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Classification of symmetric toroidal orbifolds

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Abstract

We provide a complete classification of six–dimensional symmetric toroidal orbifolds which yield $\mathcal{N} \geq 1$ supersymmetry in 4D for the heterotic string. Our strategy is based on a classification of crystallographic space groups in six dimensions. We find in total 520 inequivalent toroidal orbifolds, 162 of them with Abelian point groups such as $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$–I etc. and 358 with non–Abelian point groups such as S_3, D_4, A_4 etc. We also briefly explore the properties of some orbifolds with Abelian point groups and $\mathcal{N} = 1$, i.e. specify the Hodge numbers and comment on the possible mechanisms (local or non–local) of gauge symmetry breaking.

Contents

1 Introduction

Heterotic string model building has received an increasing attention in the past few years. The perhaps simplest heterotic compactifications are based on Abelian toroidal orbifolds [1, 2]. Unlike in the supergravity compactifications on Calabi–Yau manifolds one has a clear string theory description. In addition, the scheme is rich enough to produce a large number of candidate models that may yield a stringy completion of the (supersymmetric) standard model [3, 4] (for a review see e.g. [5]). At the same time, symmetric orbifolds have a rather straightforward geometric interpretation (cf. e.g. [6, 7, 8]). In fact, the geometric properties often have immediate consequences for the phenomenological features of the respective models. One obtains an intuitive understanding of discrete R symmetries in terms of remnants of the Lorentz group of compact space, of the appearance of matter as complete GUT multiplets due to localization properties and gauge group topographies as well as flavor structures.

Despite their simplicity, symmetric toroidal orbifolds provide us with a large number of different settings, which have, rather surprisingly, not been fully explored up to now. In the past, different attempts of classifying (parts of) these compactifications have been made [9, 10, 11, 12]. These classifications are not mutually consistent, and, as we shall see, incomplete. The perhaps most complete classification is due to Donagi and Wendland (DW) [10], who focus on $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. The main purpose of this paper is to provide a complete classification of symmetric Abelian and non–Abelian heterotic orbifolds that lead to $\mathcal{N} \geq 1$ supersymmetry (SUSY) in four dimensions.

The structure of this paper is as follows: in Section 2 we discuss the tools used to construct toroidal orbifolds. Later, in Section 3, we present a way of classifying all possible space groups that is novel in the context of string compactifications. Then, in Section 4 we impose the condition of $\mathcal{N} = 1$ SUSY in 4D. Section 5 is devoted to a survey of the resulting orbifolds, and to a comparison with previous attempts to classify Abelian symmetric orbifolds [9, 10, 11, 12]. Finally, in Section 6 we briefly discuss our results. In various appendices we collect more detailed information on our classification program. Appendix A contains some details on lattices, in Appendix B we survey the already known 2D orbifolds, and in Appendix C we provide tables of our results.

2 Construction of toroidal orbifolds

We start our discussion with the construction of toroidal orbifolds [1, 2]. There are two equivalent ways of constructing such objects: (i) one can start from the Euclidean space $\mathbb{R}^n$ and divide out a discrete group S, the so–called space group. (ii) Alternatively, one can start with an n–dimensional lattice Λ, to be defined in detail in Section 2.2, which determines a torus $\mathbb{T}^n$ and divide out some discrete symmetry group G. Note that G, the so–called orbifolding group as defined in Section 2.5, is in general not equal to the point group introduced in Section 2.3. That is, a toroidal orbifold is defined as

$$\mathbb{O} \;=\; \mathbb{R}^n/S \;=\; \mathbb{T}^n/G\,. \tag{2.1}$$

Even though we are mostly interested in the case $n = 6$ we will keep n arbitrary. In the following, we will properly define the concepts behind Equation (2.1), closely following [13].

2.1 The space group S

Let S be a discrete subgroup of the group of motions in $\mathbb{R}^n$, i.e. every element of S leaves the metric of the space invariant. If S contains n linearly independent translations, then it is called a space group of degree n. Such groups appear already in crystallography: they are the symmetry groups of crystal structures, which in turn are objects whose symmetries comprise discrete translations.

Every element g of a space group S can be written as a composition of a mapping ϑ that leaves (at least) one point invariant and a translation by some vector λ, i.e. $g = \lambda \circ \vartheta$ for $g \in S$ (one can think of ϑ as a discrete rotation or inversion). This suggests to write a space group element as[1]

$$g \;=\; (\vartheta, \lambda) \;, \tag{2.2}$$

and it acts on a vector $v \in \mathbb{R}^n$ as

$$v \;\overset{g}{\longmapsto}\; \vartheta\, v + \lambda \;. \tag{2.3}$$

Let $h = (\omega, \tau) \in S$ be another space group element. Then the composition $h \circ g$ is given by $(\omega\, \vartheta, \omega\, \lambda + \tau)$.

2.2 The lattice Λ

Let S be a space group. The subgroup Λ of S consisting of all translations in S is the lattice of the space group. Note that for a general element $g = (\vartheta, \lambda) \in S$ the vector λ does not need to be a lattice vector. Elements $g = (\vartheta, \lambda) \in S$ with $\lambda \notin \Lambda$ are called roto–translations.

Since a space group is required to contain n linear independent translations, every lattice contains a basis $\mathfrak{e} = \{e_i\}_{i \in \{1,\ldots,n\}}$ and the full lattice is spanned by the e_i (with integer coefficients), i.e. an element $\lambda \in \Lambda$ can be written as $\lambda = n_i\, e_i$, summing over $i = 1, \ldots, n$ and $n_i \in \mathbb{Z}$. Clearly, the choice of basis is not unique. For example, for a given lattice Λ take two bases $\mathfrak{e} = \{e_1, \ldots, e_n\}$ and $\mathfrak{f} = \{f_1, \ldots, f_n\}$ and define $B_\mathfrak{e}$ and $B_\mathfrak{f}$ as matrices whose columns are the basis vectors in $\mathfrak{e}$ and $\mathfrak{f}$, respectively. Then the change of basis is given by a unimodular matrix M (i.e. $M \in \mathrm{GL}(n, \mathbb{Z})$) as

$$B_\mathfrak{e}\, M \;=\; B_\mathfrak{f} \;. \tag{2.4}$$

On the other hand, one can decide whether two bases $\mathfrak{e}$ and $\mathfrak{f}$ span the same lattice by computing the matrix $M \;=\; B_\mathfrak{e}^{-1}\, B_\mathfrak{f}$ and checking whether or not it is an element of $\mathrm{GL}(n, \mathbb{Z})$.

2.3 The point group P

For a space group S with elements of the form (ϑ, λ), the set P of all ϑ forms a finite group ([13, p. 15]), the so–called point group of S. The elements of a point group are sometimes called twists or rotations. However, in general a point group can also contain inversions and reflections, i.e. $\vartheta \in \mathrm{O}(n)$.

The point group P of S maps the lattice of S to itself. Hence, similarly to the change of lattice bases, point group elements can be represented by $\mathrm{GL}(n, \mathbb{Z})$ (i.e. unimodular) matrices. When

[1] In the mathematical literature the reverse notation $g = (\lambda, \vartheta)$ is also common, since the normal subgroup element is usually written to the left, and the lattice Λ is the normal subgroup of the space group.

written in the $\mathrm{GL}(n,\mathbb{Z})$ basis, we append the twists by an index indicating the lattice basis, while the $\mathrm{O}(n)$ (or $\mathrm{SO}(n)$) representation of the twist is denoted without an index. For example, the twist $\vartheta \in \mathrm{O}(n)$ is denoted as $\vartheta_{\mathfrak{e}}$ in the lattice basis $\mathfrak{e} = \{e_1, \ldots, e_n\}$ such that $\vartheta\, e_i = (\vartheta_{\mathfrak{e}})_{ji}\, e_j$ and $\vartheta_{\mathfrak{e}} = B_{\mathfrak{e}}^{-1}\, \vartheta\, B_{\mathfrak{e}}$. Furthermore, under a change of basis as in Equation (2.4) the twist transforms according to

$$\vartheta_{\mathfrak{f}} \;=\; M^{-1}\,\vartheta_{\mathfrak{e}}\,M\ . \tag{2.5}$$

Given these definitions, and provided that the lattice is always a normal subgroup of the space group (i.e. rotation $\circ$ translation $\circ$ $(\text{rotation})^{-1}$ = translation), the space group S has a semi–direct product structure iff the point group P is a subgroup of it, i.e. $P \subset S$. In that case

$$S \;=\; P \ltimes \Lambda\,, \tag{2.6}$$

and one can write the orbifold as

$$\mathbb{O} \;=\; \mathbb{R}^n/(P \ltimes \Lambda) \;=\; \mathbb{T}^n/P\ . \tag{2.7}$$

In general, however, the point group is not a subgroup of the space group and thus the space group is not necessarily a semi–direct product of its point group with its lattice. More precisely, in general the point group P is not equal to the orbifolding group G of Equation (2.1) because of the possible presence of roto–translations, as we will see in an example in Section 2.4.

2.4 Examples: space groups with $\mathbb{Z}_2$ point group

In this section, we give two examples of space groups in two dimensions with $\mathbb{Z}_2$ point group in order to illustrate the discussion of the previous sections.

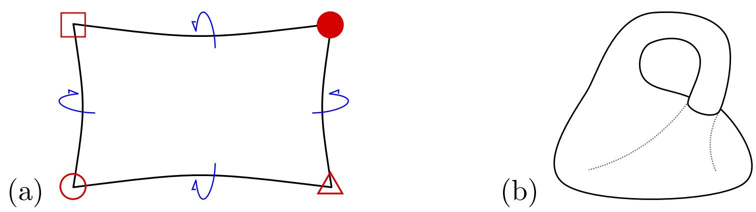

Figure 1: Two–dimensional examples: (a) "pillow" and (b) Klein bottle. In case (a) the blue arrows indicate a wrap–around and the red symbols indicate fixed points.

A simple example: the "pillow"

The first of our examples is the well known two–dimensional "pillow", see Figure 1(a). The space group S is generated as

$$S \;=\; \langle(\mathbb{1}, e_1), (\mathbb{1}, e_2), (\vartheta, 0)\rangle\ , \tag{2.8}$$

and can be realized as the semi–direct product of the oblique lattice Λ (see Appendix A.3) and the point group $P = \{\mathbb{1}, \vartheta\}$. In detail, the lattice is given as $\Lambda = \{n_1\, e_1 + n_2\, e_2, n_i \in \mathbb{Z}\}$ using the basis $\mathfrak{e} = \{e_1, e_2\}$. ϑ is a rotation by π, i.e. it acts on the lattice basis vectors as

$$\vartheta\ e_i \;=\; -e_i \qquad \text{for} \qquad i \;=\; 1, 2\ . \tag{2.9}$$

Therefore, it can be represented by a $\mathrm{GL}(2,\mathbb{Z})$ matrix

$$\vartheta_{\mathfrak{e}} \;=\; \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right) . \tag{2.10}$$

Since $\vartheta^2 = \mathbb{1}$, the point group is $\mathbb{Z}_2$.

Another example: the Klein bottle

Let us take a look at a more advanced example: the space group of a Klein bottle, see Figure 1(b). Here, the space group is generated by two orthogonal lattice vectors (which thus span a primitive rectangular lattice Λ) $\{e_1, e_2\}$, and an additional element g,

$$S \;=\; \langle(\mathbb{1}, e_1), (\mathbb{1}, e_2), g\rangle \qquad \text{with} \qquad g \;=\; \left(\vartheta, \tfrac{1}{2}e_1\right) \qquad \text{and} \qquad \vartheta_{\mathfrak{e}} \;=\; \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) . \tag{2.11}$$

g acts on a vector $v = v^1 e_1 + v^2 e_2$ as

$$v \;\overset{g}{\longmapsto}\; \vartheta\, v + \frac{1}{2}e_1 \;=\; v^1\, e_1 - v^2\, e_2 + \frac{1}{2}e_1 \,. \tag{2.12}$$

Notice that even though the point group is $\mathbb{Z}_2$ (i.e. $\vartheta^2 = \mathbb{1}$), g generates a finite group isomorphic to $\mathbb{Z}_2$ only on the torus $\mathbb{T}^2 \;=\; \mathbb{R}^2/\Lambda$, but not on the Euclidean space $\mathbb{R}^2$, because $g^2 = (\mathbb{1}, e_1) \neq (\mathbb{1}, 0)$. In other words, since the generator g also contains a translation $\frac{1}{2}e_1 \notin \Lambda$, it is not a point group element but a roto–translation.

Obviously, this space group cannot be written as a semi–direct product of a lattice and a point group, as is always the case when we have roto–translations.

2.5 The orbifolding group G

Due to the possible presence of roto–translations, it is clear that in general space groups cannot be described by lattices and point groups only. Therefore, we will need to define an additional object, the orbifolding group (see [10]). Loosely speaking, the orbifolding group G is generated by those elements of S that have a non–trivial twist part, identifying elements which differ by a lattice translation. Hence, if there are no roto–translations the orbifolding group G is equal to the point group P. In other words, the orbifolding group may contain space group elements with non–trivial, non–lattice translational parts. Combining the orbifolding group G and the torus lattice Λ generates the space group $S = \langle\{G, \Lambda\}\rangle$.

Hence, we can define the orbifold as

$$\mathbb{O} \;=\; \mathbb{R}^n/S \;=\; \mathbb{R}^n/\langle\{G,\Lambda\}\rangle \;=\; (\mathbb{R}^n/\Lambda)/G \;=\; \mathbb{T}^n/G \,. \tag{2.13}$$

Orbifolds can be manifolds (see e.g. Figure 1(b)), but in general, they come with singularities which can not be endowed with smooth maps (see e.g. Figure 1(a)).

$$\boxed{\text{affine classes}} \subset \boxed{\mathbb{Z}\text{–classes}} \subset \boxed{\mathbb{Q}\text{–classes}}$$

Figure 2: Sketch of the classification of space groups.

3 Equivalences of space groups

In the context of string orbifold compactifications, some physical properties of a given model directly depend on the choice of its space group. These features are common to whole sets of space groups and can be related to some mathematical properties. Using the latter, one can define equivalence classes of space groups. In detail, there are three kinds of equivalence classes suitable to sort space groups S with certain physical and corresponding mathematical properties. These classes are:

1. the $\mathbb{Q}$–class (see Section 3.3) determines the point group P contained in S and hence the number of supersymmetries in 4D and the number of geometrical moduli;
2. the $\mathbb{Z}$–class (see Section 3.2) determines the lattice Λ of S and hence the nature of the geometrical moduli;
3. the affine class (see Section 3.1) determines the flavor group and the nature of gauge symmetry breaking (i.e. local vs. non–local gauge symmetry breaking).

Each $\mathbb{Q}$–class can contain several $\mathbb{Z}$–classes and each $\mathbb{Z}$–class can contain several affine classes, see Figure 2. In other words, for every point group there can be several inequivalent lattices and for every lattice there can be several inequivalent choices for the orbifolding group (i.e. with or without roto–translations).

In the following, we will discuss in detail why the concept of affine classes is advantageous to classify physically inequivalent space groups. This is standard knowledge among crystallographers and can for instance be found in more detail in [13].

3.1 Affine classes of space groups

Two space groups S_1 and S_2 of degree n belong to the same affine class (i.e. $S_1 \sim S_2$) if there is an affine mapping $f \,:\, \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f^{-1}\, S_1\, f \;=\; S_2\,. \tag{3.1}$$

An affine mapping $f = (A, t)$ on $\mathbb{R}^n$ consists of a translation t and a linear mapping A, that is, it allows for rescalings and rotations. Therefore, this definition enables us to distinguish between space groups that actually describe different symmetries and space groups which are just the ones we already know, looked upon from a different angle or distance. Then, for a given representative space group of an affine class a non–trivial affine transformation A that leaves the point group invariant (i.e. $A^{-1}\, P\, A \;=\; P$) corresponds to a change of the geometrical data. In the context of superstring compactifications this corresponds to a change of values of the geometrical moduli. That is, affine transformations amount to moving in the moduli space of the respective compactification. Hence, we will only be interested in one representative for every affine class.

It turns out that, for a given dimension n, there exists only a finite number of affine classes of space groups [13, p. 10]. Hence, classifying all affine classes of space groups enables a complete classification of orbifolds for a fixed number of dimensions. In this paper, we focus on the six–dimensional case.

Example in two dimensions

Let us illustrate this at the $\mathbb{T}^2/\mathbb{Z}_2$ example with $\vartheta = -\mathbb{1}$ given in Section 2.4. As discussed there, the lattice is oblique, i.e. one can choose any linear independent vectors e_1 and e_2 as basis vectors. Define a space group S by choosing

$$e_1 = \begin{pmatrix} r_1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} r_2\cos(\alpha) \\ r_2\sin(\alpha) \end{pmatrix} . \tag{3.2}$$

This space group is in the same affine class as $\widetilde{S}$ with basis vectors

$$\widetilde{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \widetilde{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{3.3}$$

This can be seen explicitly using the affine transformation $f = (A, 0)$ with

$$A = \begin{pmatrix} r_1 & r_2\cos(\alpha) \\ 0 & r_2\sin(\alpha) \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} \frac{1}{r_1} & -\frac{1}{r_1\tan(\alpha)} \\ 0 & \frac{1}{r_2\sin(\alpha)} \end{pmatrix} . \tag{3.4}$$

Take an arbitrary element $g = (\vartheta, n_i e_i)$ with $n_i \in \mathbb{Z}$ for $i = 1, 2$. Then

$$\left(f^{-1}\, g\, f\right)(x) = \left(f^{-1}\, g\right)(Ax) = f^{-1}(\vartheta A x + n_i e_i) = \vartheta x + A^{-1}(n_i e_i) \tag{3.5a}$$
$$= \vartheta x + n_i\, \widetilde{e}_i = \widetilde{g}\, x \tag{3.5b}$$

for $x \in \mathbb{R}^2$ and $\widetilde{g} = (\vartheta, n_i\widetilde{e}_i) \in \widetilde{S}$. Therefore, $S \sim \widetilde{S}$ and there is only one affine class of $\mathbb{T}^2/\mathbb{Z}_2$ space groups with $\vartheta = -\mathbb{1}$.

This should be compared with the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold, where the angle between the basis vectors e_i and their length ratio are fixed, such that the corresponding moduli space is different. Hence, it is clear that $\mathbb{T}^2/\mathbb{Z}_2$ and $\mathbb{T}^2/\mathbb{Z}_3$ are two different orbifolds. This demonstrates the advantages of using affine classes for the classification of space groups.

3.2 $\mathbb{Z}$–classes of space groups

As discussed above, we can sort space groups into affine classes. This can be refined further by grouping affine classes according to common properties of their point groups. Following the argument in Section 2.3, the elements of the point group can be written in the lattice basis as elements of $\mathrm{GL}(n, \mathbb{Z})$. Therefore, a point group is a finite subgroup of the unimodular group on $\mathbb{Z}$.

Take two space groups S_1 and S_2. For $i = 1, 2$, the space group S_i contains a lattice Λ_i and its point group in the lattice basis is denoted by P_i, i.e. $P_i \subset \mathrm{GL}(n, \mathbb{Z})$. Then, the two space groups belong to the same $\mathbb{Z}$–class (or in other words to the same arithmetic crystal class) if there exists an unimodular matrix U (i.e. $U \in \mathrm{GL}(n,\mathbb{Z})$) such that (cf. the parallel discussion around Equation (3.1))

$$U^{-1}\, P_1\, U \;=\; P_2\;, \tag{3.6}$$

see Equation (2.5). That is, if the point groups are related by a change of lattice basis (using U), the space groups belong to the same $\mathbb{Z}$–class. Hence, $\mathbb{Z}$–classes classify the inequivalent lattices.

If two space groups belong to the same $\mathbb{Z}$–class, they have the same form space and, physically, they possess the same amount and nature of geometrical moduli. However, as we have stressed before, space groups from the same $\mathbb{Z}$–class are not necessarily equivalent because of the possible presence of roto–translations. In other words, space groups from the same $\mathbb{Z}$–class can belong to different affine classes and can hence be inequivalent.

3.3 $\mathbb{Q}$–classes of space groups

As before in Section 3.2, take two space groups S_1 and S_2. For $i = 1, 2$, the point group in the lattice basis associated to the space group S_i is denoted by P_i, i.e. $P_i \subset \mathrm{GL}(n,\mathbb{Z})$. Then, the two space groups belong to the same $\mathbb{Q}$–class (or in other words to the same geometric crystal class) if there exists a matrix $V \in \mathrm{GL}(n,\mathbb{Q})$ such that

$$V^{-1}\, P_1\, V \;=\; P_2\;. \tag{3.7}$$

Obviously, if two space groups belong to the same $\mathbb{Z}$–class they also belong to the same $\mathbb{Q}$–class, hence the inclusion sketch in Figure 2. In contrast to $\mathbb{Z}$–classes, $\mathbb{Q}$–classes do not distinguish between inequivalent lattices. However, if two space groups belong to the same $\mathbb{Q}$–class, the commutation relations and the orders of the corresponding point groups are the same. Therefore, they are isomorphic as crystallographic point groups. They also possess form spaces of the same dimension, i.e. they have the same number of moduli. What is important for physics is that all space groups in the same $\mathbb{Q}$–class share a common holonomy group (cf. Section 4). This allows us to identify settings that yield $\mathcal{N} = 1$ SUSY in 4D. In particular, in order to determine the number of SUSY generators, it is sufficient to consider only one representative from every $\mathbb{Q}$–class.

3.4 Some examples

Before going to six dimensions, let us illustrate the above definitions with some easy examples of two–dimensional $\mathbb{Z}_2$ orbifolds, taken from Appendix B.

Space groups in the same $\mathbb{Z}$–class

Consider the affine class $\mathbb{Z}_2$–II–1–1, as defined in Appendix B. As there are no roto–translations, the orbifolding group is equal to the point group and is generated by ϑ, a reflection at the horizontal axis. Now, let this reflection act on a lattice, first spanned by the basis vectors $\mathfrak{e} = \{e_1, e_2\}$ and second spanned by $\mathfrak{f} = \{f_1, f_2\}$, see Figure 3. The two corresponding space groups read

$$S_{\mathfrak{e}} \;=\; \langle(\vartheta,0),(\mathbb{1},e_1),(\mathbb{1},e_2)\rangle \qquad \text{with} \qquad \vartheta_{\mathfrak{e}} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\;, \tag{3.8}$$

$$S_{\mathfrak{f}} \;=\; \langle(\vartheta,0),(\mathbb{1},f_1),(\mathbb{1},f_2)\rangle \qquad \text{with} \qquad \vartheta_{\mathfrak{f}} \;=\; \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\;, \tag{3.9}$$

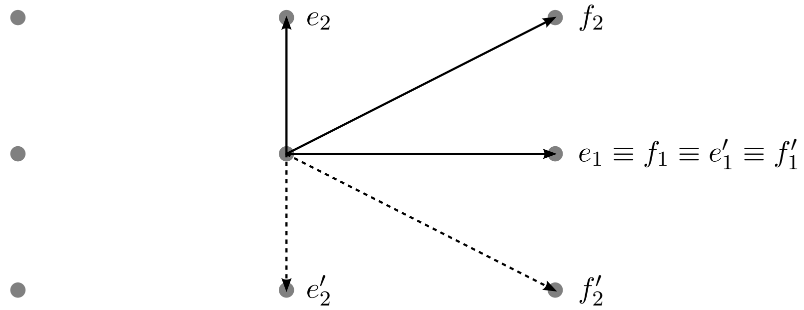

Figure 3: Two different bases for the p–rectangular lattice: $\mathfrak{e} = \{e_1, e_2\}$ and $\mathfrak{f} = \{f_1, f_2\}$, and the action of the point group generator (primed vectors).

where $\vartheta_{\mathfrak{e}} \neq \vartheta_{\mathfrak{f}}$ because they are given in their corresponding lattice bases. However, it is easy to see that they are related by the $\mathrm{GL}(2,\mathbb{Z})$ transformation

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad U^{-1}\,\vartheta_{\mathfrak{e}}\,U = \vartheta_{\mathfrak{f}}\,, \tag{3.10}$$

cf. Equation (3.6). Therefore, they belong to the same $\mathbb{Z}$–class. Hence, as we actually knew from the start, they act on the same lattice and the matrix U just defines the associated change of basis precisely as in Equation (2.4).

Space groups in the same $\mathbb{Q}$–class, but different $\mathbb{Z}$–classes

Next, consider the space groups,

$$S_{1\text{–}1} = \langle(\vartheta_{1\text{–}1}, 0), (\mathbb{1}, e_1), (\mathbb{1}, e_2)\rangle \quad \text{with} \quad \vartheta_{1\text{–}1,\mathfrak{e}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.11}$$

$$S_{2\text{–}1} = \langle(\vartheta_{2\text{–}1}, 0), (\mathbb{1}, f_1), (\mathbb{1}, f_2)\rangle \quad \text{with} \quad \vartheta_{2\text{–}1,\mathfrak{f}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.12}$$

with lattices spanned by $e_1 = (1,0)$, $e_2 = (0,1)$ and $f_1 = (1/2, 1/2)$, $f_2 = (1/2, -1/2)$, respectively. The first space group belongs to the affine class $\mathbb{Z}_2$–II–1–1 and the second one to $\mathbb{Z}_2$–II–2–1, see Appendix B. If we try to find the transformation V from Equation (3.7) that fulfills $V^{-1}\,\vartheta_{1\text{–}1,\mathfrak{e}}\,V = \vartheta_{2\text{–}1,\mathfrak{f}}$ we see that

$$V = \begin{pmatrix} x & x \\ y & -y \end{pmatrix} \quad \text{with} \quad x, y \in \mathbb{Q}\,. \tag{3.13}$$

But for all values of x and y for which V^{-1} exists, either V or V^{-1} has non–integer entries. Therefore, the space groups $\mathbb{Z}_2$–II–1–1 and $\mathbb{Z}_2$–II–2–1 belong to the same $\mathbb{Q}$–class, but to different $\mathbb{Z}$–classes. In other words, these space groups are defined with inequivalent lattices. Indeed, the first space group possesses a primitive rectangular lattice, while the second one has a centered rectangular lattice, as we will see in detail in the following.

The effect of including additional translations

There is an alternative way of seeing the relationship between the two space groups of the last example: one can amend one of the space groups by an additional translation. In general, this gives rise to a new lattice, and consequently to a different $\mathbb{Z}$–class.

In our case, let us take the $\mathbb{Z}_2$–II–1–1 affine class and add the non–lattice translation

$$\tau \;=\; \frac{1}{2}(e_1 + e_2) \tag{3.14}$$

to its space group. If we incorporate this translation into the lattice, we notice that this element changes the original primitive rectangular lattice to a centered rectangular lattice, with a unit cell of half area. The new lattice (see Figure 4) can be spanned by the basis vectors τ and $e_1 - \tau$.

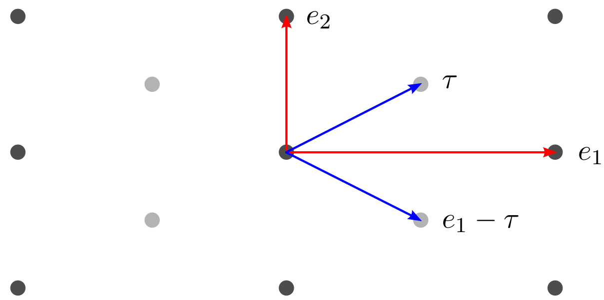

Figure 4: Change of a lattice by an additional translation: the basis of the original lattice is red, the basis of the new one blue. The additional lattice points are gray. The action of ϑ is a reflection at the horizontal axis. Therefore, it maps e_1 to itself, e_2 to its negative and interchanges τ and $e_1 - \tau$.

We can interpret the inclusion of this additional translation as a "change of basis", see Equation (2.4), but now generated by a matrix $M \in \mathrm{GL}(2,\mathbb{Q})$ instead of one from $\mathrm{GL}(2,\mathbb{Z})$. The transformation looks like

$$B_{\mathfrak{e}} M \;=\; B_\tau \quad \text{with} \quad M \;=\; \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} , \tag{3.15}$$

where $B_{\mathfrak{e}}$ and B_τ are matrices whose columns are (e_1, e_2) and $(\tau, e_1 - \tau)$, respectively. M is precisely the matrix in Equation (3.13) with values $x = y = 1/2$. Performing this basis change, the twist has to be transformed accordingly. Hence, the two $\mathbb{Z}$–classes are related by a $\mathrm{GL}(2,\mathbb{Q})$ transformation M and the new space group with lattice B_τ is $\mathbb{Z}_2$–II–2–1. The geometrical action of the twist, however, is the same in both cases: it is a reflection at the horizontal axis (see Figure 4). That is the reason for the name geometrical crystal classes for $\mathbb{Q}$–classes. A general method for including additional translations can be found in Appendix A.2.

The method of using additional translations has been used in [10] and [12] in order to classify six–dimensional space groups with point groups $\mathbb{Z}_N \times \mathbb{Z}_N$ for $N = 2, 3, 4, 6$ (the classification of [12] is not fully exhaustive). In these works, the authors start with factorized lattices, i.e. lattices which are the orthogonal sum of three two–dimensional sublattices, on which the twists act diagonally.

Then, in a second step additional translations are introduced. As we have shown here, adding such translations is equivalent to switching between $\mathbb{Z}$–classes in the same $\mathbb{Q}$–class. Hence, if one considers all possible lattices ($\mathbb{Z}$–classes) additional translations do not give rise to new orbifolds.

Space groups in different $\mathbb{Q}$–classes

Finally, consider the affine classes $\mathbb{Z}_2$–I–1–1 and $\mathbb{Z}_2$–II–1–1 defined in Appendix B. If we try to find a transformation between both space groups generators, see Equation (3.7),

$$V^{-1}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} V \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} V \;=\; V\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{3.16}$$

we obtain

$$V \;=\; \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \;\notin\; \mathrm{GL}(2,\mathbb{Q}) \qquad \forall\, x,y\;. \tag{3.17}$$

Therefore, the space groups $\mathbb{Z}_2$–I–1–1 and $\mathbb{Z}_2$–II–1–1 belong to different $\mathbb{Q}$–classes (and also to different $\mathbb{Z}$–classes). That is, the point groups are inequivalent: the twist of the first point group is a reflection at the origin and the twist of the second point group is a reflection at the horizontal axis.

4 Classification of space groups

In this section we describe our strategy to classify all inequivalent space groups for the compactification of the heterotic string to four dimensions with $\mathcal{N}=1$ SUSY.

4.1 Classification strategy

As is well known, the amount of residual supersymmetry exhibited by the 4D effective theory is related to the holonomy group of the compact space [14]. In the context of orbifolds, one can relate the holonomy group to the point group [2]. Orbifold compactifications preserve four–dimensional supersymmetry if the point group is a discrete subgroup of SU(3). Hence, the amount of unbroken SUSY is the same for all members of a given $\mathbb{Q}$–class. Therefore, we start our classification with the identification of all $\mathbb{Q}$–classes (i.e. point groups) that are subgroups of SU(3). Then, for each $\mathbb{Q}$–class we identify all $\mathbb{Z}$–classes (i.e. lattices) and finally construct for each $\mathbb{Z}$–class all affine classes (i.e. roto–translations).

In more detail, our strategy reads:

1. Choose a $\mathbb{Q}$–class and find a representative P of it.[2]

2. Check that P is a subgroup of SO(6) rather than O(6).

3. Verify that P is a subgroup of SU(3).

[2]A discussion about the possible orders of the elements of the point group, and therefore the possible point groups, can be found in Appendix B.

4. Find every possible $\mathbb{Z}$–class inside that $\mathbb{Q}$–class.

5. Find every possible affine class inside each one of those $\mathbb{Z}$–classes.

There exists a catalog of every possible affine class in up to six dimensions classified into $\mathbb{Z}$– and $\mathbb{Q}$–classes [15]. Furthermore, one can access this catalog easily using the software CARAT [16]. In detail, the command `Q_catalog` lists all $\mathbb{Q}$–classes, the command `QtoZ` lists all $\mathbb{Z}$–classes of a given $\mathbb{Q}$–class and, finally, the command `Extensions` lists all affine classes of a given $\mathbb{Z}$–class. Hence, the main open question is to decide whether a given representative of a $\mathbb{Q}$–class is a subgroup of SU(3).

4.2 Residual SUSY

We start by verifying that $P \subset \mathrm{SO}(6)$. CARAT offers representatives for all $\mathbb{Q}$–classes, i.e. it gives the generators of the point group P in some (unspecified) lattice basis $\mathfrak{e}$ as $\mathrm{GL}(6,\mathbb{Z})$ matrices $\vartheta_{\mathfrak{e}}$. In principle, one can transform them to matrices from O(6) using the (unspecified) lattice basis, i.e. $\vartheta = B_{\mathfrak{e}}\,\vartheta_{\mathfrak{e}}\,B_{\mathfrak{e}}^{-1}$. However, as the determinant is invariant under this transformation ($\det(\vartheta) = \det(\vartheta_{\mathfrak{e}})$) one can check whether or not the determinant equals $+1$ for all generators of P in the $\mathrm{GL}(6,\mathbb{Z})$ form given by CARAT. This allows us to determine whether or not $P \subset \mathrm{SO}(6)$.

Next, we recall that the matrices $\vartheta_{\mathfrak{e}} \in P$ originate from the six–dimensional representation **6** of SO(6). One way to check that P is a subgroup of SU(3) is to consider the breaking of the **6** into representations of SU(3),

$$\mathbf{6} \;\rightarrow\; \mathbf{3} \oplus \bar{\mathbf{3}}\,. \tag{4.1}$$

On the other hand, the six–dimensional representation is, in general, a reducible representation of the point group P. Hence, it can be decomposed

$$\mathbf{6} \;\rightarrow\; \boldsymbol{a} \oplus \boldsymbol{b} \oplus \ldots \tag{4.2}$$

into irreducible representations $\boldsymbol{a}, \boldsymbol{b}, \ldots$ of P. This decomposition can be computed using the character table of P as discussed in the following.

For $g \in P$, the character $\chi_{\boldsymbol{\rho}}(g)$ in the representation $\boldsymbol{\rho}$ is given by the trace of the matrix representation $\boldsymbol{\rho}(g)$ of g,

$$\chi_{\boldsymbol{\rho}}(g) \;=\; \mathrm{Tr}(\boldsymbol{\rho}(g))\,. \tag{4.3}$$

As the trace is invariant under cyclic permutations, the character $\chi_{\boldsymbol{\rho}}$ is the same for all elements of a conjugacy class, i.e.

$$\chi_{\boldsymbol{\rho}}(g) \;=\; \chi_{\boldsymbol{\rho}}(h) \quad \text{for} \quad h \;\in\; [g] \;=\; \{f\,g\,f^{-1} \text{ for all } f \in P\}\,. \tag{4.4}$$

Now, the character table of a finite group P contains one row for each irreducible representation $\boldsymbol{\rho}_i$ and one column for each conjugacy class $[g_j]$ and the entry is the corresponding character $\chi_{\boldsymbol{\rho}_i}(g_j)$. In fact, the number of irreducible representations c equals the number of conjugacy classes. Hence, the character table is a square $c \times c$ matrix. In order to decompose the **6** in Equation (4.2) we use $\chi_{\mathbf{6}}(g) = \chi_{\boldsymbol{a}}(g) + \chi_{\boldsymbol{b}}(g) + \ldots$ and the orthogonality of the rows of the character table (where the scalar product is defined over all elements of the conjugacy classes). In detail, for two irreducible representations $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we have

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \;=\; \frac{1}{|P|} \sum_{g\in P} \chi_{\boldsymbol{\alpha}}(g)\overline{\chi_{\boldsymbol{\beta}}(g)} \;=\; \begin{cases} 1 & \text{for } \boldsymbol{\alpha} = \boldsymbol{\beta}\,, \\ 0 & \text{for } \boldsymbol{\alpha} \neq \boldsymbol{\beta}\,, \end{cases} \tag{4.5}$$

where the overline indicates complex conjugation and $|P|$ is the order of P. So for each conjugacy class $[g]$ we compute the character $\chi_{\boldsymbol{\xi}}(g)$ of the six–dimensional representation $\mathbf{6}$, now denoted by $\boldsymbol{\xi}$, and determine the multiplicities n_i of the irreducible representation $\boldsymbol{\rho}_i$ in the decomposition,

$$\boldsymbol{\xi} \;\to\; \bigoplus_{i=1}^{c} n_i\,\boldsymbol{\rho}_i \quad \text{with} \quad n_i \;=\; \frac{1}{|P|} \sum_{g\in P} \chi_{\boldsymbol{\rho}_i}(g)\overline{\chi_{\boldsymbol{\xi}}(g)}\;. \tag{4.6}$$

If P is a subgroup of SU(3) this decomposition has to be of the kind

$$\mathbf{6} \;\to\; \boldsymbol{a} \oplus \bar{\boldsymbol{a}}\;, \tag{4.7}$$

where $\boldsymbol{a}$ denotes some (in general reducible) representation of P originating from the $\mathbf{3}$ of SU(3) and $\bar{\boldsymbol{a}}$ its complex conjugate (from $\bar{\mathbf{3}}$ of SU(3)). So, the first check is to see whether the decomposition (4.6) is of the form (4.7). Then we know at least $P \subset \mathrm{U}(3)$. If this is possible, then there are in general many combinations to arrange the representations $\boldsymbol{\rho}_i$ of the decomposition (4.6) into a three–dimensional representation plus its complex conjugate. But in order to see that P is a subgroup of (S)U(3) it is necessary to find only one combination. However, one needs to know the explicit matrix representation of $\boldsymbol{a}$ in order to check that the determinant is +1. Then $P \subset \mathrm{SU}(3)$ and at least $\mathcal{N} = 1$ SUSY survives the compactification of the heterotic string on the corresponding orbifold.

Let us make a short remark. If a point group is Abelian its generators can be diagonalized simultaneously. In this case, it is convenient to write them as so–called twist vectors $v = (v_1, v_2, v_3)$, three–dimensional vectors containing the three rotational angles v_i in units of 2π in the three complex planes $i = 1, 2, 3$. In this case, the check $P \subset \mathrm{SU}(3)$ is particular easy: $v_1 + v_2 + v_3 = 0 \mod 1$ so that the determinant is +1. More precisely, it is always possible to choose the signs of the v_i such that they add to 0. For example, the generator of the $\mathbb{Z}_7$ point group corresponds to the twist vector $\frac{1}{7}(1, 2, -3)$ with $\frac{1}{7}(1 + 2 - 3) = 0$ such that $\mathbb{Z}_7 \subset \mathrm{SU}(3)$.

We use the software GAP [17] and the GAP package Repsn [18] for these computations. In detail, first we use GAP to uniquely identify the discrete group P by the GAPID $[N, M]$, where N denotes the order of the group and M consecutively enumerates the discrete groups of order N. Then we perform the decomposition of the six–dimensional representation according to Equation (4.6). If the decomposition cannot be arranged according to Equation (4.7) we know that P is not a subgroup of SU(3). Otherwise, we create all combinations that fit with Equation (4.7) and compute the explicit matrix representation using the GAP package Repsn.[3] Then we can easily compute the determinant of the generators of P in the (reducible) representation $\boldsymbol{a}$.

[3] In our case, Repsn automatically created unitary representations except for one case (point group PSL(3, 2)). In this case we had to transform the representation obtained by Repsn to a unitary one by hand.

Example: S_3 point group

As an example we consider $P = S_3$ and follow the steps in order to check that $S_3 \subset \mathrm{SU}(3)$. The 2262^{nd} $\mathbb{Q}$–class obtained from CARAT is generated by two $\mathrm{GL}(6,\mathbb{Z})$ matrices, both of determinant $+1$,

$$\vartheta_{\mathfrak{e}}^{(\boldsymbol{\xi})} \;=\; \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad\text{and}\quad \omega_{\mathfrak{e}}^{(\boldsymbol{\xi})} \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.8}$$

The group generated by these (non–commuting) matrices is identified by GAP as GAPID $[6,1]$ being S_3. $\vartheta_{\mathfrak{e}}^{(\boldsymbol{\xi})}$ and $\omega_{\mathfrak{e}}^{(\boldsymbol{\xi})}$ generate the six–dimensional (reducible) representation $\boldsymbol{\xi}$ of S_3. In what follows, we figure out how this decomposes into irreducible representations of S_3.

The character table of S_3 reads (in the ordering given by GAP)

irrep	$[\mathbb{1}]$	$[\vartheta_{\mathfrak{e}}]$	$[\omega_{\mathfrak{e}}]$
$\boldsymbol{\rho}_1$	1	1	1
$\boldsymbol{\rho}_2$	1	-1	1
$\boldsymbol{\rho}_3$	2	0	-1

(4.9)

where $\boldsymbol{\rho}_1$ denotes the singlet and $\boldsymbol{\rho}_2$ and $\boldsymbol{\rho}_3$ are a one– and a two–dimensional (non–trivial) representation of S_3, respectively. Note that the conjugacy class $[\vartheta_{\mathfrak{e}}]$ contains three elements while $[\omega_{\mathfrak{e}}]$ contains two. Furthermore, the characters of the six–dimensional representation $\boldsymbol{\xi}$ generated by Equation (4.8) read

$$\chi_{\boldsymbol{\xi}} \;=\; \left(\mathrm{tr}\,\mathbb{1}_6, \mathrm{tr}\,\vartheta_{\mathfrak{e}}^{(\boldsymbol{\xi})}, \mathrm{tr}\,\omega_{\mathfrak{e}}^{(\boldsymbol{\xi})}\right) \;=\; (6,-2,0)\,. \tag{4.10}$$

Comparing this to the character table in Equation (4.9) we find that $\boldsymbol{\xi}$ decomposes into irreducible representations of S_3 as

$$\boldsymbol{\xi} \;\rightarrow\; 2\boldsymbol{\rho}_2 \oplus 2\boldsymbol{\rho}_3\;. \tag{4.11}$$

The only combination that fits into a three–dimensional representation is $\boldsymbol{\rho}_2 \oplus \boldsymbol{\rho}_3$. Using the GAP package Repsn we create the explicit matrix representation of this, resulting in

$$\vartheta^{(\mathbf{3})} \;=\; \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad\text{and}\quad \omega^{(\mathbf{3})} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp\left(-\frac{2\pi\,\mathrm{i}}{3}\right) & 0 \\ 0 & 0 & \exp\left(\frac{2\pi\,\mathrm{i}}{3}\right) \end{pmatrix} . \tag{4.12}$$

As both generators have determinant $+1$, we see that $S_3 \subset \mathrm{SU}(3)$. Furthermore, since $\mathbf{3} \rightarrow \boldsymbol{\rho}_2 \oplus \boldsymbol{\rho}_3$ does not contain the trivial singlet $\boldsymbol{\rho}_1$, we see that $\mathcal{N} = 1$ SUSY (and not more) is preserved by an S_3 orbifold compactification.

Recently, an explicit example of a non–Abelian orbifold based on S_3 has been constructed [19]. Among other things, such settings feature, unlike Abelian orbifolds, rank reduction of the gauge symmetry already at the string level.

5 Results: classification of toroidal orbifolds

We perform a systematic classification of space groups that keep (at least) $\mathcal{N}=1$ SUSY in four dimensions unbroken. As discussed in Section 3, the amount of unbroken supersymmetry depends only on the $\mathbb{Q}$–class (i.e. point group). Using CARAT we know that there are 7103 $\mathbb{Q}$–classes in six dimensions. Out of those, we find 60 $\mathbb{Q}$–classes with $\mathcal{N}\geq 1$ SUSY where 52 lead to precisely $\mathcal{N}=1$, see Table 5.1 for a summary of the results. The 60 cases split into 22 Abelian and 38 non–Abelian $\mathbb{Q}$–classes, where the Abelian cases were already known in the literature. By contrast, most of the 38 non–Abelian $\mathbb{Q}$–classes have not been used in orbifold compactifications before. Starting from these 60 $\mathbb{Q}$–classes we construct all possible $\mathbb{Z}$– and affine classes (i.e. lattices and roto–translations). In the following we discuss them in detail: Sections 5.1 and 5.2 are devoted to the Abelian and non–Abelian case, respectively.

# of generators	# of SUSY	Abelian	non–Abelian
1	$\mathcal{N}=4$	1	0
	$\mathcal{N}=2$	4	0
	$\mathcal{N}=1$	9	0
		14	0
2	$\mathcal{N}=4$	0	0
	$\mathcal{N}=2$	0	3
	$\mathcal{N}=1$	8	32
		8	35
3	$\mathcal{N}=4$	0	0
	$\mathcal{N}=2$	0	0
	$\mathcal{N}=1$	0	3
		0	3
total:	$\mathcal{N}=4$	1	0
	$\mathcal{N}=2$	4	3
	$\mathcal{N}=1$	17	35
		22	38

Table 5.1: Summary of the classification of all point groups with at least $\mathcal{N}=1$ SUSY. Out of 7103 cases obtained from CARAT there are 60 point groups with $\mathcal{N}\geq 1$ SUSY where 52 have exactly $\mathcal{N}=1$.

5.1 Abelian toroidal orbifolds

5.1.1 Our results

Restricting ourselves to Abelian point groups, we find 17 point groups with $\mathcal{N}=1$ SUSY, four cases with $\mathcal{N}=2$ and one case (i.e. the trivial point group) with $\mathcal{N}=4$ supersymmetry. Next, we classify all $\mathbb{Z}$– and affine classes. For the 17 point groups with $\mathcal{N}=1$ it turns out that there are in total 138 inequivalent space groups with Abelian point group and $\mathcal{N}=1$. Many of them were unknown before. The results are summarized in Table 5.2. More details including the generators of the orbifolding group G, the nature of gauge symmetry breaking (i.e. local or non–local) and the Hodge numbers $(h^{(1,1)}, h^{(2,1)})$ can be found in the Appendix in Table C.1. Furthermore, we have plotted the 138 pairs of Hodge numbers in Figure 7 in the Appendix, visualizing the fact that $h^{(1,1)} - h^{(2,1)}$ is always divisible by 6, except for the case $(h^{(1,1)}, h^{(2,1)}) = (20, 0)$. Note that this does not say that Standard Models with three generations of quarks and leptons are impossible, due to the possibility of introducing so–called discrete Wilson lines [2, 20] and/or discrete torsion [21, 22, 23, 24, 25, 26].

At this point, a comment on a statement in DW [10] appears appropriate. The models obtained in the free fermionic construction (such as [27]) are claimed to be related to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. DW [10] conclude from the fact that their classification does not exhibit settings with $h^{(1,1)} - h^{(2,1)}$ equal to three, that the free fermionic models, hence, cannot have a geometric interpretation. However, as pointed out in [20] and also in [26], discrete Wilson lines and/or (generalized) discrete torsion allows us to control the number of generations. We do not know whether some of the boundary conditions in the free fermionic construction correspond to such backgrounds. On the other hand, the existing three generation models based on $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds [7, 28, 29] make use of discrete Wilson lines and have, at the same time, a geometric interpretation. This might mean that the models in the free fermionic constructions may also be 'geometric'.

label of $\mathbb{Q}$–class	twist vector(s)	GAPID	CARAT symbol	CARAT index	# of $\mathbb{Z}$–classes	# of affine classes
$\mathbb{Z}_3$	$\frac{1}{3}(1,1,-2)$	[3, 1]	min.290	1965	1	1
$\mathbb{Z}_4$	$\frac{1}{4}(1,1,-2)$	[4, 1]	min.201	4667	3	3
$\mathbb{Z}_6$–I	$\frac{1}{6}(1,1,-2)$	[6, 2]	min.296	1997	2	2
$\mathbb{Z}_6$–II	$\frac{1}{6}(1,2,-3)$	[6, 2]	min.403	944	4	4
$\mathbb{Z}_7$	$\frac{1}{7}(1,2,-3)$	[7, 1]	min.665	2950	1	1
$\mathbb{Z}_8$–I	$\frac{1}{8}(1,2,-3)$	[8, 1]	min.475	5600	3	3
$\mathbb{Z}_8$–II	$\frac{1}{8}(1,3,-4)$	[8, 1]	min.467	5567	2	2
$\mathbb{Z}_{12}$–I	$\frac{1}{12}(1,4,-5)$	[12, 2]	min.562	3346	2	2
$\mathbb{Z}_{12}$–II	$\frac{1}{12}(1,5,-6)$	[12, 2]	min.553	3307	1	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{1}{2}(0,1,-1)$, $\frac{1}{2}(1,0,-1)$	[4, 2]	min.185	4625	12	35
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\frac{1}{2}(0,1,-1)$, $\frac{1}{4}(1,0,-1)$	[8, 2]	min.258	2377	10	41
$\mathbb{Z}_2 \times \mathbb{Z}_6$–I	$\frac{1}{2}(0,1,-1)$, $\frac{1}{6}(1,0,-1)$	[12, 5]	group.2702	871	2	4
$\mathbb{Z}_2 \times \mathbb{Z}_6$–II	$\frac{1}{2}(0,1,-1)$, $\frac{1}{6}(1,1,-2)$	[12, 5]	min.424	1745	4	4
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\frac{1}{3}(0,1,-1)$, $\frac{1}{3}(1,0,-1)$	[9, 2]	min.429	1964	5	15
$\mathbb{Z}_3 \times \mathbb{Z}_6$	$\frac{1}{3}(0,1,-1)$, $\frac{1}{6}(1,0,-1)$	[18, 5]	group.3567	1759	2	4
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\frac{1}{4}(0,1,-1)$, $\frac{1}{4}(1,0,-1)$	[16, 2]	min.278	2629	5	15
$\mathbb{Z}_6 \times \mathbb{Z}_6$	$\frac{1}{6}(0,1,-1)$, $\frac{1}{6}(1,0,-1)$	[36, 14]	group.3664	1859	1	1
# of Abelian $\mathcal{N} = 1$					60	138

Table 5.2: Summary of all space groups with Abelian point group and $\mathcal{N} = 1$ SUSY. Columns # 3, 4 and 5 identify the $\mathbb{Q}$–classes: "GAPID" is obtained using the command `IdGroup` in GAP, "CARAT symbol" using the CARAT command `Q_catalog` and, finally, "CARAT index" gives the index in the list of all 7103 $\mathbb{Q}$–classes obtained from CARAT.

The results are also available as input for the `orbifolder` [30], a tool to study the low energy phenomenology of heterotic orbifolds. We have created input files for the `orbifolder`, which we have made available at

http://einrichtungen.physik.tu-muenchen.de/T30e/codes/ClassificationOrbifolds/ .

There is a geometry file for each of the 138 affine classes, and one model file per $\mathbb{Q}$–class, that contains a model with standard embedding for each of the corresponding affine classes in that $\mathbb{Q}$–class.

In addition, we find 23 inequivalent space groups (i.e. affine classes) with Abelian point group and $\mathcal{N}=2$. These space groups are based on the well–known four Abelian point groups $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$. However, the inequivalent lattices and roto–translations were unknown before. They are summarized in Table 5.3.

label of $\mathbb{Q}$–class	GAPID	CARAT symbol	CARAT index	# of $\mathbb{Z}$–classes	# of affine classes
$\mathbb{Z}_2$	$[2,1]$	min.174	5	3	5
$\mathbb{Z}_3$	$[3,1]$	min.291	1968	3	5
$\mathbb{Z}_4$	$[4,1]$	min.202	4668	3	9
$\mathbb{Z}_6$	$[6,2]$	group.1611	1970	1	4
# of Abelian $\mathcal{N}=2$				10	23

Table 5.3: Summary of all space groups with $\mathcal{N}>1$ SUSY for Abelian point groups P. In addition, there is the trivial $\mathbb{Q}$–class with $\mathcal{N}=4$ SUSY (i.e. GAPID $[1,1]$, CARAT symbol min.170, CARAT index 2709) with one $\mathbb{Z}$– and one affine class.

5.1.2 Previous classifications

There are several attempts in the literature to classify six–dimensional $\mathcal{N}=1$ SUSY preserving Abelian toroidal orbifolds. For example, Bailin and Love [9] give a classification for $\mathbb{Z}_N$ orbifolds using root lattices of semi–simple Lie algebras of rank six as lattices Λ and the (generalized) Coxeter element as the generator of the point group P. However, as also discussed in Appendix A.3, they overcount the geometries and, in addition, miss a few cases. A detailed comparison to our results can be found in Table 5.4.

For $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifolds there have been two approaches for the classification of geometries. In the first one, the classification is based on Lie lattices [11], see also [31]. Again, this classification is somewhat incomplete: it misses four lattices and, in addition, neglects the possibility of roto–translations. In a second approach by DW [10] (based on [32]), a classification for $\mathbb{Z}_2\times\mathbb{Z}_2$ is given, which, as we find, is complete (but overcounts one case), see Table 5.5 for a comparison.

Furthermore, based on the strategy of DW [10], there is an (incomplete) classification of $\mathbb{Z}_N\times\mathbb{Z}_N$ for $N=3$, 4 and 6 [12]. For both $\mathbb{Z}_3\times\mathbb{Z}_3$ and $\mathbb{Z}_4\times\mathbb{Z}_4$ they identify 8 out of 15 affine classes (compare Section 2.3 of [12] to our Table C.1). Their Hodge numbers agree with our findings except for their case IV.7 (i.e. $\mathbb{Z}_4\times\mathbb{Z}_4$ with $(38,0)$). Finally, in the case of $\mathbb{Z}_6\times\mathbb{Z}_6$ they correctly identify that there is only one possible geometry but their Hodge numbers disagree with ours, i.e. they find $(80,0)$ and we have $(84,0)$.

$\mathbb{Q}$–class	$\mathbb{Z}$–class	corresponding root lattice(s)
$\mathbb{Z}_3$	1	$\mathrm{SU}(3)^3$
$\mathbb{Z}_4$	1	$\mathrm{SO}(5)^2 \times \mathrm{SU}(2)^2$
	2	$\mathrm{SO}(5) \times \mathrm{SU}(4) \times \mathrm{SU}(2)$
	3	$\mathrm{SU}(4)^2$
$\mathbb{Z}_6$–I	1	$(\mathrm{G}_2)^2 \times \mathrm{SU}(3)$ and $\left(\mathrm{SU}(3)^{[2]}\right)^2 \times \mathrm{SU}(3)$
	2	—
$\mathbb{Z}_6$–II	1	$\mathrm{G}_2 \times \mathrm{SU}(3) \times \mathrm{SU}(2)^2$ and $\mathrm{SU}(3)^{[2]} \times \mathrm{SU}(3) \times \mathrm{SU}(2)^2$
	2	—
	3	$\mathrm{SO}(8) \times \mathrm{SU}(3)$ and $\mathrm{SO}(7) \times \mathrm{SU}(3) \times \mathrm{SU}(2)$ and $\mathrm{SU}(4)^{[2]} \times \mathrm{SU}(3) \times \mathrm{SU}(2)$
	4	$\mathrm{SU}(6) \times \mathrm{SU}(2)$
$\mathbb{Z}_7$	1	$\mathrm{SU}(7)$
$\mathbb{Z}_8$–I	1	$\mathrm{SO}(9) \times \mathrm{SO}(5)$ and $\mathrm{SO}(8)^{[2]} \times \mathrm{SO}(5)$
	2	—
	3	—
$\mathbb{Z}_8$–II	1	$\mathrm{SO}(8)^{[2]} \times \mathrm{SU}(2)^2$ and $\mathrm{SO}(9) \times \mathrm{SU}(2)^2$
	2	$\mathrm{SO}(10) \times \mathrm{SU}(2)$
$\mathbb{Z}_{12}$–I	1	$\mathrm{F}_4 \times \mathrm{SU}(3)$ and $\mathrm{SO}(8)^{[3]} \times \mathrm{SU}(3)$
	2	E_6
$\mathbb{Z}_{12}$–II	1	$\mathrm{SO}(4) \times \mathrm{F}_4$ and $\mathrm{SO}(8)^{[3]} \times \mathrm{SU}(2)^2$

Table 5.4: Matching between our classification of $\mathbb{Z}_N$ space groups and the traditional notation of lattices as root lattices of semi–simple Lie algebras of rank six, see e.g. Table 3 of [9] and Table D.1 of [33]. Cases previously not known are indicated with a dash.

5.1.3 Fundamental groups

The fundamental group of a toroidal orbifold with space group S is given as [2, 34]

$$\pi_1 \;=\; S/\langle F\rangle \;, \tag{5.1}$$

where $\langle F\rangle$ is the group generated by those space group elements that leave some points fixed.

The fundamental groups of most of the Abelian orbifolds discussed here are trivial, for in those cases $\langle F\rangle \equiv S$. The only non–trivial cases are the following (see Table C.1 in the Appendix):

- 21 space groups from the $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\mathbb{Q}$–class as already calculated in [10]. See Table 5.5, where
  - 0 means a trivial fundamental group
  - S means the fundamental group equals the space group (no fixed points, hence $\langle F\rangle = \{\mathbb{1}\}$)
  - A means a $\mathbb{Z}_2 \ltimes \mathbb{Z}^2$ fundamental group

Here	Donagi et al. [10]	Förste et al. [11]	π_1		Here	Donagi et al. [10]	Förste et al. [11]	π_1
1–1	0–1	$\mathrm{SU}(2)^6$	0		5–3	1–2	—	0
1–2	0–2	—	0		5–4	1–4	—	A
1–3	0–3	—	A		5–5	1–5	—	S
1–4	0–4	—	S		6–1	2–6	$\mathrm{SU}(3)^2 \times \mathrm{SU}(2)^2$–II	0
2–1	1–6	$\mathrm{SU}(3) \times \mathrm{SU}(2)^4$	0		6–2	2–7	—	C
2–2	1–8	—	0		6–3	2–8	—	A
2–3	1–10	—	A		7–1	3–3	—	0
2–4	1–7	—	C		7–2	3–4	—	C
2–5	1–9	—	A		8–1	4–1	—	0
2–6	1–11	—	S		9–1	2–3	$\mathrm{SU}(4) \times \mathrm{SU}(3) \times \mathrm{SU}(2)$	C
3–1	2–9	—	0		9–2	2–5	—	D
3–2	2–10	—	0		9–3	2–4	—	0
3–3	2–11	—	A		10–1	3–5	—	C
3–4	2–12	—	S		10–2	3–6	—	0
4–1	2–13	$\mathrm{SU}(3)^2 \times \mathrm{SU}(2)^2$–I	0		11–1	3–1≡3–2	$\mathrm{SU}(3)^3$	0
4–2	2–14	—	D		12–1	2–1	$\mathrm{SU}(4)^2$	D
5–1	1–1	$\mathrm{SU}(4) \times \mathrm{SU}(2)^3$	C		12–2	2–2	—	C
5–2	1–3	—	C					

Table 5.5: Comparison of the affine classes of $\mathbb{Z}_2 \times \mathbb{Z}_2$ between our classification and the ones in [10] and [11]. In our case, the two numbers enumerate the $\mathbb{Z}$– and affine classes, respectively.

  - C means a $\mathbb{Z}_2$ fundamental group
  - D means a $(\mathbb{Z}_2)^2$ fundamental group

- 6 space groups from the $\mathbb{Z}_2 \times \mathbb{Z}_4$ $\mathbb{Q}$–class. In detail, the affine classes 1–6, 2–4, 3–6, 4–4, 6–5 and 8–3 posses a $\mathbb{Z}_2$ fundamental group.
- 4 space groups from the $\mathbb{Z}_3 \times \mathbb{Z}_3$ $\mathbb{Q}$–class. In detail, the affine classes 1–4, 2–4, 3–3 and 4–3 posses a $\mathbb{Z}_3$ fundamental group.

Elements of the space group that leave no fixed points are called freely acting. A non–trivial fundamental group signals the presence of non–decomposable freely acting elements in the space group, i.e. freely acting elements that cannot be written as a combination of non–freely acting elements. In the cases $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$, the non–decomposable freely acting elements belong to the orbifolding group. On the other hand, for $\mathbb{Z}_2 \times \mathbb{Z}_2$ those elements are pure lattice translations in the cases C and D, while in the cases A they are both pure lattice translations and elements of the orbifolding group.

In the context of heterotic compactifications, the phenomenologically appealing feature of non–local GUT symmetry breaking is due to the presence of non–decomposable freely acting space group elements with a non–trivial gauge embedding. In total we find 31 affine classes based on

Abelian point groups with non–trivial fundamental groups. These cases are of special interest, and their phenomenology will be studied elsewhere.

5.2 Non–Abelian toroidal orbifolds

Orbifolds with non–Abelian point groups have not been studied systematically up to now and the literature is limited to examples only. For example, in the context of free fermionic constructions compact models based on S_3, D_4 and A_4 point groups have been constructed [35]. Furthermore, non–compact examples of the form $\mathbb{C}^3/\Gamma$ with non–Abelian $\Gamma \subset \mathrm{SU}(3)$ focusing on $\Gamma = \Delta(3n^2)$ or $\Delta(6n^2)$ have been discussed in [36] and some related work has been carried out for IIB superstring theory on $\mathrm{AdS}_5 \times \mathbb{S}^5/\Gamma$ with non–Abelian $\Gamma \subset \mathrm{SU}(3)$ of order up to 31 [37].

Our classification results in 35 point groups with $\mathcal{N} = 1$ SUSY and three cases with $\mathcal{N} = 2$ SUSY, see Table C.2 in Appendix C. Surprisingly, the order of non–Abelian point groups has a much wider range compared to the Abelian case. For example, the point group $\Delta(216)$ has order 216.

Next, we classify all $\mathbb{Z}$– and affine classes. It turns out that there are in total 331 inequivalent space groups with non–Abelian point group and $\mathcal{N} = 1$ SUSY and 27 inequivalent space groups with non–Abelian point group and $\mathcal{N} = 2$. Most of them were unknown before. The results are summarized in Table 5.6 and Table 5.7.

label of $\mathbb{Q}$–class	GAPID	CARAT symbol	CARAT index	# of $\mathbb{Z}$–classes	# of affine classes
Q_8	$[8,4]$	min.487	5750	5	20
Dic_3	$[12,1]$	min.565	3374	1	3
$\mathrm{SL}(2,3)$–II	$[24,3]$	group.4493	5669	1	4
# of non–Abelian $\mathcal{N} = 2$				7	27

Table 5.6: Summary of all space groups with $\mathcal{N} > 1$ SUSY for non–Abelian P.

Example: D_6 Orbifold

Let us consider the $\mathbb{T}^6/D_6$ orbifold. D_6 is a non–Abelian finite group of order 12. The (reducible) three–dimensional representation is generated by

$$\vartheta^{(\mathbf{3})} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \omega^{(\mathbf{3})} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{6}} \end{pmatrix}, \tag{5.2}$$

and one can see that $D_6 \subset \mathrm{SU}(3)$. In terms of irreducible representations of D_6 it decomposes as $\mathbf{3} \rightarrow \mathbf{2} \oplus \mathbf{1'}$, where $\mathbf{1'}$ is a non–trivial, one–dimensional representation of D_6. Hence, we find $\mathcal{N} = 1$ SUSY in 4D.

There are two inequivalent lattices (i.e. two $\mathbb{Z}$–classes) and in total eight affine classes, see Table 5.7. For example, consider the space group generated by $\left(\vartheta^{(\mathbf{3})},0\right)$, $\left(\omega^{(\mathbf{3})},0\right)$ and the lattice

$$e_1 = (1,0,0)\ ,\quad e_2 = (\mathrm{i},0,0)\ , \tag{5.3}$$

$$e_3 = (0,1,0)\ ,\quad e_4 = \left(0,\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}},0\right)\ , \tag{5.4}$$

$$e_5 = (0,0,1)\ ,\quad e_6 = \left(0,0,\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}}\right)\ . \tag{5.5}$$

As D_6 has six conjugacy classes, the $\mathbb{T}^6/D_6$ orbifold has $6-1=5$ twisted sectors, all of them have fixed planes and hence are $\mathcal{N}=2$ subsectors.

label of $\mathbb{Q}$–class	GAPID	CARAT symbol	CARAT index	# of $\mathbb{Z}$–classes	# of affine classes
S_3	$[6,1]$	min.300	2262	6	11
D_4	$[8,3]$	min.207	4682	9	48
A_4	$[12,3]$	min.430	4893	9	15
D_6	$[12,4]$	group.1637	2258	2	8
$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$	$[16,6]$	min.506	6222	6	18
QD_{16}	$[16,8]$	group.4474	5650	4	14
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$[16,13]$	group.4469	5645	5	55
$\mathbb{Z}_3 \times S_3$	$[18,3]$	min.613	4235	6	16
Frobenius T_7	$[21,1]$	min.664	2935	3	3
$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	$[24,1]$	min.511	6266	1	1
SL$(2,3)$–I	$[24,3]$	min.536	6743	4	7
$\mathbb{Z}_4 \times S_3$	$[24,5]$	group.5943	3414	1	2
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$[24,8]$	group.5937	3408	2	6
$\mathbb{Z}_3 \times D_4$	$[24,10]$	min.616	4326	2	2
$\mathbb{Z}_3 \times Q_8$	$[24,11]$	min.528	6735	2	2
S_4	$[24,12]$	group.3770	4895	6	19
$\Delta(27)$	$[27,3]$	min.659	2864	3	10
$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$	$[32,11]$	group.5125	6337	5	30
$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$	$[36,6]$	min.620	4353	1	1
$\mathbb{Z}_3 \times A_4$	$[36,11]$	min.661	2875	3	3
$\mathbb{Z}_6 \times S_3$	$[36,12]$	group.6834	4356	2	4
$\Delta(48)$	$[48,3]$	min.651	2774	4	8
GL$(2,3)$	$[48,29]$	group.4532	5713	1	4
SL$(2,3) \rtimes \mathbb{Z}_2$	$[48,33]$	group.4531	5712	1	3
$\Delta(54)$	$[54,8]$	group.7587	2897	3	10
$\mathbb{Z}_3 \times$ SL$(2,3)$	$[72,25]$	group.5746	6988	1	2
$\mathbb{Z}_3 \times$ GAPID $[24,8]$	$[72,30]$	group.7007	4533	1	1
$\mathbb{Z}_3 \times S_4$	$[72,42]$	group.7614	2924	3	3
$\Delta(96)$	$[96,64]$	group.7498	2802	4	12
SL$(2,3) \rtimes \mathbb{Z}_4$	$[96,67]$	group.5290	6512	1	2
					continued ...

label of $\mathbb{Q}$–class	GAPID	CARAT symbol	CARAT index	# of $\mathbb{Z}$–classes	# of affine classes
$\Sigma(36\phi)$	$[108, 15]$	group.7500	2806	2	4
$\Delta(108)$	$[108, 22]$	group.7504	2810	1	1
$\mathrm{PSL}(3, 2)$	$[168, 42]$	group.7622	2934	1	3
$\Sigma(72\phi)$	$[216, 88]$	group.7540	2846	2	2
$\Delta(216)$	$[216, 95]$	group.7545	2851	1	1
# of non–Abelian $\mathcal{N} = 1$				108	331

Table 5.7: Summary of all space groups with non–Abelian point group and $\mathcal{N} = 1$ SUSY.

6 Summary and Discussion

We have classified all symmetric orbifolds that give $\mathcal{N} \geq 1$ supersymmetry in four dimensions. Our main results are as follows:

1. In total we find 60 $\mathbb{Q}$–classes (point groups) that lead to $\mathcal{N} \geq 1$ SUSY.
2. These $\mathbb{Q}$–classes decompose in
   - 22 with an Abelian point group with one or two generators, i.e. $\mathbb{Z}_N$ or $\mathbb{Z}_N \times \mathbb{Z}_M$, out of which 17 lead to exactly $\mathcal{N} = 1$ SUSY, and
   - 38 with a non–Abelian point group with two or three generators, such as S_3 or $\Delta(216)$, out of which 35 lead to exactly $\mathcal{N} = 1$ SUSY.

That is, there are 52 $\mathbb{Q}$–classes that can lead to models yielding the supersymmetric standard model.

As we have explained in detail, $\mathbb{Q}$–classes (or point groups) can come with inequivalent lattices, classified by the so–called $\mathbb{Z}$–classes. In the traditional orbifold literature, $\mathbb{Z}$–classes are given by Lie lattices and a given choice fixes an orbifold geometry. However, as we have pointed out, not all lattices can be described by Lie lattices.

Our results on $\mathbb{Q}$–classes potentially relevant for supersymmetric model building are as follows.

3. We find that there are 186 $\mathbb{Z}$–classes, or, in other words, orbifold geometries that lead to $\mathcal{N} \geq 1$ SUSY.
4. These $\mathbb{Z}$–classes decompose in
   - 71 with an Abelian point group, out of which 60 lead to exactly $\mathcal{N} = 1$ SUSY, and
   - 115 with a non–Abelian point group, out of which 108 lead to exactly $\mathcal{N} = 1$ SUSY.

Furthermore, space groups can be extended by so–called roto–translations, a combination of a twist and a (non–lattice) translation. We provide a full classification of all roto–translations in terms of affine classes, which are, as we discuss, the most suitable objects to classify inequivalent space groups.

5. We find 520 affine classes that lead to $\mathcal{N} \geq 1$ SUSY.

6. These affine classes decompose in

   - 162 with an Abelian point group, out of which 138 lead to exactly $\mathcal{N} = 1$ SUSY, and
   - 358 with a non–Abelian point group, out of which 331 lead to exactly $\mathcal{N} = 1$ SUSY.

An important aspect of our classification is that we provide the data for all 138 space groups with Abelian point group and $\mathcal{N} = 1$ SUSY required to construct the corresponding models with the C++ orbifolder [30]. Among other things, this allows one to obtain a statistical survey of the properties of the models, which has so far only been performed for the $\mathbb{Z}_6$–II orbifold [38].

Our classification also has conceivable importance for phenomenology. For instance, one of the questions is how the ten–dimensional gauge group (i.e. $\mathrm{E}_8 \times \mathrm{E}_8$ or SO(32)) of the heterotic string gets broken by orbifolding. In most of the models discussed so far, the larger symmetry gets broken locally at some fixed point. Yet it has been argued that 'non–local' GUT symmetry breaking, as utilized in the context of smooth compactifications of the heterotic string [39, 40, 41, 42], has certain phenomenological advantages [43, 44]. Explicit MSSM candidate models, based on the DW classification, featuring non–local GUT breaking have been constructed recently [28, 29]. As we have seen, there are 31 affine classes of space groups, based on the $\mathbb{Q}$–classes $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$, that lead to an orbifold with a non–trivial fundamental group, thus allowing us to introduce a Wilson line that breaks the GUT symmetry. In other words, we have identified a large set of geometries that can give rise to non–local GUT breaking. This might also allow for a dynamical stabilization of some of the moduli in the early universe, similar as in toroidal compactifications [45].

In this study, we have focused on symmetric toroidal orbifolds, which have a rather clear geometric interpretation, such that crystallographic methods can be applied in a straightforward way. We have focused on the geometrical aspects. On the other hand, it is known that background fields, i.e. the so–called discrete Wilson lines [20] and discrete torsion [21, 23, 24, 25, 26], play a crucial role in model building. It will be interesting to work out the conditions on such background fields in the geometries of our classification. Further, it is, of course, clear that there are other orbifolds, such as asymmetric and/or non–toroidal orbifolds, whose classification is beyond the scope of this study. Let us also mention, we implicitly assumed that the radii are away from the self–dual point. It might be interesting to study what happens if one sends one or more T–moduli to the self–dual values. In this case one may make contact with the free fermionic formulation, where also interesting models have been constructed [27]. In addition, our results may also be applied to compactifications of type II string theory on orientifolds (see e.g. [46, 47, 48] for some interesting models and [49] for a review).

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A Details on lattices

A.1 Bravais types and form spaces

One can classify lattices by the symmetry groups they obey. This is the concept of Bravais equivalent lattices. In more detail, denote the symmetry group of some lattice Λ as $G \subset \mathrm{GL}(n,\mathbb{Z})$. Obviously, the point group $P \subset G$, is a subgroup of it. Now, if two lattices give rise to the same finite unimodular group G, we call them Bravais equivalent. This equivalence generates a finite number of Bravais types of lattices for every dimension n. They have been classified for dimensions up to six [50].

The interesting task would now be to decide which Bravais type a given lattice belongs to. This can be done using the notion of form spaces [15]. The form space $\mathcal{F}(G)$ of some finite group $G \subset \mathrm{GL}(n,\mathbb{Z})$ is defined as the vector space of all symmetric matrices left invariant by G, i.e.

$$\mathcal{F}(G) \;=\; \{F \;\in\; \mathbb{R}^{n\times n}_{\text{sym}} \mid g^T F\, g \;=\; F \text{ for all } g \;\in\; G\}\;. \tag{A.1}$$

On the other hand, we define the Gram matrix of the lattice basis $\mathfrak{e} = \{e_1, \ldots, e_n\}$ as

$$\mathrm{Gr}(\mathfrak{e})_{ij} \;=\; (e_i, e_j) \;=\; (B_{\mathfrak{e}}^{\mathrm{T}} B_{\mathfrak{e}})_{ij}\;, \tag{A.2}$$

where the parentheses (e_i, e_j) denote the standard scalar product. By definition, the Gram matrix is a symmetric, positive definite matrix. Under a change of lattice basis, represented by a unimodular matrix M, the Gram matrix changes as $M^T \mathrm{Gr}(\mathfrak{e})\, M$, c.f. Section 2.2. By contrast, elements of the point group leave the Gram matrix invariant, i.e. for $\vartheta \in P$

$$\mathrm{Gr}(\mathfrak{e}) \;\xmapsto{\vartheta}\; \vartheta^T \mathrm{Gr}(\mathfrak{e})\, \vartheta \;=\; \mathrm{Gr}(\mathfrak{e})\;. \tag{A.3}$$

Hence, a form space is in one–to–one correspondence to a Bravais type of lattice, i.e. every lattice Λ has a basis $\mathfrak{e} = \{e_1, \ldots, e_n\}$ such that its Gram matrix $\mathrm{Gr}(\mathfrak{e})$ is an element of the form space of a finite subgroup P of $\mathrm{GL}(n,\mathbb{Z})$, i.e. $\mathrm{Gr}(\mathfrak{e}) \in \mathcal{F}(P)$ [13]. But in order to see that one lattice belongs to a given form space, it needs to be in this special basis, which is canonically chosen to be the shortest possible basis for that lattice. Fortunately, algorithms for precisely that task do exist, cf. e.g. [51] (though one should be careful: the shortest basis of a lattice is in general not unique).

Note that physically the Gram matrix is the metric of the torus defined by the lattice Λ and the dimension of the form space $\mathcal{F}(P)$ is exactly the number of (untwisted) moduli the orbifold offers.

Let us consider an example in two dimensions. Take the point group defined by

$$P \;=\; \{\mathbb{1} \;=\; \vartheta^2, \vartheta\} \;\cong\; \mathbb{Z}_2 \quad \text{with} \quad \vartheta \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\;. \tag{A.4}$$

It leaves invariant the form space

$$\mathcal{F}(P) \;=\; \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \;. \tag{A.5}$$

That form space corresponds to the Bravais type called p–rectangular lattice (cf. Appendix A.3), consisting of two arbitrarily long, orthogonal vectors.

A.2 Introducing an additional shift

DW [10] starts with an orthonormal lattice in six dimensions. Then, in a second step, additional shifts, which are linear combinations of the (orthonormal) lattice vectors with rational coefficients, are included in the space group. As we have seen in the second example in Section 3.4, those additional shifts can be incorporated to the lattice itself. Here we show in detail how to transform the space group accordingly.

The perhaps most elegant procedure is to perform a change of basis, but using transformations from $\mathrm{GL}(n,\mathbb{Q})$. Hence, we are selecting a different $\mathbb{Z}$–class from the same $\mathbb{Q}$–class, cf. Section 3. Let us list the necessary steps and illustrate them with an example:

1. The additional shift is a linear combination with rational coefficients of some of the lattice vectors. Exchange one of the old lattice vectors (that appears in the linear combination) by the new additional shift.

2. Write the transformation matrix M: start with the identity matrix and substitute the column corresponding to the exchanged vector by the coefficients of the linear combination.

3. Transform your space group using M accordingly: see Equation (2.4) and Equation (2.5).

4. (Optional) In order to see the geometry more clearly, one can perform a basis reduction (e.g. using the LLL algorithm, cf. [51]), which is a transformation from $\mathrm{GL}(n,\mathbb{Z})$.

As an example, take the $\mathbb{Z}_2\times\mathbb{Z}_2$ model named (1–1) in DW [10], which consists of an orthogonal lattice (p–cubic) with orthonormal basis $\mathfrak{e}$ and an additional shift

$$\tau \;=\; \frac{1}{2}\left(e_2+e_4+e_6\right) \;. \tag{A.6}$$

We will restrict the discussion to the three–dimensional (sub–)lattice Λ spanned by the basis $\mathfrak{e} = \{e_2, e_4, e_6\}$.

The basis matrix, Gram matrix and point group generators read

$$B_{\mathfrak{e}} \;=\; \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \;, \qquad \mathrm{Gr}(\mathfrak{e}) \;=\; \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \;, \tag{A.7a}$$

$$\vartheta_{\mathfrak{e}} \;=\; \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&-1 \end{pmatrix} \;, \qquad \omega_{\mathfrak{e}} \;=\; \begin{pmatrix} -1&0&0\\0&1&0\\0&0&-1 \end{pmatrix} \;. \tag{A.7b}$$

Let us follow the steps described above:

1. We choose to exchange the 3$^{\text{rd}}$ (originally 6$^{\text{th}}$) vector for the additional shift: the new basis $\mathfrak{f}$ is spanned by $\mathfrak{f} = \{e_2, e_4, \tau\}$. Notice that $\mathfrak{f}$ is not a basis of the lattice Λ, but one of a new, different lattice Σ.

2. In accordance with our choice, the transformation matrix is

$$M = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix} . \tag{A.8}$$

3. We perform the transformation using M. For the new lattice Σ in the new basis $\mathfrak{f}$, the quantities we are interested in look like

$$B_{\mathfrak{f}} = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}, \qquad \mathrm{Gr}(\mathfrak{f}) = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 1/2 & 1/2 & 3/4 \end{pmatrix}, \tag{A.9a}$$

$$\vartheta_{\mathfrak{f}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \omega_{\mathfrak{f}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} . \tag{A.9b}$$

4. Next, we perform a LLL reduction, which is a change of basis to a reduced one $\mathfrak{r}$, and transform the point group elements accordingly,

$$B_{\mathfrak{r}} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{pmatrix}, \qquad \mathrm{Gr}(\mathfrak{r}) = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}, \tag{A.10a}$$

$$\vartheta_{\mathfrak{r}} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \omega_{\mathfrak{r}} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} . \tag{A.10b}$$

Last, we compare the Gram matrix $\mathrm{Gr}(\mathfrak{r})$ with Table A.1. We see that introducing the additional shift τ into the p–cubic lattice is equivalent to work with the appropriately transformed point group in an i–cubic lattice.

A remark is in order. The form space left invariant by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ point group in the (reduced) basis of Equation (A.10) is

$$\mathcal{F}(P) = \begin{pmatrix} a & b & c \\ b & a & -a-b-c \\ c & -a-b-c & a \end{pmatrix} . \tag{A.11}$$

This form space is the one of a three–parametric, i–orthogonal lattice, which contains as possible realizations the i–cubic and the f–cubic lattices (both one–parametric, see table A.1). Therefore, model (1–1) in [10] corresponds to model A_4 of Förste et al. [11], i.e. to the Lie lattice $\mathrm{SU}(4)\times\mathrm{SU}(2)^3$ where the SU(4) part is an f–cubic lattice, see Table 5.5.

A.3 Bravais types and Lie lattices

It is common in the string–orbifold literature to describe lattices as root lattices of (semi–simple) Lie algebras. On the one hand, this makes it easy to identify the point group, i.e. a discrete subgroup of SU(3), using Weyl reflections and the Coxeter element. However, we find this practice to be problematic for at least three different reasons:

Redundancies

A root lattice is the lattice spanned by the simple roots of a certain (semi–simple) Lie algebra. Even if the simple roots of two non–equivalent (semi–simple) Lie algebras are different, the lattices they span might not. For example, the lattices spanned by the root systems of SU(3) and G_2 are the same (see Figure 5). Some more examples are provided in Table A.1.

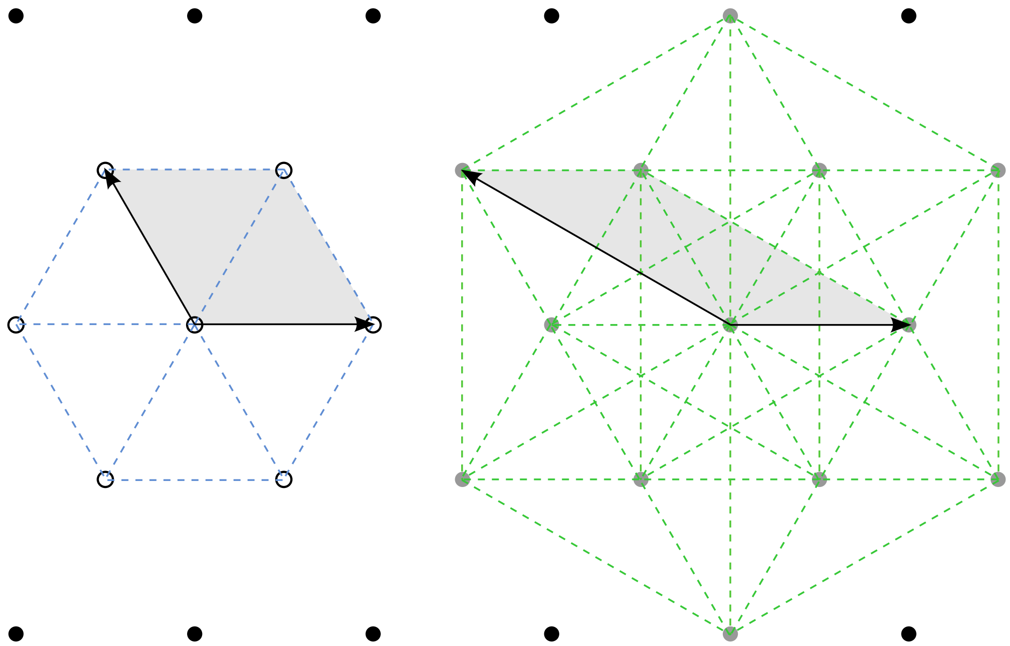


Figure 5: The hexagonal lattice: the blue lines form the SU(3) root system, and the green lines form the G_2 root system. Simple roots are also indicated, as well as the fundamental cells (shaded).

Missing lattices

When considering the redundancy of root lattices, one might think that there are more root lattices than types of lattices and that the situation could be resolved by introducing some clever convention to avoid this overcounting. But the problem exists in the other direction too: the set of all possible root lattices does not exhaust the whole family of Bravais types, i.e. there are Bravais types of lattices which are not generated by any root system. The lowest dimension in which this occurs is three and the most basic example is the body centered cubic lattice, also known as bcc or i–cubic to crystallographers (see Table A.1). The bcc lattice is a cubic lattice with an additional lattice point in the center of the unit cell. Its only free parameter is the size of the system (e.g. the edge length of the cube). One possible way to convince oneself that there is no root lattice that can generate this Bravais lattice is taking every rank three root lattice and calculating which Bravais lattice it generates. We find that the i–cubic lattice has no description as root lattice (see Table A.1).

Continuous parameters

Every Bravais type allows for a set of continuous deformations which conserve its symmetries. Those deformations are encoded and made explicit in the form space that defines that particular Bravais type (cf. Appendix A.1). The form space tells us how many deformation parameters one Bravais type allows for, and what is the effect of them (to change lengths of or angles between basis vectors). The realization of that freedom in the context of root lattices is very limited: lattices of Lie algebras allow for just one parameter, the size of the system; and if one includes semi–simple Lie algebras (direct products of simple ones), one can choose different sizes for different sublattices, but never the angles between vectors, which are fixed to a limited set of values. So, for example, a two–dimensional oblique lattice, in which the angle between the basis vectors is arbitrary, could never be unambiguously expressed in terms of Lie root lattices.

In conclusion, the language of root lattices is incomplete and ambiguous, and is lacking geometrical insight with respect to the language of Bravais types and form spaces, which is, therefore, the one used in this paper.

Nevertheless, in order to justify some of the matchings between our classification of space groups and the ones already existing in the literature, we present in Table A.1 a classification of all of the Bravais types of lattices in 1, 2 and 3 dimensions, together with their equivalent root lattices, if there are any. There, in order to overcome the discussed ambiguities in the root lattice language, some conventions have been used:

- $\oplus$ means orthogonal product. Unspecified products should be understood orthogonal.
- $\odot$ means free–angle product. The scalar product of the roots is indicated as a subindex. Notice that in the cases in which we have used this product there is actually no equivalent Lie lattice description: a non–orthogonal product of semi–simple Lie algebras is not a semi–simple Lie algebra. These possibilities are written in italics.
- $\hookleftarrow$ means a product with the leftmost factor.
- Equal subindices mean equal length of the roots or equal scalar products.
- A subindex in an algebra whose simple roots are of different length stands for the squared length of the shortest simple root, e.g. $\mathrm{G}_{2,a}$ means that the shortest simple root of G_2 has length squared a.

Gram matrix	lattice name		Lie algebra notation
1 dimension			
$\begin{pmatrix} a \end{pmatrix}$	Ruler	r	SU(2)
2 dimensions			
$\begin{pmatrix} a & 0 \\ & a \end{pmatrix}$	Square	tp	SO(5), $\mathrm{SU}(2)_a \oplus \mathrm{SU}(2)_a$
$\begin{pmatrix} a & \pm a/2 \\ & a \end{pmatrix}$	Hexagonal	hp	$\mathrm{SU}(3)_a$, $\mathrm{G}_{2,a}$
$\begin{pmatrix} a & 0 \\ & b \end{pmatrix}$	p–Rectangular	op	$\mathrm{SU}(2)_a \oplus \mathrm{SU}(2)_b$
$\begin{pmatrix} a & b \\ & a \end{pmatrix}$	c–Rectangular	oc	$SU(2)_a \odot_b SU(2)_a$
$\begin{pmatrix} a & c \\ & b \end{pmatrix}$	Oblique	mp	$SU(2)_a \odot_c SU(2)_b$
3 dimensions			
$\begin{pmatrix} a & 0 & 0 \\ & a & 0 \\ & & a \end{pmatrix}$	p–Cubic	cP	SO(7), $\mathrm{SU}(2)_a \oplus \mathrm{SU}(2)_a \oplus \mathrm{SU}(2)_a$
$\begin{pmatrix} a & a/2 & a/2 \\ & a & a/2 \\ & & a \end{pmatrix}$	f–Cubic	cF	SU(4), Sp(6)
$\begin{pmatrix} a & -a/3 & -a/3 \\ & a & -a/3 \\ & & a \end{pmatrix}$	i–Cubic	cI	(none)
$\begin{pmatrix} a & \pm a/2 & 0 \\ & a & 0 \\ & & b \end{pmatrix}$	p–Hexagonal	hP	$[\mathrm{SU}(3)_a$ or $\mathrm{G}_{2,a}] \oplus \mathrm{SU}(2)_b$
$\begin{pmatrix} a & b & b \\ & a & b \\ & & a \end{pmatrix}$	r–Hexagonal	hR	$SU(2)_a \odot_b SU(2)_a \odot_b SU(2)_a \odot_b \hookleftarrow$
$\begin{pmatrix} a & 0 & 0 \\ & a & 0 \\ & & b \end{pmatrix}$	p–Tetragonal	tP	$[\mathrm{SU}(2)_a \oplus \mathrm{SU}(2)_a$ or SO(5)$] \oplus \mathrm{SU}(2)_b$
$\begin{pmatrix} a+2b & -a & -b \\ & a+2b & -b \\ & & a+2b \end{pmatrix}$	i–Tetragonal	tI	(no simple expr.)
$\begin{pmatrix} a & 0 & 0 \\ & b & 0 \\ & & c \end{pmatrix}$	p–Orthorhombic	oP	$\mathrm{SU}(2)_a \oplus \mathrm{SU}(2)_b \oplus \mathrm{SU}(2)_c$
$\begin{pmatrix} a & c & 0 \\ & a & 0 \\ & & b \end{pmatrix}$	c–Orthorhombic	oC	$SU(2)_a \odot_c SU(2)_a \oplus SU(2)_b$
			continued . . .

Gram matrix	lattice name		Lie algebra notation
$\begin{pmatrix} a+b & a & b \\ & a+c & c \\ & & b+c \end{pmatrix}$	f–Orthorhombic	oF	(no simple expr.)
$\begin{pmatrix} a+b+c & -a & -b \\ & a+b+c & -c \\ & & a+b+c \end{pmatrix}$	i–Orthorhombic	oI	(no simple expr.)
$\begin{pmatrix} a & c & 0 \\ & b & 0 \\ & & d \end{pmatrix}$	p–Monoclinic	mP	$SU(2)_a \odot_c SU(2)_b \oplus SU(2)_d$
$\begin{pmatrix} a & c & d \\ & a & d \\ & & b \end{pmatrix}$	c–Monoclinic	mC	$SU(2)_a \odot_c SU(2)_a \odot_d SU(2)_b \odot_d \hookleftarrow$
$\begin{pmatrix} a & d & f \\ & b & e \\ & & c \end{pmatrix}$	Triclinic	aP	$SU(2)_a \odot_d SU(2)_b \odot_e SU(2)_c \odot_f \hookleftarrow$

Table A.1: List of Bravais types in 1, 2 and 3 dimensions, together with possible root lattice expressions. The following prefixes and suffixes are used for the lattice names: p primitive, c centered (in 2D) or base–centered (in 3D), f face–centered, i body–centered, and r rhombohedral.

In general, Bravais types with two or more parameters in the form space contain as specific cases other types with a lower number of parameters. For example, if we set the off diagonal parameter to zero in the two–dimensional oblique lattice (mp) (i.e. we take the basis vectors to be orthogonal), we get a p–rectangular (op) lattice. If we set now the diagonal elements of the form space to be equal (i.e. we take the basis vectors to have equal length), we get a square lattice (tp). These three lattices form the embedding chain tp$\hookrightarrow$op$\hookrightarrow$mp.

A graph containing all of the existing embeddings of that kind in two and three dimensions can be seen in Figure 6. For further information about this topic, the standard reference is [52].

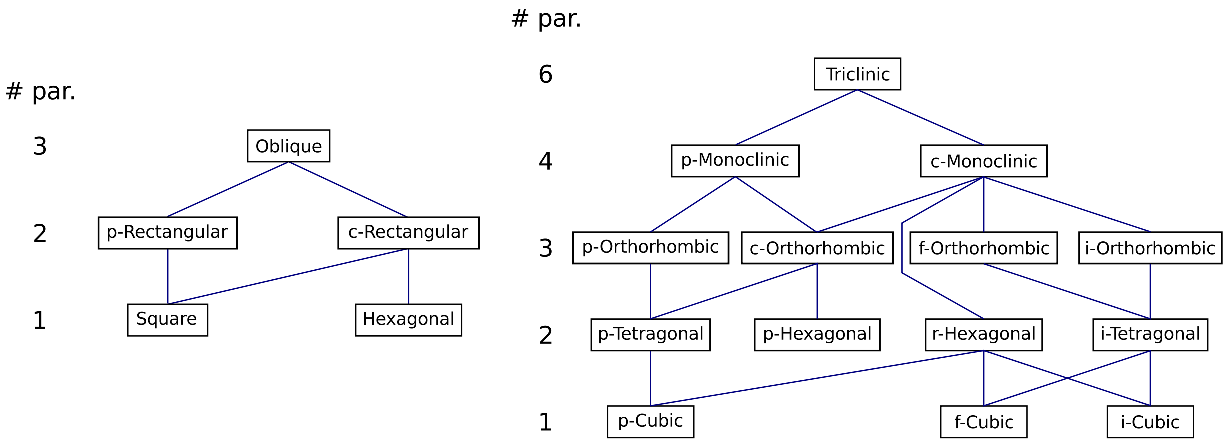


Figure 6: Graph of Bravais types embeddings in 2D and 3D.

B Two–dimensional orbifolds

In order to illustrate some of the concepts addressed in this paper, we reproduce here the list of all possible two–dimensional space groups, also known as *wallpaper groups*. They are well–known, and their classification can be found for instance in [13].

The possible orders m of point group elements in n dimensions are given by the equation

$$\phi(m) \ \leq \ n \ , \tag{B.1}$$

where ϕ is the Euler ϕ–function. For dimension two, this leaves only elements with order in $\{1, 2, 3, 4, 6\}$ as possible point group elements. In six dimensions, this gets extended to $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18\}$. Nevertheless, in dimensions $n \geq 2$, one can find point group elements with order m such that $\phi(m) > n$. This can be realized by building a point group element as the direct sum of two point group elements of dimensions that add up to n. In that case, the order of the point group element would obviously be the least common multiple of the orders of the factors. For example, in six dimensions there exist point groups with elements of order 30, which are a direct sum of a four–dimensional order 10 element and a two–dimensional order 3 element.

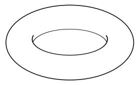
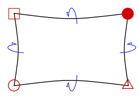
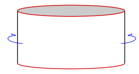
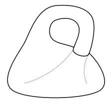
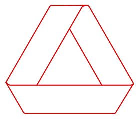
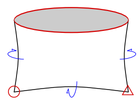
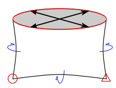
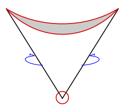
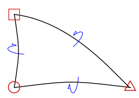
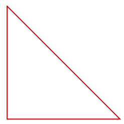
label of $\mathbb{Q}$–class	# of $\mathbb{Z}$–classes	# of affine classes
id	1	1
$\mathbb{Z}_2$–I	1	1
$\mathbb{Z}_2$–II	2	3
$\mathbb{Z}_2 \times \mathbb{Z}_2 \ \cong D_2$	2	4
$\mathbb{Z}_4$	1	1
$\mathbb{Z}_2 \ltimes \mathbb{Z}_4 \ \cong D_4$	2	2
$\mathbb{Z}_3$	1	1
$\mathbb{Z}_2 \ltimes \mathbb{Z}_3 \ \cong S_3 \cong D_3$	2	2
$\mathbb{Z}_6$	1	1
$\mathbb{Z}_2 \ltimes \mathbb{Z}_6 \ \cong D_6$	1	1

Table B.1: $\mathbb{Q}$–classes in two dimensions.

As discussed in Section 3, one can classify the 17 two–dimensional space groups by their $\mathbb{Q}$–classes. Those can be found in Table B.1. There, D_n is the dihedral group of order $2n$ and S_n is the symmetric group of order $n!$. In Table B.2 the specific information of every affine class is shown: the $\mathbb{Q}$–, $\mathbb{Z}$– and affine class to which they belong, its Bravais type of lattice (cf. Table A.1), its orbifolding group generators in augmented matrix notation and a name, description and image of its topology. The augmented matrix of some element $g_\mathfrak{e} = (\vartheta_\mathfrak{e}, \lambda_i e_i) \in S$ is given by

$$g_\mathfrak{e} \ = \ \left(\begin{array}{c|c} \vartheta_\mathfrak{e} & \lambda_i \\ \hline 0 & 1 \end{array}\right) \ , \tag{B.2}$$

using the lattice basis $\mathfrak{e}$. This matrix acts on an augmented vector $(x, 1)$ by simple matrix–vector multiplication.

$\mathbb{Q}$–$\mathbb{Z}$–aff. class lattice	generators	name & description	image
id–1–1 Oblique		Torus Manifold	
$\mathbb{Z}_2$–I–1–1 Oblique	$\left(\begin{array}{cc\|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Pillow Orbifold, 4 singularities with cone–angle π	
$\mathbb{Z}_2$–II–1–1 p–Rectangular	$\left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Pipe Manifold, 2 boundaries	
$\mathbb{Z}_2$–II–1–2 p–Rectangular	$\left(\begin{array}{cc\|c} 1 & 0 & 1/2 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Klein bottle Manifold, non–orientable	
$\mathbb{Z}_2$–II–2–1 c–Rectangular	$\left(\begin{array}{cc\|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Möbius strip Manifold, non–orientable, 1 boundary	
$\mathbb{Z}_2 \times \mathbb{Z}_2$–1–1 p–Rectangular	$\left(\begin{array}{cc\|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Rectangle Manifold, 1 boundary	
$\mathbb{Z}_2 \times \mathbb{Z}_2$–1–2 p–Rectangular	$\left(\begin{array}{cc\|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & -1 & 1/2 \\ \hline 0 & 0 & 1 \end{array}\right)$	Cut pillow Orbifold, 2 singularities with cone–angle π, 1 boundary	
$\mathbb{Z}_2 \times \mathbb{Z}_2$–1–3 p–Rectangular	$\left(\begin{array}{cc\|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 1 & 0 & 1/2 \\ 0 & -1 & 1/2 \\ \hline 0 & 0 & 1 \end{array}\right)$	Cross–cap pillow Orbifold, 2 singularities with cone–angle π	
$\mathbb{Z}_2 \times \mathbb{Z}_2$–2–1 c–Rectangular	$\left(\begin{array}{cc\|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Jester's hat Orbifold, 1 singularity with cone–angle π, 1 boundary	
$\mathbb{Z}_4$–1–1 Square	$\left(\begin{array}{cc\|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Triangle pillow Orbifold, 2 singularities with cone–angle $\pi/2$, 1 singularity with cone–angle π	
$\mathbb{Z}_2 \ltimes \mathbb{Z}_4$–1–1 Square	$\left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Triangle Manifold, one boundary, 1 angle of $\pi/2$ and 2 of $\pi/4$	

continued ...

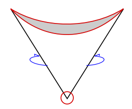
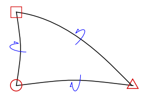
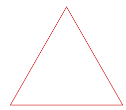
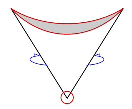
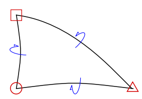
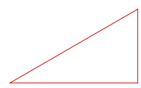
$\mathbb{Q}$–$\mathbb{Z}$–aff. class lattice	generators	name & description	image
$\mathbb{Z}_2 \ltimes \mathbb{Z}_4$–1–2 Square	$\left(\begin{array}{cc\|c} 1 & 0 & 1/2 \\ 0 & -1 & 1/2 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Jester's hat Orbifold, 1 singularity with cone–angle $\pi/2$, 1 boundary	
$\mathbb{Z}_3$–1–1 Hexagonal	$\left(\begin{array}{cc\|c} 0 & -1 & 0 \\ 1 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Triangle pillow Orbifold, 3 singularities with cone–angle $2\pi/3$	
$\mathbb{Z}_2 \ltimes \mathbb{Z}_3$–1–1 Hexagonal	$\left(\begin{array}{cc\|c} 0 & -1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 0 & -1 & 0 \\ 1 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Triangle Manifold, 3 boundary, all angles $\pi/3$	
$\mathbb{Z}_2 \ltimes \mathbb{Z}_3$–2–1 Hexagonal	$\left(\begin{array}{cc\|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 0 & -1 & 0 \\ 1 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Jester's hat Orbifold, 1 singularity with cone–angle $2\pi/3$, 1 boundary	
$\mathbb{Z}_6$–1–1 Hexagonal	$\left(\begin{array}{cc\|c} 1 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Triangle pillow Orbifold, 3 singularities with cone–angles $2\pi/3$, $\pi/3$ and π	
$\mathbb{Z}_2 \ltimes \mathbb{Z}_6$–1–1 Hexagonal	$\left(\begin{array}{cc\|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 1 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Triangle Manifold, 1 boundary, with angles $\pi/2$, $\pi/3$ and $\pi/6$	

Table B.2: List of all possible two–dimensional orbifolds. $\mathbb{Q}$–classes are separated by double lines.

Sometimes it is of interest to know the fundamental groups of the resulting orbifolds. Among the two–dimensional space groups, most of the fundamental groups are trivial with the following exceptions: the torus has a fundamental group of $(\mathbb{Z})^2$, the pipe and the Möbius strip $\mathbb{Z}$, the cross–cap pillow (a projective plane) $\mathbb{Z}_2$ and the Klein bottle's one is its own space group, with group structure

$$S = \left\{ a^n b^m \mid m, n \in \mathbb{Z}\,,\; b\,a = a^{-1}\,b \right\} \,. \tag{B.3}$$

C Tables

C.1 Abelian point groups

$\mathbb{Q}$–class (P)	$\mathbb{Z}$–class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
$\mathbb{Z}_3$	1	1	$(\theta, 0)$	
		local	$(9,0)U + (27,0)T_1$	$(36,0)$
$\mathbb{Z}_4$	1	1	$(\theta, 0)$	
		local	$(5,1)U + (16,0)T_1 + (10,6)T_2$	$(31,7)$
	2	1	$(\theta, 0)$	
		local	$(5,1)U + (16,0)T_1 + (6,2)T_2$	$(27,3)$
	3	1	$(\theta, 0)$	
		local	$(5,1)U + (16,0)T_1 + (4,0)T_2$	$(25,1)$
$\mathbb{Z}_6$–I	1	1	$(\theta, 0)$	
		local	$(5,0)U + (3,0)T_1 + (15,0)T_2 + (6,5)T_3$	$(29,5)$
	2	1	$(\theta, 0)$	
		local	$(5,0)U + (3,0)T_1 + (15,0)T_2 + (2,1)T_3$	$(25,1)$
$\mathbb{Z}_6$–II	1	1	$(\theta, 0)$	
		local	$(3,1)U + (12,0)T_1 + (6,3)T_2 + (8,4)T_3 + (6,3)T_4$	$(35,11)$
	2	1	$(\theta, 0)$	
		local	$(3,1)U + (12,0)T_1 + (6,3)T_2 + (4,0)T_3 + (6,3)T_4$	$(31,7)$
	3	1	$(\theta, 0)$	
		local	$(3,1)U + (12,0)T_1 + (3,0)T_2 + (8,4)T_3 + (3,0)T_4$	$(29,5)$
	4	1	$(\theta, 0)$	
		local	$(3,1)U + (12,0)T_1 + (3,0)T_2 + (4,0)T_3 + (3,0)T_4$	$(25,1)$
$\mathbb{Z}_7$	1	1	$(\theta, 0)$	
		local	$(3,0)U + (7,0)T_1 + (7,0)T_2 + (7,0)T_4$	$(24,0)$
$\mathbb{Z}_8$–I	1	1	$(\theta, 0)$	
		local	$(3,0)U + (4,0)T_1 + (10,0)T_2 + (6,3)T_4 + (4,0)T_5$	$(27,3)$
	2	1	$(\theta, 0)$	
		local	$(3,0)U + (4,0)T_1 + (10,0)T_2 + (4,1)T_4 + (4,0)T_5$	$(25,1)$
	3	1	$(\theta, 0)$	
		local	$(3,0)U + (4,0)T_1 + (10,0)T_2 + (3,0)T_4 + (4,0)T_5$	$(24,0)$
$\mathbb{Z}_8$–II	1	1	$(\theta, 0)$	
		local	$(3,1)U + (8,0)T_1 + (3,1)T_2 + (8,0)T_3 + (6,4)T_4$ $+(3,1)T_6$	$(31,7)$
	2	1	$(\theta, 0)$	
		local	$(3,1)U + (8,0)T_1 + (2,0)T_2 + (8,0)T_3 + (4,2)T_4$	
			continued …	

$\mathbb{Q}$–class (P)	$\mathbb{Z}$–class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
			$+(2,0)T_6$	$(27,3)$
$\mathbb{Z}_{12}$–I	1	1	$(\theta,0)$	
		local	$(3,0)U+(3,0)T_1+(3,0)T_2+(2,1)T_3+(9,0)T_4$ $+(4,3)T_6+(3,0)T_7+(2,1)T_9$	$(29,5)$
	2	1	$(\theta,0)$	
		local	$(3,0)U+(3,0)T_1+(3,0)T_2+(1,0)T_3+(9,0)T_4$ $+(2,1)T_6+(3,0)T_7+(1,0)T_9$	$(25,1)$
$\mathbb{Z}_{12}$–II	1	1	$(\theta,0)$	
		local	$(3,1)U+(4,0)T_1+(1,0)T_2+(8,0)T_3+(3,2)T_4$ $+(4,0)T_5+(4,2)T_6+(3,2)T_8+(1,0)T_{10}$	$(31,7)$
$\mathbb{Z}_2\times\mathbb{Z}_2$	1	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,3)U+(16,0)T_{0,1}+(16,0)T_{1,0}+(16,0)T_{1,1}$	$(51,3)$
		2	$\left(\theta,\frac{1}{2}e_2\right)$,$(\omega,0)$	
		local	$(3,3)U+(8,8)T_{0,1}+(8,8)T_{1,1}$	$(19,19)$
		3	$\left(\theta,\frac{1}{2}(e_2+e_6)\right)$,$(\omega,0)$	
		non–local	$(3,3)U+(8,8)T_{0,1}$	$(11,11)$
		4	$\left(\theta,\frac{1}{2}(e_2+e_6)\right)$,$\left(\omega,\frac{1}{2}e_4\right)$	
		non–local	$(3,3)U$	$(3,3)$
	2	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,3)U+(12,4)T_{0,1}+(8,0)T_{1,0}+(8,0)T_{1,1}$	$(31,7)$
		2	$\left(\theta,\frac{1}{2}e_3\right)$,$(\omega,0)$	
		local	$(3,3)U+(8,8)T_{0,1}+(4,4)T_{1,1}$	$(15,15)$
		3	$\left(\theta,\frac{1}{2}(e_3+e_6)\right)$,$(\omega,0)$	
		non–local	$(3,3)U+(8,8)T_{0,1}$	$(11,11)$
		4	$(\theta,0)$,$\left(\omega,\frac{1}{2}e_5\right)$	
		non–local	$(3,3)U+(4,4)T_{1,0}+(4,4)T_{1,1}$	$(11,11)$
		5	$\left(\theta,\frac{1}{2}e_3\right)$,$\left(\omega,\frac{1}{2}e_5\right)$	
		non–local	$(3,3)U+(4,4)T_{1,1}$	$(7,7)$
		6	$\left(\theta,\frac{1}{2}(e_3+e_6)\right)$,$\left(\omega,\frac{1}{2}e_5\right)$	
		non–local	$(3,3)U$	$(3,3)$
	3	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,3)U+(8,0)T_{0,1}+(8,0)T_{1,0}+(8,0)T_{1,1}$	$(27,3)$
		2	$\left(\theta,\frac{1}{2}e_6\right)$,$(\omega,0)$	
		local	$(3,3)U+(4,4)T_{0,1}+(4,4)T_{1,0}$	$(11,11)$
		3	$\left(\theta,\frac{1}{2}e_6\right)$,$\left(\omega,\frac{1}{2}e_5\right)$	
		non–local	$(3,3)U+(4,4)T_{1,0}$	$(7,7)$
		4	$\left(\theta,\frac{1}{2}(e_4+e_6)\right)$,$\left(\omega,\frac{1}{2}e_5\right)$	
		non–local	$(3,3)U$	$(3,3)$

continued . . .

$\mathbb{Q}$–class (P)	$\mathbb{Z}$–class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
	4	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,3)U+(10,6)T_{0,1}+(4,0)T_{1,0}+(4,0)T_{1,1}$	$(21,9)$
		2	$(\theta,0)$,$\left(\omega,\frac{1}{2}e_4\right)$	
		non–local	$(3,3)U+(2,2)T_{1,0}+(2,2)T_{1,1}$	$(7,7)$
	5	1	$(\theta,0)$,$(\omega,0)$	
		non–local	$(3,3)U+(8,0)T_{0,1}+(8,0)T_{1,0}+(8,0)T_{1,1}$	$(27,3)$
		2	$\left(\theta,\frac{1}{2}e_4\right)$,$(\omega,0)$	
		non–local	$(3,3)U+(4,4)T_{0,1}+(4,4)T_{1,1}$	$(11,11)$
		3	$\left(\theta,\frac{1}{2}(e_2+e_3)\right)$,$(\omega,0)$	
		local	$(3,3)U+(4,4)T_{0,1}+(4,4)T_{1,0}+(4,4)T_{1,1}$	$(15,15)$
		4	$\left(\theta,\frac{1}{2}e_4\right)$,$\left(\omega,\frac{1}{2}e_5\right)$	
		non–local	$(3,3)U+(4,4)T_{1,1}$	$(7,7)$
		5	$\left(\theta,\frac{1}{2}(e_4+e_6)\right)$,$\left(\omega,\frac{1}{2}e_5\right)$	
		non–local	$(3,3)U$	$(3,3)$
	6	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,3)U+(6,2)T_{0,1}+(4,0)T_{1,0}+(6,2)T_{1,1}$	$(19,7)$
		2	$(\theta,0)$,$\left(\omega,\frac{1}{2}e_5\right)$	
		non–local	$(3,3)U+(2,2)T_{1,0}+(4,4)T_{1,1}$	$(9,9)$
		3	$\left(\theta,\frac{1}{2}e_6\right)$,$\left(\omega,\frac{1}{2}e_5\right)$	
		non–local	$(3,3)U+(2,2)T_{1,0}$	$(5,5)$
	7	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,3)U+(6,2)T_{0,1}+(4,0)T_{1,0}+(4,0)T_{1,1}$	$(17,5)$
		2	$(\theta,0)$,$\left(\omega,\frac{1}{2}e_6\right)$	
		non–local	$(3,3)U+(2,2)T_{1,0}+(2,2)T_{1,1}$	$(7,7)$
	8	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,3)U+(4,0)T_{0,1}+(4,0)T_{1,0}+(4,0)T_{1,1}$	$(15,3)$
	9	1	$(\theta,0)$,$(\omega,0)$	
		non–local	$(3,3)U+(6,2)T_{0,1}+(4,0)T_{1,0}+(4,0)T_{1,1}$	$(17,5)$
		2	$(\theta,0)$,$\left(\omega,\frac{1}{2}e_6\right)$	
		non–local	$(3,3)U+(2,2)T_{1,0}+(2,2)T_{1,1}$	$(7,7)$
		3	$\left(\theta,\frac{1}{2}(e_2+e_3)\right)$,$(\omega,0)$	
		local	$(3,3)U+(4,4)T_{0,1}+(2,2)T_{1,0}+(2,2)T_{1,1}$	$(11,11)$
	10	1	$(\theta,0)$,$(\omega,0)$	
		non–local	$(3,3)U+(4,0)T_{0,1}+(4,0)T_{1,0}+(4,0)T_{1,1}$	$(15,3)$
		2	$\left(\theta,\frac{1}{2}(e_1+e_2)\right)$,$(\omega,0)$	
		local	$(3,3)U+(2,2)T_{0,1}+(2,2)T_{1,0}+(2,2)T_{1,1}$	$(9,9)$
	11	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,3)U+(3,1)T_{0,1}+(3,1)T_{1,0}+(3,1)T_{1,1}$	$(12,6)$
	12	1	$(\theta,0)$,$(\omega,0)$	

continued . . .

$\mathbb{Q}$–class (P)	$\mathbb{Z}$–class (Λ)	affine class, breaking	generators of G contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
		non–local	$(3,3)U+(4,0)T_{0,1}+(4,0)T_{1,0}+(4,0)T_{1,1}$	$(15,3)$
		2	$\big(\theta,\frac{1}{2}(e_5+e_6)\big)$,$(\omega,0)$	
		non–local	$(3,3)U+(2,2)T_{0,1}+(2,2)T_{1,0}+(2,2)T_{1,1}$	$(9,9)$
$\mathbb{Z}_2\times\mathbb{Z}_4$	1	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,1)U+(4,0)T_{0,1}+(10,0)T_{0,2}+(4,0)T_{0,3}+(12,0)T_{1,0}$ $+(16,0)T_{1,1}+(12,0)T_{1,2}$	$(61,1)$
		2	$\big(\theta,\frac{1}{2}(e_1+e_2)\big)$,$\big(\omega,\frac{1}{2}(e_1+e_2)\big)$	
		local	$(3,1)U+(2,2)T_{0,1}+(6,4)T_{0,2}+(2,2)T_{0,3}+(8,0)T_{1,1}$ $+(4,4)T_{1,2}$	$(25,13)$
		3	$\big(\theta,\frac{1}{2}(e_1+e_2+e_4+e_5)\big)$, $\big(\omega,\frac{1}{2}(e_1+e_2+e_4+e_5)\big)$	
		local	$(3,1)U+(2,2)T_{0,1}+(6,4)T_{0,2}+(2,2)T_{0,3}+(8,0)T_{1,1}$	$(21,9)$
		4	$\big(\theta,\frac{1}{2}e_4\big)$,$\big(\omega,\frac{1}{2}e_4\big)$	
		local	$(3,1)U+(10,0)T_{0,2}+(8,0)T_{1,0}+(8,0)T_{1,1}+(8,0)T_{1,2}$	$(37,1)$
		5	$\big(\theta,\frac{1}{2}(e_1+e_2+e_4)\big)$, $\big(\omega,\frac{1}{2}(e_1+e_2+e_4)\big)$	
		local	$(3,1)U+(6,4)T_{0,2}+(8,0)T_{1,0}+(4,4)T_{1,2}$	$(21,9)$
		6	$\big(\theta,\frac{1}{2}(e_1+e_2+e_4+e_5+e_6)\big)$, $\big(\omega,\frac{1}{2}(e_1+e_2+e_4+e_5+e_6)\big)$	
		non–local	$(3,1)U+(6,4)T_{0,2}+(8,0)T_{1,1}$	$(17,5)$
	2	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,1)U+(4,0)T_{0,1}+(8,2)T_{0,2}+(4,0)T_{0,3}+(8,0)T_{1,0}$ $+(16,0)T_{1,1}+(8,0)T_{1,2}$	$(51,3)$
		2	$\big(\theta,\frac{1}{2}e_6\big)$,$\big(\omega,\frac{1}{2}e_6\big)$	
		local	$(3,1)U+(8,2)T_{0,2}+(4,0)T_{1,0}+(8,0)T_{1,1}+(4,0)T_{1,2}$	$(27,3)$
		3	$\big(\theta,\frac{1}{2}(e_1+e_2)\big)$,$\big(\omega,\frac{1}{2}(e_1+e_2)\big)$	
		local	$(3,1)U+(2,2)T_{0,1}+(6,4)T_{0,2}+(2,2)T_{0,3}+(8,0)T_{1,1}$	$(21,9)$
		4	$\big(\theta,\frac{1}{2}(e_1+e_2+e_6)\big)$, $\big(\omega,\frac{1}{2}(e_1+e_2+e_6)\big)$	
		non–local	$(3,1)U+(6,4)T_{0,2}+(8,0)T_{1,1}$	$(17,5)$
		5	$\big(\theta,\frac{1}{2}(e_3+e_4)\big)$,$\big(\omega,\frac{1}{2}(e_3+e_4)\big)$	
		local	$(3,1)U+(2,2)T_{0,1}+(8,2)T_{0,2}+(2,2)T_{0,3}+(4,0)T_{1,0}$ $+(8,0)T_{1,1}+(4,0)T_{1,2}$	$(31,7)$
		6	$\big(\theta,\frac{1}{2}(e_3+e_4+e_6)\big)$, $\big(\omega,\frac{1}{2}(e_3+e_4+e_6)\big)$	
		local	$(3,1)U+(8,2)T_{0,2}+(4,0)T_{1,0}+(8,0)T_{1,1}+(4,0)T_{1,2}$	$(27,3)$
	3	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,1)U+(2,0)T_{0,1}+(6,0)T_{0,2}+(2,0)T_{0,3}+(6,0)T_{1,0}$ $+(12,0)T_{1,1}+(8,2)T_{1,2}$	$(39,3)$
		2	$\big(\theta,\frac{1}{2}(e_5+e_6)\big)$,$\big(\omega,\frac{1}{2}(e_5+e_6)\big)$	
		local	$(3,1)U+(1,1)T_{0,1}+(4,2)T_{0,2}+(1,1)T_{0,3}+(2,2)T_{1,0}$ $+(8,0)T_{1,1}$	$(19,7)$
		3	$\big(\theta,\frac{1}{2}e_4\big)$,$\big(\omega,\frac{1}{2}e_4\big)$	

continued ...

$\mathbb{Q}$–class (P)	$\mathbb{Z}$–class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
		local	$(3,1)U+(6,0)T_{0,2}+(4,0)T_{1,0}+(8,0)T_{1,1}+(6,2)T_{1,2}$	$(27,3)$
		4	$\left(\theta,\frac{1}{2}(e_4+e_5+e_6)\right),\ \left(\omega,\frac{1}{2}(e_4+e_5+e_6)\right)$	
		local	$(3,1)U+(4,2)T_{0,2}+(2,2)T_{1,0}+(8,0)T_{1,1}$	$(17,5)$
		5	$\left(\theta,\frac{1}{2}(e_1+e_3)\right),\left(\omega,\frac{1}{2}e_1\right)$	
		local	$(3,1)U+(3,1)T_{0,2}+(8,0)T_{1,1}+(4,4)T_{1,2}$	$(18,6)$
		6 non–local	$\left(\theta,\frac{1}{2}(e_1+e_3+e_5+e_6)\right),$ $\left(\omega,\frac{1}{2}(e_1+e_5+e_6)\right)$	
			$(3,1)U+(3,1)T_{0,2}+(8,0)T_{1,1}$	$(14,2)$
	4	1	$(\theta,0),(\omega,0)$	
		local	$(3,1)U+(2,0)T_{0,1}+(6,0)T_{0,2}+(2,0)T_{0,3}+(6,0)T_{1,0}$ $+(12,0)T_{1,1}+(6,0)T_{1,2}$	$(37,1)$
		2	$\left(\theta,\frac{1}{2}e_6\right),\left(\omega,\frac{1}{2}e_6\right)$	
		local	$(3,1)U+(6,0)T_{0,2}+(4,0)T_{1,0}+(8,0)T_{1,1}+(4,0)T_{1,2}$	$(25,1)$
		3 local	$\left(\theta,\frac{1}{2}(e_1+e_2+e_3+e_4)\right),$ $\left(\omega,\frac{1}{2}(e_1+e_2+e_3+e_4)\right)$	
			$(3,1)U+(1,1)T_{0,1}+(4,2)T_{0,2}+(1,1)T_{0,3}+(8,0)T_{1,1}$	$(17,5)$
		4 non–local	$\left(\theta,\frac{1}{2}(e_1+e_2+e_3+e_4+e_6)\right)$ $\left(\omega,\frac{1}{2}(e_1+e_2+e_3+e_4+e_6)\right)$	
			$(3,1)U+(4,2)T_{0,2}+(8,0)T_{1,1}$	$(15,3)$
		5	$\left(\theta,\frac{1}{2}(e_3+e_4)\right),\left(\omega,\frac{1}{2}(e_2+e_4+e_5)\right)$	
		local	$(3,1)U+(3,1)T_{0,2}+(2,2)T_{1,0}+(8,0)T_{1,1}$	$(16,4)$
	5	1	$(\theta,0),(\omega,0)$	
		local	$(3,1)U+(3,1)T_{0,1}+(7,3)T_{0,2}+(3,1)T_{0,3}+(4,0)T_{1,0}$ $+(12,0)T_{1,1}+(4,0)T_{1,2}$	$(36,6)$
		2	$\left(\theta,\frac{1}{2}e_6\right),\left(\omega,\frac{1}{2}e_6\right)$	
		local	$(3,1)U+(7,3)T_{0,2}+(2,0)T_{1,0}+(8,0)T_{1,1}+(2,0)T_{1,2}$	$(22,4)$
	6	1	$(\theta,0),(\omega,0)$	
		local	$(3,1)U+(2,0)T_{0,1}+(6,0)T_{0,2}+(2,0)T_{0,3}+(6,0)T_{1,0}$ $+(12,0)T_{1,1}+(6,0)T_{1,2}$	$(37,1)$
		2	$\left(\theta,\frac{1}{2}(e_4+e_5)\right),\left(\omega,\frac{1}{2}(e_4+e_5)\right)$	
		local	$(3,1)U+(1,1)T_{0,1}+(4,2)T_{0,2}+(1,1)T_{0,3}+(2,2)T_{1,0}$ $+(8,0)T_{1,1}+(2,2)T_{1,2}$	$(21,9)$
		3	$\left(\theta,\frac{1}{2}e_6\right),\left(\omega,\frac{1}{2}e_6\right)$	
		local	$(3,1)U+(6,0)T_{0,2}+(4,0)T_{1,0}+(8,0)T_{1,1}+(4,0)T_{1,2}$	$(25,1)$
		4	$\left(\theta,\frac{1}{2}(e_4+e_5+e_6)\right),\ \left(\omega,\frac{1}{2}(e_4+e_5+e_6)\right)$	
		local	$(3,1)U+(4,2)T_{0,2}+(2,2)T_{1,0}+(8,0)T_{1,1}+(2,2)T_{1,2}$	$(19,7)$
		5	$\left(\theta,\frac{1}{2}e_2\right),\left(\omega,\frac{1}{2}(e_1+e_3)\right)$	
		non–local	$(3,1)U+(3,1)T_{0,2}+(8,0)T_{1,1}$	$(14,2)$
	7	1	$(\theta,0),(\omega,0)$	

continued ...

$\mathbb{Q}$–class (P)	$\mathbb{Z}$–class (Λ)	affine class, breaking	generators of G contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		local	$(3,1)U + (2,0)T_{0,1} + (5,1)T_{0,2} + (2,0)T_{0,3} + (4,0)T_{1,0}$ $+(12,0)T_{1,1} + (4,0)T_{1,2}$	$(32,2)$
		2 local	$\left(\theta, \tfrac{1}{2}e_6\right),\left(\omega, \tfrac{1}{2}e_6\right)$ $(3,1)U + (5,1)T_{0,2} + (2,0)T_{1,0} + (8,0)T_{1,1} + (2,0)T_{1,2}$	$(20,2)$
		3 local	$\left(\theta, \tfrac{1}{2}(e_3+e_4+e_5)\right),\left(\omega, \tfrac{1}{2}(e_3+e_5)\right)$ $(3,1)U + (4,0)T_{0,2} + (2,0)T_{1,0} + (8,0)T_{1,1} + (2,0)T_{1,2}$	$(19,1)$
	8	1 local	$(\theta,0),(\omega,0)$ $(3,1)U + (1,0)T_{0,1} + (4,0)T_{0,2} + (1,0)T_{0,3} + (4,1)T_{1,0}$ $+(10,0)T_{1,1} + (4,1)T_{1,2}$	$(27,3)$
		2 local	$\left(\theta, \tfrac{1}{2}(e_1+e_3)\right),\left(\omega, \tfrac{1}{2}e_2\right)$ $(3,1)U + (2,0)T_{0,2} + (8,0)T_{1,1} + (2,2)T_{1,2}$	$(15,3)$
		3 non–local	$\left(\theta, \tfrac{1}{2}(e_1+e_3)\right),\ \left(\omega, \tfrac{1}{2}(e_2+e_5)\right)$ $(3,1)U + (2,0)T_{0,2} + (8,0)T_{1,1}$	$(13,1)$
	9	1 local	$(\theta,0),(\omega,0)$ $(3,1)U + (2,0)T_{0,1} + (5,1)T_{0,2} + (2,0)T_{0,3} + (4,0)T_{1,0}$ $+(12,0)T_{1,1} + (4,0)T_{1,2}$	$(32,2)$
		2 local	$\left(\theta, \tfrac{1}{2}e_6\right),\left(\omega, \tfrac{1}{2}e_6\right)$ $(3,1)U + (5,1)T_{0,2} + (2,0)T_{1,0} + (8,0)T_{1,1} + (2,0)T_{1,2}$	$(20,2)$
		3 local	$\left(\theta, \tfrac{1}{2}(e_1+e_2+e_4+e_5)\right),\ \left(\omega, \tfrac{1}{2}(e_1+e_2+e_4+e_5)\right)$ $(3,1)U + (1,1)T_{0,1} + (5,1)T_{0,2} + (1,1)T_{0,3} + (2,0)T_{1,0}$ $+(8,0)T_{1,1} + (2,0)T_{1,2}$	$(22,4)$
	10	1 local	$(\theta,0),(\omega,0)$ $(3,1)U + (1,0)T_{0,1} + (4,0)T_{0,2} + (1,0)T_{0,3} + (3,0)T_{1,0}$ $+(10,0)T_{1,1} + (3,0)T_{1,2}$	$(25,1)$
		2 local	$(\theta,0),\left(\omega, \tfrac{1}{2}e_6\right)$ $(3,1)U + (2,0)T_{0,2} + (1,1)T_{1,0} + (8,0)T_{1,1} + (1,1)T_{1,2}$	$(15,3)$
$\mathbb{Z}_2\times\mathbb{Z}_6$–I	1	1 local	$(\theta,0),(\omega,0)$ $(3,1)U + (1,0)T_{0,1} + (4,1)T_{0,2} + (6,0)T_{0,3} + (4,1)T_{0,4}$ $+(1,0)T_{0,5} + (8,0)T_{1,0} + (8,0)T_{1,1} + (8,0)T_{1,2} + (8,0)T_{1,3}$	$(51,3)$
		2 local	$\left(\theta, \tfrac{1}{2}e_4\right),\left(\omega, \tfrac{1}{2}e_4\right)$ $(3,1)U + (4,1)T_{0,2} + (4,1)T_{0,4} + (4,2)T_{1,0} + (6,0)T_{1,1}$ $+(6,0)T_{1,2} + (4,2)T_{1,3}$	$(31,7)$
	2	1 local	$(\theta,0),(\omega,0)$ $(3,1)U + (1,0)T_{0,1} + (4,1)T_{0,2} + (4,2)T_{0,3} + (4,1)T_{0,4}$ $+(1,0)T_{0,5} + (4,0)T_{1,0} + (8,0)T_{1,1} + (8,0)T_{1,2} + (4,0)T_{1,3}$	$(41,5)$
		2 local	$\left(\theta, \tfrac{1}{2}e_6\right),\left(\omega, \tfrac{1}{2}e_6\right)$ $(3,1)U + (4,1)T_{0,2} + (4,1)T_{0,4} + (2,0)T_{1,0} + (6,0)T_{1,1}$ $+(6,0)T_{1,2} + (2,0)T_{1,3}$	$(27,3)$
$\mathbb{Z}_2\times\mathbb{Z}_6$–II	1	1	$(\theta,0),(\omega,0)$	

continued ...

$\mathbb{Q}$–class (P)	$\mathbb{Z}$–class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		local	$(3,0)U + (2,0)T_{0,1} + (9,0)T_{0,2} + (6,0)T_{0,3} + (6,0)T_{1,0}$ $+(2,0)T_{1,1} + (6,0)T_{1,3} + (2,0)T_{1,4}$	$(36,0)$
	2	1 local	$(\theta,0)$,$(\omega,0)$	
			$(3,0)U + (2,0)T_{0,1} + (9,0)T_{0,2} + (2,0)T_{0,3} + (2,0)T_{1,0}$ $+(2,0)T_{1,1} + (4,2)T_{1,3} + (2,0)T_{1,4}$	$(26,2)$
	3	1 local	$(\theta,0)$,$(\omega,0)$	
			$(3,0)U + (2,0)T_{0,1} + (9,0)T_{0,2} + (2,0)T_{0,3} + (2,0)T_{1,0}$ $+(2,0)T_{1,1} + (2,0)T_{1,3} + (2,0)T_{1,4}$	$(24,0)$
	4	1 local	$(\theta,0)$,$(\omega,0)$	
			$(3,0)U + (2,0)T_{0,1} + (9,0)T_{0,2} + (2,0)T_{0,3} + (2,0)T_{1,0}$ $+(2,0)T_{1,1} + (2,0)T_{1,3} + (2,0)T_{1,4}$	$(24,0)$
$\mathbb{Z}_3 \times \mathbb{Z}_3$	1	1 local	$(\theta,0)$,$(\omega,0)$	
			$(3,0)U + (9,0)T_{0,1} + (9,0)T_{0,2} + (9,0)T_{1,0} + (27,0)T_{1,1}$ $+(9,0)T_{1,2} + (9,0)T_{2,0} + (9,0)T_{2,1}$	$(84,0)$
		2 local	$\big(\theta, \frac{1}{3}(2e_5+e_6)\big)$,$\big(\omega, \frac{1}{3}(e_5+2e_6)\big)$	
			$(3,0)U + (3,3)T_{0,1} + (3,3)T_{0,2} + (3,3)T_{1,0} + (9,0)T_{1,1}$ $+(3,3)T_{2,0}$	$(24,12)$
		3 local	$\big(\theta, \frac{1}{3}(2e_1+e_2+2e_5+e_6)\big)$,$\big(\omega, \frac{1}{3}(e_1+2e_2+e_5+2e_6)\big)$	
			$(3,0)U + (3,3)T_{0,1} + (3,3)T_{0,2} + (9,0)T_{1,1}$	$(18,6)$
		4 non–local	$\big(\theta, \frac{1}{3}(2e_1+e_2+2e_3+e_4+2e_5+e_6)\big)$, $\big(\omega, \frac{1}{3}(e_1+2e_2+e_3+2e_4+e_5+2e_6)\big)$	
			$(3,0)U + (9,0)T_{1,1}$	$(12,0)$
	2	1 local	$(\theta,0)$,$(\omega,0)$	
			$(3,0)U + (5,2)T_{0,1} + (5,2)T_{0,2} + (3,0)T_{1,0} + (15,0)T_{1,1}$ $+(3,0)T_{1,2} + (3,0)T_{2,0} + (3,0)T_{2,1}$	$(40,4)$
		2 local	$\big(\theta, \frac{1}{3}(2e_5+e_6)\big)$,$\big(\omega, \frac{1}{3}(e_5+2e_6)\big)$	
			$(3,0)U + (1,1)T_{1,0} + (9,0)T_{1,1} + (1,1)T_{1,2} + (1,1)T_{2,0}$ $+(1,1)T_{2,1}$	$(16,4)$
		3 local	$\big(\theta, \frac{1}{3}(2e_3+e_4)\big)$,$\big(\omega, \frac{2}{3}(e_1+e_2+e_4)\big)$	
			$(3,0)U + (3,3)T_{0,1} + (3,3)T_{0,2} + (9,0)T_{1,1}$	$(18,6)$
		4 non–local	$\big(\theta, \frac{1}{3}(e_1+2e_2+2e_3+e_6)\big)$, $\big(\omega, \frac{1}{3}(2e_1+e_2+e_4+e_5+e_6)\big)$	
			$(3,0)U + (9,0)T_{1,1}$	$(12,0)$
	3	1 local	$(\theta,0)$,$(\omega,0)$	
			$(3,0)U + (3,0)T_{0,1} + (3,0)T_{0,2} + (3,0)T_{1,0} + (15,0)T_{1,1}$ $+(3,0)T_{1,2} + (3,0)T_{2,0} + (3,0)T_{2,1}$	$(36,0)$
		2 local	$\big(\theta, \frac{1}{3}(e_3+2e_4)\big)$, $\big(\omega, \frac{1}{3}(2e_1+2e_2+e_3+e_4)\big)$	
			$(3,0)U + (9,0)T_{1,1} + (1,1)T_{1,2} + (1,1)T_{2,1}$	$(14,2)$
		3	$\big(\theta, \frac{1}{3}(2e_1+e_2+2e_3+e_4+e_5+2e_6)\big)$	

continued ...

$\mathbb{Q}$–class (P)	$\mathbb{Z}$–class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
		non–local	$\big(\omega,\frac{1}{3}(e_1+2e_2+e_3+2e_4+2e_5+e_6)\big)$	
			$(3,0)U+(9,0)T_{1,1}$	$(12,0)$
	4	1 local	$(\theta,0)$,$(\omega,0)$	
			$(3,0)U+(3,0)T_{0,1}+(3,0)T_{0,2}+(3,0)T_{1,0}+(15,0)T_{1,1}$ $+(3,0)T_{1,2}+(3,0)T_{2,0}+(3,0)T_{2,1}$	$(36,0)$
		2 local	$\big(\theta,\frac{1}{3}(e_2+2e_3)\big)$, $\big(\omega,\frac{1}{3}(2e_2+e_3)\big)$	
			$(3,0)U+(1,1)T_{0,1}+(1,1)T_{0,2}+(1,1)T_{1,0}+(9,0)T_{1,1}$ $+(1,1)T_{1,2}+(1,1)T_{2,0}+(1,1)T_{2,1}$	$(18,6)$
		3 non–local	$\big(\theta,\frac{1}{3}(e_1+e_3+2e_4+2e_5)\big)$, $\big(\omega,\frac{1}{3}(2e_1+2e_2+e_4+2e_5+2e_6)\big)$	
			$(3,0)U+(9,0)T_{1,1}$	$(12,0)$
	5	1 local	$(\theta,0)$,$(\omega,0)$	
			$(3,0)U+(1,0)T_{0,1}+(1,0)T_{0,2}+(1,0)T_{1,0}+(11,0)T_{1,1}$ $+(1,0)T_{1,2}+(1,0)T_{2,0}+(1,0)T_{2,1}$	$(20,0)$
$\mathbb{Z}_3\times\mathbb{Z}_6$	1	1 local	$(\theta,0)$,$(\omega,0)$	
			$(3,0)U+(1,0)T_{0,1}+(5,0)T_{0,2}+(4,1)T_{0,3}+(5,0)T_{0,4}$ $+(1,0)T_{0,5}+(6,0)T_{1,0}+(6,0)T_{1,1}+(15,0)T_{1,2}+(6,0)T_{1,3}$ $+(6,0)T_{1,4}+(6,0)T_{2,0}+(3,0)T_{2,1}+(6,0)T_{2,2}$	$(73,1)$
		2 local	$\big(\theta,\frac{1}{3}(e_3+2e_4)\big)$,$\big(\omega,\frac{1}{3}(2e_3+e_4)\big)$	
			$(3,0)U+(4,1)T_{0,3}+(2,1)T_{1,0}+(4,0)T_{1,1}+(5,0)T_{1,2}$ $+(4,0)T_{1,3}+(2,1)T_{1,4}+(2,1)T_{2,0}+(1,0)T_{2,1}+(2,1)T_{2,2}$	$(29,5)$
	2	1 local	$(\theta,0)$,$(\omega,0)$	
			$(3,0)U+(1,0)T_{0,1}+(3,1)T_{0,2}+(4,1)T_{0,3}+(3,1)T_{0,4}$ $+(1,0)T_{0,5}+(3,0)T_{1,0}+(6,0)T_{1,1}+(9,0)T_{1,2}+(6,0)T_{1,3}$ $+(3,0)T_{1,4}+(3,0)T_{2,0}+(3,0)T_{2,1}+(3,0)T_{2,2}$	$(51,3)$
		2 local	$\big(\theta,\frac{1}{3}(e_5+2e_6)\big)$,$\big(\omega,\frac{1}{3}(2e_5+e_6)\big)$	
			$(3,0)U+(4,1)T_{0,3}+(1,0)T_{1,0}+(4,0)T_{1,1}+(5,0)T_{1,2}$ $+(4,0)T_{1,3}+(1,0)T_{1,4}+(1,0)T_{2,0}+(1,0)T_{2,1}+(1,0)T_{2,2}$	$(25,1)$
$\mathbb{Z}_4\times\mathbb{Z}_4$	1	1 local	$(\theta,0)$,$(\omega,0)$	
			$(3,0)U+(4,0)T_{0,1}+(9,0)T_{0,2}+(4,0)T_{0,3}+(4,0)T_{1,0}$ $+(12,0)T_{1,1}+(12,0)T_{1,2}+(4,0)T_{1,3}+(9,0)T_{2,0}$ $+(12,0)T_{2,1}+(9,0)T_{2,2}+(4,0)T_{3,0}+(4,0)T_{3,1}$	$(90,0)$
		2 local	$\big(\theta,\frac{1}{2}(e_5+e_6)\big)$,$(\omega,0)$	
			$(3,0)U+(2,0)T_{0,1}+(7,0)T_{0,2}+(2,0)T_{0,3}+(2,0)T_{1,0}$ $+(4,0)T_{1,1}+(8,0)T_{1,2}+(7,0)T_{2,0}+(8,0)T_{2,1}+(9,0)T_{2,2}$ $+(2,0)T_{3,0}$	$(54,0)$
		3 local	$\big(\theta,\frac{1}{2}(e_1+e_2+e_5+e_6)\big)$,$(\omega,0)$	
			$(3,0)U+(2,0)T_{0,1}+(5,0)T_{0,2}+(2,0)T_{0,3}+(4,0)T_{1,1}$ $+(4,0)T_{1,2}+(7,0)T_{2,0}+(8,0)T_{2,1}+(7,0)T_{2,2}$	$(42,0)$

continued ...

$\mathbb{Q}$–class (P)	$\mathbb{Z}$–class (Λ)	affine class, breaking	generators of G contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		4 local	$\big(\theta, \frac{1}{2}(e_1+e_2+e_4+e_5+e_6)\big),\ \big(\omega, \frac{1}{2}(e_3+e_4)\big)$ $(3,0)U + (5,0)T_{0,2} + (4,0)T_{1,1} + (4,0)T_{1,2} + (5,0)T_{2,0}$ $+(4,0)T_{2,1} + (5,0)T_{2,2}$	$(30,0)$
	2	1 local	$(\theta,0)$,$(\omega,0)$ $(3,0)U + (3,0)T_{0,1} + (6,1)T_{0,2} + (3,0)T_{0,3} + (2,0)T_{1,0}$ $+(8,0)T_{1,1} + (8,0)T_{1,2} + (2,0)T_{1,3} + (6,0)T_{2,0} + (10,0)T_{2,1}$ $+(6,0)T_{2,2} + (2,0)T_{3,0} + (2,0)T_{3,1}$	$(61,1)$
		2 local	$\big(\theta, \frac{1}{2}(e_1+e_4)\big)$,$\big(\omega, \frac{1}{2}(e_1+e_3)\big)$ $(3,0)U + (1,1)T_{0,1} + (4,1)T_{0,2} + (1,1)T_{0,3} + (4,0)T_{1,1}$ $+(4,0)T_{1,2} + (3,0)T_{2,0} + (4,0)T_{2,1} + (3,0)T_{2,2}$	$(27,3)$
		3 local	$\big(\theta, \frac{1}{2}e_6\big)$,$\big(\omega, \frac{1}{2}(e_5+e_6)\big)$ $(3,0)U + (6,1)T_{0,2} + (1,0)T_{1,0} + (6,0)T_{1,1} + (6,0)T_{1,2}$ $+(1,0)T_{1,3} + (4,0)T_{2,0} + (4,0)T_{2,1} + (4,0)T_{2,2} + (1,0)T_{3,0}$ $+(1,0)T_{3,1}$	$(37,1)$
		4 local	$\big(\theta, \frac{1}{2}(e_1+e_4+e_6)\big),\ \big(\omega, \frac{1}{2}(e_1+e_3+e_5+e_6)\big)$ $(3,0)U + (4,1)T_{0,2} + (4,0)T_{1,1} + (4,0)T_{1,2} + (3,0)T_{2,0}$ $+(4,0)T_{2,1} + (3,0)T_{2,2}$	$(25,1)$
	3	1 local	$(\theta,0)$,$(\omega,0)$ $(3,0)U + (2,0)T_{0,1} + (5,0)T_{0,2} + (2,0)T_{0,3} + (2,0)T_{1,0}$ $+(8,0)T_{1,1} + (8,0)T_{1,2} + (2,0)T_{1,3} + (5,0)T_{2,0} + (8,0)T_{2,1}$ $+(5,0)T_{2,2} + (2,0)T_{3,0} + (2,0)T_{3,1}$	$(54,0)$
		2 local	$\big(\theta, \frac{1}{2}(e_1+e_2+e_3+e_5)\big),\ \big(\omega, \frac{1}{2}(e_3+e_4+e_5+e_6)\big)$ $(3,0)U + (1,0)T_{0,1} + (3,0)T_{0,2} + (1,0)T_{0,3} + (4,0)T_{1,1}$ $+(4,0)T_{1,2} + (4,0)T_{2,0} + (6,0)T_{2,1} + (4,0)T_{2,2}$	$(30,0)$
	4	1 local	$(\theta,0)$,$(\omega,0)$ $(3,0)U + (2,0)T_{0,1} + (5,0)T_{0,2} + (2,0)T_{0,3} + (2,0)T_{1,0}$ $+(8,0)T_{1,1} + (8,0)T_{1,2} + (2,0)T_{1,3} + (5,0)T_{2,0} + (8,0)T_{2,1}$ $+(5,0)T_{2,2} + (2,0)T_{3,0} + (2,0)T_{3,1}$	$(54,0)$
		2 local	$\big(\theta, \frac{1}{2}e_2\big)$,$\big(\omega, \frac{1}{2}(e_1+e_2)\big)$ $(3,0)U + (1,0)T_{0,1} + (5,0)T_{0,2} + (1,0)T_{0,3} + (1,0)T_{1,0}$ $+(6,0)T_{1,1} + (6,0)T_{1,2} + (1,0)T_{1,3} + (5,0)T_{2,0} + (6,0)T_{2,1}$ $+(5,0)T_{2,2} + (1,0)T_{3,0} + (1,0)T_{3,1}$	$(42,0)$
		3 local	$\big(\theta, \frac{1}{2}(e_2+e_5+e_6)\big),\ \big(\omega, \frac{1}{2}(e_2+e_4+e_5)\big)$ $(3,0)U + (2,1)T_{0,2} + (4,0)T_{1,1} + (4,0)T_{1,2} + (2,1)T_{2,0}$ $+(4,0)T_{2,1} + (2,1)T_{2,2}$	$(21,3)$
	5	1 local	$(\theta,0)$,$(\omega,0)$ $(3,0)U + (1,0)T_{0,1} + (3,0)T_{0,2} + (1,0)T_{0,3} + (1,0)T_{1,0}$ $+(6,0)T_{1,1} + (6,0)T_{1,2} + (1,0)T_{1,3} + (3,0)T_{2,0} + (6,0)T_{2,1}$ $+(3,0)T_{2,2} + (1,0)T_{3,0} + (1,0)T_{3,1}$	$(36,0)$

continued ...

$\mathbb{Q}$–class (P)	$\mathbb{Z}$–class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
		2 local	$\left(\theta,\frac{1}{2}(e_3+e_4)\right)$, $\left(\omega,\frac{1}{2}(e_1+e_2+e_3+e_6)\right)$	
			$(3,0)U+(1,0)T_{0,2}+(4,0)T_{1,1}+(4,0)T_{1,2}+(1,0)T_{2,0}$ $+(4,0)T_{2,1}+(1,0)T_{2,2}$	$(18,0)$
$\mathbb{Z}_6\times\mathbb{Z}_6$	1	1 local	$(\theta,0)$,$(\omega,0)$	
			$(3,0)U+(1,0)T_{0,1}+(4,0)T_{0,2}+(4,0)T_{0,3}+(4,0)T_{0,4}$ $+(1,0)T_{0,5}+(1,0)T_{1,0}+(2,0)T_{1,1}+(4,0)T_{1,2}+(4,0)T_{1,3}$ $+(2,0)T_{1,4}+(1,0)T_{1,5}+(4,0)T_{2,0}+(4,0)T_{2,1}+(9,0)T_{2,2}$ $+(4,0)T_{2,3}+(4,0)T_{2,4}+(4,0)T_{3,0}+(4,0)T_{3,1}+(4,0)T_{3,2}$ $+(4,0)T_{3,3}+(4,0)T_{4,0}+(2,0)T_{4,1}+(4,0)T_{4,2}+(1,0)T_{5,0}$ $+(1,0)T_{5,1}$	$(84,0)$

Table C.1: Summary of the classification of all six–dimensional $\mathcal{N}=1$ SUSY preserving Abelian toroidal orbifolds. The nomenclature for the $\mathbb{Q}$–classes is the common one in the literature (cf. e.g. [9]). The twists θ and ω correspond to the twist vectors listed in Table 5.2 and $T_{k,\ell}$ labels the twisted sector $\theta^k\omega^\ell$.

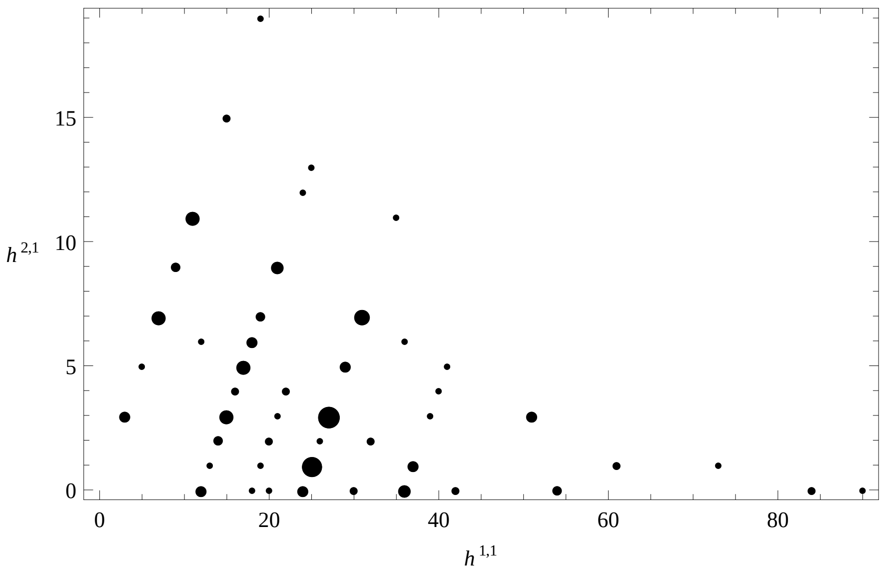


Figure 7: Statistics of the Hodge numbers for the 138 Abelian toroidal orbifolds of Table C.1.

C.2 Non–Abelian point groups

label of $\mathbb{Q}$–class GAPID	CARAT index	twists from SU(3)	# of conj. classes
S_3 $[6,1]$	2262	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}} \end{pmatrix}$	3
D_4 $[8,3]$	4682	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	5
Q_8 ($\mathcal{N}=2$) $[8,4]$	5750	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}$	5
Dic_3 ($\mathcal{N}=2$) $[12,1]$	3374	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \end{pmatrix}$	6
A_4 $[12,3]$	4893	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	4
D_6 $[12,4]$	2258	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{6}} \end{pmatrix}$	6
$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$ $[16,6]$	6222	$\begin{pmatrix} -\mathrm{i} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	10
QD_{16} $[16,8]$	5650	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{8}} \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{3}{8}} & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	7
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $[16,13]$	5645	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & -\mathrm{i} \end{pmatrix}$	10

continued ...

label of $\mathbb{Q}$–class GAPID	CARAT index	twists from SU(3)	# of conj. classes
$\mathbb{Z}_3 \times S_3$ $[18, 3]$	4235	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}} \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \end{pmatrix}$	9
Frebenius T_7 $[21, 1]$	2935	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{4}{7}} & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{7}} & 0 \\ 0 & 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{7}} \end{pmatrix}$	5
$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$ $[24, 1]$	6266	$\begin{pmatrix} -\mathrm{i} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}} \end{pmatrix}$	12
SL$(2,3)$–I $[24, 3]$	6743	$\begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} & 0 & 0 \\ 0 & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} + \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{11}{12}}) & \frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} + \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{11}{12}}) \\ 0 & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} - \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{11}{12}}) & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} - \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{11}{12}}) \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	7
SL$(2,3)$–II ($\mathcal{N}=2$) $[24, 3]$	5669	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}(1+\mathrm{i}) & \frac{1}{2}(1+\mathrm{i}) \\ 0 & -\frac{1}{2}(1-\mathrm{i}) & -\frac{1}{2}(1-\mathrm{i}) \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	7
$\mathbb{Z}_4 \times S_3$ $[24, 5]$	3414	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{5}{12}} & 0 \\ 0 & 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{12}} \end{pmatrix}$	12
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $[24, 8]$	3408	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{6}} \end{pmatrix}$	9
$\mathbb{Z}_3 \times D_4$ $[24, 10]$	4326	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \end{pmatrix}$	15
$\mathbb{Z}_3 \times Q_8$ $[24, 11]$	6735	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 \end{pmatrix}$	15

continued ...

label of $\mathbb{Q}$–class GAPID	CARAT index	twists from SU(3)	# of conj. classes
S_4 [24, 12]	4895	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	5
$\Delta(27)$ [27, 3]	2864	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \end{pmatrix}$	11
$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ [32, 11]	6337	$\begin{pmatrix} -\mathrm{i} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	14
$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ [36, 6]	4353	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}} \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 \end{pmatrix}$	18
$\mathbb{Z}_3 \times A_4$ [36, 11]	2875	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{6}} \end{pmatrix}$	12
$\mathbb{Z}_6 \times S_3$ [36, 12]	4356	$\begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{6}} \end{pmatrix}$	18
$\Delta(48)$ [48, 3]	2774	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}$	8
$\mathrm{GL}(2,3)$ [48, 29]	5713	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}(1-\mathrm{i}) & \frac{1}{2}(1-\mathrm{i}) \\ 0 & -\frac{1}{2}(1+\mathrm{i}) & -\frac{1}{2}(1+\mathrm{i}) \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{8}} + \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{3}{8}}) & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{8}} - \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{3}{8}}) \\ 0 & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{8}} - \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{3}{8}}) & \frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{8}} + \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{3}{8}}) \end{pmatrix}$	8
$\mathrm{SL}(2,3) \rtimes \mathbb{Z}_2$ [48, 33]	5712	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2}(1-\mathrm{i}) & \frac{1}{2}(1-\mathrm{i}) \\ 0 & \frac{1}{2}(1+\mathrm{i}) & \frac{1}{2}(1+\mathrm{i}) \end{pmatrix}$	14
			continued ...

label of $\mathbb{Q}$–class GAPID	CARAT index	twists from SU(3)	# of conj. classes
$\Delta(54)$ $[54,8]$	2897	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{2\pi i \frac{1}{3}} & 0 \\ 0 & 0 & e^{-2\pi i \frac{1}{3}} \\ 1 & 0 & 0 \end{pmatrix}$	10
$\mathbb{Z}_3 \times \mathrm{SL}(2,3)$ $[72,25]$	6988	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}(1+i) & \frac{1}{2}(1+i) \\ 0 & -\frac{1}{2}(1-i) & -\frac{1}{2}(1-i) \end{pmatrix}, \begin{pmatrix} e^{-2\pi i \frac{1}{3}} & 0 & 0 \\ 0 & 0 & e^{-2\pi i \frac{1}{3}} \\ 0 & e^{2\pi i \frac{1}{6}} & 0 \end{pmatrix}$	21
$\mathbb{Z}_3 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$ $= \mathbb{Z}_3 \times \mathrm{GAPID}\,[24,8]$ $[72,30]$	4533	$\begin{pmatrix} e^{2\pi i \frac{1}{6}} & 0 & 0 \\ 0 & 0 & e^{-2\pi i \frac{1}{3}} \\ 0 & e^{-2\pi i \frac{1}{3}} & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & e^{-2\pi i \frac{1}{3}} & 0 \\ 0 & 0 & e^{2\pi i \frac{1}{3}} \end{pmatrix}$	27
$\mathbb{Z}_3 \times S_4$ $[72,42]$	2924	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e^{-2\pi i \frac{1}{3}} & 0 & 0 \\ 0 & 0 & e^{-2\pi i \frac{1}{3}} \\ 0 & e^{2\pi i \frac{1}{6}} & 0 \end{pmatrix}$	15
$\Delta(96)$ $[96,64]$	2802	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}$	10
$\mathrm{SL}(2,3) \rtimes \mathbb{Z}_4$ $[96,67]$	6512	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ 0 & -\frac{1}{2}(1+i) & -\frac{1}{2}(1+i) \end{pmatrix}, \begin{pmatrix} -i & 0 & 0 \\ 0 & -\frac{1}{2}(1+i) & \frac{1}{2}(1-i) \\ 0 & \frac{1}{2}(1-i) & -\frac{1}{2}(1+i) \end{pmatrix}$	16
$\Sigma(36\phi)$ $[108,15]$	2806	$\begin{pmatrix} -\frac{1}{3}(e^{2\pi i \frac{1}{3}} + 2e^{-2\pi i \frac{1}{3}}) & -\frac{1}{3}(e^{2\pi i \frac{1}{3}} + 2e^{-2\pi i \frac{1}{3}}) & \frac{1}{3}(2e^{2\pi i \frac{1}{3}} + e^{-2\pi i \frac{1}{3}}) \\ \frac{1}{3}(2e^{2\pi i \frac{1}{3}} + e^{-2\pi i \frac{1}{3}}) & -\frac{1}{3}(e^{2\pi i \frac{1}{3}} + e^{-2\pi i \frac{1}{3}}) & \frac{1}{3}(2e^{2\pi i \frac{1}{3}} + e^{-2\pi i \frac{1}{3}}) \\ \frac{1}{3}(2e^{2\pi i \frac{1}{3}} + e^{-2\pi i \frac{1}{3}}) & -\frac{1}{3}(e^{2\pi i \frac{1}{3}} + e^{-2\pi i \frac{1}{3}}) & -\frac{1}{3}(e^{2\pi i \frac{1}{3}} + 2e^{-2\pi i \frac{1}{3}}) \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	14
$\Delta(108)$ $[108,22]$	2810	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e^{-\frac{2\pi i}{6}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i}{3}} & 0 \\ 0 & 0 & -1 \end{pmatrix}$	20
$\mathrm{PSL}(3,2)$ $[168,42]$	2934	$\begin{pmatrix} \frac{1}{18}\left(-5+4i\sqrt{7}\right) & \frac{1}{36}\left(11+5i\sqrt{7}\right) & \frac{1}{18}\left(-1-4i\sqrt{7}\right) \\ -\frac{1}{36}i\left(-25i+\sqrt{7}\right) & -\frac{1}{9}i\left(-i+\sqrt{7}\right) & \frac{1}{36}i\left(23i+\sqrt{7}\right) \\ \frac{1}{18}\left(-1+2i\sqrt{7}\right) & \frac{1}{36}\left(-5-11i\sqrt{7}\right) & \frac{1}{18}\left(7-2i\sqrt{7}\right) \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	6

continued ...

label of $\mathbb{Q}$–class GAPID	CARAT index	twists from SU(3)	# of conj. classes
$\Sigma(72\phi)$ $[216, 88]$	2846	$\begin{pmatrix} \frac{1}{6}\left(3+\mathrm{i}\sqrt{3}\right) & \frac{e^{2\pi\,\mathrm{i}\,\frac{5}{12}}}{\sqrt{3}} & -\frac{\mathrm{i}}{\sqrt{3}} \\ \frac{1}{6}\left(3+\sqrt{3}\,\mathrm{i}\right) & -\frac{\mathrm{i}}{\sqrt{3}} & \frac{e^{2\pi\,\mathrm{i}\,\frac{5}{12}}}{\sqrt{3}} \\ \frac{1}{6}\left(3+\sqrt{3}\,\mathrm{i}\right) & \frac{1}{6}\left(3+\sqrt{3}\,\mathrm{i}\right) & \frac{1}{6}\left(3+\sqrt{3}\,\mathrm{i}\right) \end{pmatrix}, \begin{pmatrix} -\frac{\mathrm{i}}{\sqrt{3}} & \frac{1}{6}\left(3+\sqrt{3}\,\mathrm{i}\right) & -\frac{\mathrm{i}}{\sqrt{3}} \\ \frac{1}{6}\left(3+\sqrt{3}\,\mathrm{i}\right) & -\frac{\mathrm{i}}{\sqrt{3}} & -\frac{\mathrm{i}}{\sqrt{3}} \\ \frac{1}{6}\left(3+\sqrt{3}\,\mathrm{i}\right) & \frac{1}{6}\left(3+\sqrt{3}\,\mathrm{i}\right) & \frac{e^{2\pi\,\mathrm{i}\,\frac{5}{12}}}{\sqrt{3}} \end{pmatrix}$	16
$\Delta(216)$ $[216, 95]$	2851	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & e^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \\ 1 & 0 & 0 \end{pmatrix}$	19

Table C.2: Summary of the classification of all non–Abelian point groups with $\mathcal{N} \geq 1$ SUSY. The GAPID $[N, M]$ consists of two numbers: the first number N gives the order of the discrete group (i.e. the number of elements) and the second number consecutively enumerates discrete groups of a certain order. The number of conjugacy classes c corresponds to $c-1$ twisted sectors for the heterotic orbifold compactification.

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