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$\mathcal{N}=2$ Superconformal Symmetry in Super Coset Models

Thomas Creutzig, Peter B. Rønne, Volker Schomerus
*DESY Theory Group, DESY Hamburg, Notkestrasse 85, D-22603 Hamburg, Germany**
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We extend the Kazama-Suzuki construction of models with $\mathcal{N}=(2,2)$ world-sheet supersymmetry to cosets S/K of supergroups. Among the admissible target spaces that allow for an extension to $\mathcal{N}=2$ superconformal algebras are some simple Lie supergroups, including PSL(N|N). Our general analysis is illustrated at the example of the $\mathcal{N}=1$ WZNW model on GL(1|1). After constructing its $\mathcal{N}=2$ superconformal algebra we determine the (anti-)chiral ring of the theory. It exhibits an interesting interplay between world-sheet and target space supersymmetry.

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1. INTRODUCTION

Sigma models with target superspaces have appeared in a large variety of physics problems, ranging from $\mathcal{N}=4$ super Yang-Mills theory to disordered electron systems. In this note we are particularly interested in theories for which an explicit $\mathcal{N}=1$ superconformal symmetry on the world-sheet gets enhanced to $\mathcal{N}=2$. A few basic examples have been discussed in the literature. These include the supersymmetric sigma model on the so-called twistorial Calabi-Yau $\mathbb{C}P^{3|4}$ that featured in Witten's work [1] on twistor string theory (see e.g. [2–7]). Sigma models on Calabi-Yau superspaces were also conjectured to describe the mirror partner of string theory on rigid Calabi-Yau manifolds [8, 9]. This makes it seem worthwhile to look for more general constructions of such models.

Quantum field theories with $\mathcal{N}=2$ superconformal symmetry possess an intimate and well known relation with topological field theories. In $\mathcal{N}=(2,2)$ superconformal models, the chiral Virasoro field T is part of a multiplet involving two fermionic fields $G^\pm$ with conformal weight $h_G=3/2$ and a bosonic U(1) current U with relations

$$G^+(z)G^-(w) \;\sim\; \frac{c/3}{(z-w)^3}+\frac{U(w)}{(z-w)^2}+\frac{(T+\frac{1}{2}\partial U)(w)}{(z-w)}$$

$$U(z)\,G^\pm(w) \;\sim\; \frac{\pm G^\pm(w)}{(z-w)} \quad , \quad U(z)\,U(w)\sim\frac{c/3}{(z-w)^2}\,.$$

The same algebra is satisfied by the anti-chiral partners $\bar{T},\bar{G}^\pm$ and $\bar{U}$. Given this structure, one may go through a process of twisting. It results in two different topological conformal field theories that are known as the A- and B-model, respectively.

In [10] (see also [11] for earlier related work), Kazama and Suzuki described a simple construction providing many key examples of world-sheet theories with $\mathcal{N}=2$ superconformal symmetry. They started from an $\mathcal{N}=1$ Wess-Zumino-Novikov-Witten (WZNW) model for the coset space S/K and investigated under which conditions the $\mathcal{N}=1$ symmetry could be extended to an $\mathcal{N}=2$ superconformal algebra. Within the list of cases they worked out are the $\mathcal{N}=2$ minimal models. These feature as building blocks for Gepner's construction of string theory on Calabi-Yau manifolds. Our aim here is to generalize the analysis of Kazama and Suzuki to the case of coset superspaces S/K where both S and K can be Lie supergroups. Following [12, 13], we shall describe the $\mathcal{N}=2$ superconformal algebras in terms of supersymmetric Manin triples. Among the resulting $\mathcal{N}=(2,2)$ theories, we find one family of particular interest: It is shown that the $\mathcal{N}=1$ WZNW models on the simple supergroups $S=\mathrm{PSL(N|N)}$ (with trivial denominator $K=\{e\}$) possess an $\mathcal{N}=2$ superconformal symmetry. A related observation for an $\mathcal{N}=(1,1)$ model on the bosonic base of PSL(2|2) was made and studied by several authors [14–16].

Let us briefly describe the content of this note. In the next section we shall outline the construction of K-gauged $\mathcal{N}=1$ WZNW models on a supergroup S. As in the case of bosonic targets, the S/K coset model can be realized within the WZNW model on the product supergroup $G=S\times K$. We continue by introducing the notion of a Manin triple for supergroups G and provide a few examples of this algebraic structure. From the data of a Manin triple we shall construct the fields $G^\pm$ and U of the $\mathcal{N}=2$ superconformal algebra in section 4. There we also discuss possible deformations of the $\mathcal{N}=2$ superconformal algebra. In section 5 we consider $S=\mathrm{GL}(1|1)$ and $K=\{e\}$ as a simple example in which we can easily determine the chiral ring. The latter is shown to consist of fields in atypical multiplets of the target space supersymmetry gl(1|1). Finally, we discuss a few extensions and open problems.

2. GAUGED $\mathcal{N}=1$ WZNW MODELS

WZNW models on coset superspaces with $\mathcal{N}=1$ world-sheet supersymmetry possess a manifestly super-

*Electronic address: thomas.creutzig@desy.de,peter.roenne@desy.de,volker.schomerus@desy.de

symmetric formulation in terms of superfields of the form

$$G \;=\; \exp(i\theta\chi)\; g\; \exp(-i\bar{\theta}\bar{\chi})\,. \tag{1}$$

Here, $g = g(z,\bar{z})$ is a field that takes values in the supergroup S and $\chi = \chi^a t_a$ is a Lie super-algebra valued field. The components χ^a are fermionic for even generators t_a, i.e. when $|a| = 0$, and they are bosonic otherwise. The multiplets χ^a and $\bar{\chi}^a$ each transform in the adjoint of the Lie superalgebra $\mathfrak{s}$ of S. One may now use the superfield G along with the covariant derivatives on the world-sheet given by

$$D \;=\; -i\frac{\partial}{\partial\theta} - 2\theta\partial \qquad \text{and} \qquad \bar{D} \;=\; -i\frac{\partial}{\partial\bar{\theta}} - 2\bar{\theta}\bar{\partial} \tag{2}$$

to build the usual action of the WZNW model on the supergroup. Writing down the action also requires fixing some non-degenerate invariant bilinear form $(\cdot,\cdot)$ on the Lie superalgebra $\mathfrak{s}$. When written in components, the action becomes

$$S^{\mathcal{N}=1}_{\rm WZNW}[G] \;=\; S'_{\rm WZNW}[g] \;+ \tag{3}$$
$$+\;\frac{1}{2\pi}\int d^2z\,(\chi,\bar{\partial}\chi) + (\bar{\chi},\partial\bar{\chi})\;.$$

In our notation, the level k of the model is absorbed into the definition of the bi-linear form $(\cdot,\cdot)$. The formula for the WZNW action on the S-valued field g has the usual form, but with the bi-linear form $(\cdot,\cdot)$ shifted by half the Killing form $\langle\cdot,\cdot\rangle$, i.e. $(\cdot,\cdot)' = (\cdot,\cdot) + \frac{1}{2}\langle\cdot,\cdot\rangle$. The Killing form is constructed and normalized in the standard fashion. In case the Killing form is proportional to $(\cdot,\cdot)$, the shift of the bilinear form simply amounts to shifting the level by the dual Coxeter number. The global target supersymmetry of the $\mathcal{N} = (1,1)$ theory gives rise to holomorphic currents J^a and $\bar{J}^a$ which satisfy the usual super-symmetric current algebra at level k. These currents include terms that are constructed out of the fields χ^a and $\bar{\chi}^a$. For a simple Lie superalgebra $\mathfrak{s}$, the total central charge of the model is

$$c \;=\; \frac{(k-h^\vee_{\mathfrak{s}})\,{\rm sdim}\,\mathfrak{s}}{k} + \frac{1}{2}\,{\rm sdim}\,\mathfrak{s} \;=\; \left(\frac{3}{2} - \frac{h^\vee_{\mathfrak{s}}}{k}\right)\,{\rm sdim}\,\mathfrak{s}\;.$$

The second term is the contribution from the fields χ^a. Note that all these fields possess conformal weight $h_\chi = 1/2$ so that each fermionic component of χ contributes $\delta c = 1/2$ to the central charge while each bosonic component subtracts the same amount.

The gauged WZNW model of Lie groups has been described in e.g. [17–23]. The formulation extends immediately to Lie supergroups. Let $A = A(z,\bar{z},\theta,\bar{\theta})$ and $\bar{A} = \bar{A}(z,\bar{z},\theta,\bar{\theta})$ be a set of gauge fields that take values in some Lie subsuperalgebra $\mathfrak{k}$ of the Lie superalgebra $\mathfrak{s}$. Then the gauged $\mathcal{N} = 1$ WZNW action is

$$S[G,A,\bar{A}] \;=\; S^{\mathcal{N}=1}_{\rm WZNW}[G] + \frac{1}{\pi}\int d^2z d^2\theta\;\big((A, G^{-1}\bar{D}G)$$
$$-(DGG^{-1},\bar{A}) + (A,\bar{A}) - (G^{-1}AG,\bar{A})\big)\,.$$

This action is invariant under the following gauge transformation

$$\begin{aligned} G \;&\to\; HGH^{-1}\,,\\ A \;&\to\; {\rm Ad}(H)A - H^{-1}DH\,,\\ \bar{A} \;&\to\; {\rm Ad}(H)\bar{A} - H^{-1}\bar{D}H \end{aligned} \tag{4}$$

for $H \in K$. Thus the above action describes an $\mathcal{N} = (1,1)$ world-sheet supersymmetric S/K supercoset. It is convenient to gauge fix this symmetry such that

$$A \;=\; DHH^{-1} \qquad , \qquad \bar{A} \;=\; \bar{D}\bar{H}\bar{H}^{-1}\;. \tag{5}$$

Thereby, we can embed our coset model into the $\mathcal{N} = 1$ WZNW model on the product supergroup $S \times K$,

$$\int \mathcal{D}G\mathcal{D}A\mathcal{D}\bar{A}\; e^{-S[G,A,\bar{A}]} \;=\; \mathcal{J}\int \mathcal{D}G\mathcal{D}H\; e^{-S[G]+S[H]}$$

for some constant $\mathcal{J}$ as explained in [23]. The gauge fixing procedure requires to introduce additional ghost fields. They come in four different kinds. There are $\dim\mathfrak{k}^{\bar{0}}$ fermionic ghosts and $\dim\mathfrak{k}^{\bar{1}}$ bosonic ones, each contributing a central charge $c = -2$ and $c = +2$, respectively. These all have $\mathcal{N} = 1$ superpartners, i.e. there are $\dim\mathfrak{k}^{\bar{0}}$ bosonic ghosts with central charge $c = -1$ and $\dim\mathfrak{k}^{\bar{1}}$ fermionic ones with central charge $c = 1$. Taking all these into account, the ghost sector contributes $c_{\rm ghosts} = -3\,{\rm sdim}\,\mathfrak{k}$ so that the total central charge is

$$\begin{aligned} c(S/K) \;&=\\ &= \left(\frac{3}{2} - \frac{h^\vee_{\mathfrak{s}}}{k}\right)\,{\rm sdim}\,\mathfrak{s} + \left(\frac{3}{2} + \frac{h^\vee_{\mathfrak{k}}}{k}\right)\,{\rm sdim}\,\mathfrak{k} - 3\;{\rm sdim}\,\mathfrak{k}\\ &= \left(\frac{3}{2} - \frac{h^\vee_{\mathfrak{s}}}{k}\right)\,{\rm sdim}\,\mathfrak{s} - \left(\frac{3}{2} - \frac{h^\vee_{\mathfrak{k}}}{k}\right)\,{\rm sdim}\,\mathfrak{k}\;\;. \end{aligned}$$

The total Virasoro field $T_{\rm total} = T_{\mathfrak{s}\times\mathfrak{k}} + T_{\rm ghost}$ possesses an $\mathcal{N} = 1$ superpartner $G_{\rm total}$. Both these fields descend to the state space of the coset model. The latter is obtained by computing the cohomology of the BRST operator Q. One may show that $T_{\rm total}$ and $G_{\rm total}$ are in the same cohomology class as the Virasoro element $T_{S/K}$ and its superpartner $G_{S/K}$ in the coset conformal field theory. Details on how this works in $\mathcal{N} = 1$ WZNW cosets S/K of bosonic groups can be found in [24, 25]. The generalization of these constructions to supergroups is entirely straightforward. In the case of Lie groups, Kazama and Suzuki used the current symmetry to show that some of the $\mathcal{N} = 1$ WZNW cosets admit an $\mathcal{N} = 2$ superconformal algebra [10]. Their construction may also be embedded into the product theory. In fact, it suffices to show that the $\mathcal{N} = 1$ superconformal algebra of the WZNW model on $S \times K$ admits an extension to $\mathcal{N} = 2$. The corresponding fields of the $\mathcal{N} = 2$ superconformal algebra receive additional contributions from the ghost sector to form a total $\mathcal{N} = 2$ algebra whose basic $G^\pm_{\rm total}$

and U_{total} reside in the same cohomology class as the associated fields in the coset model. Our goal is to extend the analysis of Kazama and Suzuki to the case in which S and K are Lie supergroups. According to the remarks we have just made, all we need to do is to exhibit an $\mathcal{N}=2$ superconformal algebra in the $\mathcal{N}=1$ WZNW model on the product $S \times K$.

3. SUPER MANIN TRIPLES

Throughout this paper, $\mathfrak{g}$ denotes a (not necessarily simple) Lie super-algebra with a non-degenerate supersymmetric invariant bilinear form $(\cdot\,,\,\cdot)$. In our application to the WZNW coset S/K, the Lie superalgebra $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{k}$. The form on $\mathfrak{g}$ is determined by the form $(\cdot\,,\,\cdot)_\mathfrak{s}$ on $\mathfrak{s}$ that we use to construct the action. On $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{k}$ it is given by

$$((X_1, Y_1), (X_2, Y_2)) \;=\; (X_1, X_2)_\mathfrak{s} - (Y_1, Y_2)_\mathfrak{s}$$

for all $X_i \in \mathfrak{s}$ and $Y_i \in \mathfrak{k} \subset \mathfrak{s}$. As we shall show below, possible $\mathcal{N}=2$ extensions of the $\mathcal{N}=1$ superconformal algebra in the S/K WZNW model are classified by special triples $(\mathfrak{g}, \mathfrak{a}_+, \mathfrak{a}_-)$. Here, $\mathfrak{a}_\pm$ denote two Lie subalgebras such that

$$\mathfrak{g} \;=\; \mathfrak{a}_+ \oplus \mathfrak{a}_- \quad . \tag{6}$$

We call such a triple $(\mathfrak{g}, \mathfrak{a}_+, \mathfrak{a}_-)$ a super Manin triple if the Lie subsuperalgebras $\mathfrak{a}_\pm$ are isotropic, i.e.

$$(\mathfrak{a}_\pm, \mathfrak{a}_\pm) \;=\; 0 \quad . \tag{7}$$

For later use we also introduce the subspace $\mathfrak{a}_\circ$ of the Lie superalgebra $\mathfrak{g}$ by

$$\mathfrak{a}_\circ := \{x \in \mathfrak{g}\,|\,(x,y) = 0\,\forall\, y \in [\mathfrak{a}_+, \mathfrak{a}_+] \cup [\mathfrak{a}_-, \mathfrak{a}_-]\}. \tag{8}$$

Super Manin triples contain all the structure constants we shall employ later to define the fields that generate the $\mathcal{N}=2$ super Virasoro algebra. Before we extract the required constants, let us discuss one series of such super Manin triples that will become particularly important below.

Example: The most important super Manin triples we shall exploit arise from Lie superalgebras $\mathfrak{g} = \mathfrak{s}$, i.e. $K = \{e\}$. Let us suppose that the even part $\mathfrak{g}^{\bar 0}$ of $\mathfrak{g}$ splits into two bosonic subalgebras $\mathfrak{g}^{\bar 0} = \mathfrak{g}^{\bar 0}_a \oplus \mathfrak{g}^{\bar 0}_b$ of equal rank. This condition applies to the Lie superalgebras $\mathfrak{g} = gl(n|n), psl(n|n), sl(n|n\pm 1)$ and $\mathfrak{g} = osp(2n+1|2n), osp(2n|2n)$. In all these examples, the bilinear form of the Cartan subalgebra of one of these subalgebras is positive definite while the other one is negative definite (with a proper choice of real form). Consequently, we can perform an isotropic decomposition of the Cartan subalgebra

$$\mathfrak{h} \;=\; \mathfrak{h}_+ \oplus \mathfrak{h}_- \,. \tag{9}$$

In order to extend the decomposition of $\mathfrak{h}$ to an isotropic decomposition of $\mathfrak{g}$ we recall that any Lie superalgebra admits a triangular decomposition into the Cartan subalgebra $\mathfrak{h}$, the subalgebra of the positive root spaces $\mathfrak{n}_+$ and the subalgebra of negative root spaces $\mathfrak{n}_-$:

$$\mathfrak{g} \;=\; \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \,. \tag{10}$$

Hence the triple $(\mathfrak{g}, \mathfrak{a}_+ = \mathfrak{h}_+ \oplus \mathfrak{n}_+, \mathfrak{a}_- = \mathfrak{h}_- \oplus \mathfrak{n}_-)$ is a super Manin triple, i.e. it satisfies the condition (7). We also note that the derived subalgebras $[\mathfrak{a}_\pm, \mathfrak{a}_\pm]$ of $\mathfrak{a}_\pm$ are contained in $\mathfrak{n}_\pm$ and consequently,

$$\mathfrak{a}_\circ \;\supseteq\; \mathfrak{h} \,. \tag{11}$$

There exist many other super Manin triples, in particular when the Lie superalgebra $\mathfrak{g}$ is not simple.

Before we can turn to the $\mathcal{N}=2$ superconformal algebra we need to extract a few structure constants that characterize the super Manin triple. Let us pick some basis x_i of the Lie superalgebra $\mathfrak{a}_+$. With the help of our bi-linear form $(.,.)$ we can then fix a dual basis x^i of $\mathfrak{a}_-$ such that $(x_i, x^j) = \delta_i^j$. Our choice of basis implies that the Lie bracket takes the following form

$$\begin{aligned} [x_i, x_j] \;&=\; c_{ij}{}^k x_k \\ [x^i, x^j] \;&=\; f^{ij}{}_k x^k \\ [x_i, x^j] \;&=\; c_{ki}{}^j x^k + f^{jk}{}_i x_k \;. \end{aligned} \tag{12}$$

Here, the first two equations involve the structure constants $c_{ij}{}^k$ and $f^{ij}{}_k$ of $\mathfrak{a}_+$ and $\mathfrak{a}_-$, respectively. The last equation follows from the first two. Let us also introduce the projection operators $\Pi_\pm : \mathfrak{g} \to \mathfrak{a}_\pm$ from the Lie superalgebra $\mathfrak{g}$ to the two summands $\mathfrak{a}_\pm$.

In addition to the structure constants c and f, our construction of the $\mathcal{N}=2$ algebra will involve a special element $\tilde\rho \in \mathfrak{g}$ that is defined through

$$\tilde\rho \;:=\; -[x^i, x_i] \;=\; (-1)^i f^{ik}{}_i x_k + (-1)^i c_{ki}{}^i x^k \,. \tag{13}$$

The Jacobi identities for the two Lie subsuperalgebras $\mathfrak{a}_\pm$ as well as for the full Lie superalgebra $\mathfrak{g}$ imply that

$$\tilde\rho \;\in\; \mathfrak{a}_\circ \qquad \text{and} \qquad [\tilde\rho_+, \tilde\rho_-] \;=\; 0\,, \tag{14}$$

where $\tilde\rho_\pm = \Pi_\pm \tilde\rho \in \mathfrak{a}_\pm$ is the image of $\tilde\rho$ under the projection map $\Pi_\pm$. The element $\tilde\rho$ determines a map $D = -\Pi_+[\tilde\rho, .] : \mathfrak{a}_+ \to \mathfrak{a}_+$. When acting on the basis elements x_i it reads

$$\begin{aligned} Dx_i \;&:=\; -\Pi_+[\tilde\rho, x_i] \;=\; D_i^l\, x_l \\ \text{where} \qquad D_i^l \;&:= (-1)^{mn}\, c_{mn}{}^l f^{mn}{}_i \;. \end{aligned} \tag{15}$$

The supertrace of the map D is related to the length of $\tilde\rho$ through

$$\mathrm{str}(D) \;=\; -(\tilde\rho, \tilde\rho) \,. \tag{16}$$

Any Lie superalgebra admits a canonical (often degenerate) graded symmetric invariant bilinear Killing form.

Since it also appears in the structure constants of the current algebra, we shall briefly evaluate the Killing form through the structure constants c and f. For any given choice of the basis, the Killing form reads

$$\langle X^a, X^b\rangle \;=\; -(-1)^n C^{na}{}_m C^{mb}{}_n\,. \qquad (17)$$

When both X^a, X^b are in the same Lie subsuperalgebra $\mathfrak{a}_\pm$, the Killing form on $\mathfrak{g}$ reduces to twice the Killing form of $\mathfrak{a}_\pm$,

$$\langle x_i, x_j\rangle \;=\; -2(-1)^n c_{ni}{}^m c_{mj}{}^n \;=\; \kappa_{ij} \qquad (18)$$

$$\langle x^i, x^j\rangle \;=\; -2(-1)^n f^{ni}{}_m f^{mj}{}_n \;=\; \kappa^{ij}\,. \qquad (19)$$

When the two elements X^a and X^b are taken from different subsuperalgebras $\mathfrak{a}_\pm$, the Killing form reads

$$\begin{aligned}\langle x_i, x^j\rangle \;&=\; \kappa_i{}^j \;=\; 2A_i^j + D_i^j \\ \text{where} \qquad A_i^j \;&=\; (-1)^{mn} c_{ni}{}^m f^{nj}{}_m\,.\end{aligned} \qquad (20)$$

The matrix D was defined in eq. (15). This terminates our preparations.

4. $\mathcal{N}=2$ SUPERCONFORMAL ALGEBRA

Let us begin by introducing the basic fields and their operator product expansions. If we denote by $J_i(z)$ and $J^i(z)$ the chiral affine currents corresponding to the generators x_i and x^i, their operator products are [26]

$$\begin{aligned}J_i(z)J_j(w) \;&\sim\; \frac{\frac{1}{2}\kappa_{ij}}{(z-w)^2} + \frac{c_{ij}{}^k J_k(w)}{(z-w)}\\ J_i(z)J^j(w) \;&\sim\; \frac{\delta_i{}^j + \frac{1}{2}\kappa_i^j}{(z-w)^2} + \frac{f^{jk}{}_i J_k(w) + c_{ki}{}^j J^k(w)}{(z-w)}\\ J^i(z)J^j(w) \;&\sim\; \frac{\frac{1}{2}\kappa^{ij}}{(z-w)^2} + \frac{f^{ij}{}_k J_k(w)}{(z-w)}\end{aligned} \qquad (21)$$

where $\langle x_i, x_j\rangle = \kappa_{ij}$ etc. are the entries of the Killing form we determined at the end of the previous section. The terms involving κ arise because we had to shift the metric by the Killing form in eq. (3). Operator product expansions of the fields χ^i and χ_i take the form

$$\begin{aligned}\chi_i(z)\,\chi_j(w) \;&\sim\; 0\\ \chi_i(z)\,\chi^j(w) \;&\sim\; \frac{\delta_i{}^j}{(z-w)}\\ \chi^i(z)\,\chi^j(w) \;&\sim\; 0\,.\end{aligned} \qquad (22)$$

All these fields have conformal weight $h(\chi_i) = 1/2 = h(\chi^i)$. The pair χ_i and χ^i form a bosonic $\beta\gamma$ system with $c=-1$ when $|i|=1$ and they generate a fermionic bc system of central charge $c=1$ when $|i|=0$.

Let $(\mathfrak{g}, \mathfrak{a}_+, \mathfrak{a}_-)$ be a super Manin triple of a Lie superalgebra $\mathfrak{g}$ such that the condition (7) holds. We now want to build a U(1) current U, the Virasoro field T and two fermionic currents $G^\pm$ of weight $h=3/2$ such that they obey the algebra of an $\mathcal{N}=2$ superconformal symmetry. We begin with the current U,

$$U(z) = \,:\!\chi^i\chi_i\!:\, + \tilde\rho^k J_k + \tilde\rho_k J^k + D_j^i \,:\!\chi^j\chi_i\!:\,. \qquad (23)$$

Here, we have extracted the numbers $\tilde\rho_i$ and $\tilde\rho^i$ from our element $\tilde\rho\in\mathfrak{g}$ through

$$\begin{aligned}\tilde\rho_k \;&:=\; (\tilde\rho, x_k) \;=\; (-1)^i c_{ki}{}^i\\ \tilde\rho^k \;&:=\; (\tilde\rho, x^k) \;=\; (-1)^i f^{ik}{}_i\,.\end{aligned}$$

The Virasoro tensor T takes the usual form

$$T(z) \;=\; \frac{1}{2}(:\!J^iJ_i\!: + (-1)^i :\!J_iJ^i\!: + :\!\partial\chi^i\chi_i\!: - :\!\chi^i\partial\chi_i\!:) \qquad (24)$$

as a sum of the Sugawara tensor of the affine superalgebra at level $k+h^\vee$ and the Virasoro tensor of the free fields χ_i and χ^i. Finally, we introduce the two super-currents by [39]

$$\begin{aligned}G^+(z) \;&=\; J_i\chi^i - \frac{1}{2}(-1)^{i+ij} c_{ij}{}^k \;:\!\chi^i\chi^j\chi_k\!:\\ G^-(z) \;&=\; J^i\chi_i - \frac{1}{2}(-1)^{j+ij} f^{ij}{}_k \;:\!\chi_i\chi_j\chi^k\!:\;.\end{aligned} \qquad (25)$$

We claim that $(U, T, G^\pm)$ form an $\mathcal{N}=2$ superconformal algebra of central charge

$$c \;=\; \frac{3}{2}\,\mathrm{sdim}\;\mathfrak{g} + 3\,\mathrm{str}D\,. \qquad (26)$$

For simple Lie supergroups $\mathfrak{g}$, $\mathrm{str}D = -h^\vee\,\mathrm{sdim}\,\mathfrak{g}/3k$ so that the value of the central charge agrees with what we had spelled out in section 2. The fields $T, G^\pm$ and U extend the $\mathcal{N}=1$ superconformal symmetry of the $S\times K$ WZNW model. In fact, the Virasoro field $T = T_{\mathfrak{s}\times\mathfrak{k}}$ and its $\mathcal{N}=1$ superpartner $G = G^+ + G^- = G_{\mathfrak{s}\times\mathfrak{k}}$ agree with the $\mathcal{N}=1$ superconformal structure of the WZNW on the product $S\times K$. As we explained at the end of section 2, all fields must be augmented by the standard contributions from the ghost sector before they descend to the desired $\mathcal{N}=2$ superconformal algebra of the coset model.

In order to prove the claim that the four currents T, U and $G^\pm$ form an $\mathcal{N}=2$ superconformal algebra one has to compute their operator products. This has been done carefully in [27]. After inserting the operator products (21) and (22) of the constituent fields $J(z)$ and $\chi(z)$, the resulting expressions can be simplified with the help of the Jacobi identity, as in the case of bosonic groups G.

For the key example of a super Manin triple that we described in the previous section, $\mathrm{str}D=0$ and hence the central charge of the associated $\mathcal{N}=2$ superconformal algebra is given by $c = \frac{3}{2}\,\mathrm{sdim}\;\mathfrak{g}$. Some of the supercosets that admit a super Manin triple are listed in a table below, along with the central charge.

S	K	$c(S/K)$
GL$(n\|n)$	GL$(n-m\|n-m)$	0
GL$(n\|n)$	SL$(n-m\|n-m\pm 1)$	0
PSL$(n\|n)$	PSL$(n-m\|n-m)$	0
PSL$(n\|n)$	SL$(n-m\|n-m\pm 1)$	-3
SL$(\tilde{n}\|n)$ $\tilde{n}>n$	SL$(\tilde{n}-m\|n-m)$	0

TABLE I: *Incomplete list of $\mathcal{N}=2$ superconformal supercosets S/K with central charge $c\big(S/K\big)$. In all cases we assume that $n>m\geq 0$.*

There exist more $\mathcal{N}=2$ superconformal algebras, which are obtained from the previous ones through a deformation by an element α in $\mathfrak{a}_o$. Consider an element $\alpha = p^i x_i + q_i x^i \in \mathfrak{a}_o$ where p^i, q_i are Grassmann elements of grade $|i|$. It follows from the very definition of $\mathfrak{a}_o$ that the components p^i and q_i must satisfy

$$c_{ij}{}^k q_k \;=\; f^{ij}{}_k p^k \;=\; 0\,. \tag{27}$$

We employ the element α to deform the fields of the $\mathcal{N}=2$ superconformal algebra as follows

$$\begin{aligned} U_\alpha(z) \;&=\; U(z) + p^i\, I_i(z) - (-1)^i q_i\, I^i(z) \\ T_\alpha(z) \;&=\; T(z) + \frac{1}{2}(p^i\,\partial I_i(z) + (-1)^i q_i\,\partial I^i(z)) \end{aligned} \tag{28}$$

where we used the following set of level k Lie superalgebra currents

$$\begin{aligned} I_i \;&=\; J_i - (-1)^{i+ij} c_{ij}{}^k :\!\chi^j\chi_k\!: -\frac{1}{2}(-1)^{ik} f^{jk}{}_i :\!\chi_j\chi_k\!: \\ I^i \;&=\; J^i - (-1)^{j+ij} f^{ij}{}_k :\!\chi_j\chi^k\!: -\frac{1}{2}(-1)^{ij} c_{jk}{}^i :\!\chi^j\chi^k\!: \,. \end{aligned}$$

The expressions for the deformed supercurrents $G^\pm$ are a bit simpler

$$\begin{aligned} G^+_\alpha \;&=\; G^+ + q_i\,\partial\chi^i \\ G^-_\alpha \;&=\; G^- + p^i\,\partial\chi_i\,. \end{aligned} \tag{29}$$

Since we want $G^\pm$ to remain fermionic under the deformation, we required α to be bosonic. The central charge of the deformed algebra is

$$c_\alpha = c - 6(-1)^i q_i p^i\,.$$

The deformed $\mathcal{N}=2$ structure extends a deformation of the original $\mathcal{N}=1$ superconformal algebra. It is relevant in particular for the discussion of models that are obtained from the WZNW model by Hamiltonian reduction.

5. THE $\mathcal{N}=1$ WZNW MODELS ON GL(1|1)

In the following section we would like to illustrate our constructions in the simplest model, the $\mathcal{N}=1$ WZNW model on the supergroup GL(1|1). The GL(1|1) WZNW model has been discussed in [28–33]. The Lie superalgebra gl(1|1) is generated by elements $E, N, \psi_\pm$ such that

$$[N,\psi_\pm] \;=\; \pm\psi_\pm \quad , \quad [\psi_+,\psi_-] \;=\; E$$

and E commutes with all other generators. It comes equipped with an invariant bilinear form $(.,.)$ whose non-vanishing entries are

$$(E,N) \;=\; k \quad , \quad (\psi_+,\psi_-) \;=\; k \;\;.$$

Written in terms of the various component fields, the action of the $\mathcal{N}=1$ GL(1|1) WZNW model is

$$\begin{aligned} S \;=\; \frac{1}{2\pi}\int d^2z\; \big(&k\partial X\bar\partial Y + k\partial Y\bar\partial X + \partial Y\bar\partial Y + \\ &+ 2e^Y\partial c_+\bar\partial c_- + \chi^N\bar\partial\chi^E + \chi^E\bar\partial\chi^N + \\ &+ \chi^+\bar\partial\chi^- - \chi^-\bar\partial\chi^+ + \bar\chi^N\partial\bar\chi^E + \\ &+ \bar\chi^E\partial\bar\chi^N + \bar\chi^+\partial\bar\chi^- - \bar\chi^-\partial\bar\chi^+\big)\,. \end{aligned} \tag{30}$$

Note the additional term $\partial Y\bar\partial Y$ which is not present in the usual $\mathcal{N}=0$ WZNW model on GL(1|1). This term is due to the shift of the bi-linear form by the Killing form (see our comment in section 2). The Lie supergroup GL(1|1) is not simple but solvable and its superalgebra has a degenerate but non-zero Killing form with the only non-vanishing entry being

$$\langle N,N\rangle \;=\; 2\;.$$

The model (30) has a gl(1|1) current algebra symmetry generated by four currents $J^E, J^N, J^\pm$. Their $\mathcal{N}=1$ superpartners will be denoted by $\chi^E, \chi^N, \chi^\pm$. We note that the Cartan algebra of gl(1|1) has two generators E and N which are isotropic. Hence, we can introduce a super Manin triple $(\text{gl}(1|1), \mathfrak{a}_+, \mathfrak{a}_-)$ through

$$\mathfrak{a}_+ \;:=\; span(E,\psi_+) \quad , \quad \mathfrak{a}_- \;:=\; span(N,\psi_-)\;. \tag{31}$$

It follows that the subspace $\mathfrak{a}_o$ is spanned by E, N and ψ_-. We shall work with the basis $x_1 = E/\sqrt{k}, x_2 = \psi_+/\sqrt{k}$ and $x^1 = N/\sqrt{k}, x^2 = \psi_-/\sqrt{k}$ such that the only non-vanishing structure constants are

$$f^{12}{}_2 \;=\; -\frac{1}{\sqrt{k}} \;=\; -f^{21}{}_2\;.$$

Consequently, the element $\tilde\rho$ takes the form $\tilde\rho = -E/k$ and hence $D=0$. According to our general formulas, the U(1)-current U and the two super-currents $G^\pm$ are given by

$$\begin{aligned} U &= \chi^N\chi^E + \chi^-\chi^+ - \frac{J^E}{\sqrt{k}} \\ G^+ &= J^E\chi^N + J^+\chi^- \qquad (32) \\ G^- &= J^N\chi^E + J^-\chi^+ - \frac{1}{\sqrt{k}}\chi^E\chi^+\chi^- . \end{aligned}$$

One can construct another anti-holomorphic $\mathcal{N}=2$ superconformal algebra out of the anti-holomorphic currents, exactly in the same way as we did in the holomorphic case.

As we have briefly reviewed in the introduction, the $\mathcal{N}=2$ superconformal algebra determines two topological conformal field theories that are obtained through $A-$ and $B-$twist. The physical states of the B-twisted model form the so-called (c,c) ring while those of the A-twisted model are in the (c,a) ring. We would like to determine these two state spaces for the example at hand. Let us recall that any representative ϕ of a (c,c) or (c,a) state must obey

$$2\Delta(\phi) + \epsilon u(\phi) \;=\; 2\bar{\Delta}(\phi) + \bar{\epsilon}'\bar{u}(\phi) \;=\; 0 \qquad (33)$$

where $\Delta(\phi), \bar{\Delta}(\phi), u(\phi)$ and $\bar{u}(\phi)$ are the conformal dimensions and U(1)-charges of the field ϕ. States in the (c,c) ring correspond to $\epsilon = 1 = \epsilon'$ while those in the (c,a) ring are associated with $\epsilon = 1 = -\epsilon'$.

All representatives of the (c,c) and (c,a) ring are based on the components of the fields

$$\Phi_{n+1} \;=\; \begin{pmatrix} e^{inY} & ic_-e^{inY} \\ ic_+e^{inY} & c_-c_+e^{inY} \end{pmatrix} \quad \text{for} \quad n \in \mathbb{R} . \qquad (34)$$

These correspond to harmonic functions on the supergroup GL(1|1), i.e. to functions that are annihilated by (some power of) the Laplacian. Only the first column is in the kernel of $Q_B = G_0^+$ and $\bar{Q}_B = \bar{G}_0^+$. The complete (c,c) ring is then spanned by products of the form

$$\left(e^{inY}, ic_+e^{inY}\right) \times \left(1, \chi^N, \bar{\chi}^N, \chi^N\bar{\chi}^N\right) .$$

Let us note that operators involving the bosonic fields χ^- and $\bar{\chi}^-$ contribute to the kernel of Q_B and $\bar{Q}_B$, but not to the cohomology since they are exact. For the (c,a) ring, a similar analysis can be performed. In this case, the kernel of $Q_A = G_0^+$ and $\bar{Q}_A = \bar{G}_0^-$ in the space of atypical fields (34) contains the constant function only. The (c,a) ring is then represented by the following four fields

$$\left(1, \chi^N, \bar{\chi}^E, \chi^N\bar{\chi}^E\right) .$$

It is not difficult to verify (see e.g. [30]) that neither the (c,c) nor the (c,a) ring depend on the level k. We also note that many states satisfying eqs. (33) are not part of the chiral ring of the model. This is in sharp contrast to the situation in unitary models [34].

6. CONCLUSIONS AND OPEN PROBLEMS

In this work we exhibited $\mathcal{N}=2$ superconformal symmetries for a large class of $\mathcal{N}=1$ WZNW models. Our constructions generalize previous studies of bosonic models [10, 11] to the case of target superspaces. One of the main new features is the existence of $\mathcal{N}=2$ superconformal symmetry in $\mathcal{N}=1$ WZNW models of simple supergroups such as PSL(N|N) or OSP(2N+1|2N). As a concrete example, we analyzed the $\mathcal{N}=1$ WZNW model on GL(1|1) and computed its (anti-)chiral ring. The contributions to the (anti-)chiral ring were all associated with states in atypical representations of the target space supersymmetry. This feature is expected to extend to higher supergroup target spaces.

The case of PSL(N|N) is particularly interesting. Since PSL(N|N) possess vanishing dual Coxeter number, the corresponding WZNW model can be deformed away from the WZ point while preserving conformal symmetry [35, 36]. In other words, the WZNW models on PSL(N|N) are special points in a one-parameter family of conformal field theories with unbroken global symmetry. The same holds for the $\mathcal{N}=1$ version of these models. Given that those deformed models still possess chiral Virasoro fields, one may wonder about the fate of the $\mathcal{N}=2$ superconformal symmetry. We believe that the fields $G_\pm$ and U also remain chiral under the deformation. The issue will be addressed in forthcoming work.

Among the coset theories with non-trivial denominator, the superspace generalization of $\mathcal{N}=2$ minimal models are of particular interest. The compact and non-compact versions are given by the two cosets PSL(1,1|2)/SL(1|2) and PSL(1,1|2)/SL(1,1|1). Both theories possess central charge $c=-3$, regardless of their level.

There are a number of other extensions of the present work that deserve a closer investigation. One of them is to incorporate world-sheets with boundary. The $\mathcal{N}=1$ WZNW models on the supergroups PSL(N|N), GL(N|N) and SL(N-1|N), for example, are all known to possess two families of maximally symmetric boundary conditions [37]. In [27], one of them was shown to descend to the A-twisted model while the other is consistent with the B-twist. Cosets with non-trivial denominator possess a richer structure. Finally, one might also wonder whether some of the $\mathcal{N}=(2,2)$ theories we discussed here allow for $\mathcal{N}=(4,4)$ superconformal symmetry. The answer turns out to be positive. We shall describe the exact conditions and consequences in a forthcoming paper.

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[1] E. Witten, Commun. Math. Phys. **252** (2004) 189 [arXiv:hep-th/0312171].
[2] M. Aganagic and C. Vafa, arXiv:hep-th/0403192.
[3] S. P. Kumar and G. Policastro, Phys. Lett. B **619** (2005) 163 [arXiv:hep-th/0405236].
[4] C. h. Ahn, Mod. Phys. Lett. A **20** (2005) 407 [arXiv:hep-th/0407009].
[5] A. Belhaj, L. B. Drissi, J. Rasmussen, E. H. Saidi and A. Sebbar, J. Phys. A **38** (2005) 6405 [arXiv:hep-th/0410291].
[6] R. Ricci, JHEP **0703** (2007) 048 [arXiv:hep-th/0511284].
[7] S. Seki, K. Sugiyama and T. Tokunaga, Nucl. Phys. B **753** (2006) 295 [arXiv:hep-th/0605021].
[8] S. Sethi, Nucl. Phys. B **430** (1994) 31 [arXiv:hep-th/9404186].
[9] A. S. Schwarz, Lett. Math. Phys. **38** (1996) 91 [arXiv:hep-th/9506070].
[10] Y. Kazama and H. Suzuki, Nucl. Phys. B **321**, 232 (1989).
[11] P. Spindel, A. Sevrin, W. Troost and A. Van Proeyen, Nucl. Phys. B **308** (1988) 662.
[12] S. E. Parkhomenko, Sov. Phys. JETP **75** (1992) 1 [Zh. Eksp. Teor. Fiz. **102** (1992) 3].
[13] E. Getzler, arXiv:hep-th/9307041.
[14] L. Rastelli and M. Wijnholt, arXiv:hep-th/0507037.
[15] A. Dabholkar and A. Pakman, Adv. Theor. Math. Phys. **13** (2009) 409 [arXiv:hep-th/0703022].
[16] M. R. Gaberdiel and I. Kirsch, JHEP **0704** (2007) 050 [arXiv:hep-th/0703001].
[17] L. D. Faddeev and S. L. Shatashvili, Theor. Math. Phys. **60** (1985) 770 [Teor. Mat. Fiz. **60** (1984) 206].
[18] K. Bardakci, E. Rabinovici and B. Saering, Nucl. Phys. B **299** (1988) 151.
[19] K. Gawedzki and A. Kupiainen, Phys. Lett. B **215** (1988) 119.
[20] K. Gawedzki and A. Kupiainen, Nucl. Phys. B **320** (1989) 625.
[21] D. Karabali and H. J. Schnitzer, Nucl. Phys. B **329** (1990) 649.
[22] D. Karabali, Q. H. Park, H. J. Schnitzer and Z. Yang, Phys. Lett. B **216** (1989) 307.
[23] A. A. Tseytlin, Nucl. Phys. B **411** (1994) 509 [arXiv:hep-th/9302083].
[24] H. Rhedin, Phys. Lett. B **373** (1996) 76 [arXiv:hep-th/9511143].
[25] J. M. Figueroa-O'Farrill and S. Stanciu, arXiv:hep-th/9511229.
[26] P. Di Vecchia, V. G. Knizhnik, J. L. Petersen and P. Rossi, Nucl. Phys. B **253** (1985) 701.
[27] T. Creutzig, arXiv:0908.1816.
[28] L. Rozansky and H. Saleur, Nucl. Phys. B **376**, 461 (1992).
[29] L. Rozansky and H. Saleur, Nucl. Phys. B **389**, 365 (1993) [arXiv:hep-th/9203069].
[30] V. Schomerus and H. Saleur, Nucl. Phys. B **734**, 221 (2006) [arXiv:hep-th/0510032].
[31] T. Creutzig, T. Quella and V. Schomerus, Nucl. Phys. B **792** (2008) 257 [arXiv:0708.0583 [hep-th]].
[32] T. Creutzig and V. Schomerus, Nucl. Phys. B **807** (2009) 471 [arXiv:0804.3469 [hep-th]].
[33] T. Creutzig and P. B. Ronne, Nucl. Phys. B **815** (2009) 95 [arXiv:0812.2835 [hep-th]].
[34] W. Lerche, C. Vafa and N. P. Warner, Nucl. Phys. B **324** (1989) 427.
[35] M. Bershadsky, S. Zhukov, A. Vaintrob, Nucl. Phys. B **559** (1999) 205 [arXiv:hep-th/9902180].
[36] T. Quella, V. Schomerus and T. Creutzig, JHEP **0810**, 024 (2008) [arXiv:0712.3549 [hep-th]].
[37] T. Creutzig, Nucl. Phys. B **812** (2009) 301 [arXiv:0809.0468 [hep-th]].
[38] G. Giribet, Y. Hikida and T. Takayanagi, JHEP **0909**, 001 (2009) [arXiv:0907.3832 [hep-th]].
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