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Liouville's Imaginary Shadow

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Abstract

N=1 super Liouville field theory is one of the simplest non-rational conformal field theories. It possesses various important extensions and interesting applications, e.g. to the AGT relation with 4D gauge theory or the construction of the OSP(1|2) WZW model. In both setups, the N=1 Liouville field is accompanied by an additional free fermion. Recently, Belavin et al. suggested a bosonization of the product theory in terms of two bosonic Liouville fields. While one of these Liouville fields is standard, the second turns out to be imaginary (or time-like). We extend the proposal to the R sector and perform extensive checks based on detailed comparison of 3-point functions involving several super-conformal primaries and descendants. On the basis of such strong evidence we sketch a number of interesting potential applications of this intriguing bozonization.

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1 Introduction

In this work we consider an interesting bosonization of $\mathcal{N}$=1 Liouville field theory that was proposed recently in [1]. $\mathcal{N}$=1 Liouville field theory contains one fermionic field ψ in addition to the Liouville field φ. These fields are coupled through the standard interaction term. For bosonization we need to add another free fermion η. The product theory appears naturally in several applications of $\mathcal{N}$=1 Liouville field theory. In particular, it has been used in [2] and [3] to compute various structure constants of the OSP(1|2) WZW model. More recently, it was considered in the context of the AGT correspondence [4] between supersymmetric 4D gauge theories and 2D conformal field theory [5, 6, 7, 8, 1].

In the bosonization, the two fermionic fields ψ and η are replaced by a single boson Y. What Belavin et al. proposed was that the two bosonic fields φ and Y can me mapped to a new set of bosonic fields, X and $\hat{X}$, where X is an ordinary (non-suspersymmetric) Liouville field and $\hat{X}$ an imaginary cousin. The latter may be thought of as a Liouville field which takes values in imaginary numbers. Because of its internal structure, we shall often refer to the fully bosonic model as *double Liouville theory* and to the factor associated with the field $\hat{X}$ as imaginary Liouville theory.

Imaginary Liouville theory is far from being an established model of 2-dimensional conformal field theory. In fact, there exist several different proposals for its structure constants but consistency (crossing symmetry) has never been established (see discussion in section 3). It is remarkable that one version of imaginary Liouville theory now appears through the bosonization of a consistent local conformal field theory.

The relation between $\mathcal{N}$=1 and double Liouville theory has a suggestive ancestor in rational conformal field theory. In that context, double Liouville theory gets replaced by a product of two minimal models and $\mathcal{N}$=1 Liouville theory by its rational counterpart. We can give a highly suggestive argument for their relation if we represent both models as coset conformal field theories. It is well known that ordinary minimal models arise through the cosets

$$\mathrm{MM}_k = (SU(2)_k \times SU(2)_1)/SU(2)_{k+1}$$

where $k = 1, 2, \ldots$. This family of rational models includes the Ising model MM_1 for a single fermion η when $k = 1$. Similarly, $\mathcal{N}$=1 supersymmetric minimal models are obtained from the coset

$$\mathrm{SMM}_k = (SU(2)_k \times SU(2)_2)/SU(2)_{k+2}\ .$$

If we allow ourselves to extend and reduce both numerator and denominator by the required additional factors we can easily see that

$$\mathrm{SMM}_{k-1} \times \mathrm{MM}_1 \sim \mathrm{MM}_k \times \mathrm{MM}_{k-1} \ . \tag{1.1}$$

Similar relations between 'generalized minimal models' and Virasoro minimal models were first discussed in [9], [10] and later (it seems independently) by [11],[12]. More recently, results for the 4D gauge theories [8] inspired Wyllard [13] to propose an extension to cosets of the type $(SU(N)_\kappa \times SU(N)_p)/SU(N)_{\kappa+p}$ where κ is a free parameter. Soon after this paper had appeared, the case of $N = 2, p = 2$ was considered in more detail by Belavin et al. [1].

Let us now describe the content of this work in more detail. We shall begin with a brief review of Liouville field theory and its $\mathcal{N}$=1 supersymmetric version in the next section. Both theories were solved long ago, see section 2 for references to the original literature. Then we turn to imaginary Liouville theory. As mentioned before, this model is very poorly understood. After a few historical comments we shall describe the 3-point functions that were proposed by Zamolodchikov in [14]. Our new results are formulated and analyzed in section 4. There we shall spell out a precise relation between an infinite tower of fields in $\mathcal{N}$=1 Liouville field theory and double Liouville theory. This relation will be checked through extensive comparison of 3-point functions on both sides of the correspondence. Applications and extensions of our results are sketched in the concluding section.

2 Review of Liouville field theory

In this section we simply review some basic facts about Liouville field theory and its $\mathcal{N}$=1 supersymmetric cousin. Most importantly, we shall discuss the spectrum of primary fields along with their 2- and 3-point functions. For a more details see the reviews [15, 16, 17].

2.1 Bosonic Liouville field theory

Liouville field theory involves a single scalar field with an exponential interaction term. On a 2-dimensional world-sheet with metric γ^{ab} and curvature R, the action of Liouville theory takes the form

$$S_L[X] \;=\; \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{\gamma}\left(\gamma^{ab}\partial_a X\partial_b X + RQX + 4\pi\mu_L e^{2bX}\right) \tag{2.1}$$

where μ_L and b are two (real) parameters of the model. The second term in this action describes the background charge of a linear dilaton. The value of the constant Q must be adjusted to the choice of b in order for S_L to define a conformal quantum field theory. We shall state the relation in a moment.

Liouville theory should be considered as a marginal deformation of the free linear dilaton theory. The Virasoro field of a linear dilaton theory is given by the familiar expression

$$T(z) \;=\; -(\partial X)^2 + Q\partial^2 X \quad .$$

The modes of this field form a Virasoro algebra with central charge $c_L = 1 + 6Q^2$. Furthermore, the usual closed string vertex operators

$$V_\alpha(z) \;=\, :\exp 2\alpha X(z,\bar{z}): \qquad \text{have} \quad h_\alpha \;=\; \alpha(Q-\alpha) \;=\; \bar{h}_\alpha \quad . \tag{2.2}$$

Here and in the following we shall not explicitly display the dependence of our vertex operators on the complex conjugate $\bar{z}$ of the world-sheet coordinate z. Note the conformal weights $h, \bar{h}$ are real if α is of the form $\alpha = Q/2 + iP$. In order for the exponential potential in the Liouville action to be marginal, i.e. $(h_b, \bar{h}_b) = (1,1)$, we must now also adjust the parameter Q to the choice of b in such a way that

$$Q = b + b^{-1} \quad .$$

Weyl invariance of the classical action S_L leads to the relation $Q_c = b^{-1}$ and the additional shift by b may be considered as a quantum correction of the classical relation. The extra term, which certainly becomes small in the semi-classical limit $b \to 0$, renders $Q = Q_c + b$ (and hence the central charge) invariant under the replacement $b \to b^{-1}$.

The solution of Liouville field theory is completely described by the 2- and 3-point functions of the model. The vertex operators V_α are introduced such that their 2-point function is canonically normalized, i.e.

$$\langle V_{\alpha_2}(z_2)V_{\alpha_1}(z_1)\rangle = |z_{12}|^{-4h_{\alpha_1}}2\pi\left(\delta(\alpha_1+\alpha_2-Q) + D_L(\alpha_1)\delta(\alpha_2-\alpha_1)\right) \tag{2.3}$$

where

$$D_L(\alpha) = \left(\pi\mu_L\gamma(b^2)\right)^{\frac{(Q-2\alpha)}{b}} \frac{\gamma(2\alpha b - b^2)}{b^2\,\gamma(2-2\alpha b^{-1}+b^{-2})} \tag{2.4}$$

Here and throughout the main text we use $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. In order to spell out the 3-point functions we need to introduce Barnes' double Γ-function $\Gamma_b(y)$. It may be defined through the following integral representation,

$$\ln \Gamma_b(y) \;=\; \int_0^\infty \frac{d\tau}{\tau}\left[\frac{e^{-y\tau}-e^{-Q\tau/2}}{(1-e^{-b\tau})(1-e^{-\tau/b})} - \frac{\left(\frac{Q}{2}-y\right)^2}{2}e^{-\tau} - \frac{\frac{Q}{2}-y}{\tau}\right] \tag{2.5}$$

for all $b \in \mathbb{R}$. The integral exists when $0 < \mathrm{Re}(y)$ and it defines an analytic function which may be extended onto the entire complex y-plane. Under shifts by $b^{\pm 1}$, the function Γ_b behaves according to

$$\Gamma_b(y+b) \;=\; \sqrt{2\pi}\,\frac{b^{by-\frac{1}{2}}}{\Gamma(by)}\,\Gamma_b(y) \quad , \quad \Gamma_b(y+b^{-1}) \;=\; \sqrt{2\pi}\,\frac{b^{-\frac{y}{b}+\frac{1}{2}}}{\Gamma(b^{-1}y)}\,\Gamma_b(y) \quad . \tag{2.6}$$

These shift equations let Γ_b appear as an interesting generalization of the usual Γ function which may also be characterized through its behavior under shifts of the argument. But in contrast to the ordinary Γ function, Barnes' double Γ function satisfies two such equations which are independent if b is not rational. We furthermore deduce from eqs. (2.6) that Γ_b has poles at

$$y_{n,m} \;=\; -nb - mb^{-1} \qquad \text{for} \qquad n,m \;=\; 0,1,2,\dots \quad . \tag{2.7}$$

From Branes' double Gamma function one may construct the following basic building block of the 3-point function,

$$\Upsilon_b(\alpha) \;:=\; \Gamma_2(\alpha|b,b^{-1})^{-1}\,\Gamma_2(Q-\alpha|b,b^{-1})^{-1} \quad . \tag{2.8}$$

The properties of the double Γ-function imply that Υ possesses the following integral representation

$$\ln \Upsilon_b(y) \;=\; \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2}-y\right)^2 e^{-t} - \frac{\sinh^2\left(\frac{Q}{2}-y\right)\frac{t}{2}}{\sinh\frac{bt}{2}\,\sinh\frac{t}{2b}}\right] \quad . \tag{2.9}$$

Moreover, we deduce from the two shift properties (2.6) of the double Γ-function that

$$\Upsilon_b(y+b) \;=\; \gamma(by)\,b^{1-2by}\,\Upsilon_b(y) \quad , \quad \Upsilon_b(y+b^{-1}) \;=\; \gamma(b^{-1}y)\,b^{-1+2b^{-1}y}\,\Upsilon_b(y) \quad . \tag{2.10}$$

Note that the second equation can be obtained from the first with the help of the self-duality property $\Upsilon_b(y) = \Upsilon_{b^{-1}}(y)$.

After this preparation it is easy to spell out the 3-point function of primary fields in Liouville field theory [18, 19],

$$\langle V_{\alpha_3}(z_3)V_{\alpha_2}(z_2)V_{\alpha_1}(z_1)\rangle = \frac{C_L(\alpha_3,\alpha_2,\alpha_1|b)}{|z_{12}|^{2h_{12}}|z_{13}|^{2h_{13}}|z_{23}|^{2h_{23}}} \tag{2.11}$$

with $h_{12} = h_{\alpha_1} + h_{\alpha_2} - h_{\alpha_3}$ etc. and coupling constants C_L of the form

$$C_L(\alpha_3,\alpha_2,\alpha_1|b) = \left[\pi\mu_L\gamma(b^2)b^{2-2b^2}\right]^{\frac{Q-\alpha}{b}} \frac{\Upsilon_b^0\,\Upsilon_b\,(2\alpha_1)\,\Upsilon_b\,(2\alpha_2)\,\Upsilon_b\,(2\alpha_3)}{\Upsilon_b\,(\alpha_{123}-Q)\ \Upsilon_b\,(\alpha_{12})\ \Upsilon_b\,(\alpha_{13})\ \Upsilon_b\,(\alpha_{23})}\ . \tag{2.12}$$

Here and in the following, the contant Υ_b^0 is given by $\Upsilon_b^0 = \Upsilon_b'(0)$. Furthermore, the parameters α_{123} and α_{ij} are certain linear combinations of α_j,

$$\alpha_{123} = \alpha_1+\alpha_2+\alpha_3\ ,\ \ \alpha_{12} = \alpha_1+\alpha_2-\alpha_3 \quad \text{etc.}$$

The solution (2.12) was first proposed by H. Dorn and H.J. Otto [18] and by A. and Al. Zamolodchikov [19], based on extensive earlier work by many authors (see e.g. the reviews [20, 15, 16] for references). Full crossing symmetry of the conjectured 3-point function was established much later in two steps by Ponsot and Teschner [21] and by Teschner [15, 22]. The proof of consistency of the DOZZ structure constants for Liouville field theory was completed recently by establishing modular invariance of 1-point functions on a torus [23].

2.2 $\mathcal{N}$=1 Liouville field theory

$\mathcal{N}$=1 supersymmetric Liouvilel field involves one real superfield that contains a real bosonic scalar φ, the two components ψ and $\bar\psi$ of a Majorana fermion and an auxiliary field F. After integrating out the latter and fixing the world-sheet metric, the action of $\mathcal{N}$=1 super Liouville field theory takes the form

$$S_{SL}[\varphi,\psi] \ = \ \frac{1}{2\pi}\int d^2z\,\left[\partial\varphi\bar\partial\varphi+\psi\bar\partial\psi+\bar\psi\partial\bar\psi\right]+2i\mu b^2\int d^2z\psi\bar\psi e^{b\varphi}\ , \tag{2.13}$$

The background charge for the boson φ is related to the parameter b by $Q = b + 1/b$. As in the case of bosonic Liouville field theory, the supersymmetric cousin is obtained by perturbing a free field theory, namely the product of a linear dilaton with a 2-dimensional Ising model. The spectrum of the Ising model contains six conformal blocks including the identity field, the two components ψ and $\bar\psi$ of the fermion and the energy density $\psi\bar\psi$, which are all part of the Neveu-Schwarz (NS) sector. In addition, there are two blocks in the Ramond (R) sector. These are generated from the spin field $\varsigma^+ = \sigma$ and the so-called disorder field $\varsigma^- = \mu$. After multiplication with the linear dilator, the model contains an $\mathcal{N}$=1 super-conformal symmetry with central charge $c_{SL} = \frac{3}{2}(1+2Q^2)$. The holomorphic half of this symmetry is generated by modes of the following fields

$$T(z) = -\frac{1}{2}\left((\partial\varphi)^2 - Q\partial^2\varphi + \psi\partial\psi\right) \quad , \quad G(z) = -i(\psi\partial\varphi - Q\partial\psi) \ . \tag{2.14}$$

Anti-holomorphic fields can be constructed similarly. The interacting theory has been solved soon after the DOZZ proposal had been put out, see [24, 25]. Vertex operators in the NS sector are super-descendents of

$$\phi_\alpha(z) =: \exp \alpha\varphi(z,\bar{z}) : \qquad \text{with} \qquad \Delta_\alpha = \alpha(Q-\alpha)/2 = \bar{\Delta}_\alpha \tag{2.15}$$

The 2-point function of these NS primary fields takes the form

$$\langle \phi_{\alpha_2}(z_2)\phi_{\alpha_1}(z_1)\rangle \ = \ |z_{12}|^{-4\Delta_{\alpha_1}} 2\pi \left[\delta(\alpha_1+\alpha_2-Q) + \delta(\alpha_2-\alpha_1)D_{NS}(\alpha_1)\right] \ , \tag{2.16}$$

with

$$D_{NS}(\alpha) \ = \ -\left(\mu\pi\gamma(\tfrac{bQ}{2})\right)^{\frac{Q-2\alpha}{b}} \frac{\Gamma(b(\alpha-\frac{Q}{2}))\Gamma(\frac{1}{b}(\alpha-\frac{Q}{2}))}{\Gamma(-b(\alpha-\frac{Q}{2}))\Gamma(-\frac{1}{b}(\alpha-\frac{Q}{2}))} \ . \tag{2.17}$$

Whereas the first term in eq. (2.16) is fixed by normalization, the second term involving D_{NS} contains dynamical information on the phase shift of tachyonic modes upon reflection off the Liouville wall.

To spell out the 3-point functions of the model we need to build two new special functions from the Υ-function we introduced in the previous subsection, see eq. (2.9). These are given by

$$\Upsilon_b^{\rm NS}(x) = \Upsilon_b(\tfrac{x}{2})\Upsilon_b(\tfrac{x+Q}{2}) \ , \qquad \Upsilon_b^{\rm R}(x) = \Upsilon_b(\tfrac{x+b}{2})\Upsilon_b(\tfrac{x+b^{-1}}{2}) \ . \tag{2.18}$$

Properties of these new functions can easily be derived from the properties of Υ_b we listed above. In particular, we note that the functions Υ_b^{NS} and $\Upsilon_b^{\rm R}$ possess the following behavior under shifts of their argument,

$$\Upsilon_b^{\rm NS}(x+b) = b^{-bx}\gamma(\tfrac{1}{2}+\tfrac{bx}{2})\Upsilon_b^{\rm R}(x) \ , \qquad \Upsilon_b^{\rm R}(x+b) = b^{1-bx}\gamma(\tfrac{bx}{2})\Upsilon_b^{\rm NS}(x) \ , \tag{2.19}$$

$$\Upsilon_b^{\rm NS}(x+\tfrac{1}{b}) = b^{\frac{x}{b}}\gamma(\tfrac{1}{2}+\tfrac{x}{2b})\Upsilon_b^{\rm R}(x) \ , \qquad \Upsilon_b^{\rm R}(x+\tfrac{1}{b}) = b^{-1+\frac{x}{b}}\gamma(\tfrac{x}{2b})\Upsilon_b^{\rm NS}(x) \ . \tag{2.20}$$

The functions $\Upsilon_b^{\rm NS}, \Upsilon_b^{\rm R}$ suffice to state the 3-point structure constants of the NS sector,

$$\langle \phi_{\alpha_3}(z_3)\phi_{\alpha_2}(z_2)\phi_{\alpha_1}(z_1)\rangle = \frac{C_{NS}(\alpha_3,\alpha_2,\alpha_1|b)}{|z_{12}|^{2h_{12}}|z_{13}|^{2h_{13}}|z_{23}|^{2h_{23}}} \tag{2.21}$$

$$\langle \phi_{\alpha_3}(z_3)\tilde\phi_{\alpha_2}(z_2)\phi_{\alpha_1}(z_1)\rangle = \frac{\tilde C_{NS}(\alpha_3,\alpha_2,\alpha_1|b)}{|z_{12}|^{2h_{12}+1}|z_{13}|^{2h_{13}-1}|z_{23}|^{2h_{23}+1}} \tag{2.22}$$

where $\tilde\phi_\alpha = \{G_{-\frac12},[\bar G_{-\frac12},\phi_\alpha]\}$, and

$$\begin{aligned} C_{NS}(\alpha_3,\alpha_2,\alpha_1|b) &= \frac12\left[\frac{\pi\mu}{2b^{b^2-1}}\gamma\left(\frac{Qb}{2}\right)\right]^{\frac{Q-\alpha_{123}}{b}} \frac{\Upsilon_b^0\,\Upsilon_b^{\rm NS}(2\alpha_1)\Upsilon_b^{\rm NS}(2\alpha_2)\Upsilon_b^{\rm NS}(2\alpha_3)}{\Upsilon_b^{\rm NS}(\alpha_{123}-Q)\Upsilon_b^{\rm NS}(\alpha_{12})\Upsilon_b^{\rm NS}(\alpha_{23})\Upsilon_b^{\rm NS}(\alpha_{13})} \\ \tilde C_{NS}(\alpha_3,\alpha_2,\alpha_1|b) &= i\left[\frac{\pi\mu}{2b^{b^2-1}}\gamma\left(\frac{Qb}{2}\right)\right]^{\frac{Q-\alpha_{123}}{b}} \frac{\Upsilon_b^0\,\Upsilon_b^{\rm NS}(2\alpha_1)\Upsilon_b^{\rm NS}(2\alpha_2)\Upsilon_b^{\rm NS}(2\alpha_3)}{\Upsilon_b^{\rm R}(\alpha_{123}-Q)\Upsilon_b^{\rm R}(\alpha_{12})\Upsilon_b^{\rm R}(\alpha_{23})\Upsilon_b^{\rm R}(\alpha_{13})} \end{aligned} \tag{2.23}$$

Any 3-point function of descendent fields can be written in terms of the correlator (2.21) or (2.22) and 3-point blocks which are completely determined by the super-conformal Ward identities (see e.g. [26]).

Let us now turn to the R sector of the model. As we recalled before, the 2-dimensional Ising model possesses two local fields of conformal weight $\Delta = 1/16 = \bar\Delta$ which we denoted by $\varsigma^+ = \sigma$ and $\varsigma^- = \mu$, see chapter 12 of [27] for more details. Using these spin fields, we can define the following two vertex operators in the R sector of $\mathcal{N}$=1 Liouville theory

$$\Sigma_\alpha^\pm(z) = \ \varsigma^\pm(z,\bar z)\ :e^{\alpha\varphi(z,\bar z)}: \quad \text{with} \quad \Delta_\alpha^{\rm R} = \frac12\alpha(Q-\alpha)+\frac1{16} = \bar\Delta_\alpha^{\rm R}\,. \tag{2.24}$$

Our conventions are the same as in [25, 28] and they imply

$$G_0\Sigma_\alpha^\pm(z) = i\beta e^{\mp i\frac{\pi}{4}}\Sigma_\alpha^\mp(z), \quad \bar G_0\Sigma_\alpha^\pm(z) = -i\beta e^{\pm i\frac{\pi}{4}}\Sigma_\alpha^\mp(z), \qquad \beta = \frac{1}{\sqrt2}\left(\frac{Q}{2}-\alpha\right)\,. \tag{2.25}$$

The 2-point functions of the vertex operators Σ_α^ϵ possess the following form

$$\langle \Sigma_{\alpha_2}^\pm(z_2)\Sigma_{\alpha_1}^\pm(z_1)\rangle \ = \ |z_{12}|^{-4\Delta_{\alpha_1}-\frac14}2\pi\left[\delta(\alpha_1+\alpha_2-Q)\pm\delta(\alpha_2-\alpha_1)D_R(\alpha_1)\right] \tag{2.26}$$

with a reflection coefficient given by

$$D_R(\alpha) \ = \ \left(\mu\pi\gamma(\tfrac{bQ}{2})\right)^{\frac{Q-2\alpha}{b}} \frac{\Gamma(\frac12+b(\alpha-\frac Q2))\Gamma(\frac12+\frac1b(\alpha-\frac Q2))}{\Gamma(\frac12-b(\alpha-\frac Q2))\Gamma(\frac12-\frac1b(\alpha-\frac Q2))}\,. \tag{2.27}$$

Let us also provide explicit expressions for the 3-point functions involving two RR fields. These were determined in [24, 25, 29] and we shall simply quote the results along with all the necessary notations,

$$\langle \phi_{\alpha_3}(z_3)\Sigma^{\pm}_{\alpha_2}(z_2)\Sigma^{\pm}_{\alpha_1}(z_1)\rangle \;=\; \frac{C^{\pm}_R(\alpha_3;\alpha_2,\alpha_1|b)}{|z_{12}|^{2\Delta_{12}+\frac{1}{4}}|z_{23}|^{2\Delta_{23}}|z_{13}|^{2\Delta_{13}}}\;. \tag{2.28}$$

The structure constants $C^{\pm}_R$ are constructed from the special functions $\Upsilon^{\rm NS}$ and $\Upsilon^{\rm R}$ as follows,

$$\begin{aligned} C^{\pm}_R(\alpha_3;\alpha_2,\alpha_1|b) \;=\;& \frac{1}{2}\left[\frac{\mu\pi}{2}\gamma(\tfrac{bQ}{2})b^{1-b^2}\right]^{\frac{Q-\alpha_{123}}{b}} \frac{\Upsilon^0_b\,\Upsilon^{\rm R}_b(2\alpha_1)\Upsilon^{\rm R}_b(2\alpha_2)\Upsilon^{\rm NS}_b(2\alpha_3)}{\Upsilon^{\rm R}_b(\alpha_{123}-Q)\Upsilon^{\rm R}_b(\alpha_{12})\Upsilon^{\rm NS}_b(\alpha_{23})\Upsilon^{\rm NS}_b(\alpha_{13})} \\ &\pm \frac{1}{2}\left[\frac{\mu\pi}{2}\gamma(\tfrac{bQ}{2})b^{1-b^2}\right]^{\frac{Q-\alpha_{123}}{b}} \frac{\Upsilon^0_b\,\Upsilon^{\rm R}_b(2\alpha_1)\Upsilon^{\rm R}_b(2\alpha_2)\Upsilon^{\rm NS}_b(2\alpha_3)}{\Upsilon^{\rm NS}_b(\alpha_{123}-Q)\Upsilon^{\rm NS}_b(\alpha_{12})\Upsilon^{\rm R}_b(\alpha_{23})\Upsilon^{\rm R}_b(\alpha_{13})}\;. \end{aligned} \tag{2.29}$$

Crossing symmetry of 4-point functions in the NS sector of $\mathcal{N}$=1 Liouville theory with structure constants (2.23) and (2.29) was first checked numerically, see [30, 31], and later proved analytically in [32, 33] using braiding and fusion properties of the 4-point blocks. In the case of 4-point functions containing R fields, crossing symmetry of $\mathcal{N}$=1 Liouville theory was verified numerically in [34]. The first step necessary for an analytical proof was presented in [35] where braiding properties of the 4-point blocks were derived.

3 Imaginary Liouville theory

Before we can state the main results of this work, we need one more ingredient, namely a version of Liouville field theory with central charge $c \leq 1$. In contrast to the models we described in the previous section, the status of the theory we are about to discuss is less clear. In particular, the issue of crossing symmetry has not been settled. We shall begin our exposition with a few historical comments in the first subsection. Then we continue by listing the proposed structure constants without much further discussion.

3.1 Some comments on history

In usual Liouville theory, the parameter b is taken to be real so that the corresponding central charge $c \geq 25$. The explicit expressions for 2- and 3-point functions admit analytic continuation to complex values of b with a non-vanishing real part. Formally, the central charge takes values $1 < c$ in this regime. Purely imaginary values of b have been a subject of several previous studies mostly because such values are relevant for time-like Liouville field theory and tachyon condensation in string theory, see e.g. [36, 37, 38, 39, 40, 41, 42] and further references in the more recent papers.

At least for $b = i$, it is possible to define the theory by taking a limit starting with $b = \epsilon + i$. The resulting theory has central charge $c = 1$ and it agrees with a certain limit of unitary minimal models. This limit was shown to satisfy crossing symmetry [43]. It is likely that similar limits can be taken for other purely imaginary values of i. But even if such limits describe consistent local quantum theories, they would at most be defined for a discrete set of b-parameters.

There is an alternative approach to defining Liouville theory for imaginary b, i.e. for $c \leq 1$. In order to describe how this works, let us recall a few facts about the usual construction of the 3-point couplings in Liouville field theory. The main idea is to evaluate crossing symmetry for 4-point functions with three physical and one degenerate field insertions. The operator product of a physical with a degenerate field involves a finite set of terms whose coefficients can be computed in free field theory. More precisely, if we take the degenerate field to be $V_{-b^{\pm 1}/2}$, then the 4-point function must satisfy a second order differential equation and hence only two terms can possibly arise on the left hand side of the operator product, e.g.

$$V_\alpha(w,\bar{w})\, V_{-b/2}(z,\bar{z}) \;=\; \sum_{\pm} \frac{c_b^{\pm}(\alpha)}{|z-w|^{h_\pm}}\; V_{\alpha\mp b/2}(z,\bar{z}) \;+\; \dots \tag{3.1}$$

where $h_\pm = \mp b\alpha + Q(-b/2 \mp b/2)$. A similar expansion for the second degenerate field is obtained by replacing $b \to b^{-1}$. We can even be more specific about the operator expansions of degenerate fields because the coefficients $c^\pm$ may be determined through a simple free field computation in the linear dilaton background. One finds that

$$\begin{aligned} c_b^-(\alpha) &= -\mu_L \int d^2z\, \langle\, V_{-b/2}(0,0)\, V_\alpha(1,1)\, V_b(z,\bar{z})\, V_{Q-b/2-\alpha}(\infty,\infty)\,\rangle_{\rm LD} \\ &= -\mu_L \pi\, \frac{\gamma(1+b^2)\,\gamma(1-2b\alpha)}{\gamma(2+b^2-2b\alpha)} \end{aligned} \tag{3.2}$$

see [16] for more details. The result in the second line is obtained using the explicit integral formulas that were derived by Dotsenko and Fateev. The corresponding field is then degenerate and it possesses an operator product consisting of two terms only.

Teschner's trick converts the crossing symmetry condition into a much simpler algebraic condition. Moreover, since we have already computed the coefficients of operator products with degenerate fields, the crossing symmetry equation is in fact linear in the unknown generic 3-point couplings. One component of these conditions for the degenerate field $V_{-b/2}$ reads as follows

$$0 \;=\; C_L(\alpha_1+\frac{b}{2},\alpha_3,\alpha_4)\,c_b^-(\alpha_1)\mathcal{P}^{--}_{+-} + C_L(\alpha_1-\frac{b}{2},\alpha_3,\alpha_4)\,c_b^+(\alpha_1)\mathcal{P}^{++}_{+-} \quad , \tag{3.3}$$

$$\text{where} \qquad \mathcal{P}^{\pm\pm}_{+-} \;=\; F_{\alpha_1\mp b/2,\alpha_3-b/2}\left[\begin{smallmatrix} -b/2 & \alpha_3 \\ \alpha_1 & \alpha_4 \end{smallmatrix}\right] F_{\alpha_1\mp b/2,\alpha_3+b/2}\left[\begin{smallmatrix} -b/2 & \alpha_3 \\ \alpha_1 & \alpha_4 \end{smallmatrix}\right] \quad .$$

Note that the combination on the right hand side must vanish because in a consistent model, the off-diagonal bulk mode $(\alpha_4 - b/2, \alpha_4 + b/2)$ does not exist and hence it cannot propagate in the intermediate channel. The required special entries of the Fusing matrix can be expressed through a combination of Γ functions. Once the expressions for $c^\pm$ and $\mathcal{P}$ are inserted (note that they only involve Γ functions), the crossing symmetry condition may be written as follows,

$$\frac{C_L(\alpha_1+b,\alpha_2,\alpha_3)}{C_L(\alpha_1,\alpha_2,\alpha_3)} \;=\; -\frac{\gamma(b(2\alpha_1+b))\gamma(2b\alpha_1)}{\pi\mu_L\gamma(1+b^2)\gamma(b(\alpha_{123}-Q))}\frac{\gamma(b(\alpha_{23}-b))}{\gamma(b\alpha_{13})\gamma(b\alpha_{12})} \tag{3.4}$$

with $\gamma(x) = \Gamma(x)/\Gamma(1-x)$, as before. The constraint takes the form of a shift equation that describes how the coupling changes if one of its arguments is shifted by b. Using the symmetry $b \leftrightarrow b^{-1}$ we obtain a second shift equation that encodes how the 3-point couplings behave under shifts by b^{-1}. For irrational values of b, the two shift equations determine the couplings completely, at least if we require that they are analytic in the momenta. The unique solution turns out to be analytic in b as well so that it may be extended to all real values of the parameter b.

We are now prepared to take a fresh look at the problem of constructing imaginary Liouville theory. While the structure constants (2.12) are not analytic in b so that their extension to imaginary b (or $c \leq 1$) may be ill-defined, the coefficients of the shift equation (3.4) involve only Γ functions so that a continuation to imaginary values of b is straight forward. If we postulate that the 3-point couplings of imaginary Liouville theory are analytic in the parameters α_i and exists for all $c \leq 1$, then there is again a unique solution [14]. We shall describe this solution in the following subsection.

3.2 Zamolodchikov's solution

Imaginary Liouville theory may be thought of as a model whose action is formally given by

$$S_{\mathcal{L}}[\hat{X}] \;=\; \frac{1}{4\pi}\int_{\Sigma} d^2\sigma\sqrt{\gamma}\left(-\gamma^{ab}\partial_a\hat{X}\partial_b\hat{X} + R\hat{Q}\hat{X} + 4\pi\mu_{\mathcal{L}}e^{-2\hat{b}\hat{X}}\right) \tag{3.5}$$

One can obtain it the usual action of ordinary Liouville theory by the formal replacements $X \to -i\hat{X}$, $b \to -i\hat{b}$ and $Q \to i\hat{Q}$. Vertex operators in this model take the form

$$\mathcal{V}_{\hat{\alpha}}(z) =: e^{2\hat{\alpha}\hat{X}(z,\bar{z})} \quad \text{with} \quad \hat{h}_{\hat{\alpha}} = -\hat{\alpha}(\hat{Q}-\hat{\alpha}) = \bar{\hat{h}}_{\hat{\alpha}} \;. \tag{3.6}$$

They are obtained from the vertex operators of ordinary Liouville theory if we replace α by $\alpha \to -\hat{\alpha}$. For conformal invariance, the parameter $\hat{Q}$ must be adjusted to the parameter $\hat{b}$ such that

$$\hat{Q} = \hat{b}^{-1} - \hat{b} \;\;. \tag{3.7}$$

In terms of these parameters, the central charge of the Virasoro algebra is now given by $c_{\mathcal{L}} = 1 - 6\hat{Q}^2$.

As we have argued in the previous subsection, it is somewhat natural to introduce the 3-point coupling of this imaginary Liouville theory such that it the shift equation (3.4) is satisfied. In terms of the real parameters $\hat{\alpha}$ and $\hat{b}$, the shift equation reads

$$\frac{C_{\mathcal{L}}(\hat{\alpha}_1-\hat{b},\hat{\alpha}_2,\hat{\alpha}_3)}{C_{\mathcal{L}}(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3)} \;=\; -\frac{\gamma(\hat{b}(2\hat{\alpha}_1-\hat{b}))\gamma(2\hat{b}\hat{\alpha}_1)}{\pi\mu_{\mathcal{L}}\gamma(1-\hat{b}^2)\gamma(\hat{b}(\hat{\alpha}_{123}-\hat{Q}))}\frac{\gamma(\hat{b}(\hat{\alpha}_{23}+\hat{b}))}{\gamma(\hat{b}\hat{\alpha}_{13})\gamma(\hat{b}\hat{\alpha}_{12})} \;. \tag{3.8}$$

Here we have simply carried out the substitutions we listed after eqs. (3.5) and (3.6). If we shift $\hat{\alpha}_1$ by $\hat{\beta}$ and invert the relation we obtain,

$$\frac{C_{\mathcal{L}}(\hat{\alpha}_1+\hat{b},\hat{\alpha}_2,\hat{\alpha}_3)}{C_{\mathcal{L}}(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3)} \;=\; -\frac{\pi\mu_{\mathcal{L}}\gamma(\hat{b}(\hat{\alpha}_{123}-\hat{b}^{-1}+2\hat{b}))}{\gamma(\hat{b}^2)\gamma(\hat{b}(2\hat{\alpha}_1))\gamma(2\hat{b}(\hat{\alpha}_1+\hat{b}))}\frac{\gamma(\hat{b}(\hat{\alpha}_{13}+\hat{b}))\gamma(\hat{b}(\hat{\alpha}_{12}+\hat{b}))}{\gamma(\hat{b}\hat{\alpha}_{23})} \;. \tag{3.9}$$

Note that all the factors that depend on linear combination of the variables $\hat{\alpha}_i$ are the same as in eq. (3.4), except for a simple shift by $\hat{b}$. Factors depending on α_1 are not universal since they are effected by the normalization of vertex operators. Following [14] we fix the normalization such that

$$\begin{aligned}
\langle \mathcal{V}_{\hat{\alpha}}(z_2)\mathcal{V}_{\hat{\alpha}}(z_1)\rangle \;&=\; |z_{12}|^{-4h_{\hat{\alpha}}}G(\hat{\alpha}) \;, \\
G(\hat{\alpha}) \;&=\; \left(\pi\mu_{\mathcal{L}}\gamma(-\hat{b}^2)\right)^{\frac{2\hat{\alpha}}{\hat{b}}}\frac{\gamma(2\hat{\alpha}\hat{b}+\hat{b}^2)\,\gamma(2-\hat{b}^{-2})}{\gamma(2+2\hat{\alpha}\hat{b}^{-1}-\hat{b}^{-2})\,\gamma(\hat{b}^2)} \;.
\end{aligned}$$

The expression on the left hand side is obtained from the second term in eq. (2.4) by our standard substitutions. Once this normalization is adopted, the associated 3-point couplings take the form

$$C_{\mathcal{L}}(\hat{\alpha}_3,\hat{\alpha}_2,\hat{\alpha}_1|\hat{b}) \;=\; \left(\pi\mu_{\mathcal{L}}\gamma\left(-\hat{b}^2\right)\right)^{\frac{\hat{\alpha}_{123}}{\hat{b}}}\; b^{2(b+b^{-1})(\hat{\alpha}_{123}+\hat{b}-\frac{1}{\hat{b}})}\,\frac{\gamma\left(2-\hat{b}^{-2}\right)}{\gamma\left(\hat{b}^2\right)}\,\hat{b}^2 \qquad (3.10)$$

$$\times\frac{\Upsilon_{\hat{b}}\left(\hat{\alpha}_{123}-\hat{b}^{-1}+2\hat{b}\right)\Upsilon_{\hat{b}}\left(\hat{\alpha}_{12}+\hat{b}\right)\Upsilon_{\hat{b}}\left(\hat{\alpha}_{23}+\hat{b}\right)\Upsilon_{\hat{b}}\left(\hat{\alpha}_{13}+\hat{b}\right)}{\Upsilon^0_{\hat{b}}\,\Upsilon_{\hat{b}}\left(2\hat{\alpha}_1+\hat{b}\right)\Upsilon_{\hat{b}}\left(2\hat{\alpha}_2+\hat{b}\right)\Upsilon_{\hat{b}}\left(2\hat{\alpha}_3+\hat{b}\right)}\;.$$

It is easy to check that these structure constants solve the shift equations (3.9), though with a different α_1-dependent prefactor. This concludes our presentation of imaginary Liouville theory.

4 Bosonization of $\mathcal{N}$=1 Liouville field theory

It is well known [27] that a certain orbifold of the product of two real fermions can be bosonized, i.e. it is equivalent to a compactified free boson with compactification radius $R = 1$. We will now show that a similar bosonization exists for an orbifold of the product of $\mathcal{N}$=1 Liouville field theory with a free fermion η. In this case, the bosonic description involves two Liouville fields, one with real and the other with imaginary parameter b. This relation was first conjectured in [1] for the Neveu-Schwarz sector of the supersymmetric Liouville field theory. We will extend the correspondence to the Ramond sector and perform extensive tests for a number of local 3-point functions.

4.1 Product of $\mathcal{N}$=1 Liouville and a fermion

Before we discuss the product of $\mathcal{N}$=1 Liouville theory and a free fermion η, let us briefly review a few things about a product of fermions. As before, we shall denote one of our fermions by $\psi,\bar{\psi}$ and the other by $\eta,\bar{\eta}$. Both ψ and η are assumed to possess the same standard operator product, i.e.

$$\psi(z)\psi(w)\sim\frac{1}{z-w}\quad,\quad\eta(z)\eta(w)\sim\frac{1}{z-w}\;.$$

While the first fermion ψ is assumed to be real, i.e. $\psi^\dagger=\psi^*=\psi$, we will modify the usual conjugation for η such that $\eta^\dagger=-\eta$. In this sense, the fermion η may be considered imaginary. Note that the usual conjugation $\eta^* = \eta$ differs from the conjugation $\dagger$ by a simple automorphism of the fermionic theory. In fact, the map $\eta \to -\eta$ preserves the operator product of the fermion η. While the algebraic properties of the two fermions are identical, we will use a different bilinear form on their state spaces, one that preserves $\dagger$ rather than the usual $*$. As one can easily see, this form is indefinite. Our choice will be motivated a posteriori through the relation with double Liouville theory (see next subsection). Alternatively, one may observe that an imaginary fermion η emerges naturally in the reduction from the OSP(1|2) WZW model to $\mathcal{N}$=1 Liouville field theory (see formula (2.17) of [2]).

The theory of a single fermion possesses six conformal primaries, namely the identity, the fermion fields, the energy density and two spin fields. The latter will be denoted by $\varsigma^\pm$ and $\sigma^\pm$ for fermions ψ and η, respectively. In order to fix our conventions for $\sigma^\pm$, let us state the analogue of the relations (2.25)

$$\eta_0 \sigma^\pm = \frac{1}{\sqrt{2}}\, e^{\mp i\frac{\pi}{4}}\, \sigma^\mp, \qquad \bar{\eta}_0 \sigma^\pm = \frac{1}{\sqrt{2}}\, e^{\pm i\frac{\pi}{4}}\, \sigma^\mp \;. \tag{4.1}$$

Here, η_0 and $\bar{\eta}_0$ denote the zero modes of the fermionic fields η and $\bar{\eta}$, respectively. Due to the conjugation rules of the fermion, i.e. $\eta^\dagger_{-n} = -\eta_n$ and $\bar{\eta}^\dagger_{-n} = -\bar{\eta}_n$, the norms of the R fields satisfy $\langle \sigma^- | \sigma^- \rangle = -\langle \sigma^+ | \sigma^+ \rangle$. Hence, one of the states $|\sigma^\pm\rangle$ has negative norm.

Coming back to the product theory between the fermion ψ and η, we note that it contains a closed subset of even local fields given by

$$1\ ,\ \psi\bar{\psi}\ ,\ \psi\eta\ ,\ \psi\bar{\eta}\ ,\ \bar{\psi}\eta\ ,\ \bar{\psi}\bar{\eta}\ ,\ \eta\bar{\eta}\ ,\ \psi\bar{\psi}\eta\bar{\eta}\ ;\ r^\pm = \frac{1}{2}(\varsigma^+\sigma^+ \pm \varsigma^-\sigma^-)\ . \tag{4.2}$$

The associated conformal blocks give rise to a modular invariant partition function

$$Z_{\text{fermion}}(q,\bar{q}) = |\chi_{(0,0)} + \chi_{(\frac{1}{2},\frac{1}{2})}|^2 + |\chi_{(0,\frac{1}{2})} + \chi_{(\frac{1}{2},0)}|^2 + 2|\chi_{(\frac{1}{16},\frac{1}{16})}|^2 \tag{4.3}$$

where $\chi_{(h,h')} = \chi^1_h \chi^2_{h'}$ are the characters of $c = 1/2$ Virasoro representations with lowest weight $h = h_\psi$ and $h' = h'_\eta$. All the fields that are included in Z_{fermion} can be bosonized through a single bosonic field Y at compactification radius $R = 1$ [27]. The exponential fields $\exp(ikY)$ possess a rather simple expression in terms of $\chi = \psi + i\eta$ along with the two spin fields $r^\pm$ we introduced in eq. (4.2). For $k \in \mathbb{Z}$ and $k \geq 0$ one finds

$$:e^{ikY}: \, = \, :\prod_{i=0}^{k-1} \frac{1}{k!}\, \partial^{(i)}\chi\, \bar{\partial}^{(i)}\bar{\chi}: \ . \tag{4.4}$$

When $k \in \mathbb{Z} + \frac{1}{2}$ we need to use the spin fields $r^{\pm}$. For $k \geq 1/2$ one has

$$:e^{ikY}: = :r^{+} \prod_{i=1}^{k-1/2} \frac{1}{k!}\, \partial^{(i)}\chi\, \bar{\partial}^{(i)}\bar{\chi}: \ . \tag{4.5}$$

We shall now replace the first fermion by $\mathcal{N}$=1 Liouville field theory with a Liouville field φ and a fermion ψ. The total central charge of our product theory is

$$c = c_{SL} + \frac{1}{2} = 2 + 3\left(b + \frac{1}{b}\right)^2 = 8 + 3b^2 + 3b^{-2} \ . \tag{4.6}$$

In our construction of fields we restrict to the even ones, just as for the free fermion model we described above. For $k = 0, 1/2, 1, 3/2, \ldots$ we set

$$\Phi_\alpha^{(k)}(z,\bar{z}) =: \exp\left(\alpha\varphi(z,\bar{z}) + ikY(z,\bar{z})\right): \ . \tag{4.7}$$

The fields $\Phi_\alpha^{(k)}$ differ from those introduced in [1] by their normalization (see also comments below). For negative $k = -1/2, -1, -3/2, \ldots$ we introduce $\Phi_\alpha^{(k)}$ through the simple prescription

$$\Phi_\alpha^{(k)}(z,\bar{z}) = \Phi_{Q-\alpha}^{(-k)}(z,\bar{z}) \ =: \exp\left((Q-\alpha)\varphi(z,\bar{z}) - ikY(z,\bar{z})\right): \ . \tag{4.8}$$

Up to the normalization we mentioned before, the fields $\Phi_\alpha^{(-|k|)}$ also agree with those defined in [1]. The conformal weight of $\Phi_\alpha^{(k)}$ is given by

$$\Delta_\alpha + \frac{k^2}{2} = \frac{1}{2}\alpha(Q-\alpha) + \frac{k^2}{2} \ .$$

We note that fields with $|k| \leq 1/2$ are primary with respect to the product of the $\mathcal{N}$=1 super-conformal algebra and the free fermion η. These primary fields are given by $\Phi_\alpha^{(0)} = \phi_\alpha$ and

$$\Phi_\alpha^{(-\frac{1}{2})} = \left(\sigma^+\Sigma_\alpha^+ - \sigma^-\Sigma_\alpha^-\right) \ , \qquad \Phi_\alpha^{(\frac{1}{2})} = \frac{1}{2i}\chi_0\bar{\chi}_0\Phi_\alpha^{(-\frac{1}{2})} = \left(\sigma^+\Sigma_\alpha^+ + \sigma^-\Sigma_\alpha^-\right) \ . \tag{4.9}$$

For all other values of k, the fields $\Phi_\alpha^{(k)}$ are descendent fields. Our explicit computations below will only involve the case of $k = \pm 1, \pm 3/2$. Using the definition (4.7) and eq. (2.14) one can rewrite the first few fields as descendents with respect to the super-conformal algebra and the fermion η, see also [1],

$$|\Phi_\alpha^{(\pm 1)}\rangle = \Omega_{\pm 1}^{-2}(\alpha)\Big[G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}} + (\tfrac{Q}{2} \pm P)^2\eta_{-\frac{1}{2}}\bar{\eta}_{-\frac{1}{2}} + (\tfrac{Q}{2} \pm P)\big(\eta_{-\frac{1}{2}}\,\bar{G}_{-\frac{1}{2}} - \bar{\eta}_{-\frac{1}{2}}\,G_{-\frac{1}{2}}\big)\Big]\,|\phi_\alpha\rangle$$

$$\begin{aligned}|\Phi_\alpha^{(\pm\frac{3}{2})}\rangle &= \chi_{-1}\bar{\chi}_{-1}|\Phi_\alpha^{(\frac{1}{2})}\rangle = \Omega_{\pm\frac{3}{2}}^{-2}(\alpha)\Big[\tfrac{2}{P^2}L_{-1}G_0\bar{L}_{-1}\bar{G}_0 + 2(\tfrac{Q}{2} \pm P)^2 G_{-1}\bar{G}_{-1} \\ &+ \sqrt{2}\,\Omega_{\pm\frac{3}{2}}(\alpha)(\tfrac{Q}{2} \pm P)(\eta_{-1}\bar{G}_{-1} - \bar{\eta}_{-1}G_{-1}) \pm \tfrac{\sqrt{2}}{P}\Omega_{\pm\frac{3}{2}}(\alpha)(\eta_{-1}\bar{L}_{-1}\bar{G}_0 - \bar{\eta}_{-1}L_{-1}G_0) \\ &+ \Omega_{\pm\frac{3}{2}}^2(\alpha)\eta_{-1}\bar{\eta}_{-1} \pm \tfrac{2}{P}(\tfrac{Q}{2} \pm P)(L_{-1}G_0\bar{G}_{-1} + G_{-1}\bar{L}_{-1}\bar{G}_0)\Big]\,|\Phi_\alpha^{(\frac{1}{2})}\rangle\end{aligned}$$

Here we wrote equations between states rather than fields by means of the usual state-field correspondence. The variable P is related to α through $\alpha = Q/2 + P$. Finally, the pre-factor $\Omega_k(\alpha)$ is given by

$$\Omega_k(\alpha) = n_k \prod_{\substack{i,j\,=\,1,\\ 2|k|-i-j\,\in\,2\mathbb{N}}}^{i+j=2|k|} (\mathit{sign}(k)(2\alpha - Q) + ib + jb^{-1}), \tag{4.10}$$

where $\mathit{sign}(k) = k/|k|$ denotes the sign of k when $k \neq 0$ and we set $\mathit{sign}(0)=1$. The first two constants take the values

$$n_1 = 2^{-1}, \quad n_{\frac{3}{2}} = 2^{-\frac{3}{2}}. \tag{4.11}$$

There exists a straightforward but cumbersome algorithm that computes the numbers n_k for higher values of k. In [1], the factors Ω_k were absorbed in the normalization of the fields $\Phi_\alpha^{(k)}$.

4.2 Relation with double Liouville theory

We are now prepared to state the main result of this work. It relates the model described in the previous subsection to a product of a Liouville field theory with $c^{(1)} \geq 25$ and an imaginary Liouville theory with $c^{(2)} \leq 1$. We shall often refer to this product as double Liouville theory. According to [1], the b-parameters of the two factors must be chosen as

$$b^{(1)} = \frac{2b}{\sqrt{2-2b^2}}, \qquad \left(\hat{b}^{(2)}\right)^{-1} = \frac{2}{\sqrt{2-2b^2}}. \tag{4.12}$$

So that the central charge is

$$c = c_L^{(1)} + c_{\mathcal{L}}^{(2)} = 2 + 6\left(b^{(1)} + \frac{1}{b^{(1)}}\right)^2 - 6\left(\hat{b}^{(2)} - \frac{1}{\hat{b}^{(2)}}\right)^2 = 8 + 3b^2 + 3b^{-2}\ .$$

Note that the sum of central charges agrees with the central charge (4.6) of the model we discussed in the previous subsection. Moreover, as was observed in [9, 10, 11], the two Virasoro algebras of double Liouville theory can actually be reconstructed from the super-conformal currents T and G along with the fermion η,

$$\begin{aligned} L_n^{(1)} &= \frac{1}{1-b^2} L_n - \frac{1+2b^2}{2-2b^2} \sum_{r=-\infty}^{\infty} r : \eta_{n-r}\eta_r : + \frac{b}{1-b^2} \sum_{r=-\infty}^{\infty} \eta_{n-r} G_r, \\ L_n^{(2)} &= \frac{1}{1-b^{-2}} L_n - \frac{1+2b^{-2}}{2-2b^{-2}} \sum_{r=-\infty}^{\infty} r : \eta_{n-r}\eta_r : + \frac{b^{-1}}{1-b^{-2}} \sum_{r=-\infty}^{\infty} \eta_{n-r} G_r. \end{aligned} \tag{4.13}$$

Similar formulas apply to the anti-holomorphic sector, of course. As anticipated in the previous subsection, we now note that the familiar relation $(L_n^{(i)})^\dagger = L_{-n}^{(i)}$ requires $\dagger$ to act as $\eta_n^\dagger = -\eta_{-n}$ on the modes of the fermion η. In other words, the Virasoro modes in double Liouville theory possess the usual conjugation rules provided that the modes L_n and G_n do and we take η to be imaginary.

Given such a close relation between their chiral algebras it seems natural to look for relations between vertex operators. Following [1] let us introduce

$$\mathbb{V}_\alpha^{(k)}(z,\bar z) = V_{\alpha^{(1)}+kb^{(1)}/2}(z,\bar z)\ \mathcal{V}_{\hat\alpha^{(2)}+k/2\hat b^{(2)}}(z,\bar z) \tag{4.14}$$

where $2k$ is an integer, α is a complex parameter and we defined

$$\alpha^{(1)} = \frac{\alpha}{\sqrt{2-2b^2}}, \qquad \hat\alpha^{(2)} = \frac{b\alpha}{\sqrt{2-2b^2}}. \tag{4.15}$$

The conformal dimension of the vertex operators (4.14) is easy to compute with the help of the expressions (2.2) and (3.6) for conformal weights in (imaginary) Liouville theory,

$$\begin{aligned} h_{(\alpha^{(1)}+kb^{(1)}/2)} + \hat h_{(\hat\alpha^{(2)}+k/2\hat b^{(2)})} &= (\alpha^{(1)}+kb^{(1)}/2)(Q^{(1)}-\alpha^{(1)}-kb^{(1)}/2) \\ -(\hat\alpha^{(2)}+k/2\hat b^{(2)})(\hat Q^{(2)}-\hat\alpha^{(2)}-k/2\hat b^{(2)}) &= \frac{1}{2}\alpha(Q-\alpha)+\frac{k^2}{2}\ . \end{aligned}$$

These weights agree with the weights of the fields $\Phi_\alpha^{(k)}$ we introduced in the previous section. Hence, with proper normalizations, the 2-point functions of the fields $\Phi_\alpha^{(k)}$ and $\mathbb{V}_\alpha^{(k)}$ agree. In addition, it is not difficult to check that the fields $\Phi_\alpha^{(k)}$ are primary with respect to Virasoro algebras (4.13) of the Liouville field theory and its imaginary cousin. Given these observations it is certainly tempting to contemplate that the relation

$$\Phi_\alpha^{(k)}(z,\bar z) = \mathcal{N}_\alpha^{(k)}\ \mathbb{V}_\alpha^{(k)}(z,\bar z) \tag{4.16}$$

might hold in arbitrary correlation functions. Through comparison of 3-point functions we shall provide very strong support in favor of this proposal. These computations determine the normalization $\mathcal{N}_\alpha^{(k)}$ to take the form

$$\mathcal{N}_\alpha^{(k)} \;=\; (-1)^k\,\tilde{\mathcal{N}}_\alpha^{(k)}, \tag{4.17}$$

when $k \in \mathbb{N}$, i.e. in the NS sector of the theory, and

$$\mathcal{N}_\alpha^{(k)} \;=\; 2^{\frac{3}{4}}\,\tilde{\mathcal{N}}_\alpha^{(k)}, \tag{4.18}$$

in R sector, i.e. when k takes the values $k \in \mathbb{N}+\frac{1}{2}$. The common factor $\tilde{\mathcal{N}}_\alpha^{(k)}$ is given by

$$\tilde{\mathcal{N}}_\alpha^{(k)} = \frac{\left[\pi\mu_L\gamma\left((b^{(1)})^2\right)\right]^{(\alpha^{(1)}+\frac{kb^{(1)}}{2})/b^{(1)}}\left[\pi M\gamma\left(-(\hat{b}^{(2)})^2\right)\right]^{-(\hat{\alpha}^{(2)}+\frac{k}{2\hat{b}^{(2)}})/\hat{b}^{(2)}}}{n_k^2\,2^{k^2}\left[\pi\mu\gamma(\frac{bQ}{2})\right]^{\frac{\alpha}{b}}\,b^{-2k}\left(\frac{1-b^2}{2}\right)^{\frac{1}{2}+2k}}. \tag{4.19}$$

The factors n_k were introduced in eq. (4.11), at least for some special values of k. In order to check that the fields $\Phi_\alpha^{(k)}$ and $\mathbb{V}_\alpha^{(k)}$ can be identified in all correlation functions, we must verify that their 3-point functions agree,

$$\varpi(k_i)\kappa(b)\langle\Phi_{\bar{\alpha}_3}^{(k_3)}\Phi_{\alpha_2}^{(k_2)}\Phi_{\alpha_1}^{(k_1)}\rangle = \mathcal{N}_{\bar{\alpha}_3}^{(k_3)}\mathcal{N}_{\alpha_2}^{(k_2)}\mathcal{N}_{\alpha_1}^{(k_1)}\langle\mathbb{V}_{\bar{\alpha}_3}^{(k_3)}\;\mathbb{V}_{\alpha_2}^{(k_2)}\;\mathbb{V}_{\alpha_1}^{(k_1)}\rangle\quad, \tag{4.20}$$

at least up to some constant $\kappa(b)$ that can be absorbed through an appropriate normalization of the vacuum state, see eq. (4.22) for a concrete formula. The factors ϖ will be shown to satisfy $\varpi^4 = 1$. Given the complexity of the fields $\Phi^{(k)}$, checking eq. (4.20) is a rather non-trivial task. We are not prepared to establish the relation (4.20) for all possible 3-point functions, but we have performed a number of highly non-trivial tests. These are described in the next subsection.

4.3 Comparison of 3-point functions

Our goal is to check relation (4.16) in a few selected examples, involving both NS and R sector fields and also super-descendent fields. Most computations are somewhat lengthy but in principle straight forward to carry out.

4.3.1 NS sector

In our first example, we take all three fields of the $\mathcal{N}$=1 Liouville theory to be super-primaries in the NS sector. These are multiplied with the identity field of the free fermion theory, i.e. we consider a 3-point correlator with $\Phi^{(k)}_\alpha = \Phi^{(0)}_\alpha = \phi_\alpha$. Since we have checked already that the conformal dimensions on both sides of the correspondence (4.16) match, we shall put the fields at the points $z_3 = \infty$, $z_2 = 1$ and $z_1 = 0$ so that we can omit all dependence on world-sheet coordinates. The 3-point function of $\Phi^{(0)}_\alpha$ is given by

$$\langle \Phi^{(0)}_{\bar\alpha_3} \Phi^{(0)}_{\alpha_2} \Phi^{(0)}_{\alpha_1} \rangle = C_{NS}(\alpha_3, \alpha_2, \alpha_1 | b)$$

with C_{NS} as given in equation (2.23). We will use the notation $\bar\alpha_i \equiv Q - \alpha_i$ for reflected momentum of the fields located at infinity. The other side of the correspondence (4.16) is given by

$$\langle \mathbb{V}^{(0)}_{\bar\alpha_3}\ \mathbb{V}^{(0)}_{\alpha_2}\ \mathbb{V}^{(0)}_{\alpha_1} \rangle \;\; = \;\; C_L(\alpha_3^{(1)}, \alpha_2^{(1)}, \alpha_1^{(1)} | b^{(1)})\, C_{\mathcal{L}}(\hat\alpha_3^{(2)}, \hat\alpha_2^{(2)}, \hat\alpha_1^{(2)} | \hat b^{(2)}).$$

The arguments $\alpha_i^{(\nu)}$ and $b^{(\nu)}$ that appear in the arguments of the structure constants were introduced in eqs. (4.15) and (4.12). Explicit expressions for the structure constants can be found in eqs. (2.12) and (3.10). Using the identities

$$\frac{\Upsilon_{b^{(1)}}\left(\alpha^{(1)}\right)}{\Upsilon_{\hat b^{(2)}}\left(\hat\alpha^{(2)} + \hat b^{(2)}\right)} = B(\alpha)\Upsilon_b^{\rm NS}(\alpha), \tag{4.21}$$

where

$$B(\alpha) = \frac{\Upsilon^0_{b^{(1)}}}{\Upsilon^0_{\hat b^{(2)}}\Upsilon^0_b}\, b^{\frac{b^2\alpha(Q-\alpha)}{2-2b^2}} \left(\frac{1-b^2}{2}\right)^{\frac{\alpha(Q-\alpha)-2}{4}},$$

stated in (A.9) of [1], one can check that

$$C_L(\alpha_3^{(1)}, \alpha_2^{(1)}, \alpha_1^{(1)} | b^{(1)})\, C_{\mathcal{L}}(\hat\alpha_3^{(2)}, \hat\alpha_2^{(2)}, \hat\alpha_1^{(2)} | \hat b^{(2)}) = A_1\, C_{NS}(\alpha_3, \alpha_2, \alpha_1 | b)$$

with

$$A_1 = \frac{2\left(\pi\mu_L\gamma\left(\frac{2b^2}{1-b2}\right)\right)^{\frac{Q-\alpha}{2b}}\left(\pi M\gamma\left(\frac{b^2-1}{2}\right)\right)^{\frac{ba}{1-b^2}}\gamma\left(\frac{-2b^2}{1-b^2}\right)\gamma\left(\frac{b^2+1}{2}\right)}{\left(\left(\frac{\pi\mu}{2}\right)\gamma\left(\frac{b^2+1}{2}\right)\right)^{\frac{Q-\alpha}{b}}\ \left(\frac{2}{1-b^2}\right)^{\frac{3}{2}}}.$$

Comparison of this α-dependent factor with the product of the three normalizations $\mathcal{N}^{(0)}_{\alpha_i}$ we introduced in the previous subsection gives

$$\mathcal{N}^{(0)}_{\bar\alpha_3}\mathcal{N}^{(0)}_{\alpha_2}\mathcal{N}^{(0)}_{\alpha_1} A_1 = \kappa(b)\ .$$

The function $\kappa(b)$ depends on the parameter b, but it is independent of the labels α_i. Explicitly, it is given by

$$\kappa(b) = \frac{2\left[\pi\mu_L\gamma\left((b^{(1)})^2\right)\right]^{\frac{Q^{(1)}}{b^{(1)}}}}{\left[\pi\mu\gamma(\frac{bQ}{2})\right]^{\frac{Q}{b}}}\gamma\left(\frac{-2b^2}{1-b^2}\right)\gamma\left(\frac{b^2+1}{2}\right) \ . \tag{4.22}$$

In conclusion, we have established eq. (4.20) for $k_i = 0$ with $\varpi(0,0,0) = 1$.

Let us now proceed to the next and slightly more complicated example of the relation (4.16) in which at least one of the vertex operators involves super-descendents in the $\mathcal{N}$=1 Liouville field. More specifically, let us insert one of the operators $\Phi^{(\pm1)}_\alpha$ along with two of the operators $\Phi^{(0)}_\alpha$. Looking back at the explicit formulas we spelled out at the end of section 4.1, we observe that only the second term from these expressions can contribute since $\langle\eta\rangle = \langle\bar\eta\rangle = \langle\eta\bar\eta\rangle = 0$. Hence we obtain

$$\langle\Phi^{(0)}_{\bar\alpha_3}\ \Phi^{(1)}_{\alpha_2}\ \Phi^{(0)}_{\bar\alpha_1}\rangle = \Omega_1^{-2}(\alpha_2)\langle\phi_{\bar\alpha_3}G_{-\frac{1}{2}}\bar G_{-\frac{1}{2}}\phi_{\alpha_2}\phi_{\alpha_1}\rangle = \alpha_2^{-2}\,\tilde C_{NS}(\alpha_3,\alpha_2,\alpha_1|b)$$

where the evaluation of the correlator in the $\mathcal{N}$=1 Liouville theory uses the structure constants (2.23). On the other side of our correspondence (4.16) one finds

$$\langle\mathbb{V}^{(0)}_{\bar\alpha_3}\ \mathbb{V}^{(1)}_{\alpha_2}\ \mathbb{V}^{(0)}_{\alpha_1}\rangle = C_L(\alpha_3^{(1)},\alpha_2^{(1)}+\frac{b^{(1)}}{2},\alpha_1^{(1)}|b^{(1)})\ C_{\mathcal{L}}(\hat\alpha_3^{(2)},\hat\alpha_2^{(2)}+\frac{1}{2\hat b^{(2)}},\hat\alpha_1^{(2)}|\hat b^{(2)})$$

If we insert the explicit formulas (2.12) and (3.10) for the structure constants C_L and $C_{\mathcal{L}}$ along with the shift properties (2.6) and the identity

$$\Upsilon_R(\alpha) = \frac{b^{b\alpha}}{\gamma\left(\frac{b\alpha+1}{2}\right)}\ \Upsilon_{NS}(\alpha+b) = B^{-1}(\alpha+b)\,\frac{b^{b\alpha}}{\left(\frac{1-b^2}{2}\right)^{\frac{b\alpha}{2}}}\frac{\Upsilon_{b^{(1)}}\left(\alpha^{(1)}+\frac{b^{(1)}}{2}\right)}{\Upsilon_{\hat b^{(2)}}\left(\hat\alpha^{(2)}+\frac{1}{2\hat b^{(2)}}+\hat b^{(2)}\right)} \tag{4.23}$$

where $B(\alpha)$ is the function defined after eq. (4.21), we can check that

$$C_L(\alpha_3^{(1)},\alpha_2^{(1)}+\frac{b^{(1)}}{2},\alpha_1^{(1)}|b^{(1)})\ C_{\mathcal{L}}(\hat\alpha_3^{(2)},\hat\alpha_2^{(2)}+\frac{1}{2\hat b^{(2)}},\hat\alpha_1^{(2)}|\hat b^{(2)}) = A_2\,\tilde C_{NS}(\alpha_3,\alpha_2,\alpha_1|b)$$

equation where A_2 are given by

$$A_2 = -\frac{\left(\pi\mu_L\gamma\left(\frac{2b^2}{1-b2}\right)\right)^{\frac{Q-\alpha-b}{2b}}\ \left(\pi M\gamma\left(\frac{b^2-1}{2}\right)\right)^{\frac{ba+1}{1-b^2}}\ \gamma\left(\frac{-2b^2}{1-b^2}\right)\gamma\left(\frac{b^2+1}{2}\right)}{i\left(\left(\frac{\pi\mu}{2}\right)\ \gamma\left(\frac{b^2+1}{2}\right)\right)^{\frac{Q-\alpha}{b}}\ b^2\ \left(\frac{2}{1-b^2}\right)^{\frac{7}{2}}\ \alpha_2^2} \ .$$

As in the previous subsection, it is not difficult to see that the functions A_2 may be factorized into a product of three α-dependent factors $\mathcal{N}$, i.e.

$$\mathcal{N}^{(0)}_{\bar\alpha_3}\mathcal{N}^{(1)}_{\alpha_2}\mathcal{N}^{(0)}_{\alpha_1}A_2 = i\kappa(b)\,\alpha_2^{-2} \ .$$

The constant $\kappa(b)$ was introduced in eq. (4.22) above. Combining these results we conclude that eq. (4.20) also holds for $k_1 = k_3 = 0$ and $k_2 = 1$ with the same constant $\kappa(b)$ as in the previous computation and $\varpi(0,1,0) = i$.

As a final check for fields from the NS sector we want to consider a correlation function in which two fields have $k = 1$,

$$\begin{aligned}\langle \Phi^{(0)}_{\bar\alpha_3}\ \Phi^{(1)}_{\alpha_2}\ \Phi^{(1)}_{\alpha_1}\rangle &= \Omega_1^{-2}(\alpha_1)\Omega_1^{-2}(\alpha_2)\\ &\times\ \Big((\tfrac{Q}{2}+P_1)^2(\tfrac{Q}{2}+P_2)^2\langle \phi_{\alpha_3}\,\eta_{-\frac12}\bar\eta_{-\frac12}\phi_{\alpha_2}\,\eta_{-\frac12}\bar\eta_{-\frac12}\phi_{\alpha_1}\rangle + \langle \phi_{\alpha_3}\,G_{-\frac12}\bar G_{-\frac12}\phi_{\alpha_2}\,G_{-\frac12}\bar G_{-\frac12}\phi_{\alpha_1}\rangle\\ &+\ (\tfrac{Q}{2}+P_1)(\tfrac{Q}{2}+P_2)\Big(\langle \phi_{\alpha_3}\,\eta_{-\frac12}\bar G_{-\frac12}\phi_{\alpha_2}\,\eta_{-\frac12}\bar G_{-\frac12}\phi_{\alpha_1}\rangle + \langle \phi_3\,\bar\eta_{-\frac12}G_{-\frac12}\phi_2\,\bar\eta_{-\frac12}G_{-\frac12}\phi_1\rangle\Big)\Big)\end{aligned}$$

It can be reduced to the basic structure constants with the help of the super-conformal Ward identities [26]

$$\begin{aligned}\langle\varphi_3\,G_k\varphi_2(z,\bar z)\,\varphi_1\rangle &= \sum_{m=0}^{k+\frac12}\begin{pmatrix} k+\frac12\\ m\end{pmatrix}(-z)^m\,(\langle G_{m-k}\varphi_3\,\varphi_2(z,\bar z)\,\varphi_1\rangle\\ &\qquad -\epsilon\ \langle\varphi_3\,\varphi_2(z,\bar z)\,G_{k-m}\varphi_1\rangle)\,,\qquad k\geqslant -\tfrac12,\\ \langle\varphi_3\,G_{-k}\varphi_2(z,\bar z)\,\varphi_1\rangle &= \sum_{m=0}^{\infty}\begin{pmatrix} k-\frac32+m\\ m\end{pmatrix} z^m\langle G_{k+m}\varphi_3\,\varphi_2(z,\bar z)\,\varphi_1\rangle\\ &\quad-\epsilon(-1)^{k+\frac12}\sum_{m=0}^{\infty}\begin{pmatrix} k-\frac32+m\\ m\end{pmatrix} z^{-k-m+\frac12}\langle\varphi_3\,\varphi_2(z,\bar z)\,G_{m-\frac12}\varphi_1\rangle,\qquad k>\tfrac12,\end{aligned}$$

$$\langle G_{-k}\varphi_3\,\varphi_2(z,\bar z)\,\varphi_1\rangle = \epsilon\langle\varphi_3\,\varphi_2(z,\bar z)\,G_k\varphi_1\rangle + \sum_{m=-1}^{l(k-\frac12)}\begin{pmatrix} k+1/2\\ m+1\end{pmatrix} z^{k-\frac12-m}\langle\varphi_3\,G_{m+\frac12}\varphi_2(z,\bar z)\,\varphi_1\rangle,$$

where ϵ denotes the parity of the field φ_2 and $l(n) = n$ for $n+1\geq 0$ while $l(n)=\infty$ for $n+1<0$. The result reads

$$\begin{aligned}\langle \Phi^{(0)}_{\bar\alpha_3}\ \Phi^{(1)}_{\alpha_2}\ \Phi^{(1)}_{\alpha_1}\rangle &= \Omega_1^{-2}(\alpha_1)\Omega_1^{-2}(\alpha_2)\big((\tfrac{Q}{2}+P_1)^2(\tfrac{Q}{2}+P_2)^2 + (\Delta_3-\Delta_2-\Delta_1)^2\\ &\qquad +2(\tfrac{Q}{2}+P_1)(\tfrac{Q}{2}+P_2)(\Delta_3-\Delta_2-\Delta_1)\big)C_{NS}(\alpha_3,\alpha_2,\alpha_1|b)\\ &= \frac{(\frac{Q}{2}+P_1+P_2-P_3)^2\,(\frac{Q}{2}+P_1+P_2+P_3)^2}{4\alpha_1^2\,\alpha_2^2}\,C_{NS}(\alpha_3,\alpha_2,\alpha_1|b)\,.\end{aligned}$$

On the other side we find

$$\begin{aligned}&\langle \mathbb{V}^{(0)}_{\bar\alpha_3}\ \mathbb{V}^{(1)}_{\alpha_2}\ \mathbb{V}^{(1)}_{\alpha_1}\rangle\\ &\quad = C_L(\alpha_3^{(1)},\alpha_2^{(1)}+\frac{b^{(1)}}{2},\alpha_1^{(1)}+\frac{b^{(1)}}{2}|b^{(1)})\ C_{\mathcal{L}}(\hat\alpha_3^{(2)},\hat\alpha_2^{(2)}+\frac{1}{2\hat b^{(2)}},\hat\alpha_1^{(2)}+\frac{1}{2\hat b^{(2)}}|\hat b^{(2)})\end{aligned}$$

As before, comparing structure constants and using eqs. (4.21) and (4.23) we can check that

$$C_L(\alpha_3^{(1)},\alpha_2^{(1)}+\frac{b^{(1)}}{2},\alpha_1^{(1)}+\frac{b^{(1)}}{2}|b^{(1)})C_{\mathcal{L}}(\hat{\alpha}_3^{(2)},\hat{\alpha}_2^{(2)}+\frac{1}{2\hat{b}^{(2)}},\hat{\alpha}_1^{(2)}+\frac{1}{2\hat{b}^{(2)}}|\hat{b}^{(2)}) = A_3 C_{NS}(\alpha_3,\alpha_2,\alpha_1|b),$$

where

$$\begin{aligned} A_3 \;&=\; \frac{\left(\pi\mu_L\gamma\left(\frac{2b^2}{1-b2}\right)\right)^{\frac{Q-\alpha_{123}-2b}{2b}}\left(\pi M\gamma\left(\frac{b^2-1}{2}\right)\right)^{\frac{b\alpha_{123}+2}{1-b^2}}\gamma\left(\frac{-2b^2}{1-b^2}\right)\gamma\left(\frac{b^2+1}{2}\right)}{\left(\left(\frac{\pi\mu}{2}\right)\gamma\left(\frac{b^2+1}{2}\right)\right)^{\frac{Q-\alpha_{123}}{b}}} \\ &\times\; 2b^{-4}\left(\frac{2}{1-b^2}\right)^{-\frac{11}{2}}(2\alpha_2)^{-2}(2\alpha_1)^{-2}\,(\alpha_1+\alpha_2-\alpha_3)^2\,(\alpha_1+\alpha_2+\alpha_3-Q)^2. \end{aligned}$$

Thus we have

$$\mathcal{N}^{(0)}_{\bar{\alpha}_3}\mathcal{N}^{(1)}_{\alpha_2}\mathcal{N}^{(1)}_{\alpha_1}A_3 = \kappa(b)\,\frac{(\frac{Q}{2}+P_1+P_2-P_3)^2\,(\frac{Q}{2}+P_1+P_2+P_3)^2}{4\alpha_2^2\,\alpha_1^2}.$$

Once more we have established an instance of eq. (4.20), this time for $k_3 = 0$ and $k_1 = k_2 = 1$. The constant factor $\kappa(b)$ is given by the same expression as in the previous two cases and $\varpi(0,1,1) = 1$.

4.3.2 R sector

So far we have only looked at operators $\Phi^{(k)}_\alpha$ with $k \in \mathbb{Z}$ that involve fields from the NS sector of the $\mathcal{N}$=1 Liouville field theory. The correspondence (4.16) we have formulated also involves fields from the R sector. These appear for values $k \in \mathbb{Z}+\frac{1}{2}$. Actually, correlators of fields in the R sector have been one of the crucial motivations for this work, see next section. Therefore, we would like to perform a few tests involving $\Phi^{(k)}_\alpha$ with $k \in \mathbb{Z}+\frac{1}{2}$.

The simplest possible 3-point function involving R sector fields is the correlation function

$$\langle\Phi^{(0)}_{\alpha_3}\;\Phi^{(\pm\frac{1}{2})}_{\alpha_2}\;\Phi^{(\pm\frac{1}{2})}_{\alpha_1}\rangle = \langle\phi_{\alpha_3}\,\sigma_2^+\Sigma^+_{\alpha_2}\sigma_1^+\Sigma^+_{\alpha_1}\rangle + \langle\phi_{\alpha_3}\,\sigma_2^-\Sigma^-_{\alpha_2}\sigma_1^-\Sigma^-_{\alpha_1}\rangle \tag{4.24}$$

involving two fields from the R sector along with one from the NS sector. The primary fields $\Sigma^\pm_\alpha$ of the $\mathcal{N}$=1 Liouville field theory are accompanied by the spin fields $\sigma^\pm$ of the free fermion. We added a subscript to these fields in order to keep track of the insertion points. The fields $\sigma_2^\pm$ and $\sigma_1^\pm$ are inserted at $z = 1$ and $z = 0$, respectively. The field inserted at $z = \infty$ involves the identity field of the free fermion model. Hence, for the

3-point function we consider, we only need to insert 2-point functions of the free fermion model. In passing from the left hand side of eq. (4.24) we have inserted the definition of $\Phi_\alpha^{(\pm\frac{1}{2})}$ and we used that $\langle\sigma_2^{\pm}\sigma_1^{\mp}\rangle = 0$. Assuming that $\sigma^{\pm}$ have been normalized, the remaining 2-point functions are $\langle\sigma_2^{\pm}(1)\sigma_1^{\pm}(0)\rangle = \pm 1$ so that we obtain

$$\langle\Phi_{\bar{\alpha}_3}^{(0)}\ \Phi_{\alpha_2}^{(\pm\frac{1}{2})}\ \Phi_{\alpha_1}^{(\pm\frac{1}{2})}\rangle = \langle\phi_{\bar{\alpha}_3}\,\Sigma_{\alpha_2}^{+}\Sigma_{\alpha_1}^{+}\rangle + \langle\phi_{\bar{\alpha}_3}\,\Sigma_{\alpha_2}^{-}\Sigma_{\alpha_1}^{-}\rangle = 2\,C_R^{(+)}(\alpha_3;\alpha_2,\alpha_1|b). \tag{4.25}$$

The relevant correlation functions in $\mathcal{N}$=1 Liouville field theories were spelled out after eq. (2.28). With their help we find

$$\begin{aligned} C_R^{(+)}(\alpha_3;\alpha_2,\alpha_1|b) &= \frac{1}{2}\left(C_R^{+}(\alpha_3,\alpha_2;\alpha_1|b) + C_R^{-}(\alpha_3;\alpha_2,\alpha_1|b)\right) \\ &= \frac{1}{2}\left(\frac{\pi\mu}{2}\gamma\left(\frac{Qb}{2}\right)b^{1-b^2}\right)^{\frac{Q-\alpha_{123}}{b}}\frac{\Upsilon_b^0\,\Upsilon_b^{\rm R}(2\alpha_1)\Upsilon_b^{\rm R}(2\alpha_2)\Upsilon_b^{\rm NS}(2\alpha_3)}{\Upsilon_b^{\rm R}(\alpha_{123}-Q)\Upsilon_b^{\rm R}(\alpha_{12})\Upsilon_b^{\rm NS}(\alpha_{23})\Upsilon_b^{\rm NS}(\alpha_{13})}\end{aligned}$$

Similarly, we can compute

$$\langle\Phi_{\bar{\alpha}_3}^{(0)}\ \Phi_{\alpha_2}^{(\frac{1}{2})}\ \Phi_{\alpha_1}^{(-\frac{1}{2})}\rangle = 2\,C_R^{(-)}(\alpha_3;\alpha_2,\alpha_1|b) \tag{4.26}$$

where

$$\begin{aligned} C_R^{(-)}(\alpha_3;\alpha_2,\alpha_1|b) &= \frac{1}{2}\left(C_R^{+}(\alpha_3;\alpha_2,\alpha_1|b) - C_R^{-}(\alpha_3;\alpha_2,\alpha_1|b)\right) \\ &= \frac{1}{2}\left(\frac{\pi\mu}{2}\gamma\left(\frac{Qb}{2}\right)b^{1-b^2}\right)^{\frac{Q-\alpha_{123}}{b}}\frac{\Upsilon_b^0\,\Upsilon_b^{\rm R}(2\alpha_1)\Upsilon_b^{\rm R}(2\alpha_2)\Upsilon_b^{\rm NS}(2\alpha_3)}{\Upsilon_b^{\rm NS}(\alpha_{123}-Q)\Upsilon_b^{\rm NS}(\alpha_{12})\Upsilon_b^{\rm R}(\alpha_{23})\Upsilon_b^{\rm R}(\alpha_{13})}\end{aligned}$$

On the other hand we can compute the 3-point functions of the corresponding fields in double Liouville theory. Using the explicit formulae (2.12) and the relations (4.21), (4.23) one may check that

$$\begin{aligned}\langle\mathbb{V}_{\bar{\alpha}_3}^{(0)}\ \mathbb{V}_{\alpha_2}^{(\pm\frac{1}{2})}\ \mathbb{V}_{a_1}^{(-\frac{1}{2})}\rangle &= C_L(\alpha_3^{(1)},\alpha_2^{(1)}\pm\frac{b^{(1)}}{4},\alpha_1^{(1)}-\frac{b^{(1)}}{4}|b^{(1)})C_{\mathcal{L}}(\hat{\alpha}_3^{(2)},\hat{\alpha}_2^{(2)}\pm\frac{1}{4\hat{b}^{(2)}},\hat{\alpha}_1^{(2)}-\frac{1}{4\hat{b}^{(2)}}|\hat{b}^{(2)}) \\ &= A_4^{\pm}C_R^{(\mp)}(\alpha_3;\alpha_2,\alpha_1|b)\end{aligned}$$

where

$$A_4^{\pm} = \frac{2\left(\pi\mu_L\gamma\left(\frac{2b^2}{1-b2}\right)\right)^{\frac{Q-\alpha_{123}+b/2\pm b/2}{2b}}\left(\pi M\gamma\left(\frac{b^2-1}{2}\right)\right)^{\frac{b(\alpha_{123}-1/2\pm 1/2)}{1-b^2}}\gamma\left(\frac{-2b^2}{1-b^2}\right)\gamma\left(\frac{b^2+1}{2}\right)}{\left(\left(\frac{\pi\mu}{2}\right)\ \gamma\left(\frac{b^2+1}{2}\right)\right)^{\frac{Q-\alpha}{b}}b^{-(1\mp 1)}\ \left(\frac{2}{1-b^2}\right)^{\frac{1}{2}\pm 1}}\,.$$

As in all our previous computations, it is straight forward to show that the functions $A_4^{\pm}$ may be factorized into a product of three α-dependent factors $\mathcal{N}$, up to the familiar α-independent term (4.22) , i.e.

$$\mathcal{N}_{\bar{\alpha}_3}^{(0)} \mathcal{N}_{\alpha_2}^{(\pm\frac{1}{2})} \mathcal{N}_{\alpha_1}^{(-\frac{1}{2})} A_4^{\pm} = 2\kappa(b) \ .$$

Combining these results we conclude once more that eq. (4.20) holds, with the familiar $\kappa(b)$ and a factor $\varpi(0, \pm 1/2, -1/2) = 1$.

Next let us consider two correlation functions containing the field $\Phi_\alpha^{(\frac{3}{2})}$. In this case in order to express the correlators in terms of the structure constants (4.25), (4.26) one should use the Ward identities [44, 28]

$$\langle G_{-n} \varphi_3^{\mathrm{R}} \, \varphi_2(z,\bar{z}) \, \varphi_1^{\mathrm{R}} \rangle = \epsilon \, \langle \varphi_3^{\mathrm{R}} \, \varphi_2(z,\bar{z}) \, G_n \varphi_1^{\mathrm{R}} \rangle + \sum_{k=-\frac{1}{2}}^{\infty} \binom{n+1/2}{k+1/2} z^{n-k} \langle \varphi_3^{\mathrm{R}} \, G_k \varphi_2(z,\bar{z}) \, \varphi_1^{\mathrm{R}} \rangle,$$

$$\begin{aligned}
\sum_{p=0}^{\infty} \binom{\frac{1}{2}}{p} \; z^{\frac{1}{2}-p} \; \langle \varphi_3^{\mathrm{R}} \, G_{p-k} \varphi_2(z,\bar{z}) \, \varphi_1^{\mathrm{R}} \rangle = & \sum_{p=0}^{\infty} \binom{\frac{1}{2}-k}{p} (-z)^p \; \langle G_{p+k-\frac{1}{2}} \varphi_3^{\mathrm{R}} \, \varphi_2(z,\bar{z}) \, \varphi_1^{\mathrm{R}} \rangle \\
& -\epsilon \sum_{p=0}^{\infty} \binom{\frac{1}{2}-k}{p} (-z)^{\frac{1}{2}-k-p} \langle \varphi_3^{\mathrm{R}} \, \varphi_2(z,\bar{z}) \, G_p \varphi_1^{\mathrm{R}} \rangle \, ,
\end{aligned}$$

where φ_i^{R} denotes a R field and ϵ is the parity of the NS field. Similar Ward identities apply to the fermion η. With the help of these identities one can see that the simplest correlator with $\Phi^{(\frac{3}{2})}$ has only a few non-vanishing terms

$$\begin{aligned}
\langle \Phi_{\bar{\alpha}_3}^{(\pm\frac{1}{2})} \; \Phi_{\alpha_2}^{(0)} \; \Phi_{\alpha_1}^{(\frac{3}{2})} \rangle \;\; = \;\; & \Omega_{\frac{3}{2}}^{-2}(\alpha_1) \, \langle \Phi_{\bar{\alpha}_3}^{(\pm\frac{1}{2})} \Phi_{\alpha_2}^{(0)} \Big(2P_1^{-2} L_{-1} G_0 \bar{L}_{-1} \bar{G}_0 + \frac{1}{2}(Q+2P_1)^2 G_{-\frac{1}{2}} \bar{G}_{-\frac{1}{2}} \\
& + P_1^{-1}(Q+2P_1)(L_{-1}G_0\bar{G}_{-1} + G_{-1}\bar{L}_{-1}\bar{G}_0) \Big) \Phi_{\alpha_1}^{(\frac{1}{2})} \rangle,
\end{aligned}$$

so that

$$\begin{aligned}
\langle \Phi_{\bar{\alpha}_3}^{(\pm\frac{1}{2})} \Phi_{\alpha_2}^{(0)} \Phi_{\alpha_1}^{(\frac{3}{2})} \rangle & = \frac{i}{\Omega_{\frac{3}{2}}^2(\alpha_1)} \left((\Delta_3 - \Delta_2 - \Delta_1) - (Q+2P_1)(P_1+P_3)/2 \right)^2 \langle \Phi_{\bar{\alpha}_3}^{(\pm\frac{1}{2})} \Phi_{\alpha_2}^{(0)} \Phi_{\alpha_1}^{(-\frac{1}{2})} \rangle \\
& = \frac{i(Q+2P_1+2P_2 \pm 2P_3)^2 (Q+2P_1-2P_2 \pm 2P_3)^2}{4(2P_1+2b+b^{-1})^2(2P_1+b+2b^{-1})^2} \, C^{(\mp)}(\alpha_2; \alpha_3, \alpha_1 | b)
\end{aligned}$$

The second correlator is more complicated,

$$\langle \Phi^{(\pm\frac{1}{2})}_{\bar{\alpha}_3}\, \Phi^{(1)}_{\alpha_2}\, \Phi^{(\frac{3}{2})}_{\alpha_1} \rangle = \Omega^{-2}_{\frac{3}{2}}(\alpha_1)\Omega^{-2}_{1}(\alpha_2)$$

$$\Big\langle \Phi^{(\pm\frac{1}{2})}_{\bar{\alpha}_3} \Big(G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}} + (\tfrac{Q}{2}+P_2)^2 \eta_{-\frac{1}{2}}\bar{\eta}_{-\frac{1}{2}} + (\tfrac{Q}{2}+P_2)(\eta_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}} - \bar{\eta}_{-\frac{1}{2}}G_{-\frac{1}{2}}\Big)\Phi^{(0)}_{\alpha_2}$$

$$\times\Big(2^{-1}(Q+2P_1)^2 G_{-1}\bar{G}_{-1} + 2^{-\frac{1}{2}}\Omega_{\frac{3}{2}}(\alpha_1)(Q+2P_1)(\eta_{-1}\bar{G}_{-1} - \bar{\eta}_{-1}G_{-1})$$

$$+2P_1^{-2}L_{-1}G_0\bar{L}_{-1}\bar{G}_0 + \Omega^2_{\frac{3}{2}}(\alpha_1)\eta_{-1}\bar{\eta}_{-1} + \sqrt{2}\Omega_{\frac{3}{2}}(\alpha_1)P_1^{-1}(\eta_{-1}\bar{L}_{-1}\bar{G}_0 - \bar{\eta}_{-1}L_{-1}G_0)$$

$$+P_1^{-1}(Q+2P_1)(L_{-1}G_0\bar{G}_{-1} + S_{-1}\bar{L}_{-1}\bar{G}_0)\Big)\Phi^{(\frac{1}{2})}_{\alpha_1}\Big\rangle\ .$$

Using the Ward identities we arrive at

$$\langle \Phi^{(\pm\frac{1}{2})}_{\bar{\alpha}_3}\ \Phi^{(1)}_{\alpha_2}\ \Phi^{(\frac{3}{2})}_{\alpha_1} \rangle = \Omega^{-2}_{\frac{3}{2}}(\alpha_1)\Omega^{-2}_{1}(\alpha_2)\Big((b+2b^{-1}+2P_1)\left(2b+b^{-1}+2P_1\right)\left(P_2+\tfrac{Q}{2}\right)$$

$$-2(P_1\mp P_3)\left(\Delta_3-\Delta_2-\Delta_1-1/2\right) + 2(2P_1+Q)(\Delta_3-2\Delta_2-\Delta_1)$$

$$+2\left(P_2+\tfrac{Q}{2}\right)(\Delta_3-\Delta_2-\Delta_1) - (P_1\mp P_3)(2P_1+Q)\left(P_2+\tfrac{Q}{2}\right)\Big)^2 \langle \Phi^{(\pm\frac{1}{2})}_{\alpha_3}\ \Phi^{(0)}_{\alpha_2}\ \Phi^{(\frac{1}{2})}_{\alpha_1} \rangle$$

$$= \frac{(2P_1+2P_2\pm 2P_3+3b+b^{-1})^2(2P_1+2P_2\pm 2P_3+b+3b^{-1})^2\ C^{(\pm)}(\alpha_2;\alpha_3,\alpha_1|b)}{8(2P_1+2P_2\mp 2P_3+Q)^{-2}(2P_2+b+b^{-1})^2(2P_1+2b+b^{-1})^2(2P_1+b+2b^{-1})^2}$$

Within double Liouville theory we find

$$\begin{aligned}
\langle \mathbb{V}^{(\pm\frac{1}{2})}_{\bar{\alpha}_3}\ \mathbb{V}^{(0)}_{\alpha_2}\ \mathbb{V}^{(\frac{3}{2})}_{\alpha_1} \rangle &= C_L(\alpha_3^{(1)} \pm \frac{b^{(1)}}{4}, \alpha_2^{(1)}, \alpha_1^{(1)} + \frac{3b^{(1)}}{4}|b^{(1)}) \\
&\quad C_{\mathcal{L}}(\hat{\alpha}_3^{(2)} \pm \frac{1}{4\hat{b}^{(2)}}, \hat{\alpha}_2^{(2)}, \hat{\alpha}_1^{(2)} + \frac{3}{4\hat{b}^{(2)}}|\hat{b}^{(2)}) = A_5^{\pm} C_R^{(\mp)}(\alpha_2;\alpha_3,\alpha_1|b) \\
\langle \mathbb{V}^{(\pm\frac{1}{2})}_{\bar{\alpha}_3}\ \mathbb{V}^{(1)}_{\alpha_2}\ \mathbb{V}^{(\frac{3}{2})}_{\alpha_1} \rangle &= C_L(\alpha_3^{(1)} \pm \frac{b^{(1)}}{4}, \alpha_2^{(1)} + \frac{b^{(1)}}{2}, \alpha_1^{(1)} + \frac{3b^{(1)}}{4}|b^{(1)}) \\
&\quad C_{\mathcal{L}}(\hat{\alpha}_3^{(2)} \pm \frac{1}{4\hat{b}^{(2)}}, \hat{\alpha}_2^{(2)} + \frac{1}{2\hat{b}^{(2)}}, \hat{\alpha}_1^{(2)} + \frac{3}{4\hat{b}^{(2)}}|\hat{b}^{(2)}) = A_6^{\pm} C_R^{(\pm)}(\alpha_2;\alpha_3,\alpha_1|b)
\end{aligned}$$

where

$$A_5^{\pm} = \frac{\left(\pi\mu_L\gamma\left(\frac{2b^2}{1-b2}\right)\right)^{\frac{Q-\alpha-3b/2\mp b/2}{2b}} \left(\pi M\gamma\left(\frac{b^2-1}{2}\right)\right)^{\frac{b(a+3/2\pm1/2)}{1-b^2}} \gamma\left(\frac{-2b^2}{1-b^2}\right)\gamma\left(\frac{b^2+1}{2}\right)}{\left(\left(\frac{\pi\mu}{2}\right)\,\gamma\left(\frac{b^2+1}{2}\right)\right)^{\frac{Q-\alpha}{b}} b^{3\pm1}\,\left(\frac{2}{1-b^2}\right)^{\frac{9}{2}\pm1}}$$
$$\frac{(Q+2P_1+2P_2\pm 2P_3)^2(Q+2P_1-2P_2\pm 2P_3)^2}{8\,(2P_1+2b+b^{-1})^2(2P_1+b+2b^{-1})^2}$$

$$A_6^{\pm} = -\frac{\left(\pi\mu_L\gamma\left(\frac{2b^2}{1-b2}\right)\right)^{\frac{Q-\alpha-5b/2\mp b/2}{2b}} \left(\pi M\gamma\left(\frac{b^2-1}{2}\right)\right)^{\frac{b(a+5/2\pm1/2)}{1-b^2}} \gamma\left(\frac{-2b^2}{1-b^2}\right)\gamma\left(\frac{b^2+1}{2}\right)}{\left(\left(\frac{\pi\mu}{2}\right)\,\gamma\left(\frac{b^2+1}{2}\right)\right)^{\frac{Q-\alpha}{b}} b^{5\pm1}\,\left(\frac{2}{1-b^2}\right)^{\frac{13}{2}\pm1}}$$
$$\frac{(2P_1+2P_2\mp 2P_3+Q)^2(2P_1+2P_2\pm 2P_3+3b+b^{-1})^2(2P_1+2P_2\pm 2P_3+b+3b^{-1})^2}{32\,(2P_2+b+b^{-1})^2(2P_1+2b+b^{-1})^2(2P_1+b+2b^{-1})^2}$$

Comparing with the correlators from the first part of the computation in $\mathcal{N}$=1 Liouville theory we obtain,

$$\mathcal{N}_{\bar{\alpha}_3}^{(\pm\frac{1}{2})}\mathcal{N}_{\alpha_2}^{(0)}\mathcal{N}_{\alpha_1}^{(\frac{3}{2})}A_5^{\pm} = 2\kappa(b)\,\frac{(Q+2P_1+2P_2\pm 2P_3)^2(Q+2P_1-2P_2\pm 2P_3)^2}{8\,(2P_1+2b+b^{-1})^2(2P_1+b+2b^{-1})^2}\ ,$$
$$\mathcal{N}_{\bar{\alpha}_3}^{(\pm\frac{1}{2})}\mathcal{N}_{\alpha_2}^{(1)}\mathcal{N}_{\alpha_1}^{(\frac{3}{2})}A_6^{\pm} = 4\kappa(b)$$
$$\frac{(2P_1+2P_2\mp 2P_3+Q)^2(2P_1+2P_2\pm 2P_3+3b+b^{-1})^2(2P_1+2P_2\pm 2P_3+b+3b^{-1})^2}{32\,(2P_2+b+b^{-1})^2(2P_1+2b+b^{-1})^2(2P_1+b+2b^{-1})^2}$$

so that we verified two additional cases of eq. (4.20) with $\varpi(\pm1/2,0,3/2) = -i$ and $\varpi(\pm1/2,1,3/2) = 1$. This concludes the tests of our main correspondence (4.16).

5 Outlook and Conclusions

The main result of this work is our formula (4.16) that relates fields in the product of $\mathcal{N}$=1 Liouville field theory with a free fermion to primaries in double Liouville field theory. We have tested this proposal through a number of non-trivial calculations. The correspondence (4.16) extends related observations in [1] to the R sector. In addition, we have been able to normalize the fields in both R and NS sector such that the 3-point functions agree up to a simple b-dependent factor $\sim\kappa$. Since this factor does not depend on the fields we insert, it can be absorbed through a normalization of the vacuum state.

Our results may be extended in a number of different directions. It clearly seems worthwhile to study the correspondence (4.16) for correlation functions e.g. on discs with non-trivial boundary conditions or higher genus surfaces. $\mathcal{N}$=1 Liouville field theory possesses one continuous family of boundary conditions which preserve the $\mathcal{N}$=1 super-conformal algebra. Though boundary conditions in imaginary Liouville theory have not received as much attention as the bulk model, see however [39], it seems likely that double Liouville theory admits conformal boundary conditions that are parametrized by two continuous labels. A subset of these boundary conditions should preserve the larger $\mathcal{N}$=1 super-conformal symmetry along with simple gluing conditions for the fermion η.

In an interesting recent paper [45] Gaiotto engineers a conformal interface between the minimal models MM_k and MM_{k-1}. Gaiotto's construction makes essential use of the relation (1.1) between the product theory and supersymmetric minimal models. It seems likely that a similar interface between Liouville theory and its imaginary version also exists. Constructing this interface explicitly might be of some interest as it could provide more insight into the relation between standard Liouville field theory and its imaginary cousin.

The main motivation for this work, however, came from the results of [2] which relate correlators of the OSP(1|2) WZW model at level k to those of $\mathcal{N}$=1 Liuoville theory with $b^{-2} = 2k - 3$. In order to compute N-point functions of primaries $V_j^\epsilon(\mu|z)$ in the WZW model, one needs to calculate higher correlators in a product of $\mathcal{N}$=1 Liouville field theory with a free fermion. The latter involve N fields from the physical spectrum of the supersymmetric Liouville theory along with N-2 degenerate ones whose insertion points $y_i = y_i(\mu_\nu)$ depend on the complex parameters μ_ν. It turns out that all these fields must be takes from the R sector of the model. More precisely, one finds

$$\langle \prod_{\nu=1}^{N} V_{j_\nu}^{\epsilon_\nu}(\mu_\nu|z_\nu)\rangle \sim \delta^2(\sum_{\nu=1}^{N}\mu_\nu)\langle \prod_{\nu=1}^{N} \frac{1}{2}\left(\Phi_{\alpha_\nu}^{(-\frac{1}{2})}(z_\nu) - i\epsilon_\nu \Phi_{\alpha_\nu}^{(\frac{1}{2})}(z_\nu)\right) \prod_{j=1}^{N-2} \Phi_{-\frac{1}{2b}}^{(\frac{1}{2})}(y_j)\rangle \ . \tag{5.1}$$

up to some simple factors. In the present article we have argued that the correlation functions on the right hand side can be calculated in double Liouville theory. We believe that such a relation between the OSP(1|2) WZW model and double Liouville theory could become a crucial ingredient in finding a supersymmetric analogue of the celebrated FZZ-duality between the SL(2)/U(1) black hole sigma model and sine-Liouville field theory, much along the lines of [46]. In this context it is crucial to observe that, according to eq. (4.16), the degenerate fields we have to insert at points y_j on the right hand side of the correspondence (5.1) are trivial in imaginary Liouville theory, i.e.

$$\Phi^{(\frac{1}{2})}_{-\frac{1}{2b}}(y) \sim V_{-\frac{1}{2b}}(y) \ .$$

Hence, the imaginary Liouville theory is merely a spectator throughout most of the computations performed in [46]. Consequently, we can express correlation functions in the OSP(1|2) WZW model through a product of sine-Liouville and imaginary Liouville theory. It then remains to rewrite the latter in terms of a more conventional theory. We shall return to these issues in a forthcoming paper.

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