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On semiclassical analysis of pure spinor superstring in an $AdS_5 \times S^5$ background

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Abstract

Relation between semiclassical analyses of Green-Schwarz and pure spinor formalisms in an $AdS_5 \times S^5$ background is clarified. It is shown that the two formalisms have identical semiclassical partition functions for a simple family of classical solutions. It is also shown that, when the classical string is furthermore rigid, this in turn implies that the two formalisms predict the same one-loop corrections to spacetime energies.

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1 Introduction

Over the last decade, the semiclassical study of string theory in an $AdS_5 \times S^5$ background has been a central tool for exploring the AdS/CFT correspondence [1, 2, 3] beyond a supergravity approximation. To date, an enormous amount of works has been done extending the basic picture laid in [4, 5, 6], matching quantum corrections to string energies to anomalous dimensions of gauge invariant operators in the $\mathcal{N} = 4$ super Yang-Mills theory.

Since AdS geometries that appear in the AdS/CFT correspondence are supported by Ramond-Ramond flux, it is hard to make use of the Ramond-Neveu-Schwarz formalism. For an $AdS_5 \times S^5$ background, one may either use the Green-Schwarz formalism [7] or Berkovits' pure spinor formalism [8]. However, most of the works in the area have been done only in the former. This is a pity because the pure spinor formalism has many aspects that are simpler than the Green-Schwarz formalism, and is potentially more powerful especially if one wants more than the fluctuation spectrum around a given classical solution.

The purpose of this article is to provide support for an equivalence of the Green-Schwarz and pure spinor formalisms at a semiclassical level. Using the pure spinor formalism we perform a semiclassical analysis around a simple family of classical solutions in an $AdS_5 \times S^5$ background and show that the formalism reproduces the one-loop anomalous dimensions known from the Green-Schwarz formalism. It would be useful to exploit integrability methods for a more systematic comparison, but in this article we stick to a down-to-earth explicit comparison.

In the rest of this introduction, we would like to put our study into context by briefly summarizing what has been known about the pure spinor formalism. For a more complete list, we refer the reader to a recent review [9].

Pure spinor formalism in a flat background is defined as a worldsheet conformal field theory with a BRST symmetry and it allows one to quantize a string in a super-Poincaré covariant manner. Its basics and validity have been established quite adequately. The formalism reproduces the superstring spectrum correctly [10][11], and is capable of computing tree and multi-loop amplitudes in a covariant manner [8, 12]. There remains some subtleties at three-loops and higher [13], but the formalism has been very successful going far beyond (e.g. [14, 15]) what have been done in other formalisms. Also, in a flat background, it is known how the BRST symmetry of the formalism arises from the classical Green-Schwarz action [16][17].

In generic supergravity backgrounds, both Green-Schwarz and pure spinor formalisms can be used to describe a string at a classical level. Equations of motion for the background fields are implied by the kappa symmetry [18] in the former (e.g. [19][20]) and by the BRST symmetry in the latter [21]. Preservation of these symmetries in worldsheet perturbation theories are expected to characterize stringy α' corrections to the background equations of motion. However, kappa symmetry is a complicated gauge symmetry and it is difficult to discuss them quantum mechanically. In pure spinor formalism, kappa symmetry is replaced by a BRST symmetry and it is straightforward to identify the conditions for conservation and nilpotency of the BRST charge at a quantum level [21]. By exploiting this simplicity, one-loop conformal invariance in generic supergravity backgrounds has been shown in [22, 23].

Specializing to an $AdS_5 \times S^5$ background, a Green-Schwarz action with kappa symmetry was constructed explicitly as a supercoset model by Metsaev and Tseytlin [24]. The key to their construction was that the $AdS_5 \times S^5$ space can be realized as the bosonic body of a supercoset $PSU(2,2|4)/(SO(4,1) \times SO(5))$ with 32 fermionic directions. The supercoset has a $\mathbb{Z}_4$-structure (a natural extension of the notion of the symmetric coset space) which makes it possible to rewrite the Metsaev-Tseytlin action as a bilinear form of currents [25]. A classical action for the pure spinor formalism can be explicitly written down by applying the same technique and by introducing pure spinor variables adopted to $AdS_5 \times S^5$ [8]. Presumably, the pure spinor action can be understood as a BRST reformulation of the Metsaev-Tseytlin action but to date the expectation has not been shown explicitly. Although these actions are constructed from currents on a group manifold, these currents are not holomorphic. Therefore, unlike the Wess-Zumino-Witten models, it is not known how to solve the models based on symmetry principles. On the other hand, both models are known to possess an integrable structure [26][27] and one may hope to eventually solve these models by combining integrability and conformal field theory techniques.

Although exact quantizations of Green-Schwarz and pure spinor superstrings in the $AdS_5 \times S^5$ backgrounds are not within a reach at the moment, there are no problems in performing classical and semiclassical analyses. In the Green-Schwarz formalism, basics of semiclassical analysis (in particular subtleties arising from gauge fixing Virasoro and kappa symmetries) have been clarified in [28] and concrete analyses around very many classical solutions have been performed, providing strong supports in favour of the AdS/CFT conjecture. In the pure spinor formalism, there are no complicated gauge symmetries to be fixed and the semiclassical analysis is straightforward. One-loop conformal invariance in the $AdS_5 \times S^5$ background has been shown in [29] and later extended to an all-loop proof [30]. Although the pure spinor formalism has not been used much for computing concrete quantities in the AdS/CFT context, it has been used in [31] to compute the anomalous dimensions of the Konishi multiplet at strong coupling, and the result of [31] is in accord with the ones predicted from the Green-Schwarz formalism [32] and integrability techniques [33].

So, all in all, parallel developments have been made in the Green-Schwarz and pure spinor formalisms, but it has never been clarified why or how the two are equivalent at a (semi)classical level. It is this relation of the two formalisms we wish to address in this article.

The plan of this article is as follows. In section 2 we review the classical mechanics of the pure spinor formalism in an $AdS_5 \times S^5$ background. Section 3 contains the body of the article. After a general discussion on semiclassical analyses in the pure spinor formalism, we introduce a simple family of classical solutions and show that one-loop corrections to spacetime energies are related to the expectation values of the fluctuation Hamiltonians on the worldsheet. We then compare the one-loop partition functions in the Green-Schwarz and pure spinor formalisms and argue that they agree. We conclude in section 4 and point out some future directions. An appendix is added to summarize our notation and conventions.

2 Classical pure spinor superstring in $AdS_5 \times S^5$ background

We start with a brief review of the pure spinor formalism in an $AdS_5 \times S^5$ background, with some emphases on comparison with the Green-Schwarz formalism. To motivate the definition of the pure spinor superstring action in the $AdS_5 \times S^5$ background, we start from an explanation of the pure spinor formalism in trivial and generic supergravity backgrounds.

2.1 Trivial background

In contrast to conventional approaches to string theory, the pure spinor formalism in a trivial background starts off by postulating a quadratic worldsheet action with a BRST symmetry [8]. For type II superstring the action is given as[1]

$$S_{\text{flat}} = \frac{1}{\pi\alpha'}\int \mathrm{d}^2 z\Big(\frac{1}{2}\partial x^a \overline{\partial} x_a + p_\alpha \overline{\partial}\theta^\alpha + \widehat{p}_{\hat{\alpha}}\partial\widehat{\theta}^{\hat{\alpha}} - w_\alpha \overline{\partial}\lambda^\alpha - \widehat{w}_{\hat{\alpha}}\partial\widehat{\lambda}^{\hat{\alpha}}\Big) \tag{2.1}$$

where $(x^a, \theta^\alpha, \widehat{\theta}^{\hat{\alpha}})$ $(a = 0, \ldots, 9;\ \alpha, \hat{\alpha} = 1, \ldots, 16)$ are the standard type II superspace variables, $(p_\alpha, \widehat{p}_{\hat{\alpha}})$ are conjugate momenta of $(\theta^\alpha, \widehat{\theta}^{\hat{\alpha}})$, and the rest are "ghost" variables consisting of pure spinors $(\lambda^\alpha, \widehat{\lambda}^{\hat{\alpha}})$ and their conjugates $(w_\alpha, \widehat{w}_{\hat{\alpha}})$. As can be seen from the action, $(p_\alpha, \theta^\alpha, w_\alpha, \lambda^\alpha)$ are left moving (holomorphic) and $(\widehat{p}_{\hat{\alpha}}, \widehat{\theta}^{\hat{\alpha}}, \widehat{w}_{\hat{\alpha}}, \widehat{\lambda}^{\hat{\alpha}})$ are right moving (antiholomorphic), and $(x^a, \theta^\alpha, \widehat{\theta}^{\hat{\alpha}}, \lambda^\alpha, \widehat{\lambda}^{\hat{\alpha}})$ are all understood to carry conformal weight 0.

The left and right moving ghosts $\lambda^\alpha(z)$ and $\widehat{\lambda}^{\hat{\alpha}}(\overline{z})$ are subject to quadratic "pure spinor constraints" [34]

$$\lambda^\alpha \gamma^a_{\alpha\beta}\lambda^\beta(z) = 0, \quad \widehat{\lambda}^{\hat{\alpha}}\gamma^a_{\hat{\alpha}\hat{\beta}}\widehat{\lambda}^{\hat{\beta}}(\overline{z}) = 0 \tag{2.2}$$

and their conjugates $(w_\alpha, \widehat{w}_{\hat{\alpha}})$ are defined only up to "gauge transformations"

$$\delta_\Omega w_\alpha(z) = (\gamma^a\lambda)_\alpha \Omega_a(z), \quad \delta_\Omega \widehat{w}_{\hat{\alpha}}(\overline{z}) = (\gamma^a\widehat{\lambda})_{\hat{\alpha}}\widehat{\Omega}_a(\overline{z}). \tag{2.3}$$

The constraints of (2.2) seems to imply 10 constraints for each λ^α and $\widehat{\lambda}^{\hat{\alpha}}$, but actually one half of them is ineffective and a pure spinor has $16 - 5 = 11$ independent components. The ghost sector therefore is a collection of 11×2 bosonic $\beta\gamma$ systems of weight $(1, 0)$ and has $c = 22 \times 2$. Note that the value is exactly what one needs to compensate the central charge $c = (10 - 32) \times 2$ from the matter sector.

Because of the non-linear nature of the constraints of (2.2), the simplicity of the ghost action in (2.1) appears deceptive, but there is a nice formalism called the "theory of curved $\beta\gamma$ systems" (or the "theory of chiral differential operators") that can be used to rigorously define the first order systems on certain non-trivial spaces such as the pure spinor cone (2.2). For more on this, we refer the reader to the literature [35][36, 37, 38][11].

The other input to the formalism, the BRST operator, is given by

$$Q_{\mathrm{B}} = Q + \overline{Q}, \quad Q = \int \mathrm{d}z \lambda^\alpha d_\alpha(z), \quad \overline{Q} = \int \mathrm{d}\overline{z}\widehat{\lambda}^{\hat{\alpha}}\widehat{d}_{\hat{\alpha}}(\overline{z}) \tag{2.4}$$

[1] See appendix A for a summary of the notation.

where

$$d_\alpha = p_\alpha + (\gamma^a\theta)_\alpha(\partial x_a - \frac{1}{2}(\theta\gamma^a\partial\theta)), \quad \widehat{d}_{\hat\alpha} = \widehat{p}_{\hat\alpha} + (\gamma^a\widehat{\theta})_{\hat\alpha}(\partial x_a - \frac{1}{2}(\widehat{\theta}\gamma^a\partial\widehat{\theta})) \tag{2.5}$$

are left and right moving supersymmetric fermionic momenta satisfying simple operator product expansions

$$d_\alpha(z)d_\beta(w) = \frac{\gamma^a_{\alpha\beta}\Pi_a(w)}{z-w}, \quad \Pi^a = \partial x^a - \theta\gamma^a\partial\theta, \tag{2.6}$$

$$\widehat{d}_{\hat\alpha}(\overline{z})\widehat{d}_{\hat\beta}(\overline{w}) = \frac{\gamma^a_{\hat\alpha\hat\beta}\widehat{\Pi}_a(\overline{w})}{\overline{z}-\overline{w}}, \quad \widehat{\Pi}^a = \overline{\partial}x^a - \widehat{\theta}\gamma^a\overline{\partial}\widehat{\theta}. \tag{2.7}$$

Thanks to the pure spinor constraint (2.2), the BRST operator Q_{B} of (2.4) is nilpotent and it makes sense to talk of its cohomology. Q_{B} acts on operators via free field operator product expansions and physical states are found as cohomologies with ghost numbers $(1,1)$, where λ and $\widehat{\lambda}$ are defined to carry ghost numbers $(1,0)$ and $(0,1)$. Cohomologies at other ghost numbers are interpreted as spacetime ghosts and antifields. The cohomology has been rather thoroughly investigated and there is no doubt that it reproduces the well-known superstring spectrum in the trivial background.[2]

Of course, there have been attempts to explain how "natural" the BRST structure is. Works taking a conventional viewpoint have explained how the BRST structure arises from the classical Green-Schwarz superstring [16, 17]. In these approaches, pure spinor "ghosts" in the BRST operator are literally interpreted as the BRST ghosts for the kappa symmetry of the classical Green-Schwarz action. Less conventional (but potentially useful) interpretations of the BRST structure include its relation to the so-called superembedding formalism [39], and recent "twistorial" interpretation of Berkovits [40].

Note that the pure spinor formalism does not have the reparameterization bc ghosts as fundamental fields. However, one may define composite operators $b(z)$ and $\widehat{b}(\overline{z})$ that makes left and right moving stress tensors $T(z)$ and $\overline{T}(\overline{z})$ BRST trivial [12]:

$$Qb(z) = T(z), \quad \overline{Q}\widehat{b}(\overline{z}) = \overline{T}(\overline{z}), \quad Q\widehat{b}(\overline{z}) = \overline{Q}b(z) = 0. \tag{2.8}$$

Although one cannot define the c ghosts conjugate to b's, presence of b ghosts is just enough for defining higher-loop amplitudes [12], Siegel gauge vertex operators [41] etc.

At any rate, the combination of the free field action of (2.1) and the BRST symmetry of (2.4) is arguably much simpler than the classical Green-Schwarz formalism with the troublesome kappa symmetry, and the pure spinor formalism has been proved very useful for computing amplitudes in a flat spacetime (see e.g. [14, 15] and references therein).

[2] To be more precise, the theory of curved $\beta\gamma$ systems demands that the BRST operator be supplemented by a small extra term that takes care of fine global issues on the pure spinor space [38]. This modification is crucial for defining a composite b-ghost [12] and for correctly reproducing the higher massive spectrum [11].

2.2 Generic supergravity background

Since the pure spinor formalism is super-Poincaré covariant, it is straightforward to generalize the flat action of (2.1) to a non-linear sigma model describing a string propagating in a generic supergravity background [21].

Linearized coupling to a supergravity background is described by an integrated massless vertex operator. In the pure spinor formalism, this can be constructed from left-right products of supersymmetric currents $(\partial\theta^\alpha, \Pi^a, d_\alpha, N^{ab})$ and $(\overline{\partial}\widehat{\theta}^{\hat{\alpha}}, \overline{\Pi}^a, \widehat{d}_{\hat{\alpha}}, \widehat{N}^{ab})$ as

$$
\begin{aligned}
V = \frac{1}{2\pi\alpha'}\int \mathrm{d}^2 z \Big(& \partial\theta^\alpha \overline{\partial}\widehat{\theta}^{\hat{\beta}} A_{\alpha\hat{\beta}} + \partial\theta^\alpha \overline{\Pi}^b A_{\alpha b} + \Pi^a \overline{\partial}\widehat{\theta}^{\hat{\beta}} A_{a\hat{\beta}} + \Pi^a \overline{\Pi}^b A_{ab} \\
& + d_\alpha (\overline{\partial}\widehat{\theta}^{\hat{\beta}} E^\alpha_{\hat{\beta}} + \overline{\Pi}^b E^\alpha_b) + \widehat{d}_{\hat{\alpha}} (\partial\theta^\beta E^{\hat{\alpha}}_\beta + \Pi^b E^{\hat{\alpha}}_b) \\
& + \frac{1}{2} N^{ab} (\overline{\partial}\widehat{\theta}^{\hat{\gamma}} \Omega_{ab\hat{\gamma}} + \overline{\Pi}^c \Omega_{abc}) + \frac{1}{2} \widehat{N}^{ab} (\partial\theta^\gamma \widehat{\Omega}_{ab\gamma} + \Pi^c \widehat{\Omega}_{abc}) \\
& + d_\alpha \widehat{d}_{\hat{\beta}} P^{\alpha\hat{\beta}} + N^{ab} \hat{d}_{\hat{\gamma}} C^{\hat{\gamma}}_{ab} + d_\alpha \widehat{N}^{cd} \widehat{C}^\alpha_{cd} + \frac{1}{4} N^{ab} \widehat{N}^{cd} R_{abcd} \Big)
\end{aligned}
\tag{2.9}
$$

where

$$
\begin{aligned}
& A_{\alpha\hat{\beta}},\ A_{\alpha b},\ A_{a\hat{\beta}},\ A_{ab},\ E^\alpha_{\hat{\beta}},\ E^\alpha_b,\ E^{\hat{\alpha}}_\beta,\ E^{\hat{\alpha}}_b,\ \Omega_{ab\hat{\gamma}},\ \Omega_{abc},\ \widehat{\Omega}_{ab\gamma},\ \widehat{\Omega}_{abc}, \\
& P^{\alpha\hat{\beta}},\ C^{\hat{\gamma}}_{ab},\ \widehat{C}^\alpha_{cd},\ R_{abcd}
\end{aligned}
\tag{2.10}
$$

are superfields (functions of the zero-modes of $(x^a, \theta^\alpha, \widehat{\theta}^{\hat{\alpha}})$) representing fluctuations of type IIB supergravity. Physical state condition and gauge invariance for integrated vertex operators are given by $QV = \overline{Q}V = 0$ and $\delta_{\Lambda,\Lambda'} V = Q\Lambda + \overline{Q}\Lambda'$ and these indeed imply linearized equations of motion and gauge invariances for the superfields of (2.10) [21]. For example, the superpotential $A_{\alpha\hat{\beta}}$ of lowest dimension is found to satisfy the correct constraints and gauge invariances

$$
(\gamma_{abcde})^{\alpha\beta} D_\alpha A_{\beta\hat{\beta}} = (\gamma_{abcde})^{\hat{\alpha}\hat{\beta}} \widehat{D}_{\hat{\alpha}} A_{\beta\hat{\beta}} = 0 \tag{2.11}
$$

$$
\delta_{\Lambda,\Lambda'} A_{\alpha\hat{\alpha}} = D_\alpha \Lambda_{\hat{\alpha}} + \widehat{D}_{\hat{\alpha}} \Lambda'_\alpha \tag{2.12}
$$

where

$$
D_\alpha = \partial_\alpha - (\gamma^a\theta)_\alpha \partial_a, \quad \widehat{D}_{\hat{\alpha}} = \partial_{\hat{\alpha}} - (\gamma^a\widehat{\theta})_{\hat{\alpha}} \partial_a \tag{2.13}
$$

are the supercovariant derivatives of type IIB superspace. Other superfields of higher dimensions can be constructed from $A_{\alpha\hat{\alpha}}$ and $(D_\alpha, \widehat{D}_{\hat{\alpha}})$.

To construct a non-linear action whose linearization gives the vertex operator of (2.9), one covariantizes as usual $S_{\text{flat}} + V$ with respect the target space reparameterization by introducing the supervielbein E_M^A $(M = (m, \mu, \hat{\mu}),\ A = (a, \alpha, \hat{\alpha}))$ and the curved spacetime coordinate

$Z^M = (x^m, \theta^\mu, \widehat{\theta}^{\hat{\mu}})$:

$$S = \frac{1}{\pi\alpha'} \int \mathrm{d}^2 z \Big(\frac{1}{2}(G_{MN} + B_{MN}) \partial Z^M \overline{\partial} Z^N \\ + d_\alpha \overline{\partial} Z^M E_M^\alpha + \partial Z^M \widehat{d}_{\hat{\alpha}} E_M^{\hat{\alpha}} + d_\alpha \widehat{d}_{\hat{\beta}} P^{\alpha\hat{\beta}} \\ + (w_\alpha \overline{\partial} \lambda^\alpha + \frac{1}{2} \overline{\partial} Z^M N_{ab} \Omega_M^{ab}) + (\widehat{w}_{\hat{\alpha}} \partial \widehat{\lambda}^{\hat{\alpha}} + \frac{1}{2} \partial Z^M \widehat{N}_{ab} \widehat{\Omega}_M^{ab}) \\ + d_\alpha \widehat{N}_{ab} C^{\alpha,ab} + \widehat{d}_{\hat{\beta}} N_{ab} \widehat{C}^{\hat{\beta},ab} + \frac{1}{4} N^{ab} \widehat{N}^{cd} R_{abcd} \Big). \tag{2.14}$$

First line is just the standard non-linear sigma model of the Green-Schwarz formalism in a conformal gauge, where the term with $G_{MN} = \eta_{ab} E_M^a E_N^b$ is the kinetic term and the one with B_{MN} is the Wess-Zumino term (possibly with an integration over an extra dimension). It is useful to remember that $P^{\alpha\hat{\alpha}}$ is a superfield whose lowest component is the Ramond-Ramond fieldstrength, and R_{abcd} is a superfield whose lowest component is the spacetime curvature.

The BRST operator is still given by the expression of the form (2.4), but its action on fields is defined via commutation relations between $(Z^M, \lambda^\alpha, \widehat{\lambda}^\alpha)$ and their canonical conjugates. Conditions for this definition to make sense, namely the conservation of the BRST currents $\overline{\partial}(\lambda^\alpha d_\alpha) = \partial(\widehat{\lambda}^{\hat{\alpha}} \widehat{d}_{\hat{\alpha}}) = 0$ and nilpotency of the BRST charge, actually imply supergravity equations of motion for the background superfields [21]. Since requiring the kappa symmetry in a generic supergravity puts the background superfields on-shell in the Green-Schwarz formalism [19][20], this is consistent with the expectation that the kappa symmetry is replaced by the BRST symmetry in the pure spinor formalism.

Also, note that the action of (2.14) can be checked to be BRST invariant if the first line (the "Green-Schwarz part") is assumed to be kappa symmetric [42]. This is not entirely obvious and means that a Green-Schwarz action in any supergravity background can be consistently extended to a pure spinor action of the form (2.14). This observation, on the other hand, does not explain the equivalence of the two formalisms even at a classical level.

When the Ramond-Ramond superfield $P^{\alpha\hat{\alpha}}$ is invertible as a 16×16 matrix, $(d_\alpha, \widehat{d}_{\hat{\beta}})$ becomes auxiliary and the action (2.14) can be simplified to

$$S = \frac{1}{\pi\alpha'} \int \mathrm{d}^2 z \Big(\frac{1}{2}(G_{MN} + B_{MN}) \partial Z^M \overline{\partial} Z^N \\ + (w_\alpha \overline{\nabla} \lambda^\alpha + \frac{1}{2} \overline{\partial} Z^M N_{ab} \Omega_M^{ab}) + (\widehat{w}_{\hat{\alpha}} \partial \widehat{\lambda}^{\hat{\alpha}} + \frac{1}{2} \partial Z^M \widehat{N}_{ab} \widehat{\Omega}_M^{ab}) + \frac{1}{4} N^{ab} \widehat{N}^{cd} R_{abcd} \Big) \tag{2.15}$$

for some shifted background superfields. The action (2.15) still has a BRST symmetry and the corresponding charge reads

$$Q_{\mathrm{B}} = \int \mathrm{d}z \lambda^\alpha \partial Z_M E_\alpha^M + \int \mathrm{d}\overline{z} \widehat{\lambda}^{\hat{\alpha}} \overline{\partial} Z_M E_{\hat{\alpha}}^M. \tag{2.16}$$

It is this form of the action that we shall be using in our analysis of strings in an $AdS_5 \times S^5$ background, since the Ramond-Ramond flux is non-degenerate (and constant) in the background.

2.3 $AdS_5 \times S^5$ background

For a maximally supersymmetric $AdS_5 \times S^5$ background with constant Ramond-Ramond flux, one may use the Metsaev-Tseytlin construction [24] to explicitly write down the background superfields in the action of (2.15) [8]. A reason why it works is that an appropriate superspace can be written as a supercoset of the form $G/H = PSU(2,2|4)/(SO(4,1) \times SO(5))$.

2.3.1 Metsaev-Tseytlin coset construction of Green-Schwarz action for $AdS_5 \times S^5$

The basic building block for the Metsaev-Tseytlin coset construction is the left invariant Maurer-Cartan 1-form $\widetilde{J} = \widetilde{g}^{-1}\mathrm{d}\widetilde{g}$ ($\widetilde{g} \in G$) on G, or more precisely its pull-back to G/H via a section $g\colon G/H \to G$:

$$J = g^{-1}\mathrm{d}g. \tag{2.17}$$

To construct an action on the coset G/H using J, an H gauge invariance shall be introduced to make the choice of the section g irrelevant. For an application to the $AdS_5 \times S^5$ superstring relevant groups are $G = PSU(2,2|4)$ and $H = SO(4,1) \times SO(5)$ and J takes values in the Lie algebra $\mathfrak{g} = \mathfrak{psu}(2,2|4)$.[3]

If one regards $g = g(\tau, \sigma)$ as a function on a worldsheet with values in the section $G/H \subset G$, the 1-form J becomes a current on the worldsheet. The Maurer-Cartan equation can then be pulled back to the worldsheet and it implies that J satisfies

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0 \tag{2.18}$$

where $J_\pm = \frac{1}{2}(J_\tau \pm J_\sigma)$ are lightcone components of the current J.

The current J carries a local H action and a global G action that are inherited from the section $g\colon G/H \to G$. Namely, under a local H transformation of g defined by

$$g \to gh(\tau,\sigma), \quad h = h(\tau,\sigma) \in H \tag{2.19}$$

J transforms as

$$J \to h^{-1}\mathrm{d}h + h^{-1}Jh \tag{2.20}$$

and under a global G transformation of g defined by

$$g(x) \to g(xa) = ag(x)h(a;\tau,\sigma)^{-1}, \quad x \in G/H,\ a \in G,\ h(a;\tau,\sigma) \in H \tag{2.21}$$

J transforms as

$$J \to hJh^{-1} - (\mathrm{d}h)h^{-1}. \tag{2.22}$$

So, J is invariant under the global G transformation up to a compensating H gauge transformation.

[3] See appendix A for our conventions for $\mathfrak{psu}(2,2|4)$.

For the case at hand, Lie algebra $\mathfrak{g}$ of $G = PSU(2,2|4)$ admits a $\mathbb{Z}_4$ grading,

$$\mathfrak{g} = \bigoplus_{i=0}^{3} \mathfrak{g}^i, \quad [\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}, \quad i,j \in \mathbb{Z}_4 \tag{2.23}$$

and the degree zero piece $\mathfrak{g}^0$ is nothing but the Lie algebra of the denominator $H = SO(4,1) \times SO(5)$. Hence, if one decomposes the Metsaev-Tseytlin current by the $\mathbb{Z}_4$ grading as

$$\begin{aligned} J = J^A T_A = J^0 + J^1 + J^2 + J^3, \quad J^i \in \mathfrak{g}^i, \\ J^0 = J^{ab} L_{ab}, \quad J^1 = J^a Q_\alpha, \quad J^2 = J^a P_a, \quad J^3 = J^{\hat{\alpha}} Q_{\hat{\alpha}} \end{aligned} \tag{2.24}$$

the local H transformations can be refined as

$$J^0 \to h^{-1}\mathrm{d}h + h^{-1}J^0 h \quad \text{and} \quad J^i \to h^{-1}J^i h \quad (i = 1,2,3). \tag{2.25}$$

This refinement facilitates the construction of a G-invariant action on a supercoset G/H, just like in the case of a symmetric coset space.

Since the currents J^i $(i = 1,2,3)$ transforms homogeneously under the H gauge transformation of (2.19) an action of the form

$$\int \mathrm{d}^2\sigma \,\mathrm{str}\Big(\frac{1}{2}J^2_+ J^2_- + aJ^1_+ J^3_- + bJ^3_+ J^1_-\Big) \tag{2.26}$$

for any constants a, b is invariant under the global G action of (2.21) and the local H action of (2.19). However, the coset G/H has 32 (too many) fermionic dimensions and one does not expect (2.26) to describe a superstring except perhaps at some special values of (a, b). Just as in a flat superspace, to construct a superstring model using the coset action of (2.26), one has to kill a half of fermionic coordinates either by introducing a fermionic local symmetry (kappa symmetry) [7], or by coupling it to appropriate bosonic ghosts (like pure spinors) [8]. Remarkably, both can be done.

In the works of Metsaev and Tseytlin [24] and Berkovits et al. [25], it was found that a kappa symmetric Green-Schwarz action in a conformal gauge can indeed be written in the form (2.26) and is essentially unique $(a = -b = \pm 1/4)$:

$$S_{\mathrm{GS}} = \frac{R^2}{\pi\alpha'} \int \mathrm{d}^2\sigma \,\mathrm{str}\Big(\frac{1}{2}J^2_+ J^2_- - \frac{1}{4}(J^1_+ J^3_- - J^3_+ J^1_-)\Big). \tag{2.27}$$

That the Wess-Zumino term can be written as an integration over the two dimensional worldsheet follows from the fact that $\mathfrak{psu}(2,2|4)$ admits a $\mathbb{Z}_4$ automorphism [25]. The "radius" parameter R is related to the number N of D3-branes that source the Ramond-Ramond flux supporting $AdS_5 \times S^5$, but the integrality of N cannot be probed by an elementary string. From now on we set the radius R in the unit of $\sqrt{\alpha'}$ to be one. In the context of the AdS/CFT correspondence, the semiclassical parameter α' then is related to the 't Hooft coupling λ of the $\mathcal{N} = 4$ super Yang-Mills theory as $\alpha' \sim 1/\sqrt{\lambda}$.

Since the Green-Schwarz action of (2.27) is written in a conformal gauge, it is understood to be accompanied by Virasoro constraints

$$T = \frac{1}{2\alpha'}\,\mathrm{str}(J^2_+ J^2_+) \approx 0, \quad \bar{T} = \frac{1}{2\alpha'}\,\mathrm{str}(J^2_- J^2_-) \approx 0. \tag{2.28}$$

Note that the second term of (2.27) is a topological Wess-Zumino term (i.e. does not couple to worldsheet metric) and hence does not contribute to the stress tensors. However, the Green-Schwarz action have more constraints than the Virasoro constraints of (2.28) and separation of the first and second class constraints makes it more natural to improve the naive Virasoro constraints so that they become first class. The improved Virasoro constraints are then closely related to the stress tensor of the pure spinor formalism.

2.3.2 Pure spinor action for $AdS_5 \times S^5$

In subsection 2.2 we explained a relation between Green-Schwarz action and pure spinor action in an arbitrary supergravity background. One can find a pure spinor action in an $AdS_5 \times S^5$ background by applying the argument to the Metsaev-Tseytlin action. In the "second order" form it reads[4]

$$\begin{aligned} S &= S_{\mathrm{GS}} + \frac{1}{\pi\alpha'}\int \mathrm{d}^2\sigma\, \mathrm{str}\Big(J^3_+ J^1_-\Big) + S_{\mathrm{gh}} && (2.29)\\ &= \frac{1}{\pi\alpha'}\int \mathrm{d}^2\sigma\, \mathrm{str}\Big(\frac{1}{2}J^2_+ J^2_- + \frac{1}{4}J^1_+ J^3_- + \frac{3}{4}J^3_+ J^1_-\Big) + S_{\mathrm{gh}}. && (2.30)\end{aligned}$$

where

$$\begin{aligned} &S_{\mathrm{gh}} = \frac{1}{\pi\alpha'}\int \mathrm{d}^2\sigma\, \mathrm{str}\Big(w^3[D_-,\lambda^1] + \widehat{w}^1[D_+,\widehat{\lambda}^3] - N\widehat{N}\Big),\\ &(D_\pm = [\partial_\pm + J^0_\pm,\ \cdot\ ]) \end{aligned} \qquad (2.31)$$

describes the contribution of pure spinor ghosts and their coupling to the "matter" sector. The second term in (2.29) comes from integrating out the auxiliary fields $(d_\alpha, \widehat{d}_{\hat{\alpha}})$ as explained at the end of subsection 2.2. In the "ghost" action S_{gh} (2.31) we have introduced pure spinor variables as supermatrices

$$\lambda^1 = \lambda^\alpha T_\alpha \in \mathfrak{g}^1, \quad \widehat{\lambda}^3 = \widehat{\lambda}^{\hat{\alpha}} T_{\hat{\alpha}} \in \mathfrak{g}^3 \qquad (2.32)$$

satisfying $SO(4,1) \times SO(5)$ pure spinor constraint

$$\{\lambda^1, \lambda^1\} = \lambda^\alpha \gamma^a_{\alpha\beta}\lambda^\beta = 0, \quad \{\widehat{\lambda}^3, \widehat{\lambda}^3\} = \widehat{\lambda}^{\hat{\alpha}}\gamma^a_{\hat{\alpha}\hat{\beta}}\widehat{\lambda}^{\hat{\beta}} = 0. \qquad (2.33)$$

Since the pure spinor ghosts are bosonic, supermatrices λ^1 and $\widehat{\lambda}^3$ have a wrong Grassmann parity. We have also introduced the conjugates to λ^1 and $\widehat{\lambda}^3$

$$w^3 = \eta^{\alpha\hat{\alpha}} w_\alpha T_{\hat{\alpha}} \in \mathfrak{g}^3, \quad \widehat{w}^1 = \eta^{\hat{\alpha}\alpha}\widehat{w}_{\hat{\alpha}} T_\alpha \in \mathfrak{g}^1 \qquad (2.34)$$

and Lorentz $(SO(4,1) \times SO(5))$ generators of the pure spinor sector

$$N = -\{w^3, \lambda^1\}, \quad \widehat{N} = -\{\widehat{w}^1, \widehat{\lambda}^3\}. \qquad (2.35)$$

[4]Here, we have judiciously used the opposite sign for the Wess-Zumino term in S_{GS} with respect to the one given in (2.27) because it is the variables in (2.30) that have a simple relation to the Green-Schwarz variables of (2.27). Otherwise the relation between the variables of the two formalisms gets twisted by an automorphism $\mathfrak{g}^1 \leftrightarrow \mathfrak{g}^3$ of $\mathfrak{psu}(2,2|4)$. Of course, this is a matter of convention but we find it prettier this way.

Note that the matter sector of pure spinor superstring action (2.30) is not kappa symmetric since Green-Schwarz action of (2.27) is the unique such action. Another important difference is that the pure spinor action is not accompanied by Virasoro constraints even though it is written in a "conformal gauge". In pure spinor formalism, both the kappa symmetry and the Virasoro constraint are replaced by a BRST symmetry.

2.4 $PSU(2,2|4)$ symmetry and Noether current

The local $H = SO(4,1) \times SO(5)$ transformation of (2.19) and the global $G = PSU(2,2|4)$ transformation of (2.21) can be extended to the pure spinor sector in a way that the action is invariant. The coupling of pure spinors to the connection J^0 implies that the former is

$$g \to gh(\tau,\sigma), \quad (w,\lambda,\widehat{w},\widehat{\lambda}) \to h(\tau,\sigma)^{-1}(w,\lambda,\widehat{w},\widehat{\lambda})h(\tau,\sigma), \quad h(\tau,\sigma) \in H \tag{2.36}$$

and the latter is

$$g \to agh(a;\tau,\sigma)^{-1}, \quad (w,\lambda,\widehat{w},\widehat{\lambda}) \to h(a;\tau,\sigma)(w,\lambda,\widehat{w},\widehat{\lambda})h(a;\tau,\sigma)^{-1}, \quad a \in G,\ h(a;\tau,\sigma) \in H. \tag{2.37}$$

The Noether current associated with the $PSU(2,2|4)$ symmetry can be computed in a standard manner, and is given by

$$\begin{aligned} &j = (j_+, j_-) = j^A T_A \in \mathfrak{psu}(2,2|4), \\ &j_+ = g(J_+^2 + \frac{1}{2}J_+^1 + \frac{3}{2}J_+^3 + 2N)g^{-1}, \quad j_- = g(J_-^2 + \frac{3}{2}J_-^1 + \frac{1}{2}J_-^3 + 2\widehat{N})g^{-1}. \end{aligned} \tag{2.38}$$

The normalization of j here is such that the corresponding conserved charge is given by $\frac{1}{4\pi\alpha'}\int d\sigma j_\tau^A$. Individual components for each $\mathfrak{psu}(2,2|4)$ generator can be extracted as

$$j^A = \eta^{AB}\,\mathrm{str}(T_B\, j) \tag{2.39}$$

where η^{AB} is the inverse of the trace metric $\eta_{AB} = \mathrm{str}(T_A T_B)$. Of particular importance for us is the components for $T_0, T_9 \in \mathfrak{g}^2$. Conserved charges associated with them are the AdS energy and an angular momentum in S^5

$$E = \frac{1}{4\pi\alpha'}\int d\sigma j_\tau^0, \quad J = \frac{1}{4\pi\alpha'}\int d\sigma j_\tau^9. \tag{2.40}$$

2.5 BRST symmetry, composite b-ghost and stress tensor

The pure spinor action of (2.30) is invariant under an on-shell BRST transformation defined by[5]

$$\delta_B g = g(\lambda^1 + \widehat{\lambda}^3), \quad \delta_B w^3 = -J_+^3, \quad \delta_B \widehat{w}^1 = -J_-^1, \quad \delta_B \lambda^1 = \delta_B \widehat{\lambda}^3 = 0. \tag{2.41}$$

[5] The BRST symmetry can be promoted to an off-shell symmetry by adding some auxiliary fields [43, 44].

On Metsaev-Tseytlin currents, it acts as

$$\delta_{\rm B} J^0 = [J^3,\lambda^1] + [J^1,\widehat{\lambda}^3], \quad \delta_{\rm B} J^1 = [D,\lambda^1] + [J^2,\widehat{\lambda}^3], \tag{2.42}$$

$$\delta_{\rm B} J^2 = [J^1,\lambda^1] + [J^3,\widehat{\lambda}^3], \quad \delta_{\rm B} J^3 = [D,\widehat{\lambda}^3] + [J^2,\lambda^1]. \tag{2.43}$$

Associated BRST charge can be written as a sum of left-moving and right-moving components

$$Q_{\rm B} = Q + \bar{Q}, \quad Q = \int {\rm d}\sigma^+ \,{\rm str}(\lambda^1 J^3_+), \quad \bar{Q} = \int {\rm d}\sigma^- \,{\rm str}(\widehat{\lambda}^3 J^1_-) \tag{2.44}$$

where $\partial_- \,{\rm str}(\lambda^1 J^3_+) = \partial_+ \,{\rm str}(\widehat{\lambda}^3 J^1_-) = 0$ because of the equations of motion.

In any BRST formulation of string theory, it is crucial to have b ghost fields that make stress tensors BRST trivial as in $\{Q_{\rm B}, b\} = T$, $\{Q_{\rm B}, \widehat{b}\} = \bar{T}$. Since the stress tensors

$$T = \frac{1}{\alpha'}{\rm str}(\frac{1}{2}J^2_+ J^2_+ + J^1_+ J^3_+ + w^3[D_+,\lambda^1]), \quad \bar{T} = \frac{1}{\alpha'}{\rm str}(\frac{1}{2}J^2_- J^2_- + J^1_- J^3_- + \widehat{w}^1[D_-,\widehat{\lambda}^3]) \tag{2.45}$$

carry ghost number $(0,0)$ while Q and $\bar{Q}$ carry ghost numbers $(1,0)$ and $(0,1)$, one needs operators of negative ghost numbers to construct the b ghosts. In an $AdS_5 \times S^5$ background $(\lambda\widehat{\lambda}) \equiv {\rm str}(\lambda^1\widehat{\lambda}^3)$ is in the cohomology of $Q_{\rm B}$, and it has been argued that it is consistent to allow inverse powers of $(\lambda\widehat{\lambda})$ [45]. One can utilize this observation to construct composite b ghosts with negative ghost numbers $(-1,0)$ and $(0,-1)$ as [45, 46]

$$\begin{aligned} b &= \frac{1}{\alpha'}{\rm str}\Big(\frac{\widehat{\lambda}^3[J^2_+, J^3_+]}{(\lambda\widehat{\lambda})} - w^3 J^1_+ + \frac{\{w^3,\widehat{\lambda}^3\}[\lambda^1, J^1_+]}{(\lambda\widehat{\lambda})}\Big), \\ \widehat{b} &= \frac{1}{\alpha'}{\rm str}\Big(\frac{\lambda^1[J^2_-, J^1_-]}{(\lambda\widehat{\lambda})} - \widehat{w}^1 J^3_- + \frac{\{\widehat{w}^1,\lambda^1\}[\widehat{\lambda}^3, J^3_-]}{(\lambda\widehat{\lambda})}\Big) \end{aligned} \tag{2.46}$$

and it can be checked that these satisfy

$$\{Q, b\} = T, \quad \{\bar{Q}, \widehat{b}\} = \bar{T}, \quad \{Q, \widehat{b}\} = \{\bar{Q}, b\} = 0. \tag{2.47}$$

Note that b and $\widehat{b}$ are actually invariant under $\delta_\Omega w^3 = \{\Omega^2, \lambda^1\}$ and $\delta_\Omega \widehat{w}^1 = \{\Omega^2, \widehat{\lambda}^3\}$ for an arbitrary operator Ω^2 and that, although b is not purely left-moving and $\widehat{b}$ is not purely right-moving, $\partial_- b$ and $\partial_+ \widehat{b}$ are BRST trivial [46].

A remark is in order. The action of (2.30) can be naively coupled to worldsheet gravity and the stress tensor of (2.45) are the ones that one would obtain from this coupling. However, as mentioned earlier, the action of (2.30) should not be regarded as arising from gauge fixing this naive reparameterization invariant action, for that would imply that the stress tensor is a constraint. If one wishes to start from a reparameterization invariant action, the correct starting point should rather be the classical Green-Schwarz action. Studies along this line in a flat background tell us that the pure spinor variables arise as bosonic ghosts for the kappa symmetry, and that one should think of the fundamental bc-ghosts to be "integrated out" from the theory, effectively getting replaced by one of the pure spinor constraints [10, 17].

2.6 Classical equations of motion

Equations of motion for both Green-Schwarz and pure spinor superstrings can be readily computed from their actions (2.27) and (2.30).

Green-Schwarz Classical equations of motion for the Green-Schwarz superstring in an $AdS_5 \times S^5$ background is well known. In a conformal gauge they read

$$[D_-, J^2_+] + [J^1_-, J^1_+] = 0, \qquad [D_+, J^2_-] + [J^3_+, J^3_-] = 0, \tag{2.48}$$

$$[J^2_-, J^3_+] = 0, \qquad [J^2_+, J^1_-] = 0 \tag{2.49}$$

where, as before, the spin covariant derivatives are defined as $D_\pm = \partial_\pm + [J^0_\pm, \,\cdot\,]$. These are understood to be supplemented by the Maurer-Cartan equations

$$\partial_+ J^i_- - \partial_- J^i_+ + \sum_{j+k=i} [J^j_+, J^k_-] = 0, \qquad (i \in \mathbb{Z}_4) \tag{2.50}$$

and by the Virasoro constraint coming from a choice of the conformal gauge

$$\mathrm{str}(J^2_+ J^2_+) = \mathrm{str}(J^2_- J^2_-) = 0. \tag{2.51}$$

Pure spinor The currents from the matter sector of the pure spinor formalism satisfy the same set of Maurer-Cartan equations as the ones in the Green-Schwarz formalism, but their equations of motion are different:

$$[D_- - \widehat{N}, J^2_+] + [J^1_-, J^1_+] = [J^2_-, N], \qquad [D_+ - N, J^2_-] + [J^3_+, J^3_-] = [J^2_+, \widehat{N}], \tag{2.52}$$

$$[D_- - \widehat{N}, J^3_+] = [J^3_-, N], \qquad [D_+ - N, J^1_-] = [J^1_+, \widehat{N}]. \tag{2.53}$$

If one ignores ghost contributions, the equations of motion for the bosonic current $J^2_\pm$ reduce to that of the Green-Schwarz formalism. On the other hand, the equations of motion for the fermionic currents $J^1_\pm$ and $J^3_\pm$ take the forms of covariant constancy conditions even after dropping the ghost contributions and do not reduce to the "algebraic" equations of motions of the Green-Schwarz formalism.

Equations of motion for the pure spinor ghost variables are

$$[D_- - \widehat{N}, \lambda^1] = 0, \qquad [D_+ - N, \widehat{\lambda}^3] = 0, \tag{2.54}$$

$$[D_- - \widehat{N}, w^3] = 0, \qquad [D_+ - N, \widehat{w}^1] = 0. \tag{2.55}$$

The equations for $(w^3, \widehat{w}^1)$ can be replaced by that for the gauge invariant Lorentz currents

$$[D_- - \widehat{N}, N] = 0, \qquad [D_+ - N, \widehat{N}] = 0. \tag{2.56}$$

Unlike in the Green-Schwarz formalism, the Virasoro condition is not a part of the equations of motion. Nevertheless, in a semiclassical setup, it is still true that the "classical solution" around which one studies small fluctuations should have vanishing worldsheet energy and momentum $(L_0 \pm \overline{L}_0)$, since the Virasoro currents T and $\overline{T}$ are BRST exact.

3 Semiclassical pure spinor superstring in $AdS_5 \times S^5$ background

We now turn to the main topic of the present article. Our primary goal is to explain the reason why the one-loop correction to classical string energy computed using the pure spinor formalism agrees with that from the Green-Schwarz formalism. For simplicity, we shall restrict ourselves to a simple family of classical solutions (defined in section 3.3), but we believe that the pattern that connects the two formalisms stay the same for a broader class of solutions.

The structure of our argument is as follows. After developing some semiclassical formulas for the pure spinor superstring around a generic classical solution, we show that, for a certain class of solutions, the one-loop correction to spacetime energy comes entirely from the zero-point "energy" of worldsheet fluctuations. The zero-point "energy" is the normal ordering constant in the Hamiltonian of quadratic fluctuations, and can be computed from the one-loop partition function on the worldsheet. To argue that the one-loop partition functions of Green-Schwarz and pure spinor formalisms agree, we analyze the equations of motion for fluctuations of the latter and identify Green-Schwarz like degrees of freedom. Morally speaking, those degrees of freedom are related to the BRST cohomology of fluctuations and yield the same zero-point "energy" as the Green-Schwarz fluctuations. The remaining degrees of freedom, which are decoupled from the Green-Schwarz like ones, have a trivial partition function and do not contribute to the zero-point "energy".

3.1 Comparison of semiclassical analyses for Green-Schwarz and pure spinor formalisms

As we have reviewed in the previous section, compared to the Green-Schwarz formalism, the pure spinor formalism has an extended set of fields and the Virasoro and kappa symmetries are replaced by a BRST symmetry. To compare semiclassical analyses in Green-Schwarz and pure spinor formalisms, one has to identify classical solutions of both sides and compare the structure of small fluctuations around them.

From the forms of classical equations of motion (subsection 2.6), one finds that a purely bosonic solution of the Green-Schwarz formalism is automatically a solution of the pure spinor formalism (with a trivial ghost profile). However, it is not clear if all classical solutions of the pure spinor formalism can be obtained in this way. In this article, we shall leave the complete comparison of the space of classical solutions along the line of [47] as an interesting open question.

So in the discussion that follows, we pick a solution of the Green-Schwarz formalism and regard it as the solution of the pure spinor formalism describing the same classical string.

Since the Green-Schwarz action in a conformal gauge comes with Virasoro and kappa symmetries, fluctuations around a classical solution have to respect certain constraints. The presence of the kappa symmetry manifests itself in the semiclassical analysis as a degeneracy of fermionic propagators. Namely, one half of the fermionic fluctuations does not propagate and one may simply freeze these fluctuations to deal with the kappa symmetry. The Virasoro constraint implies that two of ten bosonic fluctuations are functionals of others, and normally the two fluctuations are removed by either imposing a lightcone gauge or a static gauge condition.

After properly dealing with the constraints, one may in principle quantize the quadratic fluctuations and compute semiclassical quantities. The classical solution is identified with the ground state $|\Omega\rangle$ of the worldsheet Hamiltonian H_2 for the quadratic fluctuations, and a semiclassical correction to the spacetime energy of the solution can be computed as

$$\Delta E(\Omega) = \langle\Omega|(E - \underline{E})|\Omega\rangle. \tag{3.1}$$

Here, E on the right hand side is the Noether charge for the AdS time translation written in terms of fluctuations and $\underline{E}$ denotes its classical value. For the class of solutions defined in section 3.3, this quantity can be related to the expectation value of the worldsheet Hamiltonian H_2 by imposing Virasoro constraint on fluctuations [6]. This is a good fortune because one can bypass the explicit quantization of fluctuations when computing $\Delta E(\Omega)$.

As an aside, let us mention that one may ignore the fluctuations of Goldstone modes to the one-loop approximation and that a quantum state $|\Psi\rangle$ with some excitations over $|\Omega\rangle$ represents a string state with slightly higher energy. Quantization of Goldstone modes is interesting (this should turn the ground state to a multiplet of spontaneously broken global symmetries), and is certainly important for two-loops and beyond. We, however, do not inquire into these issues in this article.

In the pure spinor formalism, the procedure for the semiclassical analysis is similar but now the Virasoro and kappa symmetries are replaced by a BRST symmetry.

When performing a semiclassical analysis for a BRST system in general, it is useful to keep the following geometric picture in mind (cf. [48]). Presence of a (on-shell) nilpotent BRST symmetry implies that a critical point of the action in the space of fields belongs either to a trivial orbit (BRST singlet) or a non-trivial orbit with zero volume (BRST doublet). A "classical solution" around which one performs a semiclassical analysis has to be a solution to the equations of motion and at the same time a BRST singlet. When a solution is a BRST singlet, the BRST symmetry induces a nilpotent action on fluctuations around the solution. So one gets a new BRST system of fluctuations and the ground state $|\Omega\rangle$ and excited states $|\Psi\rangle$ are defined as BRST cohomologies. Semiclassical quantization of fluctuations of a BRST system around a "classical solution" is conceptually simpler than that of a gauge invariant system because all the problems with degenerate phase space of the latter are already taken care of by the BRST symmetry.

Coming back to the relation between Green-Schwarz and pure spinor formalisms, one expects that a quantum state $|\Psi\rangle$ of the former can be mapped to a BRST cohomology class of the latter. This mapping should allow one to directly compare the one-loop corrections $\Delta E(\Psi) = \langle\Psi|(E - \underline{E})|\Psi\rangle$ in the two formalisms. Unfortunately, however, it is not necessarily easy to show the equivalence in this way, just because quantization of fluctuations around a given classical solution could be too hard. In general, both kinetic and mass terms are not constant and moreover have complicated mixing, so quantization is not easy even for the lightcone Green-Schwarz formalism.

But if one is mainly interested in comparing one-loop corrections $\Delta E(\Omega)$ to the energies of the classical solution, explicit quantization can be sometimes circumvented. As mentioned above, there is a family of classical solutions for which one-loop energy corrections are related to expectation values of their worldsheet Hamiltonians H_2, both in Green-Schwarz and pure spinor formalisms. Then, the equivalence of the two formalisms (as far as $\Delta E(\Omega)$ is concerned) is reduced to a simpler problem of comparing one-loop partition functions. In subsections 3.5 and 3.6 we study equations of motions for fluctuations in Green-Schwarz and pure spinor formalisms and argue that their one-loop partition functions around the classical solutions of subsection 3.3 do agree.

3.2 Quadratic fluctuations

Computations of semiclassical quantities can be done by using a background field method. For a sigma model on a group manifold, a convenient way to separate the worldsheet variable $g(\tau, \sigma)$ to its background value $\underline{g}(\tau, \sigma)$ and small fluctuations $X(\tau, \sigma) \in \mathfrak{g}$ around it is as

$$g = \underline{g} \mathrm{e}^X . \tag{3.2}$$

To perform a consistent semiclassical analysis, X is understood to be a quantity of order $\sqrt{\alpha'}$. When the sigma model is on a coset G/H, $\underline{g}$ is a coset representative and the small fluctuation X takes values in a subspace of $\mathfrak{g}$. Identification (3.2) may require a compensating H gauge transformation which, however, is irrelevant for gauge invariant quantities like action. For the case at hand, the fluctuation X can be split according to the $\mathbb{Z}_4$ grading of $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$ and we choose it to have the components orthogonal to $\mathfrak{g}^0 (= \mathfrak{h})$:

$$X = \bigoplus_{i=1}^{3} X^i , \quad X^i \in \mathfrak{g}^i . \tag{3.3}$$

For simplicity, we assume the background to be purely bosonic and ghost free (i.e. no background values for the fermionic currents (J^1, J^3) and the ghosts).

3.2.1 Quadratic action

Expansion of the coset action of the form (2.26) to quadratic order in fluctuations is straightforward. Vast simplification for the end result occur precisely when the relative coefficients of $J^1_+ J^3_-$ and $J^3_+ J^1_-$ with respect to $\frac{1}{2} J^2_+ J^2_-$ are either as in the Green-Schwarz action (2.27) or as in the pure spinor action (2.30). Moreover, the fluctuation actions for these two cases bear a striking resemblance to each other.

Green-Schwarz To the quadratic order, there is no mixing of bosonic and fermionic fluctuations, so the quadratic action is of the form

$$S_2^{\mathrm{GS}} = S_{2B}^{\mathrm{GS}} + S_{2F}^{\mathrm{GS}} \tag{3.4}$$

where

$$S_{2B}^{\mathrm{GS}} = \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\,\mathrm{str}\big([D_+, X^2][D_-, X^2] - [\underline{J}_+^2, X^2][\underline{J}_-^2, X^2]\big), \tag{3.5}$$

$$S_{2F}^{\mathrm{GS}} = -\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\,\mathrm{str}\big([D_+, X^1][\underline{J}_-^2, X^1] + [\underline{J}_+^2, X^3][D_-, X^3] + 2[\underline{J}_+^2, X^3][\underline{J}_-^2, X^1]\big). \tag{3.6}$$

Here and hereafter, $\underline{J}_\pm \equiv \underline{g}^{-1}\partial_\pm \underline{g}$ denotes the background values of the current $J_\pm$.

A characteristic feature of S_{2F}^{GS} is that it has a first order kinetic term. On a slightly closer inspection one finds that actually one half of the fermionic fluctuation modes are absent from S_{2F}^{GS}. (Roughly speaking, the classical Virasoro constraint implies that matrices representing $[\underline{J}_\pm^2,\ \cdot\ ]$ have half maximal rank and project out one halves of X^1 and X^3.) Of course, this reflects the fact that the Green-Schwarz action has a kappa symmetry.

Pure spinor Since we are assuming that the background values for pure spinor ghosts are trivial, the quadratic action for the fluctuations is of the form

$$S_2^{\mathrm{PS}} = S_{2B}^{\mathrm{PS}} + S_{2F}^{\mathrm{PS}} + S_{2G}^{\mathrm{PS}} \tag{3.7}$$

where S_{2B}^{PS} is the same as S_{2B}^{GS} of Green-Schwarz formalism (3.5) and

$$S_{2F}^{\mathrm{PS}} = \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\,\mathrm{str}\big(2[D_+, X^3][D_-, X^1] + [\underline{J}_+^2, X^1][D_-, X^1] + [D_+, X^3][\underline{J}_-^2, X^3]\big), \tag{3.8}$$

$$S_{2G}^{\mathrm{PS}} = \frac{1}{\pi\alpha'}\int \mathrm{d}^2\sigma\,\mathrm{str}\big(w[D_-, \lambda] + \widehat{w}[D_+, \widehat{\lambda}]\big). \tag{3.9}$$

Since the fluctuation actions for the bosonic modes X^2 in Green-Schwarz and pure spinor formalisms are the same, their contributions to the semiclassical partition functions of the Green-Schwarz and pure spinor formalisms can be related trivially. Of course, constraint structures for the fluctuations are different (Virasoro in Green-Schwarz and BRST in pure spinor), but it just implies that contributions of unphysical fluctuations along "lightcone directions" to physical quantities get neutralized by different fermionic fluctuations (reparameterization ghosts in Green-Schwarz and unphysical fermionic fluctuations in pure spinor). We therefore focus on more interesting fermionic fluctuations (X^1, X^3) in the following discussions.

Note that the kinetic term for the fermionic fluctuations in S_{2F}^{PS} is of second order and non-degenerate. This is in sharp contrast to the case of Green-Schwarz. On the other hand, the appearance of S_{2F}^{PS} here is rather similar to S_{2F}^{GS} (3.6) of the Green-Schwarz formalism and can be obtained by formally replacing the "mass term" in S_{2F}^{GS} by the second order kinetic term.

3.2.2 Linearized equations of motion

To compare the structures of fluctuations of Green-Schwarz and pure spinor formalisms, it is useful to compare their equations of motions. We record them here for future use. We also introduce a component notation by choosing a basis of $\mathfrak{g}^1$ and $\mathfrak{g}^3$.

Bosonic fluctuations Equations of motion for bosonic fluctuation $X^2 \in \mathfrak{g}^2$ are the same for Green-Schwarz and pure spinor formalisms:

$$[D_+, [D_-, X^2]] - [\underline{J}^2_+, [\underline{J}^2_-, X^2]] = 0. \tag{3.10}$$

Those modes contribute the same amount to one-loop corrections in two formalisms and hence are not of primary interest to us.

Green-Schwarz By using the classical equations of motion (2.48) for the backgrounds and the Maurer-Cartan equation, the equations of motion for X^1 and X^3 are found to be

$$[D_+, [\underline{J}^2_-, X^1]] + [\underline{J}^2_-, [\underline{J}^2_+, X^3]] = 0, \quad [D_-, [\underline{J}^2_+, X^3]] + [\underline{J}^2_+, [\underline{J}^2_-, X^1]] = 0. \tag{3.11}$$

To study these equations further, it is convenient to take an explicit basis for $\mathfrak{g}^1$ and $\mathfrak{g}^3$ and denote

$$X^1 = \theta^\alpha T_\alpha, \quad X^3 = \widehat{\theta}^{\hat{\alpha}} T_{\hat{\alpha}}. \tag{3.12}$$

Actions of $D_\pm$ and $\underline{J}^2_\pm$ on $(\theta^\alpha, \widehat{\theta}^{\hat{\alpha}})$ can be understood by noting that the bosonic currents $\underline{J}^0$ and $\underline{J}^2$ are related to the spacetime spin connection $\omega_m{}^{ab}$ and vielbein e^a_m respectively. We denote

$$(\rho_\pm)_{\alpha\beta} \equiv \partial_\pm x^m e^a_m (\gamma_a)_{\alpha\beta}, \quad (\rho_\pm)^{\alpha\beta} \equiv \partial_\pm x^m e^a_m (\gamma_a)^{\alpha\beta} \tag{3.13}$$

where γ's are $SO(4,1) \times SO(5)$ gamma matrices. Spinor indices can be raised and lowered using the invariant spinor metric $\eta_{\alpha\hat{\alpha}} = -\eta_{\hat{\alpha}\alpha}$ coupling $\mathfrak{g}^1$ and $\mathfrak{g}^3$ and its inverse. We often omit spinor indices assuming that they are contracted appropriately. It is useful to remember that the classical equations of motion for the background implies $[D_\pm, \rho_\mp] = 0$ and that the Virasoro condition implies $\rho_+\rho_+ = \rho_-\rho_- = 0$. Actually, $\rho_\pm$ have half the maximal ranks so they act as projectors on spinors.

In terms of $(\theta^\alpha, \widehat{\theta}^{\hat{\alpha}})$ the equations of motion can be written as

$$D_+(\eta\rho_-\theta)^\alpha - \frac{1}{2}(\eta\rho_-)^\alpha{}_{\hat{\beta}}(\eta\rho_+\widehat{\theta})^{\hat{\beta}} = 0\,, \quad D_-(\eta\rho_+\widehat{\theta})^{\hat{\alpha}} + \frac{1}{2}(\eta\rho_+)^{\hat{\alpha}}{}_\beta(\eta\rho_-\theta)^\beta = 0 \tag{3.14}$$

where $D_\pm = \partial_\pm - \frac{1}{4}\omega_\pm{}^{ab}\gamma_{ab}$ denotes the action of the covariant derivative $[D_\pm, \,\cdot\,]$ on spinors. Since $\rho_\pm$ behave as projectors, one halves of θ^α and $\widehat{\theta}^{\hat{\alpha}}$ are absent from the equations of motion.

Pure spinor Equations of motion for the fermionic fluctuations X^1 and X^3 are

$$[D_+, [D_-, X^1]] + [J^2_-, [D_+, X^3]] = 0, \quad [D_-, [D_+, X^3]] + [J^2_+, [D_-, X^1]] = 0 \tag{3.15}$$

or in the component notation

$$D_+(D_-\theta)^\alpha - \frac{1}{2}(\eta\rho_-)^\alpha{}_{\hat{\beta}}(D_+\widehat{\theta})^{\hat{\beta}} = 0, \quad D_-(D_+\widehat{\theta})^{\hat{\alpha}} + \frac{1}{2}(\eta\rho_+)^{\hat{\alpha}}{}_\beta(D_-\theta)^\beta = 0. \tag{3.16}$$

Note well the difference and resemblance of these to the corresponding equations in the Green-Schwarz formalism (3.14). Unlike in the Green-Schwarz formalism, equations of motion (3.16) for fermionic fluctuations here are of second order and non-degenerate. On the other hand, if one defines $S = (\eta\rho_-\theta)$ and $\widehat{S} = (\eta\rho_+\widehat{\theta})$ in Green-Schwarz formalism and $\Theta = (D_-\theta)$ and $\widehat{\Theta} = (D_+\widehat{\theta})$ in pure spinor formalism, the equations here can be obtained by formally replacing $(S, \widehat{S})$ in (3.14) by $(\Theta, \widehat{\Theta})$. Since $(\Theta, \widehat{\Theta})$ do not contain the projectors $\rho_\pm$ as $(S, \widehat{S})$ do, one cannot immediately identify them with $(S, \widehat{S})$, but we shall show in the subsection 3.5 that one can further split $(\Theta, \widehat{\Theta})$ to the Green-Schwarz like degrees of freedom $(S, \widehat{S})$ and the rest, at least around the classical solutions contained in an $\mathbb{R}_t \times S^2 \subset AdS_5 \times S^5$.

Equations of motion for the pure spinor ghosts are simply

$$[D_-, \lambda^1] = [D_-, w^3] = 0, \quad [D_+, \widehat{\lambda}^3] = [D_+, \widehat{w}^1] = 0 \tag{3.17}$$

or

$$D_-\lambda^\alpha = D_-w_\alpha = 0, \quad D_+\widehat{\lambda}^{\hat{\alpha}} = D_+\widehat{w}_{\hat{\alpha}} = 0. \tag{3.18}$$

Note that $(D_-\theta) = (D_+\widehat{\theta}) = 0$ is a solution to the equations of motion (3.16). So there are 22×2 bosonic modes and 16×2 fermionic modes satisfying the same equations of motion, and one already expects a huge cancellation of zero-point energies.

3.2.3 BRST transformations of fluctuations

Although we will not need it in this article, the action of the BRST symmetry on fluctuations $X = X^1 + X^2 + X^3$ can be computed from the "finite" BRST transformation

$$g = \underline{g}\mathrm{e}^X \to \underline{g}\mathrm{e}^X\mathrm{e}^{\lambda^1+\widehat{\lambda}^3} \tag{3.19}$$

by using the Baker-Campbell-Hausdorff formula. To the second order in fluctuations, they are given by

$$\begin{aligned}
&\delta_{\mathrm{B}}X^2 = 0 + \frac{1}{2}([X^1, \lambda^1] + [X^3, \widehat{\lambda}^3]) + \cdots, \\
&\delta_{\mathrm{B}}X^1 = \lambda^1 + \frac{1}{2}[X^2, \lambda^3] + \cdots, \quad \delta_{\mathrm{B}}X^3 = \lambda^1 + \frac{1}{2}[X^2, \lambda^3] + \cdots.
\end{aligned} \tag{3.20}$$

Note that, because of pure spinor constraints $\{\lambda^1, \lambda^1\} = \{\widehat{\lambda}^3, \widehat{\lambda}^3\} = 0$, the right hand sides of these equations are linear in $(\lambda^1, \widehat{\lambda}^3)$. Pure spinors λ^1 and $\widehat{\lambda}^3$ are BRST invariant and the conjugates w^3 and $\widehat{w}^1$ transform as

$$\delta_{\mathrm{B}}w^3 = -[D_+, X^3] - [\underline{J}_+^2, X^1] + \cdots, \quad \delta_{\mathrm{B}}\widehat{w}^1 = -[D_-, X^1] - [\underline{J}_+^2, X^3] + \cdots. \tag{3.21}$$

3.3 A family of classical solutions in $AdS_5 \times S^5$

For simplicity, we from now on restrict ourselves to a rather simple family of classical solutions in which the string sits at the center of AdS_5 and (possibly) extended in an $S^2 \subset S^5$. Moreover, we assume that the string is rigid, meaning that the coefficients of fluctuation action is τ-independent.[6] More concretely, if one denotes AdS time by t and azimuthal and polar angles of S^2 by (ψ, ϕ) with $\psi = 0 \sim \pi$ and $\phi = 0 \sim 2\pi$, a solution in the family can be written as

$$t = \kappa\tau, \quad \psi = \psi(\sigma), \quad \phi = \nu\tau + \phi_0(\sigma) \tag{3.22}$$

for some constants κ and ν, and τ-independent functions $\psi(\sigma)$ and $\phi_0(\sigma)$. Solutions in this class include the point-like rotating BMN string [4], the folded spinning string [5], and if the periodicity in σ direction is relaxed, the giant magnon [49].

We shall identify (t, ψ, ϕ) directions to the directions generated by $(T_0, T_8, T_9) \in \mathfrak{g}^2$. The parameterization of the coset representative $g(\tau, \sigma)$ in terms of (t, ψ, ϕ) is then

$$g = \mathrm{e}^{tT_0}\mathrm{e}^{\phi T_9}\mathrm{e}^{(\psi-\pi/2)T_8}. \tag{3.23}$$

The non-vanishing components of the Metsaev-Tseytlin current are

$$J_\pm \equiv g^{-1}\partial_\pm g = \partial_\pm t T_0 + \partial_\pm \psi T_8 + \partial_\pm \phi \sin\psi T_9 - \partial_\pm \phi \cos\psi T_{89}. \tag{3.24}$$

Components of the current $J_\pm$ are just the pullbacks of vielbein and spin connection on S^2

$$e_t^0 = 1, \quad e_\psi^8 = 1, \quad e_\phi^9 = \sin\psi, \tag{3.25}$$

$$\omega_\phi{}^{89} = \cos\psi. \tag{3.26}$$

3.4 Relation between ΔE and worldsheet Hamiltonian H_2

For the class of solutions described in the previous subsection, the one-loop correction to the spacetime energy $\langle\Omega|(E - \underline{E})|\Omega\rangle$ has a rather simple relation to a properly defined worldsheet Hamiltonian H_2 for fluctuations. This is well-known in the Green-Schwarz formalism (both in conformal and static gauges) and it will be shown here that the same is true for the pure spinor formalism as well. To be more specific, it will now be shown that the relation[7]

$$\langle\Psi|\big(\kappa(E - \underline{E}) - \nu(J - \underline{J})\big)|\Psi\rangle = \langle\Psi|H_2|\Psi\rangle \tag{3.27}$$

holds for any quantum state $|\Psi\rangle$ in the BRST cohomology built on the ground state $|\Omega\rangle$. Moreover, since J is a compact generator with discrete eigenvalues, the ground state $|\Omega\rangle$ is supposed to have the same eigenvalue $\underline{J}$ as the classical solution. Exploiting the relation (3.27) is useful because the expectation value of H_2 (zero-point energy) for the ground state $|\Omega\rangle$ can be computed from the one-loop partition function of fluctuations.

A proof of a relation of the type (3.27) in the Green-Schwarz formalism in a conformal gauge is given [6] by noting

$$\kappa(E - \underline{E}) - \nu(J - \underline{J}) + (L_0 + \bar{L}_0) \approx H_2 \tag{3.28}$$

[6] The rigidity assumption is for facilitating the proof of a relation between the one-loop correction to spacetime energy and the expectation value of worldsheet Hamiltonian (see next subsection); it is unnecessary for the comparison of semiclassical partition functions of the Green-Schwarz and pure spnior formalisms.

[7] Here, $(J, \underline{J})$ are an angular momentum in S^5 and its classical value, and have nothing to do with the Metsaev-Tseytlin current J.

where $L_0 + \overline{L}_0$ is the zero-mode of the Green-Schwarz Virasoro operator (including contributions from reparameterization ghosts) expanded to quadratic order in fluctuations and the equality holds up to fermionic constraints of the Green-Schwarz formalism. In (3.28) both $\kappa(E - \underline{E}) - \nu(J - \underline{J})$ and $L_0 + \overline{L}_0$ contain terms linear in fluctuations along a lightcone direction, but the linear terms cancel in the sum and the remaining expression quadratic in fluctuations coincides with H_2. In simple situations where one can take a lightcone gauge, the Hamiltonian H_2 can be decomposed into three pieces $H_{\rm phys} + H_{\rm lc} + H_{bc}$ each representing the Hamiltonian for physical transverse directions, lightcone directions ($x^{\pm} = t \pm \phi$), and reparameterization ghosts. Contributions from $H_{\rm lc} + H_{bc}$ cancel out from the expectation value $\langle\Psi|H_2|\Psi\rangle$ in the right hand side of (3.27) and leaves a result identical to the one in a lightcone gauge.

In the pure spinor formalism, even though the Virasoro operator is not a constraint, a cohomology of the BRST operator has to have a vanishing eigenvalue of $L_0 + \overline{L}_0$ since there is a composite b-ghost that makes the Virasoro operator trivial. So one hopes that the expression of the form (3.27) with $L_0 + \overline{L}_0 = \{Q_{\rm B}, b_0 + \overline{b}_0\}$ is also true in the pure spinor formalism. Although the appearance of Virasoro operators as well as the charges (E, J) and Hamiltonians in Green-Schwarz and pure spinor formalisms are quite different, this hope turns out to be true.

The rest of this subsection is devoted to some details of the proof of (3.27). First we note that the proper definition of the quadratic Hamiltonian (written in terms of "velocity variables") should be

$$H_2 = H_{2B} + H_{2F} + H_{2G} = \frac{1}{2\pi\alpha'}\int \mathrm{d}\sigma(\mathcal{H}_{2B} + \mathcal{H}_{2F} + \mathcal{H}_{2G}) \tag{3.29}$$

where

$$\begin{aligned}
\mathcal{H}_{2B} &= \frac{1}{4}\,\mathrm{str}\Big(([\partial_\tau, X^2])^2 - ([\underline{J}^0_\tau, X^2])^2 + ([\underline{J}^2_\tau, X^2])^2 + ([D_\sigma, X^2])^2 - ([\underline{J}^2_\sigma, X^2])^2\Big),\\
\mathcal{H}_{2F} &= \mathrm{str}\Big([D_+ - \underline{J}^0_\tau, X^1][D_+, X^3] - [\underline{J}^0_-, X^1][\underline{J}^2_+, X^1] + \frac{1}{2}[\partial_\sigma, X^1][\underline{J}^2_\tau, X^1]\\
&\qquad + [D_- - \underline{J}^0_\tau, X^3][D_-, X^1] - [\underline{J}^0_+, X^3][\underline{J}^2_-, X^3] - \frac{1}{2}[\partial_\sigma, X^3][\underline{J}^2_\tau, X^3]\Big),\\
\mathcal{H}_{2G} &= \mathrm{str}\Big(w^3[D_+, \lambda^1] + \widehat{w}^1[D_-, \widehat{\lambda}^3] - N\underline{J}^0_\tau - \widehat{N}\underline{J}^0_\tau\Big).
\end{aligned}$$

The bosonic Hamiltonian H_{2B} is nothing but the canonical Hamiltonian computed from the quadratic Lagrangian L_{2B} of (3.5),

$$H_{2B} = P_2\partial_\tau X^2 - L_{2B}, \quad P_2 \equiv \frac{\partial L_{2B}}{\partial(\partial_\tau X^2)} = \frac{1}{4\pi\alpha'}[D_\tau, X^2]. \tag{3.30}$$

The Hamiltonians for fermions H_{2F} and ghosts H_{2G} are not in a naive canonical form, but they reduce to the standard Hamiltonians for the second order fermions and the left and right moving $\beta\gamma$ systems of weight $(1, 0)$ when the coupling to the background currents $\underline{J}^0$ and $\underline{J}^2$ is dropped. The coupling to the background currents is fixed by the BRST symmetry up to an addition of BRST trivial terms so we claim that (3.29) is the correct Hamiltonian for the quadratic fluctuations.

As mentioned above, in order to relate the one-loop correction to the spacetime energy to the expectation value of H_2, it is convenient to look at the quantity

$$\kappa E - \nu J = -\frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\,\mathrm{str}\Big(j_\tau\big((\partial_\tau \underline{t})T_0 + (\partial_\tau\underline{\phi})T_9\big)\Big) \tag{3.31}$$

where

$$j_\tau = j_+ + j_- = g(J^2_\tau + J^1_\tau + J^3_\tau - \frac{1}{2}(J^1_\sigma - J^3_\sigma) + 2N + 2\widehat{N})g^{-1} \tag{3.32}$$

is the τ-component of the $PSU(2,2|4)$ Noether current defined in (2.38). Classical values $(\underline{E},\underline{J})$ of (E,J) are given by

$$\underline{E} = \frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma \partial_\tau \underline{t} = \frac{\kappa}{2\alpha'}, \quad \underline{J} = \frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma \sin^2\underline{\psi}\,\partial_\tau\underline{\phi} = \frac{\nu}{4\pi\alpha'}\int \mathrm{d}\sigma\sin^2\underline{\psi} \tag{3.33}$$

and semiclassical expressions for (E,J) can be computed by separating $g = g(\tau,\sigma)$ and the currents $(J_\tau, J_\sigma, N, \widehat{N})$ in (3.32) to their background values and fluctuations. (Recall that we are expanding around a trivial ghost profile so N and $\widehat{N}$ are understood to be quadratic in fluctuations.) It is useful to note that the rigidity assumption $\partial_\tau\underline{\psi} = 0$ implies

$$\underline{g}^{-1}((\partial_\tau\underline{t})T_0 + (\partial_\tau\underline{\phi})T_9)\underline{g} = (\partial_\tau\underline{t}T_0 + \partial_\tau\underline{\phi}\sin\underline{\psi}T_9) - \partial_\tau\underline{\phi}\cos\underline{\psi}T_{89} = \underline{J}^2_\tau + \underline{J}^0_\tau. \tag{3.34}$$

Computation of $\kappa E - \nu J$ is then straightforward and to the quadratic order in fluctuations it is given by

$$\kappa E - \nu J = \kappa\underline{E} - \nu\underline{J} - \frac{1}{2\pi\alpha'}\int \mathrm{d}\sigma(\mathcal{C}_1 + \mathcal{C}_{2B} + \mathcal{C}_{2F} + \mathcal{C}_{2G}) \tag{3.35}$$

where

$$\begin{aligned}
\mathcal{C}_1 &= \frac{1}{2}\mathrm{str}\Big(([D_\tau, X^2] + [\underline{J}^0_\tau, X^2])\underline{J}^2_\tau\Big),\\
\mathcal{C}_{2B} &= \frac{1}{2}\mathrm{str}\Big([D_\tau, X^2][\underline{J}^0_\tau, X^2] - ([\underline{J}^2_\tau, X^2])^2\Big),\\
\mathcal{C}_{2F} &= \frac{1}{4}\mathrm{str}\Big(- [D_\sigma, X^1][\underline{J}^2_\tau, X^1] + 2[\underline{J}^0_\tau, X^1][\underline{J}^2_\tau, X^1] + [\underline{J}^0_\tau, X^1][\underline{J}^2_\sigma, X^1]\\
&\quad + [D_\sigma, X^3][\underline{J}^2_\tau, X^3] + 2[\underline{J}^0_\tau, X^3][\underline{J}^2_\tau, X^3] - [\underline{J}^0_\tau, X^3][\underline{J}^2_\sigma, X^3]\\
&\quad - [\underline{J}^2_\tau, X^1][\underline{J}^2_\sigma, X^3] + [\underline{J}^2_\sigma, X^1][\underline{J}^2_\tau, X^3] + 2[D_\tau, X^1][\underline{J}^0_\tau, X^3] + 2[\underline{J}^0_\tau, X^1][D_\tau, X^3]\\
&\quad - [D_\sigma, X^1][\underline{J}^0_\tau, X^3] + [\underline{J}^0_\tau, X^1][D_\sigma, X^3]\Big),\\
\mathcal{C}_{2G} &= \mathrm{str}\Big((N + \widehat{N})\underline{J}^0_\tau\Big).
\end{aligned}$$

The semiclassical expression for the worldsheet energy $L_0 + \overline{L}_0$ (which is BRST trivial) can be computed in a similar manner. To quadratic order in fluctuations it is given by

$$L_0 + \overline{L}_0 = \frac{1}{2\pi\alpha'}\int \mathrm{d}\sigma(\mathcal{L}_1 + \mathcal{L}_{2B} + \mathcal{L}_{2F} + \mathcal{L}_{2G}) \tag{3.36}$$

where

$$\begin{aligned}
\mathcal{L}_1 &= \mathrm{str}(\underline{J}^2_+[D_+, X^2] + \underline{J}^2_-[D_-, X^2]), \\
\mathcal{L}_{2B} &= \frac{1}{2}\,\mathrm{str}\Big(([D_+, X^2])^2 + ([D_-, X^2])^2 - ([\underline{J}^2_+, X^2])^2 - ([\underline{J}^2_-, X^2])^2\Big), \\
\mathcal{L}_{2F} &= \mathrm{str}\Big([D_+, X^1][D_+, X^3] + [D_-, X^1][D_-, X^3] \\
&\qquad + \frac{1}{2}\sum_{i=1,3}([\underline{J}^0_+, X^i][\underline{J}^2_+, X^i] + [\underline{J}^0_-, X^i][\underline{J}^2_-, X^i])\Big), \\
\mathcal{L}_{2G} &= \mathrm{str}\Big(w[D_+, \lambda] + \widehat{w}[D_-, \widehat{\lambda}]\Big).
\end{aligned}$$

Upon integrating a σ-derivative by parts and using the Maurer-Cartan equation as well as $\partial_\tau \underline{J}^2_\mu = 0$ (the rigidity assumption on the classical solution), $\mathcal{L}_1$ is found to be equal to $\mathcal{C}_1$. Then, one finds that the sum of $\kappa(E - \underline{E}) - \nu(J - \underline{J})$ and $(L_0 + \overline{L}_0)$ only contains terms quadratic in fluctuations and is nothing but the worldsheet Hamiltonian H_2:

$$\kappa(E - \underline{E}) - \nu(J - \underline{J}) + (L_0 + \overline{L}_0) = H_2. \tag{3.37}$$

This is the analogue of (3.28) for the pure spinor formalism that we wanted to show. Note that this incidentally shows that H_2 is BRST invariant, since both $PSU(2,2|4)$ and Virasoro charges are BRST invariant.

3.5 Disentangling fermionic fluctuations

Here, we study in detail the fermionic fluctuations around the family of classical solutions (3.22) but with the rigidity assumption relaxed:

$$t = \kappa\tau, \quad \psi = \psi(\tau, \sigma), \quad \phi = \phi(\tau, \sigma), \quad (\psi, \phi) \in S^2 \subset S^5.$$

For notational simplicity we set $\kappa = 2\alpha'\underline{E} = 1$ by adjusting α'.

Green-Schwarz We first study the fermionic fluctuations $(\theta^\alpha, \widehat{\theta}^{\hat\alpha})$ in the Green-Schwarz formalism whose equations of motion are (3.14)

$$D_+(\eta\rho_-\theta)^\alpha - \frac{1}{2}(\eta\rho_-)^\alpha{}_{\hat\beta}(\eta\rho_+\widehat{\theta})^{\hat\beta} = 0\,, \quad D_-(\eta\rho_+\widehat{\theta})^{\hat\alpha} + \frac{1}{2}(\eta\rho_+)^{\hat\alpha}{}_\beta(\eta\rho_-\theta)^\beta = 0.$$

For the class of solutions at hand, matrices $\rho_\pm$ and covariant derivatives $D_\pm$ can be diagonalized neatly.

It will be convenient to take our basis of 16×16 γ-matrices to have

$$\gamma_8 = (-\sigma_2 \otimes 1_8), \quad \gamma_9 = (\sigma_1 \otimes 1_8) \tag{3.38}$$

so that the spin connection becomes diagonal:

$$\omega_\pm = \partial_\pm\phi\,\omega_\phi^{89}\gamma_{89}, \quad \gamma_{89} \equiv \frac{1}{2}(\gamma_8\gamma_9 - \gamma_9\gamma_8) = \frac{\mathrm{i}}{2}(\sigma_3 \otimes 1_8). \tag{3.39}$$

Below, we shall often display γ-matrices in a 2×2 format and leave the trivial factor of 1_8 implicit. In this basis, $\rho_\pm = \partial_\pm x^m e^a_m \gamma_a$ takes the form

$$(\rho_\pm)_{\alpha\beta} = \gamma_0 + a_\pm(-\sigma_2 \otimes 1_8) + b_\pm(\sigma_1 \otimes 1_8) = \begin{pmatrix} 1 & c_\pm \\ c^*_\pm & 1 \end{pmatrix} \tag{3.40}$$

where

$$a_\pm = \partial_\pm \underline{\psi}, \quad b_\pm = \partial_\pm \underline{\phi} \sin \underline{\psi}, \quad c_\pm = \mathrm{i} a_\pm + b_\pm. \tag{3.41}$$

Note that the Virasoro condition implies

$$c^*_\pm c_\pm = a^2_\pm + b^2_\pm = 1 \tag{3.42}$$

so $c_\pm$ are complex numbers of modulus 1. We denote by $\alpha_\pm$ the phase of $c_\pm$:

$$c_\pm = \mathrm{e}^{\mathrm{i}\alpha_\pm}. \tag{3.43}$$

Classical equations of motion for the background field implies

$$(\partial_\pm - \mathrm{i}\omega_\pm)c_\mp = 0, \quad \partial_\pm \alpha_\mp = \omega_\pm. \tag{3.44}$$

With these notational preparation, it is straightforward to find a basis in which $\rho_\pm$ and $D_\pm$ simplify simultaneously. Namely, for

$$U = \begin{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha_+} & \mathrm{e}^{+\frac{\mathrm{i}}{2}\alpha_+} \\ -\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha_+} & \mathrm{e}^{+\frac{\mathrm{i}}{2}\alpha_+} \end{pmatrix}, \quad V = \begin{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha_-} & \mathrm{e}^{+\frac{\mathrm{i}}{2}\alpha_-} \\ -\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha_-} & \mathrm{e}^{+\frac{\mathrm{i}}{2}\alpha_-} \end{pmatrix} \tag{3.45}$$

one finds that

$$\rho_+ = U^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} U, \quad \rho_- = V^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} V, \tag{3.46}$$

$$D_+ = V^{-1}\partial_+ V \equiv \partial_+ + V^{-1}(\partial_+ V), \quad D_- = U^{-1}\partial_- U \equiv \partial_- + U^{-1}(\partial_+ U). \tag{3.47}$$

Substituting these into the equations of motion, one finds

$$\begin{aligned} &\partial_+ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V\theta - \eta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} VU^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U\widehat{\theta} = 0, \\ &\partial_- \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U\widehat{\theta} + \eta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} UV^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V\theta = 0. \end{aligned} \tag{3.48}$$

This clearly shows that one-halves of $V\theta$ and $U\widehat{\theta}$ do not propagate.

To be more concrete, introduce variables $(S, \widehat{S}, T, \widehat{T})$ and β via

$$V\theta = \begin{pmatrix} S \\ T \end{pmatrix}, \quad U\widehat{\theta} = \begin{pmatrix} \widehat{S} \\ \widehat{T} \end{pmatrix}, \tag{3.49}$$

$$\beta = \frac{1}{2}(\alpha_+ - \alpha_-) \;\rightarrow\; UV^{-1} = \begin{pmatrix} \cos\beta & \mathrm{i}\sin\beta \\ \mathrm{i}\sin\beta & \cos\beta \end{pmatrix}. \tag{3.50}$$

Then, T and $\widehat{T}$ decouple from the equations of motion and S and $\widehat{S}$ obey

$$\nabla_{\rm GS}\begin{pmatrix} S \\ \widehat{S}\end{pmatrix}=0, \quad \nabla_{\rm GS}\equiv\begin{pmatrix}\partial_+ & -\eta\cos\beta \\ \eta\cos\beta & \partial_-\end{pmatrix}. \tag{3.51}$$

It is amusing to note that the combination $\phi_s=2\beta=(\alpha_+-\alpha_-)$ is the solution to the sine-Gordon equation $4\partial_+\partial_-(\phi_s)=\sin(\phi_s)$ which determines our solution (t,ψ,ϕ) completely. For example, $\beta=0$ corresponds to the rotating point-like string $t=\phi=\kappa\tau,\ \psi=\pi/2$ of [4] and (3.51) reduces to the well-known equations of motion for the lightcone fermions in a Ramond-Ramond plane-wave background [50].

Pure spinor Recall that the coupled equations of motion for fluctuations are

$$D_+(D_-\theta)^\alpha-\frac{1}{2}(\eta\rho_-)^\alpha{}_{\hat\beta}(D_+\widehat{\theta})^{\hat\beta}=0, \quad D_-(D_+\widehat{\theta})^{\hat\alpha}+\frac{1}{2}(\eta\rho_+)^{\hat\alpha}{}_\beta(D_-\theta)^\beta=0. \tag{3.52}$$

These have two "branches" of solutions. First branch is given by

$$D_-\theta=D_+\widehat{\theta}=0 \tag{3.53}$$

where $D_-\theta=0$ implies $D_+\widehat{\theta}=0$ and vice versa. To show that $D_-\theta=0$ implies $D_+\widehat{\theta}=0$, denote for convenience

$$\Psi=U\theta, \quad \widehat{\Psi}=V\widehat{\theta}. \tag{3.54}$$

Note that $D_-\theta=0$ is equivalent to $\partial_-\Psi=0$ and that $D_+\widehat{\theta}=0$ is equivalent to $\partial_+\widehat{\Psi}=0$. Now, assuming $\partial_-\Psi=0$, the equations of (3.52) imply that $\widehat{\Psi}$ satisfies

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\partial_+\widehat{\Psi}=0, \quad \partial_-\begin{pmatrix}\cos\beta & \mathrm{i}\sin\beta\\ \mathrm{i}\sin\beta & \cos\beta\end{pmatrix}\partial_+\widehat{\Psi}=0. \tag{3.55}$$

In terms of the $8+8$ splitting $\widehat{\Psi}=\begin{pmatrix}\widehat{\Psi}_1\\ \widehat{\Psi}_2\end{pmatrix}$ these are equivalent to

$$\partial_+\widehat{\Psi}_1=0, \quad \partial_+\partial_-\widehat{\Psi}_2=-(\partial_-\beta)\cot\beta\,\partial_+\widehat{\Psi}_2, \quad \partial_+\partial_-\widehat{\Psi}_2=(\partial_-\beta)\tan\beta\,\partial_+\widehat{\Psi}_2. \tag{3.56}$$

Thus for non-constant β one finds $\partial_+\widehat{\Psi}_2=0$ as well. When β is a constant its only possible values are $0 \bmod \pi/2$ since 2β is a solution to the sine-Gordon equation. Then equations of motion for $\widehat{\Psi}_2$ is just $\partial_+\partial_-\widehat{\Psi}_2=0$ and one can include a half of the solutions $\partial_+\widehat{\Psi}_2=0$ in the present branch, and the other half $\partial_-\widehat{\Psi}_2=0$ in the other branch described shortly. This completes the proof that $D_-\theta=0$ implies $D_+\widehat{\theta}=0$, and we have learnt that this solution branch consists of 16 left-moving fields $\Psi^\alpha(\sigma^-)$ and 16 right-moving fields $\widehat{\Psi}^{\hat\alpha}(\sigma^+)$.

To describe the other branch, it is useful to introduce the variables $(S,\widehat{S},T,\widehat{T})$ via

$$V(D_-\theta)=\begin{pmatrix}S\\ T\end{pmatrix}, \quad U(D_+\widehat{\theta})=\begin{pmatrix}\widehat{S}\\ \widehat{T}\end{pmatrix}. \tag{3.57}$$

Since we have already taken care of the branch $D_-\theta = D_+\widehat{\theta} = 0$, one may assume that neither (S,T) nor $(\widehat{S},\widehat{T})$ is identically zero. Equations of motion for $(S,\widehat{S},T,\widehat{T})$ are found to be

$$\begin{aligned} \partial_+ \begin{pmatrix} S \\ T \end{pmatrix} - \eta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta & -\mathrm{i}\sin\beta \\ -\mathrm{i}\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \widehat{S} \\ \widehat{T} \end{pmatrix} = 0, \\ \partial_- \begin{pmatrix} \widehat{S} \\ \widehat{T} \end{pmatrix} + \eta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta & \mathrm{i}\sin\beta \\ \mathrm{i}\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} S \\ T \end{pmatrix} = 0. \end{aligned} \tag{3.58}$$

Compared to the Green-Schwarz equations in the same basis (3.48), one here does not have projections to $(S,\widehat{S})$ so there remains a mixing between $(S,\widehat{S})$ and $(T,\widehat{T})$:

$$\nabla_F \begin{pmatrix} S \\ \widehat{S} \\ T \\ \widehat{T} \end{pmatrix} = 0, \quad \nabla_F \equiv \begin{pmatrix} \partial_+ & -\eta\cos\beta & 0 & \mathrm{i}\eta\sin\beta \\ \eta\cos\beta & \partial_- & \mathrm{i}\eta\sin\beta & 0 \\ 0 & 0 & \partial_+ & 0 \\ 0 & 0 & 0 & \partial_- \end{pmatrix}. \tag{3.59}$$

However, the mixing is minor as can be seen from the block triangular structure of the matrix differential operator ∇_F in (3.59). In particular, equations of motion for T and $\widehat{T}$ are simply $\partial_+ T = \partial_- \widehat{T} = 0$ and the functional determinant of ∇_F factorize as

$$\det\nabla_F = (\det\nabla_{\mathrm{GS}})(\det\partial_+)(\det\partial_-) \tag{3.60}$$

where $\det\nabla_{\mathrm{GS}}$ is the functional determinant of the matrix differential operator appeared in the equations of motion (3.51) for the Green-Schwarz fermions. Although we do not quite pretend to have shown the factorization (3.60) rigorously, we believe that it is possible to do so for example by employing the technique of [51].

3.6 Comparison of 1-loop corrections

Partition function Based on the analyses made thus far, it will now be shown that one-loop partition functions of fluctuations in Green-Schwarz and pure spinor formalisms agree for any classical solution in the family of subsection 3.3. The following table summarizes the contributions of various fluctuations to the partition functions (the partition function for the Green-Schwarz formalism is for a conformal semilightcone gauge in which non-propagating fermionic fluctuations are dropped):

	Bosons	Fermions		Ghosts	
Green-Schwarz	$t,\psi,\phi,\ \ x^i$	$S^A,\widehat{S}^A$	–	$b,c,\overline{b},\overline{c}$	–
(conf. gauge)	$(\det\Delta_3)^{-1}(\det\Delta_7)^{-1}$	$\det\nabla_{\mathrm{GS}}$	–	$(\det\Box)^2$	–
Pure spinor	$t,\psi,\phi,\ \ x^i$	$S^A,\widehat{S}^A$	$T^{\dot{A}},\widehat{T}^{\dot{A}},\Psi^\alpha,\widehat{\Psi}^{\hat{\alpha}}$	–	$w_\alpha,\lambda^\alpha,\widehat{w}_{\dot{\alpha}},\widehat{\lambda}^\alpha$
	$(\det\Delta_3)^{-1}(\det\Delta_7)^{-1}$	$\det\nabla_{\mathrm{GS}}$	$(\det\Box)^{8+16}$	–	$(\det\Box)^{-22}$

Table 1. Contribution of fluctuation modes to partition functions

As can be immediately seen, the products of relevant factors do agree in the two formalisms, so to complete our proof it only remains to explain individual factors. Basically, the only factors which we have not explained are those for ghosts.

Recall that the fluctuation action for the pure spinor ghosts to the quadratic order is simply $\int(w_\alpha D_- \lambda^\alpha + \widehat{w}_{\hat{\alpha}} D_+ \widehat{\lambda}^{\hat{\alpha}})$ so it can be diagonalized just as fermionic fluctuations by using the matrices U and V of (3.45). Then, the pure spinor ghosts and their conjugates become $(11 + 11) \times 2$ left and right moving fields so their contributions to the partition function combine into $(\det \Box)^{-22}$ as claimed. Here, $\Box = 4\partial_+\partial_-$ is the massless Klein-Gordon operator. Similarly, the reparameterization ghosts in the Green-Schwarz formalism consists of 2 left movers (b, c) and 2 right movers $(\overline{b}, \overline{c})$ as usual so they contribute $(\det \Box)^2$.

Contributions from the fermionic coordinates $(\theta^\alpha, \widehat{\theta}^{\hat{\alpha}})$ can be inferred from the analysis of the previous subsection. In the Green-Schwarz formalism, only a half of $(\theta^\alpha, \widehat{\theta}^{\hat{\alpha}})$ are propagating because of the kappa symmetry, and their partition function can be written as $(\det \nabla_{\mathrm{GS}})$ where ∇_{GS} is defined in (3.51). In the pure spinor formalism, partition function of $(\theta^\alpha, \widehat{\theta}^{\hat{\alpha}})$ can be written as $(\det \nabla_{\mathrm{GS}})(\det \Box)^{24}$ and is interpreted as coming from Green-Schwarz like degrees of freedom $(S^A, \widehat{S}^A)$ and the rest consiting of $(8+16)\times 2$ left and right moving variables $(\widehat{T}^{\dot{A}}, \widehat{\Psi}^{\hat{\alpha}})$ and $(T^{\dot{A}}, \Psi^\alpha)$. Actual computation of $(\det \nabla_{\mathrm{GS}})$ is not necessarily easy, but the difficulty does not hamper the comparison of the Green-Schwarz and pure spinor formalisms.

As for bosonic fluctuations $(\tilde{t}, \tilde{x}^i, \tilde{\psi}, \tilde{\phi})$, $(i = 1, \ldots, 7;\ \tilde{t} = t - \underline{t}$ etc.), recall that they are governed by the same quadratic action (3.5) in the two formalisms, so the detailed study of their partition functions is not really necessary for showing the equivalence. However, it is of some interest to look into their structures. For a classical solution of the type discussed in this article partition function factorizes into a product of functional determinants as $(\det \Delta_3)^{-1}(\det \Delta_7)^{-1}$ where Δ_3 and Δ_7 are some second order matrix differential operators that act on $(\tilde{t}, \tilde{\psi}, \tilde{\phi}) \in \mathbb{R}_t \times S^2$ and on the remaining bosonic fluctuations $\tilde{x}^i$ $(i = 1, \ldots, 7)$. Actually, the operator Δ_7 is diagonal in the present setting and $(\det \Delta_7) = (\det \Box_\kappa)^7$ where $\Box_\kappa$ is the Klein-Gordon operator with mass κ. The other factor Δ_3 acts as $\Box$ on $\tilde{t}$ and does not mix it with $(\tilde{\psi}, \tilde{\phi})$, but its action on $(\tilde{\psi}, \tilde{\phi})$ is complicated in general. Nevertheless, if one believes in the equivalence of the conformal gauge computation to a static gauge $(\tilde{t} = \tilde{\phi} = 0)$ one, the functional determinant of Δ_3 should further factorize as $(\det \Delta_3) = (\det \Box)^2(\det \Delta_\psi)$ where Δ_ψ is the second order differential operator acting on $\tilde{\psi}$ in the static gauge. In connection with this, note that it has been argued quite convincingly that $(\det \Delta_3)$ actually can be factorized in this way when a folded string is spinning rigidly in an $AdS_3 \subset AdS_5$ instead of $\mathbb{R}_t \times S^2$ [51].

Putting everything together, we have learnt that the one-loop partition functions of Green-Schwarz and pure spinor formalisms agree for the classical solutions of subsection 3.3, and that the partition function is given as

$$Z = (\det \Delta_\psi)^{-1}(\det \Box_\kappa)^{-7}(\det \nabla_{\mathrm{GS}}). \tag{3.61}$$

Since the one-loop partition function is related to the one-loop correction ΔE to spacetime energy in the present setup, this amounts to a proof of the equivalence of ΔE computed in the two formalisms.

Fluctuation spectra It is tempting to interprete the agreement of the partition functions as indicating that the pure spinor partition function receives non-trivial contributions only from physical fluctuations, i.e. from BRST cohomologies. Such an interpretation is possible if, after a quantization, one can construct transverse DDF operators [52] that generate the BRST cohomologies. The DDF operators should be in one-to-one correspondence with the transverse oscillators of the lightcone Green-Schwarz formalism, and completeness of the DDF operators implies that the remaining degrees of freedom form BRST quartets with a BRST trivial Hamiltonian.

Although an explicit quantization of fluctuation is not easy in general even in the Green-Schwarz formalism, it is straightforward around a point-like rotating string of Berenstein, Maldacena and Nastase [4]. The semiclassical analysis around the BMN string in the pure spinor formalism is just a linearization of the formalism in a Ramond-Ramond plane-wave background [53, 45]. We here wish to explain briefly how a physical state of the lightcone Green-Schwarz formalism is mapped to a BRST cohomology in this case.

In the plane-wave background, physical states of lightcone Green-Schwarz formalism are described by 8 massive bosonic fields x^I and 8 pairs of massive fermionic fields $(S^A, \widehat{S}^A)$, where I and A are the vector and chiral spinor of $SO(4)\times SO(4)$ [50]. As explained in subsection 3.5, it is easy to identify the fields with same properties in the pure spinor formalism at a linearlized level. Remaining degrees of freedom are lightcone coordinates $x^{\pm}$, extra fermionic coordinates $(T^A, \widehat{T}^A)$ and $(\theta^{\dot{A}}, \widehat{\theta}^{\dot{A}})$, and pure spinor ghosts $(w_\alpha, \lambda^\alpha, \widehat{w}_{\hat{\alpha}}, \widehat{\lambda}^{\hat{\alpha}})$. Although the modes of $(x^I, S^A, \widehat{S}^A)$ do not directly generate the BRST cohomology, it should be able to show that elements in their Fock space are in one-to-one correspondence with BRST cohomologies at ghost number $(1,1)$ by adopting the methods of [10] or [54] developed for a flat background.

4 Conclusion

In this article we have explained how the one-loop semiclassical analyses of Green-Schwarz and pure spinor superstrings in an $AdS_5 \times S^5$ background are related. In particular, we have shown that one-loop corrections to spacetime energies of a classical solution is the same when the solution is rigid and contained in an $\mathbb{R}_t \times S^2 \subset AdS_5 \times S^5$. We would like to interprete the result as a support for the equivalence of the two formalisms at a semiclassical level.

Let us recapture the main points:

1. Any purely bosonic classical solution of the Green-Schwarz formalism can be regarded as a classical solution of the pure spinor formalism describing the same classical string.

2. To the quadratic order, actions for bosonic fluctuations around a generic classical solution are the same for the two formalisms. (Structures at higher orders are different because of their coupling to fermionic fluctuations.) By contrast, quadratic actions for fermionic fluctuations are different, yet their structures are strikingly similar. See equations (2.27) and (2.30).

3. When a classical string is rigid and contained in an $\mathbb{R}_t \times S^2 \subset AdS_5 \times S^5$, the one-loop correction ΔE to its spacetime energy is given by the zero point energy of the worldsheet Hamiltonian H_2, both in Green-Schwarz and pure spinor formalisms. To show that ΔE are the same in two formalisms, it therefore suffices to show that the one-loop partition

functions are the same. Moreover, in view of the second item, it is enough to compare the partition functions of fermions and ghosts.

4. Even if the rigidity assumption in the previous item is dropped, fermionic fluctuations in pure spinor formalisms can be separated into the Green-Schwarz fermions $(S^A, \widehat{S}^A)$ and the rest consisting of $(8+16) \times 2$ left and right movers. There is a minor coupling between the two types of degrees of freedom, but the partition function factorizes to the contributions from the two.

5. Reparameterization bc ghosts in Green-Schwarz formalism in a conformal gauge consists of $(1+1) \times 2$ left and right movers.

   Pure spinor ghosts are also massless and consists of $(11+11) \times 2$ left and right movers.

6. The combined partition function of the extra fermions and ghosts in the pure spinor formalism coincides with that of the bc ghosts in the Green-Schwarz formalism. This shows that the total partition functions of the two formalisms are the same. Hence, if the string is rigid, the one-loop correction ΔE to the spacetime energy computed in the two formalisms agree.

It is natural to ask how far does the equivalence above can be generalized. As a matter of fact, we believe that the agreement of one-loop partition functions holds quite generally. Indeed, around any classical configuration, $D_-\theta^\alpha = D_+\widehat{\theta}^{\hat{\alpha}} = 0$ gives a solution to fluctuation equations of motion for the pure spinor formalism and the equations of motion for the combination $(\Theta^\alpha, \widehat{\Theta}^{\hat{\alpha}}) \equiv (D_-\theta^\alpha, D_+\widehat{\theta}^{\hat{\alpha}})$ is closely related to that of Green-Schwarz formalism. Decoupling between the $D_-\theta^\alpha = D_+\widehat{\theta}^{\hat{\alpha}} = 0$ sector and the $(\Theta^\alpha, \widehat{\Theta}^{\hat{\alpha}})$ sector, and splitting of $(\Theta^\alpha, \widehat{\Theta}^{\hat{\alpha}})$ into the Green-Schwarz $(S^A, \widehat{S}^A)$ variables and the rest depend on some details of the classical solution in concern, but it appears reasonable to expect that the combined partition function of fermions and ghosts in the pure spinor formalism just gives $(\det \nabla_{\mathrm{GS}})(\det \Box_{bc})^2$ whenever the Green-Schwarz partition function factorizes as in table 1. It would be interesting to explicitly check these expectations by studying classical strings in $\mathbb{R}_t \times S^3 \in AdS_5 \times S^5$ and $AdS_3 \times S^1 \in AdS_5 \times S^5$ (so-called $SU(2)$ and $SL(2)$ sectors).

The interpretation of the agreement of partition functions requires additional consideration. In this article, to obtain a simple relation between ΔE and the worldsheet Hamiltonian H_2, we have put a rigidity assumption on our strings in $\mathbb{R}_t \times S^2$. Presumably, the simple relation continues to hold as long as $\underline{t} = \kappa\tau$ (targetspace time is proportional to worldsheet time classically) and the classical motion is periodic in time. However, a direct proof purely within a conformal gauge is not necessarily easy.

Extension along another obvious direction, namely, comparison of semiclassical Green-Schwarz and pure spinor formalisms at two-loops and higher is of course important. At higher loops, structures of bosonic fluctuations in the two formalisms are no longer the same due to their coupling to fermions (and ghosts). Also, quartic self-coupling of ghosts $N\widehat{N}$, which is essential for the conformal invariance of the model, should play an important role to establish an equivalence. It would be interesting to understand the relation explicitly.

True power of the pure spinor formalism, however, should be in its generality. The fact that one may treat all classical solutions uniformly without being bothered with gauge fixing appears to make it more suitable for exploiting integrability. As is well known, both Green-Schwarz and pure spinor superstrings in the $AdS_5 \times S^5$ background possess Lax connections whose flatness imply classical equations of motion [26][27]. To show the integrals of motion generated by the flat connection to be mutually commutative, one wishes to check that the connection satisfies a certain exchange algebra introduced by Maillet [55]. In [56][57], the Green-Schwarz flat connection of Bena-Polchinski-Roiban [26] have been investigated within the Dirac-Hamiltonian formalism, and it have been found that the flat connection have to be improved by adding phase space constraints to satisfy the exchange property. Moreover, the flat connection after the improvement have been found to be the one in the pure spinor formalism constructed by one of the authors [27] (minus the ghost contribution). This indicates that the pure spinor formalism is a properly gauge fixed version of the Green-Schwarz formalism. It would be reasonable and interesting, therefore, to exploit the integrability of the pure spinor formalism systematically.

Ultimately, one would like to solve the string theory in the $AdS_5 \times S^5$ background by an exact quantization. In supercoset models describing Ramond-Ramond backgrounds, currents J are not holomorphic unlike in the Wess-Zumino-Witten models, so their operator product expansions are difficult to control. It is just about hopeless to find a good theory for arbitrary non-holomorphic currents, but one could hope that the Ramond-Ramond supercoset models form a good class of conformal field theories. For example, the currents J are actually covariantly holomorphic as in (2.52) indicating an enhancement of chiral algebra from the Virasoro algebra [58].

We would like to come back to some of these issues in the near future.

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A Appendix: Notation and conventions

Worldsheet

Worldsheet of a string is assumed to be a cylinder. We keep the worldsheet to be Minkowskian except in subsection 2.1 where we review the pure spinor formalism in a flat background.

- Coordinates on worldsheet cylinder:

$$\sigma^\mu = (\tau, \sigma), \quad \sigma + 2\pi = \sigma, \quad (\eta^{\tau\tau} = -1,\ \eta^{\sigma\sigma} = 1) \tag{A.1}$$

- Lightcone:

$$\sigma^\pm = \tau \pm \sigma, \quad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \tag{A.2}$$

- Coordinates on a (Euclidean) complex plane:

$$z = \mathrm{e}^{\tilde\tau + \mathrm{i}\sigma}, \quad \overline{z} = \mathrm{e}^{\tilde\tau - \mathrm{i}\sigma}, \quad (\tilde\tau = \mathrm{i}\tau) \tag{A.3}$$

Gamma matrices

- $SO(9,1)$ and $SO(4,1)\times SO(5)$ gamma matrices of size 16×16 are denoted by $(\gamma_a)_{\alpha\beta}$ and $(\gamma_a)^{\alpha\beta}$. They satisfy $\{(\gamma_a)_{\alpha\beta}, (\gamma_b)^{\beta\gamma}\} = 2\eta_{ab}\delta_a{}^\gamma$. We assume that a basis for spinors is chosen so that $(\gamma_0)_{\alpha\beta} = -(\gamma_0)^{\alpha\beta} = 1_{16}$.

- For $SO(4,1)\times SO(5)$ there is an invariant tensor given by an antisymmetric product γ^{01234} of gamma matrices:

$$\eta_{\hat\alpha\alpha} \equiv -\eta_{\alpha\hat\alpha} \equiv (\gamma^{01234})_{\hat\alpha\alpha}, \quad \eta^{\hat\alpha\alpha} \equiv -\eta^{\alpha\hat\alpha} \equiv (\gamma_{01234})^{\hat\alpha\alpha} \tag{A.4}$$

$$\eta_{\alpha\hat\alpha}\eta^{\hat\alpha\beta} = \delta_\alpha{}^\beta, \quad \eta_{\hat\alpha\alpha}\eta^{\alpha\hat\beta} = \delta_{\hat\alpha}{}^{\hat\beta} \tag{A.5}$$

  We use η to define spinor indices with hats. In particular, gamma matrices with hatted indices are defined via

$$(\gamma_a)^{\hat\alpha\hat\beta} = \eta^{\hat\alpha\alpha}(\gamma_a)_{\alpha\beta}\eta^{\beta\hat\beta}, \quad (\gamma_a)_{\hat\alpha\hat\beta} = \eta_{\hat\alpha\alpha}(\gamma_a)^{\alpha\beta}\eta_{\beta\hat\beta} \tag{A.6}$$

- In the context of $\mathfrak{psu}(2,2|4)$, $\eta_{\alpha\hat\alpha}$ can be identified as the "spinor metric" coupling $\mathfrak{g}^1$ and $\mathfrak{g}^3$. See below.

$\mathfrak{psu}(2,2|4)$

- Generators:

$$T_A = (T_a, T_{ab}; T_\alpha, T_{\hat\alpha}) = (P_a, L_{ab}; Q_\alpha, \widehat{Q}_{\hat\alpha}), \quad A = (a, ab; \alpha, \hat\alpha) \tag{A.7}$$

- $\mathbb{Z}_4$ structure:

$$\mathfrak{psu}(2,2|4) = \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2 + \mathfrak{g}^3 \tag{A.8}$$

$$L_{ab} \in \mathfrak{g}^0, \quad P_a \in \mathfrak{g}^2, \quad Q_\alpha \in \mathfrak{g}^1, \quad \widehat{Q}_{\hat\alpha} \in \mathfrak{g}^3 \tag{A.9}$$

  Both commutation relations and inner product below respect $\mathbb{Z}_4$:

$$[\mathfrak{g}^i, \mathfrak{g}^j] = \mathfrak{g}^{i+j}, \quad \mathrm{str}(\mathfrak{g}^i\mathfrak{g}^j) \neq 0 \text{ only when } i+j=0 \tag{A.10}$$

- Trace metric:

$$\eta_{AB} \equiv \mathrm{str}(T_A T_B), \tag{A.11}$$

$$\mathrm{str}(P_a P_b) = \eta_{ab} \tag{A.12}$$

$$\mathrm{str}(L_{ab} L_{cd}) = -R_{abcd} = \begin{cases} -(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad}) & AdS_5 \\ \delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad} & S^5 \end{cases} \tag{A.13}$$

$$\mathrm{str}(\widehat{Q}_{\hat{\alpha}} Q_\beta) = -\,\mathrm{str}(Q_\beta \widehat{Q}_{\hat{\alpha}}) = \gamma^{01234}_{\hat{\alpha}\beta} \tag{A.14}$$

- Commutation relations:
It is convenient to split $a = (a'', a')$ where $a'' = 0, \ldots, 4$ are AdS_5 directions and $a' = 5, \ldots, 9$ are S^5 directions. Non-trivial commutation relations are then

$$[P_{aa''}, P_{b''}] = L_{a''b''}, \quad [P_{a'}, P_{b'}] = -L_{a'b'} \tag{A.15}$$

$$[L_{a''b''}, P_{c''}] = \eta_{b''c''} P_{a''} - \eta_{a''c''} P_{b''}, \quad [L_{a'b'}, P_{c'}] = \eta_{b'c'} P_{a'} - \eta_{a'c'} P_{b'} \tag{A.16}$$

$$[L_{a''b''}, L_{c''d''}] = \eta_{b''c''} L_{a''d''} \pm \cdots, \quad [L_{a'b'}, L_{c'd'}] = \eta_{b'c'} L_{a'd'} \pm \cdots \tag{A.17}$$

$$[L_{ab}, Q_\alpha] = -\frac{1}{2}(\gamma_{ab})_\alpha{}^\beta Q_\beta, \quad [L_{ab}, \widehat{Q}_{\hat{\alpha}}] = -\frac{1}{2}(\gamma_{ab})_{\hat{\alpha}}{}^{\hat{\beta}} \widehat{Q}_{\hat{\beta}} \tag{A.18}$$

$$[P_a, Q_\alpha] = \frac{1}{2}(\eta\gamma_a)_\alpha{}^{\hat{\beta}} \widehat{Q}_{\hat{\beta}}, \quad [P_a, \widehat{Q}_{\hat{\alpha}}] = -\frac{1}{2}(\eta\gamma_a)_{\hat{\alpha}}{}^\beta Q_\beta \tag{A.19}$$

$$\{Q_\alpha, Q_\beta\} = \gamma^a_{\alpha\beta} P_a, \quad \{\widehat{Q}_{\hat{\alpha}}, \widehat{Q}_{\hat{\beta}}\} = \gamma^a_{\hat{\alpha}\hat{\beta}} P_a \tag{A.20}$$

$$\{Q_\alpha, Q_{\hat{\beta}}\} = \frac{1}{2}(\eta\gamma^{a''b''})_{\alpha\hat{\beta}} L_{a''b''} - \frac{1}{2}(\eta\gamma^{a'b'})_{\alpha\hat{\beta}} L_{a'b'} \tag{A.21}$$

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