HP, FP models)} \\ c_{\tilde{\phi}} W^\pm_{1,\mu} - s_{\tilde{\phi}} W^\pm_{2,\mu}. & \text{(UU, NU models)} \end{cases} \quad (12)$$

After the first stage of symmetry breaking, there is still an unbroken $SU(2)_L \times U(1)_Y$, which may be identified as the Standard Model gauge group. The gauge bosons $\hat{W}^{\pm,3}$ and $\hat{B}$ are massless, and only $\hat{Z}'$ and $\hat{W}'^\pm$ are massive, with TeV-scale masses. The Lagrangian representing the heavy gauge boson masses has the form

$$\mathcal{L}^{\text{stage-1}} = \frac{1}{2} \widetilde{M}^2_{Z'} \hat{Z}'_\mu \hat{Z}'^\mu + \widetilde{M}^2_{W'} \hat{W}'^+_\mu \hat{W}'^{-\mu}, \quad (13)$$

where $\widetilde{M}^2_{Z'}$ and $\widetilde{M}^2_{W'}$ are given in Table VI.

Before discussing the second stage of symmetry breaking, it is convenient to define, similarly to the Standard Model, A_μ (which will turn out to be the photon) and $\hat{Z}_\mu$ (approximately the physical Z-boson) in terms of the massless gauge bosons $\hat{W}^3_\mu$ and $\hat{B}_\mu$

$$\begin{aligned} A_\mu &\equiv \left(\frac{\tilde{e}}{\tilde{g}_1} W^3_{1,\mu} + \frac{\tilde{e}}{\tilde{g}_2} W^3_{2,\mu} + \frac{\tilde{e}}{\tilde{g}_X} \hat{X}_\mu \right), \\ &= s_{\tilde{\theta}} \hat{W}^3_\mu + c_{\tilde{\theta}} \hat{B}_\mu, \\ \hat{Z}_\mu &\equiv c_{\tilde{\theta}} \hat{W}^3_\mu - s_{\tilde{\theta}} \hat{B}_\mu, \end{aligned} \quad (14)$$

At the electroweak scale, the second stage of symmetry breaking takes place, breaking $SU(2) \times U(1) \to U(1)_{\text{em}}$. The Higgs vacuum expectation value (VEV) at the second stage not only gives masses to $\hat{Z}$ and $\hat{W}^\pm$, but also induces further mixing among the gauge bosons $\hat{W}^\pm$, $\hat{Z}$, $\hat{W}'$ and $\hat{Z}'$. The masses of the gauge bosons depend not only on the breaking pattern, but also on the group representations of the Higgs bosons whose VEV's trigger the symmetry breaking. For simplicity, for breaking pattern I, we consider only models with either a doublet or triplet under $SU(2)_2$, and do not consider models with *both* doublets and triplets. Introducing additional Higgses and VEVs would modify the masses of the W' and Z' [26]. For breaking pattern II, since the first stage of symmetry breaking breaks $SU(2)_1 \times SU(2)_2$ to the diagonal subgroup, the masses of W' and Z' are degenerate at this stage, and we only consider the case of an $SU(2)_1 \times SU(2)_2$ bi-doublet. For the convenience of typesetting, we also denote, for example, a left-right model with first-stage symmetry breaking triggered by an $SU(2)$-doublet(-triplet) as LR-D (LR-T).

Although different breaking patterns and different group representations of the Higgs bosons will lead to different Lagrangians, we can write down the Lagrangian involving the gauge boson masses and fermionic gauge interactions in a general form

$$\begin{aligned}\mathcal{L}_{\rm fund} = &\frac{1}{2}\widetilde{M}_Z^2 \hat{Z}_\mu \hat{Z}^\mu + \frac{1}{2}(\widetilde{M}_{Z'}^2 + \Delta\widetilde{M}_{Z'}^2)\hat{Z}'_\mu \hat{Z}'^\mu + \delta\widetilde{M}_Z^2 \hat{Z}'_\mu \hat{Z}^\mu \\ &+ \widetilde{M}_W^2 \hat{W}_\mu^+ \hat{W}^{-\mu} + (\widetilde{M}_{W'}^2 + \Delta\widetilde{M}_{W'}^2)\hat{W}'^+_\mu \hat{W}'^{-\mu} + \delta\widetilde{M}_W^2 (\hat{W}'^+_\mu \hat{W}^{-\mu} + \hat{W}'^-_\mu \hat{W}^{+\mu}) \\ &+ \hat{W}'^+_\mu K^{+\mu} + \hat{W}'^-_\mu K^{-\mu} + \hat{Z}'_\mu K^{0\mu} \\ &+ \hat{W}^+_\mu J^{+\mu} + \hat{W}^-_\mu J^{-\mu} + \hat{Z}_\mu J^{0\mu} + A_\mu J^\mu, \end{aligned} \tag{15}$$

where we have denoted the currents that couple to the primed gauge bosons ($\hat{W}'$ and $\hat{Z}'$) as K^0_μ and $K^\pm_\mu$, and the currents that couple to the SM gauge bosons as J_μ, J^0_μ and $J^\pm_\mu$. The SM-like currents have the familiar forms

$$J^\mu = \tilde{e} \sum_f Q^f \overline{f} \gamma^\mu f, \tag{16}$$

$$J^0_\mu = \sqrt{\tilde{g}_L^2 + \tilde{g}_Y^2} \sum_f \left(T^f_{3L} \overline{f}_{\rm L} \gamma_\mu P_{\rm L} f_{\rm L} - s^2_{\tilde{\theta}} Q^f \ \overline{f} \gamma_\mu f \right), \tag{17}$$

$$J^+_\mu = \frac{\tilde{g}_L}{\sqrt{2}} \left(\overline{d}_{\rm L} \gamma_\mu P_{\rm L} u_{\rm L} + \overline{e}_{\rm L} \gamma_\mu P_{\rm L} \nu_{\rm L} \right), \tag{18}$$

with an implicit sum over the three generations of fermions. The neutral currents (K^0_μ) and charged currents ($K^\pm_\mu$), for the various models are summarized in Tables IV and V. We note the following features:

- The residual $SU(2)_L \times U(1)_Y$ is broken to the $U(1)_{\rm em}$, and there are now mass terms for the $\hat{Z}$ and $\hat{W}$ bosons, denoted as $\widetilde{M}^2_{Z,W}$. These masses have the familiar form

$$\widetilde{M}_Z^2 = \frac{1}{4}(\tilde{g}_L^2 + \tilde{g}_Y^2)\tilde{v}^2, \tag{19}$$

$$\widetilde{M}_W^2 = \frac{1}{4}\tilde{g}_L^2 \tilde{v}^2, \tag{20}$$

  where the couplings $\tilde{g}_L$ and $\tilde{g}_Y$ are defined as in Eq. (3) for the two different breaking patterns.

- There are mass-mixing contributions $\delta\widetilde{M}^2_{Z,W}$ that induce $\hat{Z} - \hat{Z}'$ and $\hat{W} - \hat{W}'$ mixing. They are dependent on the breaking pattern and are given in Table VI.

	$\overline{u}\gamma^\mu u$	$\overline{d}\gamma^\mu d$	$\overline{\nu}\gamma^\mu \nu$	$\overline{e}\gamma^\mu e$
LR	$(\frac{1}{2}c_{\tilde{\phi}}\tilde{g}_2 - \frac{1}{6}s_{\tilde{\phi}}\tilde{g}_X)P_{\rm R}$ $-\frac{1}{6}s_{\tilde{\phi}}\tilde{g}_X P_{\rm L}$	$(-\frac{1}{2}c_{\tilde{\phi}}\tilde{g}_2 - \frac{1}{6}s_{\tilde{\phi}}\tilde{g}_X)P_{\rm R}$ $-\frac{1}{6}s_{\tilde{\phi}}\tilde{g}_X P_{\rm L}$	$(\frac{1}{2}c_{\tilde{\phi}}\tilde{g}_2 + \frac{1}{2}s_{\tilde{\phi}}\tilde{g}_X)P_{\rm R}$ $+\frac{1}{2}s_{\tilde{\phi}}\tilde{g}_X P_{\rm L}$	$(-\frac{1}{2}c_{\tilde{\phi}}\tilde{g}_2 + \frac{1}{2}s_{\tilde{\phi}}\tilde{g}_X)P_{\rm R}$ $+\frac{1}{2}s_{\tilde{\phi}}\tilde{g}_X P_{\rm L}$
LP	$(\frac{1}{2}c_{\tilde{\phi}}\tilde{g}_2 - \frac{1}{6}s_{\tilde{\phi}}\tilde{g}_X)P_{\rm R}$ $-\frac{1}{6}s_{\tilde{\phi}}\tilde{g}_X P_{\rm L}$	$(-\frac{1}{2}c_{\tilde{\phi}}\tilde{g}_2 - \frac{1}{6}s_{\tilde{\phi}}\tilde{g}_X)P_{\rm R}$ $-\frac{1}{6}s_{\tilde{\phi}}\tilde{g}_X P_{\rm L}$	$\frac{1}{2}s_{\tilde{\phi}}\tilde{g}_X P_{\rm L}$	$s_{\tilde{\phi}}\tilde{g}_X(\frac{1}{2}P_{\rm L} + P_{\rm R})$
HP	$-s_{\tilde{\phi}}\tilde{g}_X(\frac{1}{6}P_{\rm L} + \frac{2}{3}P_{\rm R})$	$-s_{\tilde{\phi}}\tilde{g}_X(\frac{1}{6}P_{\rm L} - \frac{1}{3}P_{\rm R})$	$(\frac{1}{2}c_{\tilde{\phi}}\tilde{g}_2 + \frac{1}{2}s_{\tilde{\phi}}\tilde{g}_X)P_{\rm R}$ $+\frac{1}{2}s_{\tilde{\phi}}\tilde{g}_X P_{\rm L}$	$(-\frac{1}{2}c_{\tilde{\phi}}\tilde{g}_2 + \frac{1}{2}s_{\tilde{\phi}}\tilde{g}_X)P_{\rm R}$ $+\frac{1}{2}s_{\tilde{\phi}}\tilde{g}_X P_{\rm L}$
FP	$-s_{\tilde{\phi}}\tilde{g}_X(\frac{1}{6}P_{\rm L} + \frac{2}{3}P_{\rm R})$	$-s_{\tilde{\phi}}\tilde{g}_X(\frac{1}{6}P_{\rm L} - \frac{1}{3}P_{\rm R})$	$\frac{1}{2}s_{\tilde{\phi}}\tilde{g}_X P_{\rm L}$	$s_{\tilde{\phi}}\tilde{g}_X(\frac{1}{2}P_{\rm L} + P_{\rm R})$
UU	$\frac{1}{2}c_{\tilde{\phi}}\tilde{g}_1 P_{\rm L}$	$-\frac{1}{2}c_{\tilde{\phi}}\tilde{g}_1 P_{\rm L}$	$-\frac{1}{2}s_{\tilde{\phi}}\tilde{g}_2 P_{\rm L}$	$\frac{1}{2}s_{\tilde{\phi}}\tilde{g}_2 P_{\rm L}$
NU	$\frac{1}{2}\begin{pmatrix} c_{\tilde{\phi}}\tilde{g}_1 \\ -s_{\tilde{\phi}}\tilde{g}_2 \end{pmatrix} P_{\rm L}$	$-\frac{1}{2}\begin{pmatrix} c_{\tilde{\phi}}\tilde{g}_1 \\ -s_{\tilde{\phi}}\tilde{g}_2 \end{pmatrix} P_{\rm L}$	$\frac{1}{2}\begin{pmatrix} c_{\tilde{\phi}}\tilde{g}_1 \\ -s_{\tilde{\phi}}\tilde{g}_2 \end{pmatrix} P_{\rm L}$	$-\frac{1}{2}\begin{pmatrix} c_{\tilde{\phi}}\tilde{g}_1 \\ -s_{\tilde{\phi}}\tilde{g}_2 \end{pmatrix} P_{\rm L}$

TABLE IV: This table displays the couplings $\tilde{g}(\overline{f}, f, \hat{Z}')$ of the current $K^{0\mu} = \overline{f}\gamma^\mu \tilde{g}(\overline{f}, f, \hat{Z}')f$. For the top four models (LR, LP, HP, and FP), $\tan\phi \equiv \tilde{g}_X/\tilde{g}_2$. For the lower two models (UU and NU), $\tan\phi \equiv \tilde{g}_2/\tilde{g}_1$. For the NU model (last row), the top values denote the couplings to the first two generations of fermions, and the bottom values denote the couplings to the third generation.

	$\overline{d}\gamma^\mu u$	$\overline{e}\gamma^\mu \nu$
LR	$\frac{1}{\sqrt{2}}\tilde{g}_2 P_{\rm R}$	$\frac{1}{\sqrt{2}}\tilde{g}_2 P_{\rm R}$
LP	$\frac{1}{\sqrt{2}}\tilde{g}_2 P_{\rm R}$	0
HP	0	$\frac{1}{\sqrt{2}}\tilde{g}_2 P_{\rm R}$
FP	0	0
UU	$\frac{1}{\sqrt{2}}c_{\tilde{\phi}}\tilde{g}_1 P_{\rm L}$	$-\frac{1}{\sqrt{2}}s_{\tilde{\phi}}\tilde{g}_2 P_{\rm L}$
NU	$\frac{1}{\sqrt{2}}\begin{pmatrix} c_{\tilde{\phi}}\tilde{g}_1 \\ -s_{\tilde{\phi}}\tilde{g}_2 \end{pmatrix} P_{\rm L}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} c_{\tilde{\phi}}\tilde{g}_1 \\ -s_{\tilde{\phi}}\tilde{g}_2 \end{pmatrix} P_{\rm L}$

TABLE V: This table displays the couplings $\tilde{g}(\overline{\psi}, \xi, \hat{W}'^{+})$ of the current $K^{+\mu} = \overline{\psi}\gamma^\mu \tilde{g}(\overline{\psi}, \xi, \hat{W}'^{+})\xi$. For the top four models (LR, LP, HP, and FP), $\tan\phi \equiv \tilde{g}_X/\tilde{g}_2$. For the lower two models (UU and NU), $\tan\phi \equiv \tilde{g}_2/\tilde{g}_1$. For the NU model (last row), the top values denote the couplings to the first two generations of fermions, and the bottom values denote the couplings to the third generation.

- There are additional contributions to the masses of the $\hat{Z}'$ and $\hat{W}'$ after the second stage of symmetry breaking, which we denote as $\Delta\widetilde{M}^2_{Z',W'}$. They are also dependent on the breaking pattern and are given in Table VI.

	$\widetilde{M}^2_{Z'}$	$\widetilde{M}^2_{W'}$	$\Delta\widetilde{M}^2_{Z'}$	$\Delta\widetilde{M}^2_{W'}$	$\delta\widetilde{M}^2_{Z}$	$\delta\widetilde{M}^2_{W}$
LR-D, LP-D HP-D, FP-D	$\frac{1}{4}(\tilde{g}_2^2+\tilde{g}_X^2)\tilde{u}_{\rm D}^2$	$\frac{1}{4}\tilde{g}_2^2\tilde{u}_{\rm D}^2$	$\frac{c^2_{\tilde{\phi}}}{4}\tilde{g}_2^2\tilde{v}^2$	$\frac{1}{4}\tilde{g}_2^2\tilde{v}^2$	$-\frac{c^2_{\tilde{\phi}}}{4\tilde{e}}\tilde{g}_1\tilde{g}_2\tilde{g}_X\tilde{v}^2$	$-\frac{1}{4}\tilde{g}_1\tilde{g}_2\tilde{v}^2 s_{2\tilde{\beta}}$
LR-T, LP-T HP-T, FP-T	$(\tilde{g}_2^2+\tilde{g}_X^2)\tilde{u}_{\rm T}^2$	$\frac{1}{2}\tilde{g}_2^2\tilde{u}_{\rm T}^2$	$\frac{c^2_{\tilde{\phi}}}{4}\tilde{g}_2^2\tilde{v}^2$	$\frac{1}{4}\tilde{g}_2^2\tilde{v}^2$	$-\frac{c^2_{\tilde{\phi}}}{4\tilde{e}}\tilde{g}_1\tilde{g}_2\tilde{g}_X\tilde{v}^2$	$-\frac{1}{4}\tilde{g}_1\tilde{g}_2\tilde{v}^2 s_{2\tilde{\beta}}$
UU, NU	$\frac{1}{4}(\tilde{g}_1^2+\tilde{g}_2^2)\tilde{u}^2$	$\frac{1}{4}(\tilde{g}_1^2+\tilde{g}_2^2)\tilde{u}^2$	$\frac{s^2_{\tilde{\phi}}}{4}\tilde{g}_2^2\tilde{v}^2$	$\frac{s^2_{\tilde{\phi}}}{4}\tilde{g}_2^2\tilde{v}^2$	$-\frac{s^2_{\tilde{\phi}}}{4\tilde{e}}\tilde{g}_1\tilde{g}_2\tilde{g}_X\tilde{v}^2$	$-\frac{1}{4}\tilde{g}_1\tilde{g}_2\tilde{v}^2 s^2_{\tilde{\phi}}$

TABLE VI: This table displays the model-dependent parameters $\widetilde{M}^2_{Z',W'}$ in Eq. (13), and $\Delta\widetilde{M}^2_{Z',W'}$ and $\delta\widetilde{M}^2_{Z,W}$ in Eq. (15).

Therefore, the gauge boson mass terms can be written as

$$\begin{aligned}\mathcal{L}_{\rm mass} &= \begin{pmatrix} \hat{W}^+_\mu & \hat{W}'^+_\mu \end{pmatrix} \begin{pmatrix} \widetilde{M}^2_W & \delta\widetilde{M}^2_W \\ \delta\widetilde{M}^2_W & \widetilde{M}^2_{W'}+\Delta\widetilde{M}^2_{W'} \end{pmatrix} \begin{pmatrix} \hat{W}^{-\mu} \\ \hat{W}'^{-\mu} \end{pmatrix} \\ &+ \frac{1}{2}\begin{pmatrix} A & \hat{Z}_\mu & \hat{Z}'_\mu \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \widetilde{M}^2_Z & \delta\widetilde{M}^2_Z \\ 0 & \delta\widetilde{M}^2_Z & \widetilde{M}^2_{Z'}+\Delta\widetilde{M}^2_{Z'} \end{pmatrix} \begin{pmatrix} A \\ \hat{Z}^\mu \\ \hat{Z}'^\mu \end{pmatrix}. \end{aligned} \tag{21}$$

In Table III, we expect that the scale $\tilde{u}^2$ of the first-stage breaking is much larger than the electroweak scale $\tilde{v}^2$. We work to leading order in $\tilde{v}^2/\tilde{u}^2$, and so if we take the approximation

$$\widetilde{M}^2_{Z',W'} \gg \widetilde{M}^2_{Z,W}, \delta\widetilde{M}^2_{Z,W}, \Delta\widetilde{M}^2_{Z,W}, \tag{22}$$

we can expand in large $\widetilde{M}^2_{Z',W'}$. To order $\mathcal{O}(\widetilde{M}^{-2}_{W',Z'})$, the mass eigenstates, denoted without the hats (ˆ), are given by (similarly for the charged gauge bosons):

$$Z_\mu \equiv \hat{Z}_\mu - \frac{\delta\widetilde{M}^2_Z}{\widetilde{M}^2_{Z'}-\widetilde{M}^2_Z}\hat{Z}'_\mu, \tag{23}$$

$$Z'_\mu \equiv \frac{\delta\widetilde{M}^2_Z}{\widetilde{M}^2_{Z'}-\widetilde{M}^2_Z}\hat{Z}_\mu + \hat{Z}'_\mu. \tag{24}$$

Now we can rewrite the fundamental Lagrangian in terms of the mass eigenstates for both neutral and charged gauge bosons

$$\begin{aligned}
\mathcal{L}^{\text{mass}}_{\text{fund}} &= \frac{1}{2}\left(\widetilde{M}_Z^2 - \frac{\delta\widetilde{M}_Z^4}{\widetilde{M}_{Z'}^2}\right) Z_\mu Z^\mu + \left(\widetilde{M}_W^2 - \frac{\delta\widetilde{M}_W^4}{\widetilde{M}_{W'}^2}\right) W^{+\mu}W^-_\mu \\
&+ \frac{1}{2}\left(\widetilde{M}_{Z'}^2 + \Delta\widetilde{M}_{Z'}^2 + \frac{\delta\widetilde{M}_Z^4}{\widetilde{M}_{Z'}^2}\right) Z'_\mu Z'^\mu + \left(\widetilde{M}_{W'}^2 + \Delta\widetilde{M}_{W'}^2 + \frac{\delta\widetilde{M}_W^4}{\widetilde{M}_{W'}^2}\right) W'^{+\mu}W'^-_\mu \\
&+ Z_\mu\left(J^{0\mu} - \frac{\delta\widetilde{M}_Z^2}{\widetilde{M}_{Z'}^2}K^{0\mu}\right) + Z'_\mu\left(K^{0\mu} + \frac{\delta\widetilde{M}_Z^2}{\widetilde{M}_{Z'}^2}J^{0\mu}\right) + A_\mu J^\mu \\
&+ \left[W^+_\mu\left(J^{+\mu} - \frac{\delta\widetilde{M}_W^2}{\widetilde{M}_{W'}^2}K^{+\mu}\right) + W'^+_\mu\left(K^{+\mu} + \frac{\delta\widetilde{M}_W^2}{\widetilde{M}_{W'}^2}J^{+\mu}\right) + (+\leftrightarrow -)\right]. \qquad (25)
\end{aligned}$$

We can now obtain the effective Lagrangian by successively integrating out the massive gauge bosons. In the basis of the mass eigenstates, integrating out Z' and W' (whose masses are expected to be at or above the TeV scale) results in an effective Lagrangian valid at the electroweak scale:

$$\begin{aligned}
\mathcal{L}^{\text{EW}}_{\text{eff}} &= \frac{1}{2}\left(\widetilde{M}_Z^2 - \frac{\delta\widetilde{M}_Z^4}{\widetilde{M}_{Z'}^2}\right) Z_\mu Z^\mu + \left(\widetilde{M}_W^2 - \frac{\delta\widetilde{M}_W^4}{\widetilde{M}_{W'}^2}\right) W^{+\mu}W^-_\mu \\
&+ Z_\mu\left(J^{0\mu} - \frac{\delta\widetilde{M}_Z^2}{\widetilde{M}_{Z'}^2}K^{0\mu}\right) + \left[W^+_\mu\left(J^{+\mu} - \frac{\delta\widetilde{M}_W^2}{\widetilde{M}_{W'}^2}K^{+\mu}\right) + (+\leftrightarrow -)\right] \\
&- \frac{1}{2\widetilde{M}_{Z'}^2}K^{0\mu}K^0_\mu - \frac{1}{\widetilde{M}_{W'}^2}K^{+\mu}K^-_\mu + A_\mu J^\mu. \qquad (26)
\end{aligned}$$

From Eq. (26), we see that the low-energy effects of the heavy gauge bosons are parameterized by the shifts in the masses of the W and Z gauge bosons, and in the shifts of their couplings to the fermions, and additional four-fermion interactions.

We can further integrate out the Z and $W^\pm$ gauge bosons (again to leading order in $\widetilde{M}^{-2}_{W',Z'}$). We then have the four-fermion interactions

$$\begin{aligned}
\mathcal{L}^{4f}_{\text{eff}} &= -\frac{1}{2\widetilde{M}_Z^2}\left[J^{0\mu}J^0_\mu + \frac{\widetilde{M}_Z^2}{\widetilde{M}_{Z'}^2}\left(\frac{\delta\widetilde{M}_Z^4}{\widetilde{M}_Z^4}J^0_\mu J^{0\mu} - 2\frac{\delta\widetilde{M}_Z^2}{\widetilde{M}_Z^2}J^0_\mu K^{0\mu} + K^0_\mu K^{0\mu}\right)\right] \\
&- \frac{1}{\widetilde{M}_W^2}\left[J^{+\mu}J^-_\mu + \frac{\widetilde{M}_W^2}{\widetilde{M}_{W'}^2}\left(\frac{\delta\widetilde{M}_W^4}{\widetilde{M}_W^4}J^+_\mu J^{-\mu} - \frac{\delta\widetilde{M}_W^2}{\widetilde{M}_W^2}(J^+_\mu K^{-\mu} + J^-_\mu K^{+\mu})\right.\right. \\
&\left.\left. + K^+_\mu K^{-\mu}\right)\right]. \qquad (27)
\end{aligned}$$

Before we can compare the predictions of Eq. (27) with experimental results for the different $G(221)$ models, we first have to properly define some experimental input values (for example, the Fermi constant G_F) for the $G(221)$ models under study. We will discuss this in Section IV.

2. Triple Gauge Boson Couplings

In the basis defined through Eqs. (11), (12) and (14), the triple gauge boson couplings (TGCs) $g(\hat{Z}\hat{W}^+\hat{W}^-)$ and $g(A\hat{W}^+\hat{W}^-)$ have the standard forms

$$g(\hat{Z}\hat{W}^+\hat{W}^-) = -\tilde{g}_L \cos\tilde{\theta}, \tag{28}$$

$$g(A\hat{W}^+\hat{W}^-) = \tilde{e}. \tag{29}$$

In the basis of mass eigenstates, however, we expect there to be a shift to these couplings because the mass eigenstate Z (W) now is a mixture of $\hat{Z}$ ($\hat{W}$) and $\hat{Z}'$ ($\hat{W}'$). However, because of QED gauge invariance, the AW^+W^- coupling does not receive a shift. On the other hand, the ZW^+W^- coupling does shift, and we shall discuss in turn this shift for the two breaking patterns.

In breaking pattern I (LR, LP, HP, and FP models), in the hat (ˆ) basis of the gauge bosons, the Lagrangian contains $\hat{Z}\hat{W}'\hat{W}'$ and $\hat{Z}'\hat{W}'\hat{W}'$ vertices in addition to the typical $\hat{Z}\hat{W}\hat{W}$ vertex. Since the overlap between $\hat{W}'$ and the light mass eigenstate W is of order $\mathcal{O}(\widetilde{M}_{W'}^{-2})$, contributions from $g\left(\hat{Z}\hat{W}'\hat{W}'\right)$ and $g\left(\hat{Z}'\hat{W}'\hat{W}'\right)$ to $g\left(ZWW\right)$ are at least of order $\mathcal{O}(\widetilde{M}_{W'}^{-4})$. As we are only working to leading order in $\mathcal{O}(\widetilde{M}_{W'}^{-2})$, there is no shift due to these additional interactions at this order.

For breaking pattern II, the story is similar. There are no $\hat{Z}\hat{W}\hat{W}'$ nor $\hat{Z}'\hat{W}\hat{W}$ vertices, only $\hat{Z}\hat{W}'\hat{W}'$ and $\hat{Z}'\hat{W}'\hat{W}$ interactions. The contributions to the ZWW coupling are suppressed by fourth powers of the heavy masses $\widetilde{M}_{W',Z'}^{-4}$, and thus of higher order than those kept in the effective theory.

In both breaking patterns, however, there will be a shift to the ZWW-vertex due to a shift in $\tilde{\theta}$ (cf. Eq. (47)) , the counterpart of the Standard Model weak mixing angle θ , as defined in our fitting scheme. The LEP-II experiments, however, do not directly probe the ZWW-vertex, but instead infer the ZWW-vertex through the process $e^+e^- \to W^+W^-$ assuming SM couplings for all other vertices. To properly compare the relationship between the experimental measurement of the ZWW-vertex and the theoretical shifts in the $G(221)$ models, we would have to take into account all the other shifts in the couplings that enter the process $e^+e^- \to W^+W^-$. We will discuss this in further detail in Section V.

3. The Yukawa and Higgs Sectors

We complete our discussion of the effective Lagrangians of the $G(221)$ models with a brief discussion of the Higgs sectors and the Yukawa interactions. It is important to stress, however, that despite the complexity of the Higgs sectors and Yukawa interactions, our results of the global analysis only depend on the gauge interactions of the fermions, and not on the details of the Yukawa interactions. This is because we work only with those observables involving gauge interactions (which excludes, for example, the branching ratio $\mathrm{Br}(b \to s\gamma)$), and keep only tree-level contributions originated from the new physics.

We discuss the Higgs sectors of the two breaking patterns separately. In breaking pattern I, we take as an example the left-right model, where the electroweak symmetry is broken by a bi-doublet (LR-D). This is necessary because the VEV's of the bi-doublet should generate the fermion masses, and the right-handed fermions now are doublets under the $SU(2)_2$. With the bi-doublet H in Table III, we may have Yukawa couplings (similarly for leptons) such as:

$$-\mathcal{L} \supset \overline{Q}_{\mathrm{R}} \left(\mathcal{Y}_Q H + \tilde{\mathcal{Y}}_Q \tilde{H} \right) Q_{\mathrm{L}} + \mathrm{h.c.}, \qquad (30)$$

with $\tilde{H} = -i\tau_2 H^* \tau_2$, and where $\mathcal{Y}_Q$ and $\tilde{\mathcal{Y}}_Q$ have flavor structures that may be related by imposing additional symmetries (for example, left-right parity) on the model. In any case, unlike the Standard Model where we can solve for Yukawa couplings in terms of fermion masses and the Higgs VEV, in $G(221)$ models there are more free parameters in the Yukawa sectors. These parameters can lead to interesting flavor phenomena, particularly in the arena of neutrino physics, and have been studied in detail in the literature (see, for example, Mohapatra *et. al.* [27]). On the other hand, the details of the Yukawa sectors do not affect the gauge couplings of the fermions at leading order and therefore do not affect the results of our analysis.

In breaking pattern II, in addition to those Higgs bosons that are required to break the electroweak symmetry, it may be the case that the Higgs sector needs to be extended to generate fermion masses. This is because, with the current set-up, the Higgs boson that generates EWSB can couple only to leptons (in the case of un-unified model) or fermions of the third generation (in the case of the non-universal model). With additional Higgs bosons, the structure of the Higgs potential may mimic that of the two-Higgs doublet models. Again, as with breaking pattern I, there are more degrees of freedom than can be determined from the fermion masses, but the details of the Yukawa interactions do not affect the results of our paper, which only depend on the fermionic gauge interactions.

IV. THE GLOBAL FIT ANALYSIS

In this section we illustrate our procedure for performing the global-fit analysis to obtain constraints on new physics contributions. From Tables III and IV, we see that the $G(221)$ models contains six (five) parameters for the first (second) breaking pattern: three (two) VEV's $\{\tilde{u}_{\mathrm{D,T}}, \tilde{v}\sin\tilde{\beta}, \tilde{v}\cos\tilde{\beta}\}$ in Table III and three gauge couplings $\{\tilde{g}_1, \tilde{g}_2, \tilde{g}_X\}$ in Table IV. (For breaking pattern II, there are only two VEV's $\{\tilde{u}, \tilde{v}\}$.) Compared to the gauge sector of the SM, which contains only three parameters (two gauge couplings and one VEV; g_{L}, g_{Y} and v), there are three (two) additional parameters, and our goal is to:

- find a useful parameterization of these three additional parameters so as to parameterize the effects of *new* physics, and

- determine the constraints on these parameters from electroweak precision measurements through a global-fit analysis.

We discuss these two steps in detail in turn.

A. Parameterization

As stated above, the $G(221)$ models contain six (five) parameters in the gauge sector:

$$\{\tilde{g}_1,\ \tilde{g}_2,\ \tilde{g}_X,\ \tilde{u}_{\rm D}(\tilde{u}_{\rm T},\ \text{or}\ \tilde{u}),\ \tilde{v}^2,\ \tilde{\beta}\}, \tag{31}$$

where the parameter $\tilde{\beta}$ only exists in models with breaking pattern I. Using Eqs. (6),(7), and (8), an equivalent set of parameters is

$$\{\tilde{\alpha}_e,\ \tilde{\theta},\ \tilde{\phi},\ \tilde{x},\ \tilde{v}^2,\ s_{2\tilde{\beta}}\}, \tag{32}$$

where $\tilde{x}$ is defined as

$$\tilde{x} \equiv \begin{cases} \tilde{u}_{\rm D}^2/\tilde{v}^2 & \text{for LR-D, LP-D, HP-D, and FP-D} \\ \tilde{u}_{\rm T}^2/\tilde{v}^2 & \text{for LR-T, LP-T, HP-T, and FP-T} \\ \tilde{u}^2/\tilde{v}^2 & \text{for UU and NU.} \end{cases} \tag{33}$$

As we expect $\tilde{x}$ to be large ($\tilde{x} \gtrsim 100$), we work to leading order in $\tilde{x}^{-1}$.

In addition to these parameters, the loop-level predictions will require the values of the masses of the top quark (m_t) and the Higgs boson (M_H). For each $G(221)$ model, we perform two separate analyses with regard to these parameters. In one analysis, we fit these two parameters, m_t and M_H, in addition to the model parameters. In a second analysis, we fix these two parameters at the best-fit SM values.

With regard to the parameters in Eq. (32), we will take three reference observables to constrain three combinations of the parameters and perform a global-fit over $\{\tilde{x},\ \tilde{\phi},\ s_{2\tilde{\beta}},\ \overline{m}_t,\ M_H\}$. The bar ($^{-}$) over m_t indicates that we will use the top quark mass as defined in the $\overline{\rm MS}$-scheme. We take as reference observables the experimental measurements of

- the mass of the Z boson ($M_Z = 91.1876$ GeV), determined from the Z-line shape at LEP-I.

- the Fermi constant ($G_F = 1.16637 \times 10^{-5}$ GeV^{-2}), determined from the lifetime of the muon,

- the fine structure constant ($\alpha_e^{-1} = 127.918$ at the scale M_Z).

Our task then is to express the model parameters, cf Eq. (32)

$$\{\tilde{\alpha}_e,\ \tilde{\theta},\ \tilde{v}^2,\ \tilde{x},\ \tilde{\phi},\ s_{2\tilde{\beta}},\ \overline{m}_t,\ M_H\},$$

in terms of the reference and fit parameters

$$\{\alpha_e,\ M_Z,\ G_F,\ \tilde{x},\ \tilde{\phi},\ s_{2\tilde{\beta}},\ \overline{m}_t,\ M_H\}. \tag{34}$$

That is, we want the relationships

$$\overbrace{\{\tilde{\alpha}_e,\ \tilde{\theta},\ \tilde{v}^2,\ \tilde{x},\ \tilde{\phi},\ s_{2\tilde{\beta}}, \bar{m}_t, M_H\}}^{\text{model parameters}} \Leftrightarrow \{\ \overbrace{\alpha_e,\ M_Z,\ G_F,}^{\text{reference parameters}}\ \ \overbrace{\tilde{x},\ \tilde{\phi},\ s_{2\tilde{\beta}},\ \overline{m}_t,\ M_H\}}^{\text{fit parameters}} \tag{35}$$

Since $\{\tilde{x}, \tilde{\phi}, \tilde{\beta},\ \overline{m}_t, M_H\}$ appear in both the model and fit parameters (by construction), we only have to solve for $\{\tilde{\alpha}_e, \tilde{\theta}, \tilde{v}^2\}$ in terms of the reference and fit parameters. This can be done by analyzing how the reference parameters are related to the model parameters.

1. Electric Charge

The electric charge in the $G(221)$ models is the gauge coupling of the unbroken $U(1)_{\rm em}$ group, which we have parameterized as $\tilde{e}$ in Eq. (5). There are no tree-level modifications to the wavefunction renormalization of the photon, so we then simply have the relationship

$$\tilde{\alpha}_e = \alpha_e. \tag{36}$$

2. The Fermi Constant

The Fermi constant, G_F, is experimentally determined from the muon lifetime as [28]

$$\tau_\mu^{-1} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[1+\mathcal{O}\left(\frac{m_e^2}{m_\mu^2}\right)\right]\left[1+\mathcal{O}\left(\frac{m_\mu^2}{M_W^2}\right)\right]\left[1+\mathcal{O}\left(\frac{1}{16\pi^2}\right)\right], \tag{37}$$

where the precise forms of the higher-order corrections are given in Ref. [28]. Neglecting these higher-order corrections, the SM contribution to the muon lifetime is

$$\tau_\mu^{-1} = \frac{g_L^4}{192\cdot 32\pi^3 M_W^4} m_\mu^5, \tag{38}$$

and, using the SM relation $4M_W^2 = g_L^2 v^2$, we obtain

$$G_F = \frac{1}{\sqrt{2}v^2}. \tag{39}$$

In the $G(221)$ models, we have extra contributions to the four-fermion charged-current effective theory below the electroweak scale, cf Eq. (27),

$$\mathcal{L}_{\rm eff}^{CC,4f} = -\frac{1}{\widetilde{M}_W^2}J^+J^- - \frac{1}{\widetilde{M}_{W'}^2}\left(K^+K^- - \frac{\delta\widetilde{M}_W^2}{\widetilde{M}_W^2}(J^+K^- + K^-J^+) + \frac{\delta\widetilde{M}_W^4}{\widetilde{M}_W^4}J^+J^-\right),$$

and these contributions will modify the SM relation in Eq. (39). In principle, the fermionic contributions to K_μ^+ can have both left- and right-handed components and differ among the different generations. However, for the $G(221)$ models we consider here, $K_\mu^\pm$ couples universally to the first two generations. Furthermore, $K_\mu^\pm$ is either purely right-handed (the LR, HP, LP, FP models) or purely left-handed (the UU and NU models). We therefore focus on these special cases instead of performing the general analysis.

We first consider the case that $K_\mu^\pm$ is purely right-handed. The contributions to the amplitude come from JJ, JK, and KK operators that do not interfere with one another in the limit of neglecting the masses of electrons and neutrinos. The squared-amplitudes from the JK and KK operators are of order $\mathcal{O}(M_{W'}^{-4}) \sim \mathcal{O}(x^{-2})$ at leading order, and we do not keep these contributions. The Fermi constant is then given by

$$\frac{G_F}{\sqrt{2}} = \frac{\tilde{g}_L^2}{8\widetilde{M}_W^2}\left(1 + \frac{\delta\widetilde{M}_W^4}{\widetilde{M}_W^2\widetilde{M}_{W'}^2}\right), \text{ (for breaking pattern I)}, \tag{40}$$

independent of the details of $K_\mu^\pm$. The expression of G_F, which depends on the details of the Higgs representation, is written in terms of model parameters as

$$G_F = \begin{cases} \frac{1}{\sqrt{2}\tilde{v}^2}\left(1 + \frac{s_{2\tilde{\beta}}^2}{\tilde{x}}\right), & \text{(for LR-D, LP-D, HP-D, and FP-D)} \\ \frac{1}{\sqrt{2}\tilde{v}^2}\left(1 + \frac{s_{2\tilde{\beta}}^2}{2\tilde{x}}\right), & \text{(for LR-T, LP-T, HP-T, and FP-T)} \end{cases} \tag{41}$$

Though the left-right and right-right current operators do not contribute to the total muon decay rate at the order $\mathcal{O}(\tilde{x}^{-1})$, they do contribute at leading order to the Michel parameters (for a detailed discussion of the Michel parameters, see the Muon Decay Parameters article in the Particle Data Group (PDG) [28]).

In the case that $K_\mu^\pm$ is purely left-handed, all the charged-current operators in Eq. (27) contribute, and G_F is given by

$$\frac{G_F}{\sqrt{2}} = \frac{\tilde{g}_L^2}{8\widetilde{M}_W^2}\left[1 + \frac{\widetilde{M}_W^2}{M_{W'}^2}\left(\frac{\tilde{g}_{W'}^2}{\tilde{g}_L^2} - 2\frac{\delta\widetilde{M}_W^2}{M_W^2}\frac{\tilde{g}_{W'}}{\tilde{g}_L} + \frac{\delta\widetilde{M}_W^4}{M_W^4}\right)\right], \quad \text{(for UU and NU)} \tag{42}$$

where $\tilde{g}_{W'}$ can be looked up in Table V. For the UU and NU models, these contributions cancel each other, and we are simply left with

$$G_F = \frac{1}{\sqrt{2}\tilde{v}^2} \quad \text{(for UU and NU)}. \tag{43}$$

We can rewrite our results in a more suggestive manner by defining the SM VEV (v^2 without tilde ˜) through the Fermi constant

$$v^2 \equiv \frac{1}{\sqrt{2}G_F}. \tag{44}$$

We then have

$$\tilde{v}^2 = \begin{cases} v^2\left(1+\dfrac{s^2_{2\tilde{\beta}}}{\tilde{x}}\right), & \text{(for LR-D, LP-D, HP-D, and FP-D)} \\ v^2\left(1+\dfrac{s^2_{2\tilde{\beta}}}{2\tilde{x}}\right), & \text{(for LR-T, LP-T, HP-T, and FP-T)} \\ v^2. & \text{(for UU and NU)} \end{cases} \tag{45}$$

3. *Z-Mass*

In our effective theory approach, the mass eigenvalue of the Z-boson is given by (using Eq. (25), Table VI, and $\tilde{\alpha}_e = \alpha_e$)

$$\begin{aligned} M_Z^2 &= \widetilde{M}_Z^2 - \frac{\delta\widetilde{M}_Z^4}{\widetilde{M}_{Z'}^2} \left(\begin{array}{c}\text{general form from the} \\ \text{fundamental } G(221) \text{ Lagrangian}\end{array}\right) \\ &= \begin{cases} \dfrac{\alpha_e\pi\tilde{v}^2}{s^2_{\tilde{\theta}}c^2_{\tilde{\theta}}}\left(1-\dfrac{c^4_{\tilde{\phi}}}{\tilde{x}}\right), & \text{(for LR-D, LP-D, HP-D, and FP-D)} \\ \dfrac{\alpha_e\pi\tilde{v}^2}{s^2_{\tilde{\theta}}c^2_{\tilde{\theta}}}\left(1-\dfrac{c^4_{\tilde{\phi}}}{4\tilde{x}}\right), & \text{(for LR-T, LP-T, HP-T, and FP-T)} \\ \dfrac{\alpha_e\pi\tilde{v}^2}{s^2_{\tilde{\theta}}c^2_{\tilde{\theta}}}\left(1-\dfrac{s^4_{\tilde{\phi}}}{\tilde{x}}\right), & \text{(for UU and NU)} \end{cases} . \end{aligned} \tag{46}$$

Solving Eq. (46) for $c^2_{\tilde{\theta}}s^2_{\tilde{\theta}}$, and using Eqs. (41) and (43), we can solve for $\tilde{\theta}$ in terms of the reference and fit parameters

$$s^2_{\tilde{\theta}}c^2_{\tilde{\theta}} = \begin{cases} s^2_\theta c^2_\theta\left[1-\dfrac{1}{x}\left(c^4_{\tilde{\phi}}-s^2_{2\tilde{\beta}}\right)\right], \text{(for LR-D, LP-D, HP-D, and FP-D)} \\ s^2_\theta c^2_\theta\left[1-\dfrac{1}{x}\left(\frac{1}{4}c^4_{\tilde{\phi}}-\frac{1}{2}s^2_{2\tilde{\beta}}\right)\right], \text{(for LR-T, LP-T, HP-T, and FP-T)} \\ s^2_\theta c^2_\theta\left[1-\dfrac{s^4_{\tilde{\phi}}}{x}\right], \text{(for UU and NU)}, \end{cases} \tag{47}$$

where θ (without a tilde ˜) is defined in terms of the reference parameters

$$\sin^2\theta\cos^2\theta \equiv \frac{\pi\alpha_e}{\sqrt{2}M_Z^2 G_F}. \tag{48}$$

Eqs. (36), (45), and (47) then enable us to translate all the model parameters to reference and fit parameters.

B. Corrections to Observables

In this subsection we illustrate the corrections to several example observables that we include in our global analysis. These examples elucidate the procedures we had outlined earlier, and we will refer to these results when we discuss the observables included in our global analysis.

1. The Z-Partial Widths $\Gamma(Z \to f\overline{f})$

As a first example, we can then consider the $Z \to f\overline{f}$ partial width, which at tree-level has the expression in the Standard Model

$$\Gamma(Z \to f\overline{f}) = \frac{n_c}{12\pi} M_Z \left(g_V^2 + g_A^2\right), \tag{49}$$

where $n_c = 3$ if f is s quark, and $n_c = 1$ for leptons, and

$$g_V = \frac{e}{2s_\theta c_\theta}\left(T_{3L}^f - 2Q^f \sin^2\theta\right), \tag{50}$$

$$g_A = \frac{e}{2s_\theta c_\theta} T_{3L}^f, \tag{51}$$

where T_{3L}^f and Q^f are respectively the weak-isospin and electric charge of the fermion f.

In the $G(221)$ models, the partial decay width can be written in terms of model parameters as

$$\Gamma(Z \to f\overline{f}) = \frac{n_c}{12\pi}\widetilde{M}_Z\left(1 - \frac{\delta\widetilde{M}_Z^4}{2\widetilde{M}_Z^2\widetilde{M}_{Z'}^2}\right)\left([\tilde{g}_V^z(f)]^2 + [\tilde{g}_A^z(f)]^2\right), \tag{52}$$

where $\delta\widetilde{M}_Z^2, \widetilde{M}_{Z'}^2, \tilde{g}_V^z(f)$, and $\tilde{g}_A^z(f)$ depend on the details of the model. For models that follow the breaking pattern I (LR-D, LP-D, HP-D, FP-D), the couplings have the form (to order $\mathcal{O}(\tilde{x}^{-1})$)

$$\tilde{g}_V^z(f) = \frac{e}{2s_{\tilde{\theta}}c_{\tilde{\theta}}}\left((T_{3L}^f - 2Q^f s_{\tilde{\theta}}^2) + \frac{c_{\tilde{\phi}}^2}{2\tilde{x}}\left[T_{3R}^f c_{\tilde{\phi}}^2 - (X_L^f + X_R^f)s_{\tilde{\phi}}^2\right]\right), \tag{53}$$

$$\tilde{g}_A^z(f) = \frac{e}{2s_{\tilde{\theta}}c_{\tilde{\theta}}}\left(T_{3L}^f - \frac{c_{\tilde{\phi}}^2}{2\tilde{x}}\left[T_{3R}^f c_{\tilde{\phi}}^2 - (X_R^f - X_L^f)s_{\tilde{\phi}}^2\right]\right), \tag{54}$$

where X_L^f, and X_R^f, and T_{3R} are respectively the left- and right-handed fermion charges under the $U(1)_X$, and the z-component isospin under the $SU(2)_2$ (which is identified as $SU(2)_R$ in left-right models). Expressing $\tilde{\theta}$ in terms of the reference and the model parameters through

Eq. (47) and collecting terms of $\mathcal{O}(\tilde{x}^{-1})$, we have (in units of GeV)

$$\begin{aligned}
\Gamma(Z \to f\overline{f}) = \Gamma(Z \to f\overline{f})_{\rm SM} \\
+\frac{n_f}{\tilde{x}} \Big[s_{\tilde{\theta}}^4 \Big(& -0.446\ (Q^f)^2 + 1.773\ Q^f T_{3L}^f - 0.310\ Q^f T_{3R}^f \\
& -0.310\ Q^f X_R^f - 0.664\ (T_{3L}^f)^2 \Big) \\
& + s_{\tilde{\theta}}^2 \Big(0.582\ (Q^f)^2 - 1.91\ Q^f T_{3L}^f + 0.620\ Q^f T_{3R}^f + 0.310\ Q^f X_R^f \Big) \\
& + s_{2\tilde{\beta}}^2 \Big(0.136\ (Q^f)^2 - 0.136\ Q^f T_{3L}^f - 0.664\ (T_{3L}^f)^2 \Big) \\
& -0.136\ (Q^f)^2 + 0.136\ Q^f T_{3L}^f - 0.310\ Q^f T_{3R}^f + 0.664\ (T_{3L}^f)^2 \Big] , \\
& \text{(for LR-D, LP-D, HP-D, and FP-D)}
\end{aligned} \tag{55}$$

where $\Gamma(Z \to \overline{f} f)_{\rm SM}$ is given by Eq. (49), and we have used the numerical values of the reference parameters.

2. The Mass of the W-boson

As a second example, we compute the mass of the W-boson in the $G(221)$ models. The SM expression, for the same set of reference parameters $\{\alpha,\ M_Z,\ G_F\}$, is given by

$$M_W = M_Z c_\theta, \tag{56}$$

where θ is defined in terms of the reference parameters in Eq. (48). In the $G(221)$ models, the mass of the W-boson has the general form

$$M_W = \widetilde{M}_W \left(1 - \frac{\delta \widetilde{M}_W^4}{2 \widetilde{M}_W^2 \widetilde{M}_{W'}^2} \right). \tag{57}$$

More specifically, in terms of the model parameters for the individual models, we have

$$M_W = \begin{cases} \dfrac{\tilde{e}\tilde{v}}{2 s_{\tilde{\theta}}} \left(1 - \dfrac{s_{\tilde{\beta}}^2}{2\tilde{x}} \right) & \text{(for LR-D, LP-D, HP-D, FP-D)}, \\ \dfrac{\tilde{e}\tilde{v}}{2 s_{\tilde{\theta}}} \left(1 - \dfrac{s_{\tilde{\beta}}^2}{4\tilde{x}} \right) & \text{(for LR-T, LP-T, HP-T, FP-T)}, \\ \dfrac{\tilde{e}\tilde{v}}{2 s_{\tilde{\theta}}} \left(1 - \dfrac{s_{\tilde{\phi}}^4}{2\tilde{x}} \right) & \text{(for UU, NU)}. \end{cases} \tag{58}$$

Using Eqs. (36), (45), and (47), we can convert all the model parameters to reference and fit parameters

$$M_W = \begin{cases} M_Z\cos\theta\left[1+\frac{1}{2\tilde{x}}\frac{c_\theta^2}{c_\theta^2-s_\theta^2}\left(c_{\tilde{\phi}}^4-s_{2\tilde{\beta}}^2\right)\right] & \text{(for LR-D, LP-D, HP-D, FP-D)},\\ M_Z\cos\theta\left[1+\frac{1}{2\tilde{x}}\frac{c_\theta^2}{c_\theta^2-2s_\theta^2}\left(\frac{c_{\tilde{\phi}}^4}{4}-\frac{s_{2\tilde{\beta}}^2}{2}\right)\right] & \text{(for LR-T, LP-T, HP-T, FP-T)},\\ M_Z\cos\theta\left[1+\frac{1}{2\tilde{x}}\frac{s_\theta^2}{c_\theta^2-s_\theta^2}s_{\tilde{\phi}}^4\right] & \text{(for UU, NU)}.\end{cases} \tag{59}$$

C. Implementation of the Global Fit and List of Observables

For a measured observable $O^{\rm exp}$, the SM prediction can be broken down into the tree- and loop-level components

$$O^{\rm th}_{\rm SM} = O^{\rm th,tree}_{\rm SM} + O^{\rm th,loop}_{\rm SM}(\overline{m}_t, M_H), \tag{60}$$

where $O^{\rm th}$ is expressed in terms of the reference parameters. Since the top quark mass ($\overline{m}_t$) and the mass of the Higgs boson (M_H) enter into the loop-calculations in the SM, a global analysis of precision data and direct detection data can be used to constrain M_H. In the $G(221)$ models, we can express the theoretical prediction as

$$O^{\rm th} = O^{\rm th,tree}_{\rm SM} + O^{\rm th,loop}_{\rm SM}(\overline{m}_t, M_H) + O^{\rm th,tree}_{\rm NP}(\tilde{x},\tilde{\phi},\tilde{\beta}), \tag{61}$$

where $O^{\rm th,tree}_{\rm NP}$ is of the order $\mathcal{O}(1/\tilde{x})$, and we assume that

$$\tilde{x}^{-1} \sim \frac{1}{16\pi^2} \sim O^{\rm th,loop}_{\rm SM}. \tag{62}$$

That is, the Born-level new physics contributions from the $G(221)$ models are numerically of one-loop order, and loop corrections involving new physics are of two-loop order $\mathcal{O}\left(\frac{1}{16\pi^2\tilde{x}}\right)$, which we discard in our analysis.

To compare with precision data (from LEP-1 and SLD) and low-energy observables, we calculate the shifts in observables $O^{\rm th,tree}_{\rm NP}(\tilde{x},\tilde{\phi},\tilde{\beta})$, as in the previous examples of the partial decay widths of the Z-boson and the mass of the W-boson, and we adapt these corrections into a numerical package GAPP [9]. GAPP then computes $O^{\rm th,tree}_{\rm SM}$ and $O^{\rm th,loop}_{\rm SM}(\bar{m}_t, M_H)$[42], together with the $O^{\rm th,tree}_{\rm NP}(\tilde{x},\tilde{\phi},\tilde{\beta})$ to find the best-fit values of the fit parameters and the confidence level contours using the CERN library MINUIT [10].

We perform a global fit over the following classes of observables

- LEP-I Z-pole observables: the total Z-width (Γ_Z), left-right asymmetries (A_{LR}), and related observables.
- the mass (M_W) and decay width (Γ_W) of the W-boson,

- the tau lifetime τ_τ,
- the ratios of neutral-to-charged current cross sections measured from neutrino-hadron deep-inelastic scattering (DIS) experiments ($R_\nu \equiv \sigma^{\rm NC}_{\nu N}/\sigma^{\rm CC}_{\nu N}$ and similarly defined for $\overline{\nu}$),
- effective vector and axial-vector neutrino-electron couplings ($g_V^{\nu e}$ and $g_A^{\nu e}$),
- weak charges (Q_W) of atoms and the electron measured from atomic parity experiments.

Detailed information on these observables can be found in PDG [28], and here we only briefly summarize the observables. The set of the observables included in our analysis is the same as that used in the PDG analysis [28], with two exceptions.

- First, we do not include the anomalous magnetic moment of the muon and the decay branching ratio $b \to s\gamma$. At leading order, these observables are of one-loop order, and they depend on the details of the extended flavor structure of the $G(221)$ models. In this work, we assume W' bosons only couple to fermions in the same generation.
- Second, we include the measurements of the decay width of the W-boson, which are not included in the PDG analysis. However, because of the comparatively low precision of these measurements, this observable turns out to be insensitive to the new physics contributions from the $G(221)$ models.

In total, we include a set of 37 experimental observables in our global-fit analysis.

Before we give a brief discussion on each of these classes of observables, we note that for some low-energy observables, such as the measurements from the atomic parity violation and neutrino-neucleus DIS experiments, we implement the shifts in the coefficients of the relevant four-fermion interactions, and rely on GAPP to compute the theoretical predictions based on these modified coefficients. The expressions of the coefficients of the four-fermion interactions are given in the Appendix.

For the ease of typesetting in the following subsections, we introduce the abbreviation for the various forms of the fermionic currents

$$\begin{aligned}
\left(\bar{f}_1 f_2\right)^\mu_L &\equiv \bar{f}_1\gamma^\mu\left(1-\gamma_5\right) f_2,\\
\left(\bar{f}_1 f_2\right)^\mu_R &\equiv \bar{f}_1\gamma^\mu\left(1+\gamma_5\right) f_2,\\
\left(\bar{f}_1 f_2\right)^\mu_V &\equiv \bar{f}_1\gamma^\mu f_2,\\
\left(\bar{f}_1 f_2\right)^\mu_A &\equiv \bar{f}_1\gamma^\mu\gamma_5 f_2.
\end{aligned} \tag{63}$$

1. Precision Measurements at the Z-Pole

The precision measurements at the Z-pole (including LEP-1 and SLD experiments) fall into two broad classes: observables that can be constructed from the partial widths (for example, in Eq. (55)) and the asymmetry (constructed from the couplings in Eq. (50) and (51)). We discuss these two classes in turn.

In addition to the total width Γ_Z, there are also the following measurements:

$$\sigma_{\text{had}} = \frac{12\pi}{M_Z^2\Gamma_Z^2}\cdot\Gamma_Z\left(e^-e^+\right)\Gamma_Z\left(\text{had.}\right), \tag{64}$$

$$R(\ell) = \frac{\Gamma_Z\left(\text{had.}\right)}{\Gamma_Z\left(\ell\bar{\ell}\right)}, \quad \text{for } \ell = e,\mu,\tau, \tag{65}$$

$$R(q) = \frac{\Gamma_Z\left(q\bar{q}\right)}{\Gamma_Z\left(\text{had.}\right)}, \quad \text{for } q = u,d,c,s,b, \tag{66}$$

$$\mathcal{R}(s) = \frac{R(s)}{R(u)+R(d)+R(s)}, \tag{67}$$

where $\Gamma_Z(f\overline{f})$ is the partial decay width $\Gamma(Z\to f\overline{f})$, and

$$\Gamma_Z\left(\text{had.}\right) = \sum_{q=u,d,c,s,b}\Gamma_Z(q\bar{q}). \tag{68}$$

The left-right asymmetry $A_{LR}(f)$ is defined as

$$A_{LR}(f) \equiv \frac{\left[g_L^Z(f)\right]^2-\left[g_R^Z(f)\right]^2}{\left[g_L^Z(f)\right]^2+\left[g_R^Z(f)\right]^2}, \tag{69}$$

where $g_L^Z(f)$ and $g_R^Z(f)$ are the couplings of the fermion f to the Z-boson:

$$\mathcal{L}\supset Z_\mu(g_{\text{L}}^Z(f)\overline{f}_{\text{L}}\gamma^\mu f_{\text{L}}+g_{\text{R}}^Z(f)\overline{f}_{\text{R}}\gamma^\mu f_{\text{R}}). \tag{70}$$

From the quark branching ratios $R(q)$ defined above, the hadronic left-right asymmetry Q_{LR} can be defined as [9] [29]

$$Q_{LR}\equiv\sum_{q=d,s,b}R(q)A_{LR}(q)-\sum_{q=u,c}R(q)A_{LR}(q). \tag{71}$$

A second class of asymmetries, the forward-backward asymmetries $A_{FB}(f)$, emerges from the convolution of the $A_{LR}(f)$ asymmetries with the polarization asymmetry $A_{LR}(e)$ of the electron. The hadronic charge asymmetry Q_{FB} is defined accordingly [9] [29]

$$A_{FB}(f)\equiv\frac{3}{4}A_{LR}(e)A_{LR}(f), \tag{72}$$

$$Q_{FB}\equiv\frac{3}{4}A_{LR}(e)Q_{LR}. \tag{73}$$

2. The Tau Lifetime

In terms of model parameters, the expression of the tau (τ) lifetime is similar to the muon (μ) lifetime in the $G(221)$ models, cf. Eq.(38), with the obvious replacement of m_μ in the μ lifetime by m_τ in the τ lifetime. This is true even in the non-universal (NU) model, in which third generation fermions transform under a different gauge group compared to the first two generations. In the four-fermion effective theory of the NU model, only interactions involving two pairs of third-generation fermions receive new physics contributions, and the interactions involving one pair of third-generation fermions with one pair of light-flavor fermions (those responsible for the decay of the τ) are the same as those between two pairs of first two generations of fermions (those responsible for the decay of μ). This is similar to the case of the un-unified model, where only interactions involving two pairs of quarks $(\overline{q}q)(\overline{q}q)$ receive new physics contributions, while the $(\overline{q}q)(\overline{\ell}\ell)$ interactions are the same as the $(\overline{\ell}\ell)(\overline{\ell}\ell)$. The lifetime τ_τ can be calculated at tree level as

$$\tau_\tau^{-1} \simeq \frac{G_F^2 m_\tau^5}{192\pi^3}\left(1+3\frac{m_\tau^2}{M_W^2}\right), \tag{74}$$

in the SM. The dominant new physics contribution from $G(221)$ models can be captured in the shift of M_W as shown in Eq. (59).

3. νN Deep Inelastic Scattering

The νN deep inelastic scattering experiments probe the coefficients $\varepsilon_L(q)$ and $\varepsilon_R(q)$ (for q being u or d) that parameterize the neutral current $\overline{\nu}\nu\overline{q}q$ interactions below the electroweak scale

$$\mathcal{L} \supset -\frac{G_F}{\sqrt{2}}\left(\overline{\nu}\nu\right)_{L,\mu}\sum_{q=u,d}\left[\varepsilon_L(q)\left(\bar{q}q\right)_L^\mu+\varepsilon_R(q)\left(\bar{q}q\right)_R^\mu\right]. \tag{75}$$

The DIS experiments measure the ratios of neutral-to-charged current cross sections

$$R_\nu \equiv \sigma_{\nu N}^{\rm NC}/\sigma_{\nu N}^{\rm CC}, \quad R_{\overline{\nu}} \equiv \sigma_{\overline{\nu}N}^{\rm NC}/\sigma_{\overline{\nu}N}^{\rm CC}, \tag{76}$$

which can be written in terms of $\varepsilon_L(q)$ and $\varepsilon_R(q)$ as

$$R_\nu = (1-\delta)\left[a_L(u)\varepsilon_L^2(u)+a_L(d)\varepsilon_L^2(d)+a_R(u)\varepsilon_R^2(u)+a_R(u)\varepsilon_R^2(d)\right], \tag{77}$$

$$R_{\bar{\nu}} = \left(1-\bar{\delta}\right)\left[\bar{a}_L(u)\varepsilon_L^2(u)+\bar{a}_L(d)\varepsilon_L^2(d)+\bar{a}_R(u)\varepsilon_R^2(u)+\bar{a}_R(u)\varepsilon_R^2(d)\right]. \tag{78}$$

The coefficients δ and $a_{L,R}$ are related to the nuclei form factors that are experiment specific. These coefficients are included in GAPP, and we implement only the corrections to $\varepsilon_L(q)$ and $\varepsilon_R(q)$.

4. νe Scattering

The most precise data on neutrino-electron scattering comes from the CHARM II [30] experiment at CERN that utilized ν_μ and $\overline{\nu}_\mu$. The relevant parameters $\varepsilon_L(e)$ and $\varepsilon_R(e)$ are defined similarly as in the νN scattering

$$\mathcal{L} \supset -\frac{G_F}{\sqrt{2}} \left(\bar{\nu}\nu\right)_{L,\mu} \left[\varepsilon_L(e)\left(\bar{e}e\right)_L^\mu + \varepsilon_R(e)\left(\bar{e}e\right)_R^\mu\right]. \tag{79}$$

We can further define

$$g_V^{\nu e} \equiv \varepsilon_R(e) + \varepsilon_L(e), \tag{80}$$

$$g_A^{\nu e} \equiv \varepsilon_R(e) - \varepsilon_L(e), \tag{81}$$

which are related to the measured total cross sections $\sigma_{\nu e}^{\rm NC}$ and $\sigma_{\bar{\nu} e}^{\rm NC}$ or their ratio $\sigma_{\nu e}^{\rm NC}/\sigma_{\bar{\nu} e}^{\rm NC}$. In the limit of large incident neutrino energies, $E_\nu \gg m_e$, the cross sections are given as

$$\sigma_{\nu e}^{\rm NC} = \frac{G_F^2 m_e E_\nu}{2\pi}\left[\left(g_V^{\nu e} + g_A^{\nu e}\right)^2 + \frac{1}{3}\left(g_V^{\nu e} - g_A^{\nu e}\right)^2\right], \tag{82}$$

$$\sigma_{\bar{\nu} e}^{\rm NC} = \frac{G_F^2 m_e E_\nu}{2\pi}\left[\left(g_V^{\nu e} - g_A^{\nu e}\right)^2 + \frac{1}{3}\left(g_V^{\nu e} + g_A^{\nu e}\right)^2\right]. \tag{83}$$

We implement corrections to the couplings due to new physics in GAPP and compute the cross sections that are used in the global-fit analysis.

5. Parity Violation Experiments

We consider observables from three different measurements: atomic parity violation (APV), Møller scattering ($e^-e^- \to e^-e^-$) [31], and eN DIS. These experiments measure the weak charge (Q_W) of the electron [31], caesium-133 [32][33] and thallium-205 nuclei [34][35]. Before defining the weak charge, it is useful to parameterize the coefficients of the $(\overline{e}e)(\overline{q}q)$ and $(\overline{e}e)(\overline{e}e)$ interactions in terms of C_{1q}, C_{2q}, and C_{1e} as

$$\mathcal{L} \supset -\frac{G_F}{\sqrt{2}}\sum_q \left[C_{1q}\left(\bar{e}e\right)_{A,\mu}\left(\bar{q}q\right)_V^\mu + C_{2q}\left(\bar{e}e\right)_{V,\mu}\left(\bar{q}q\right)_A^\mu\right] - \frac{G_F}{\sqrt{2}} C_{1e}\left(\bar{e}e\right)_{A,\mu}\left(\bar{e}e\right)_V^\mu \tag{84}$$

The weak charges of the quark and electron are defined as

$$Q_W(q) = 2C_{1q}, \qquad Q_W(e) = 2C_{1e}. \tag{85}$$

We can express the SM tree-level couplings of quarks to the Z-boson as $\mathcal{L} \supset Z^\mu J_\mu^Z$, where

$$J_\mu^Z = |g_A^Z(q)| \cdot \left[Q_W(q)\left(\bar{q}q\right)_{V,\mu} \pm \left(\bar{q}q\right)_{A,\mu}\right], \tag{86}$$

and the ± on the axial-vector term is the opposite sign of the T_L^{3q}. Hence $Q_W(q)$ can be interpreted as the ratio of the vector current to axial-vector current coupling of quark q to the Z-boson:

$$Q_{W,\mathrm{SM}}(q) = \frac{g_V^Z(q)}{|g_A^Z(q)|}. \tag{87}$$

The weak charges of the nucleons and nuclei can be built up from those of the quarks

$$Q_W(p) = 2Q_W(u) + Q_W(d), \tag{88}$$
$$Q_W(n) = Q_W(u) + 2Q_W(d), \tag{89}$$

and for nucleus ^{A}Z (with atomic number Z and mass number A), which contains Z protons and $N(= A - Z)$ neutrons,

$$Q_W\left({}^A\mathrm{Z}\right) = Z \cdot Q_W(p) + N \cdot Q_W(n) \tag{90}$$
$$= 2\left[(Z + A) \cdot C_{1u} + (2A - Z) \cdot C_{1d}\right]. \tag{91}$$

There are also measurements of certain linear combinations of the coupling coefficients C_{1u} and C_{1d} from polarized electron-hadron scattering data [36]. The particular linear combinations, determined by the experimental data,

$$\mathcal{C}_1 = 9C_{1u} + 4C_{1d},$$
$$\mathcal{C}_2 = -4C_{1u} + 9C_{1d}, \tag{92}$$

are included in our global analysis.

V. RESULTS

A. Global Analysis

In this section, we present the allowed regions of parameter space based on the global-fit analysis. A testament to the success of the SM is that, for all the $G(221)$ models, the global fitting pushes $\tilde{x}$ to large values, decoupling the effects of the new physics. This is presented in Figs. 1 and 2, where we show the 95% confidence level (C.L.) contours on the $\tilde{x} - c_{\tilde{\phi}}$ plane.

In addition to the constraints from the precision and low-energy data, we also require $\cos\phi$ ($\sin\phi$) to be greater than 0.1 (0.18) for the first (second) breaking pattern so that all the gauge couplings in Eq. (6), (7) and (8) are perturbative and do not exceed $\sqrt{4\pi}$. These constraints are shown as horizontal dotted lines in the figures.

Since $\tilde{x}$ and $\tilde{\phi}$ are defined in a model-dependent manner, it is also useful to show the corresponding contours on the $M_{Z'}$-$M_{W'}$ plane to compare different $G(221)$ models. We