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Rigorous construction and Hadamard property of the Unruh state in Schwarzschild spacetime

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Abstract. The discovery of the radiation properties of black holes prompted the search for a natural candidate quantum ground state for a massless scalar field theory on Schwarzschild spacetime, here considered in the Eddington-Finkelstein representation. Among the several available proposals in the literature, an important physical role is played by the so-called Unruh state which is supposed to be appropriate to capture the physics of a black hole formed by spherically symmetric collapsing matter. Within this respect, we shall consider a massless Klein-Gordon field and we shall rigorously and globally construct such state, that is on the algebra of Weyl observables localised in the union of the static external region, the future event horizon and the non-static black hole region. Eventually, out of a careful use of microlocal techniques, we prove that the built state fulfils, where defined, the so-called Hadamard condition; hence, it is perturbatively stable, in other words realizing the natural candidate with which one could study purely quantum phenomena such as the role of the back reaction of Hawking's radiation.

¿From a geometrical point of view, we shall make a profitable use of a bulk-to-boundary reconstruction technique which carefully exploits the Killing horizon structure as well as the conformal asymptotic behaviour of the underlying background. From an analytical point of view, our tools will range from Hörmander's theorem on propagation of singularities, results on the role of passive states, and a detailed use of the recently discovered peeling behaviour of the solutions of the wave equation in Schwarzschild spacetime.

Contents

1 Introduction

In the wake of Hawking's discovery of the radiating properties of black holes [Haw74], several investigations on the assumptions leading to such result were prompted. In between them, that of Unruh [Un76] caught the attention of the scientific community, since he first emphasised the need to identify a physically sensible candidate quantum state which could be called the vacuum for a quantum massless scalar field theory on the Schwarzschild spacetime. This is especially true when such spacetime is viewed as that of a real black hole obtained out of the collapse of spherically symmetric matter.

If we adopt the standard notation (*e.g.*, see [Wa94]), this spacetime can be identified with the union of the regions I and III in the Kruskal manifold including the future horizon, though we must omit the remaining two regions together with their boundaries [Wa84, Wa94]. To the date, in the literature, three candidate background states are available, going under the name of *Boulware* (for the external region), *Hartle-Hawking* (for the complete Kruskal manifold) and *Unruh* state (for the union of both the external and black hole region, including the future event horizon). The goal of this paper is to focus on the latter, mostly due to its remarkable physical properties. As a matter of fact, earlier works (see for example [Ca80, Ba84, Ba01]) showed that such a state could be employed to compute the expectation value of the regularised stress-energy tensor for a massless scalar field in the physical region of Schwarzschild spacetime, above pointed out. The outcome is a regular expression on the future event horizon while, at future null infinity, it appears an outgoing flux of radiation compatible with that of a blackbody at the black hole temperature. As pre-announced, this result, together with Birkhoff's theorem, lead to the conjecture that the very same Unruh state, say ω_U, as well as its smooth perturbations, is the natural candidate to be used in the description of the gravitational collapse of a spherically symmetric star. However, to this avail, one is also lead to assume that ω_U fulfils the so-called *Hadamard property* [KW91, Wa94], a prerequisite for states on curved background to be indicated as physically reasonable. As a matter of fact, in between the many properties, it is noteworthy to emphasise that such condition assures the existence of a well-behaved averaged stress energy tensor [Wa94]. Therefore, from a heuristic point of view, this condition is tantamount to require that the ultraviolet behaviour mimics that of the Minkowski vacuum, leading to a physically clear prescription on how to remove the singularities of the averaged stress-energy tensor; this comes at hand whenever one needs to compute the back-reaction of the quantum matter on the gravitational background through Einstein's equations.

The relevance of the Hadamard condition is further borne out by the analysis in [FH90], where the description of the gravitational collapse of a spherically symmetric star is discussed and, under the assumption of the existence of suitable algebraic states of Hadamard form, it is shown that the appearance of the Hawking radiation, brought, at large times, by any of the said states, is precisely related to the scaling-limit behaviour of the underlying two-point function of the state computed on the 2-sphere determined by the locus where the star radius crosses the Schwarzschild one.

It is therefore manifest the utmost importance to verify whether ω_U satisfies or not the Hadamard property, a condition which appears reasonable to assume at least in the static region of Schwarzschild spacetime also in view both of the former analysis in [Ca80] and of the general results achieved in [SV00] applied to those in [DK86-87]. Indeed such a check is one of the main purposes to write this paper. Our goals are, however, broader, as we shall make a novel use of the Killing and conformal structure of Schwarzschild spacetime in order to construct rigorously and unambiguously the Unruh state, contemporary in the static region, inside the internal region and on the future event horizon. To this avail, we shall exploit some techniques which in the recent past have been successfully applied to manifolds with Killing horizons, asymptotically flat spacetimes (see also the recent [Sc09]) as well as cosmological backgrounds [MP05, DMP06, Mo06, Da08a, Da08b, Mo08, DMP09a, DMP09b]. That mathematical technology also relies upon some ideas essentially due to Ashtekar [As87] and that have also received attention for applications to electrodynamics [Her97, Her08] in asymptotically flat spacetimes. Finally, a mathematically similar procedure was employed in [Ho00] to prove the Hadamard property of some relevant states in a different physical context.

From the perspective of this manuscript, the above cited papers by the authors of the present work are most notable for their underlying common "philosophy". To wit, as a first step, one always identifies a preferred codimension 1 null submanifolds of the background, one is interested in. Afterwards, the classical solutions of the bulk dynamical system, one wishes to consider, are projected on a suitable function set living on the chosen submanifold. The most notable property of this set is that one can associate to it a Weyl algebra of observable, which carries a corresponding distinguished quantum algebraic state which can be pulled-back to bulk via the above projection map. On the one hand this procedure induces a state for the bulk algebra of observables and, on the other hand, such new state enjoys several important physical properties, related both with the symmetries of the spacetime and with suitable notions of uniqueness and energy positivity.

Particularly, although at a very first glance, one would be tempted to conclude that the Hadamard property is automatically satisfied as a consequence of the construction itself and of the known results for the microlocal composition of the wave front sets, actually we face an harsher reality. To wit, this feature has to be verified via a not so tantalising case by case analysis since it is strictly intertwined to the geometrical details of the background. Unfortunately the case, we analyse in this paper, is no exception and, thus, we shall be forced to use an novel different procedure along the lines below outlines.
As a starting point, we shall remark that, in the Schwarzschild background, the role of the distinguished null codimension one hypersurface, on which to encode the bulk data, will be played by the union of the complete Killing past horizon and of null past infinity. Afterwards, as far as the state is concerned, it will be then defined on the selected hypersurfaces just following the original recipe due to Unruh: a vacuum defined with respect to the affine parameter of the null geodesics forming the horizon and a vacuum with respect to the Schwarzschild Killing vector ∂_t at past null infinity. At a level of two-point function, the end point of our construction takes a rather distinguished shape whenever restricted to the subalgebra smeared by compactly supported functions, which coincides with the one already noticed in [Sw82, DK86-87, KW91]. Nonetheless, from our perspective, the most difficult technical step will consist of the extension of the methods employed in our previous papers, the reason being that the full algebra both on the horizon and on null infinity is subject to severe constraints whose origin can be traced back to

some notable recent achievements by Dafermos and Rodnianski [DR08]. To make things worse, a similar problem will appear for the state constructed for the algebra at the null infinity. Nonetheless we shall display a way to overcome both potential obstructions and the full procedure will ultimately lead to the implementation of a fully mathematically coherent Unruh state, ω_U for the spacetime under analysis.
Despite these hard problems, the bright side of the approach, we advocate, lies in the possibility to develop a global definition for ω_U for the spacetime which encompasses the future horizon, the external as well as the internal region. Furthermore our approach will be advantageous since it allows to avoid most of the technical cumbersomeness, encountered in the earlier approaches, the most remarkable in [DK86-87] (see also [Ka85a]), where the Unruh state was defined via an S-matrix out of the solutions of the corresponding field equation of motion in asymptotic Minkowski spacetimes. Alas, the definition was established only for the static region and the Hadamard condition was not checked, hence leaving open several important physical questions.
Differently, our boundary-to-bulk construction, as pre-announced, will allow us to make a full use of the powerful techniques of microlocal analysis, thus leading to a verification of the Hadamard condition using the global microlocal characterisation discovered by Radzikowski [Ra96a, Ra96b] and fruitfully exploited in all the subsequent literature (see also [BFK96]). Differently from the proofs of the Hadamard property presented in [Mo08] and [DMP09b] here we shall adopt a more indirect procedure, which has the further net advantage to avoid potentially complicated issues related to the null geodesics reaching i^- from the interior of the Schwarzschild region. The Hadamard property will be first established in the static region making use of an extension of the formalism and the results presented in [SV00] valid for *passive states*. The black hole region together with the future horizon will be finally encompassed by a profitable use of the Hörmander's propagation of singularity theorem joined with a direct computation of the relevant remaining part of wavefront set of the involved distributions, all in view of well-established results of microlocal analysis.
¿From a mathematical point of view, it is certainly worth acknowledging that the results we present in this paper are obtainable thanks to several remarkable achievements presented in a recent series of papers due to Dafermos and Rodnianski [DR05, DR07, DR08, DR09], who discussed in great details the behaviour of a solution φ of the Klein-Gordon equation in Schwarzschild spacetime improving a classical result of Kay and Wald [KW87]. Particularly we shall benefit from the obtained peeling estimates for φ both on the horizons and at null infinity, thus proving the long-standing conjecture known as Price law [DR05].

In detail, the paper will be divided as follows.
In section 2.1, we recall the geometric properties of Schwarzschild spherically symmetric solution of Einstein's equations. Particularly, we shall introduce, characterise and discuss all the different regions of the background which will play a distinguished role in the paper.
Subsequently, in section 2.2 and 2.3, we shall define the relevant Weyl C^*-algebras of observables respectively in the bulk and in the codimension 1 submanifolds, we are interested in, namely the past horizon and null infinity.
Eventually, in section 2.4, we shall relate bulk and boundary data by means of an certain isometric $*$-homomorphism whose existence will be asserted and, then, discussed in detail.
Section 3 will be instead devoted to a detailed analysis on the relation between bulk and boundary states. Particularly we shall focus on the state defined by Kay and Wald for a (smaller) algebra associated with the past horizon $\mathcal{H}$ [KW91], showing that that state can be extended to the (larger) algebra relevant for our purposes.
The core of our results will be in section 4 where we shall first define the Unruh state and, then, we will prove that it fulfils the Hadamard property. Eventually we draw some conclusions.
Appendix A contains further geometric details on the conformal structure of Schwarzschild spacetime,

while Appendix C encompasses the proofs of most propositions. At the same time Appendix B is noteworthy because it summarises several different definitions of the KMS condition and their mutual relation is briefly sketched.

1.1 Notation, mathematical conventions

Throughout, $A \subset B$ (or $A \supset B$) includes the case $A = B$, moreover $\mathbb{R}_+ \doteq [0, +\infty)$, $\mathbb{R}^*_+ \doteq (0, +\infty)$, $\mathbb{R}_- \doteq (-\infty, 0]$, $\mathbb{R}^*_- \doteq (-\infty, 0)$ and $\mathbb{N} \doteq \{1, 2, \ldots\}$. For smooth manifolds $\mathscr{M}, \mathscr{N}$, $C^\infty(\mathscr{M}; \mathscr{N})$ is the space of smooth functions $f : \mathscr{M} \to \mathscr{N}$. $C^\infty_0(\mathscr{M}; \mathscr{N}) \subset C^\infty(\mathscr{M}; \mathscr{N})$ is the subspace of compactly-supported functions. If $\chi : \mathscr{M} \to \mathscr{N}$ is a diffeomorphism, χ^* is the natural extension to tensor bundles (counter-, co-variant and mixed) from $\mathscr{M}$ to $\mathscr{N}$ (Appendix C in [Wa84]). A **spacetime** $(\mathscr{M}, g)$ is a Hausdorff, second-countable, smooth, four-dimensional connected manifold $\mathscr{M}$, whose smooth metric has signature $-+++$. We shall also assume that a spacetime is *oriented* and *time oriented*. The symbol $\Box_g$ denotes the standard **D'Alembert operator** associated with the unique metric, torsion free, affine connection $\nabla_{(g)}$ constructed out of the metric g. $\Box_g$ is locally individuated by $g_{ab}\nabla^a_{(g)}\nabla^b_{(g)}$. We adopt definitions and results about causal structures as in [Wa84, O'N83], but we take recent results [BS03-05, BS06] into account, too. If $(\mathscr{M}, g)$ and $(\mathscr{M}', g')$ are spacetimes and $S \subset \mathscr{M} \cap \mathscr{M}'$, then $J^\pm(S; \mathscr{M})$ $(I^\pm(S; \mathscr{M}))$ and $J^\pm(S; \mathscr{M}')$ $(I^\pm(S; \mathscr{M}'))$ indicate the **causal** (resp. **chronological**) **sets** generated by S in the spacetime $\mathscr{M}$ or $\mathscr{M}'$, respectively. An (anti)symmetric bilinear map over a real vector space $\sigma : V \times V \to \mathbb{R}$ is **nondegenerate** when $\sigma(u, v) = 0$ for all $v \in V$ entails $u = 0$.

2 Quantum Field theories - bulk to boundary relations

2.1 Schwarzschild-Kruskal spacetime

In this paper we will be interested in the analysis of a Klein-Gordon scalar massless field theory on Schwarzschild spacetime and, therefore, we shall first recall the main geometric properties of the background we shall work with. Within this respect, we shall follow section 6.4 of [Wa84] and we will focus on the *physical region* $\mathscr{M}$ of the full Kruskal manifold $\mathscr{K}$ (represented in figure 2 in the appendix), associated with a black hole of mass $m > 0$.

$\mathscr{M}$ is made of the union of three pairwisely disjoint parts, $\mathscr{W}, \mathscr{B}$ and $\mathcal{H}_{ev}$ which we shall proceed to describe. According to figure 1 (and figure 2 in the appendix), we individuate $\mathscr{W}$ as the (open) **Schwarzschild wedge**, the (open) **black hole** region is denoted by $\mathscr{B}$ while their common boundary, the **event horizon**, is indicated by $\mathcal{H}_{ev}$.

The underlying metric is easily described if we make use of the standard **Schwarzschild coordinates** t, r, θ, ϕ, where $t \in \mathbb{R}$, $r \in (r_S, +\infty)$, $(\theta, \phi) \in \mathbb{S}^2$ in $\mathscr{W}$, whereas $t \in \mathbb{R}$, $r \in (0, r_S)$, $(\theta, \phi) \in \mathbb{S}^2$ in $\mathscr{B}$. Within this respect the metric in both $\mathscr{W}$ and $\mathscr{B}$ assumes the standard Schwarzschild form:

$$-\left(1 - \frac{2m}{r}\right) dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr + r^2 h_{\mathbb{S}^2}(\theta, \phi)\,, \tag{1}$$

where $h_{\mathbb{S}^2}$ is the standard metric on the unit 2-sphere. Here, per direct inspection, one can recognise that the locus $r = 0$ corresponds to proper metrical singularity of this spacetime, whereas $r = r_S = 2m$ individuates the apparent singularity on the event horizon.

It is also convenient to work with the **Schwarzschild light** or **Eddington-Finkelstein coordinates**