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Solutions of type IIB and D=11 supergravity with Schrödinger(z) symmetry

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Abstract

We construct families of supersymmetric solutions of type IIB and $D = 11$ supergravity that are invariant under the non-relativistic Schrödinger(z) algebra for various values of the dynamical exponent z. The new solutions are based on five- and seven-dimensional Sasaki-Einstein manifolds, respectively, and include supersymmetric solutions with $z = 2$.

1 Introduction

An interesting development in string/M-theory is the possibility of using holographic ideas to study condensed matter systems. Starting with [1, 2], one focus has been on non-relativistic systems with Schrödinger symmetry, a non-relativistic version of conformal symmetry. The corresponding Schrödinger algebra is generated by Galilean transformations, an anisotropic scaling of space ($\mathbf{x}$) and time (x^+) coordinates given by $\mathbf{x} \to \mu\mathbf{x}$, $x^+ \to \mu^2 x^+$, and an additional special conformal transformation. More generally, one can consider systems invariant under what we shall call Schrödinger(z) (or Sch(z)) symmetry, where one maintains the Galilean transformations, but allows for other scalings, $\mathbf{x} \to \mu\mathbf{x}$, $x^+ \to \mu^z x^+$, with z the "dynamical exponent", and, in general, sacrifices the special conformal transformations. In this notation the Schrödinger algebra is Sch(2). The full set of commutation relations for $Sch\,(z)$ are written down in e.g. [2].

Various solutions of type IIB supergravity and $D = 11$ supergravity have been constructed that are invariant under Sch(z) symmetry, for different values of z. The type IIB solutions of [3]-[8] can be viewed as deformations of the supersymmetric $AdS_5 \times SE_5$ solutions, where SE_5 is a five-dimensional Sasaki-Einstein space, and should be holographically dual to non-relativistic systems with two spatial dimensions. Similarly, there are deformations of the $AdS_4 \times SE_7$ solutions[1] of $D = 11$ supergravity, where SE_7 is a seven-dimensional Sasaki-Einstein space, that are invariant under Sch(z) and these should be dual to non-relativistic systems with a single spatial dimension [7, 8].

The type IIB solutions constructed in [3, 4, 5] with $z = 2$, and hence invariant under the larger Schrödinger algebra, are based on a deformation in the three-form flux and do not preserve any supersymmetry [4]. In [6] supersymmetric solutions of type IIB with various values of z were constructed which are based on a metric deformation and include supersymmetric solutions with $z = 2$. However, it was argued that these supersymmetric solutions are unstable. On the other hand it was shown that the instability can be removed by also switching on the three-form flux deformation, which then breaks supersymmetry. In a more recent development a rich class of supersymmetric solutions of both type IIB and $D = 11$ supergravity were constructed in [8] which have various values of $z \geq 4$ and $z \geq 3$, respectively (particular examples of the $z = 4$ and $z = 3$ solutions were first constructed in [4] and [7], respectively).

[1] Note that deformations of AdS_5 solutions of $D = 11$ supergravity were studied in [9].

In this short note, we generalise the constructions in [8] for both type IIB and $D = 11$ supergravity, finding new classes of supersymmetric solutions with various values of z including $z = 2$.

Note Added: In the process of writing this paper up, we became aware of [10], which also constructs some of the supersymmetric solutions of type IIB supergravity that we present in section 2.

2 Solutions of type IIB supergravity

Consider the general ansatz for the bosonic fields of type IIB supergravity given by

$$\begin{aligned} ds^2_{10} &= \Phi^{-1/2}\left[2dx^+dx^- + h(dx^+)^2 + 2C dx^+ + dx_1^2 + dx_2^2\right] + \Phi^{1/2}ds^2(CY_3) \\ F_5 &= dx^+ \wedge dx^- \wedge dx_1 \wedge dx_2 \wedge d\Phi^{-1} + *_{CY_3} d\Phi \\ &\quad - dx^+ \wedge \left[*_{CY_3} dC + d\left(\Phi^{-1}C\right) \wedge dx_1 \wedge dx_2\right] \\ G &= dx^+ \wedge W \end{aligned} \tag{2.1}$$

where G is the complex three-form and the axion and dilaton are set to zero. Here Φ, h are functions, C is a one-form and W is a complex two-form all defined on the Calabi-Yau three-fold, CY_3. Our conventions for type IIB supergravity [11, 12] are as in [13]. One finds that all the equations of motion are satisfied provided that

$$\begin{aligned} \nabla^2_{CY}\Phi &= 0 \\ d *_{CY} dC &= 0 \\ dW = d *_{CY} W &= 0 \\ \nabla^2_{CY} h &= -|W|^2_{CY} \end{aligned} \tag{2.2}$$

where $|W|^2_{CY} \equiv (1/2!)W^{ij}W^*_{ij}$ with indices raised with respect to the CY metric. Observe that when $C = h = W = 0$ we have the standard D3-brane class of solutions with a transverse CY_3 space.

If we choose the two-form W to be primitive and have no $(0,2)$ pieces (i.e. just $(2,0)$ and/or $(1,1)$ components), on CY_3 then the solutions generically preserve 2 supersymmetries[2], which is enhanced to 4 supersymmetries if the CY_3 is flat. More

[2]We note that one can add a closed, primitive $(1,2)$-form A on CY_3 to the three-form G while still preserving the same amount of supersymmetry. This changes two of the equations to $\nabla^2_{CY}\Phi = -(1/2)|A|^2_{CY}$ and $d *_{CY} dC = i/2(W \wedge A^* - W^* \wedge A)$. Such solutions will not, in general, admit a scaling symmetry, so we shall not consider them further here, however we note that solutions with $W = 0$ and Galilean symmetry were presented in [14].

specifically, we introduce the orthonormal frame $e^+ = \Phi^{-1/4} dx^+, e^- = \Phi^{-1/4}(dx^- + C + \frac{h}{2} dx^+), e^2 = \Phi^{-1/4} dx^1$, etc. and choose positive orientation to be given by $e^{+-23} \wedge \mathrm{Vol}_{\mathrm{CY}}$, where $\mathrm{Vol}_{\mathrm{CY}}$ is the volume element on CY_3. Consider first the special case that $C = h = W = 0$. Then, as usual, a generic CY_3 breaks 1/4 of the supersymmetry, while the harmonic function Φ leads to a further breaking of 1/2, the Killing spinors satisfying the additional projection $\Gamma^{+-23}\epsilon = i\epsilon$. Switching on C, h, W we find that generically we need to also impose $\Gamma^+\epsilon = 0$ and $\Gamma^{ij} W_{ij} \epsilon^c = 0$.

We now specialise to the case that the CY_3 is a metric cone over a five-dimensional Sasaki-Einstein manifold SE_5, $ds^2(CY_3) = dr^2 + r^2 ds^2(SE_5)$. In order to get solutions with $\mathrm{Sch}(z)$ symmetry we now set

$$\begin{aligned} \Phi &= r^{-4} \\ C &= r^{\lambda_1}\beta \\ h &= r^{\lambda_2} q \\ W &= d\left(r^{\lambda_3}\sigma\right) \end{aligned} \tag{2.3}$$

where q is a function, β and σ are, respectively, a real and a complex one-form on SE_5, and λ_i are constants which we will take to be positive. The full solution now reads

$$\begin{aligned} ds^2_{10} &= r^2\left[2dx^+dx^- + r^{\lambda_2} q\,(dx^+)^2 + 2r^{\lambda_1}\,dx^+\beta + dx_1^2 + dx_2^2\right] + \frac{dr^2}{r^2} + ds^2\,(SE_5) \\ F_5 &= 4r^3\,dx^+\wedge dx^-\wedge dx_1\wedge dx_2\wedge dr + 4\,\mathrm{Vol}_{\mathrm{SE}_5} \\ &\quad - dx^+\wedge\left[r^{\lambda_1+1}\,dr\wedge *_{SE_5} d\beta + \lambda_1 r^{\lambda_1+2}\ *_{SE_5}\beta + d\left(r^{4+\lambda_1}\beta\right)\wedge dx_1\wedge dx_2\right] \\ G &= dx^+\wedge d\left(r^{\lambda_3}\sigma\right)\ . \end{aligned} \tag{2.4}$$

Generically, when $C, h, W \neq 0$, solutions with $\lambda_1 + 2 = 1 + \lambda_2/2 = \lambda_3 \equiv z$ will be $\mathrm{Sch}(z)$ invariant. In particular, the scaling acts on the coordinates via $(x^+, x^-, x_i, r) \to (\mu^z x^+, \mu^{2-z} x^-, \mu x_i, \mu^{-1} r)$ (for other transformations see [2]). Observe that if we set $C = h = W = 0$ then we have the standard $AdS_5 \times SE_5$ solution of type IIB. Generically, when $C, h, W \neq 0$, we still need to impose the projections mentioned above in order to preserve supersymmetry. Note in particular that, generically, half of the Poincaré supersymmetries of the $AdS_5 \times SE_5$ solution are preserved, while none of the special conformal supersymmetries are. It would be interesting to explore special subclasses of solutions with enhanced supersymmetry, which occur, for example, when the CY_3 is flat.

In [6], supersymmetric solutions with $W = C = 0$, $h \neq 0$ were constructed with

$$\nabla^2_{SE} q + \lambda_2(4 + \lambda_2) q = 0 \tag{2.5}$$

and give rise to solutions with $z = 1 + \lambda_2/2 \geq 3/2$, with the bound only achievable for $SE_5 = S^5$. In particular supersymmetric solutions with $z = 2$ were found, but, because the solutions have the metric component g_{++} positive in some regions of the SE_5, the solutions were argued to be unstable. In [8], supersymmetric solutions with $W = h = 0$, $C \neq 0$ were constructed with

$$\triangle_{SE}\beta = \lambda_1 (\lambda_1 + 2) \beta, \qquad d^\dagger \beta = 0 \tag{2.6}$$

where $\triangle_{SE} = dd^\dagger + d^\dagger d$ is the Hodge-deRahm operator on SE_5, and give rise to solutions with $z = 2 + \lambda_1 \geq 4$, with the bound achievable for any SE_5 space. More specifically, the bound is achieved when β is a one-form dual to a Killing vector on the SE_5 space; the class of such $z = 4$ solutions using the one-form dual to the Reeb vector on the SE_5 space were first constructed in [4]. It was also shown in [8] that one can combine these classes of solutions with $h, C \neq 0$ (still with $W = 0$), and providing that one can solve for q, β so that $2 + \lambda_1 = 1 + \lambda_2/2$ then the solutions have dynamical exponent $z = 2 + \lambda_1 \geq 4$.

We now consider $W \neq 0$. This implies that $h \neq 0$ and we need to set $\lambda_2 = 2(\lambda_3 - 1)$. In addition to (2.6) we also need to solve

$$\begin{aligned} \triangle_{SE}\sigma = \lambda_3 (\lambda_3 + 2) \sigma, \qquad d^\dagger \sigma = 0 \\ \nabla^2_{SE} q + 4(\lambda_3^2 - 1) q = -\lambda_3^2 |\sigma|^2_{SE} - |d\sigma|^2_{SE} \ . \end{aligned} \tag{2.7}$$

The solutions for which $\lambda_3 = 2 + \lambda_1$ are invariant under Sch(z)with $z = \lambda_3$. If $C \neq 0$ then since $\lambda_1 \geq 2$, necessarily we have $z \geq 4$.

If we set $C = 0$, which is needed to obtain supersymmetric solutions with $z = 2$ for example, then we just need to solve (2.7). The first equation implies that $z = \lambda_3 \geq 2$, with the bound being saturated when σ is a one-form dual to a Killing vector on the SE_5 space. A simple solution is obtained by taking $\sigma = c\eta$ for some constant c, where η is the canonical one-form dual to the Reeb vector on SE_5 and $q = -|c|^2$. This solution has $z = \lambda_3 = 2$ and was first constructed in [3, 4, 5]. Observe that for this solution $W = 2cJ_{CY}$. Thus while W is $(1,1)$ it is not primitive and so this solution does not preserve any supersymmetry as previously pointed out in [4]. On the other hand it is straightforward to construct solutions with $z = 2$ that are supersymmetric. For example, we can take any Killing vector on the SE_5 space that leaves invariant the Killing spinors on SE_5. It is straightforward to construct such solutions explicitly when the metric for the SE_5 is known explicitly as it is for the S^5, $T^{1,1}$ [15], $Y^{p,q}$ [16] and $L^{a,b,c}$ [17] spaces. For the case of S^5 it is also easy to construct explicit solutions for all values of z using spherical harmonics. It is worth noting that the

$z = 2$ solutions for the S^5 case can have q constant and negative and hence do not suffer from the instability discussed in [6]. This is easy to see since W must be a constant linear combination of the 15 harmonic two-forms on $\mathbb{R}^6$, $dx^i \wedge dx^j$, or, if we demand supersymmetry, of the eight primitive $(1,1)$ forms and three $(2,0)$ forms. Then, in general, q will be the sum of a negative constant with a scalar harmonic on S^5 with eigenvalue 12. It would be interesting to investigate the issue of stability further for all of the new solutions we have constructed. Some additional comments about the solutions are presented in appendix A.

3 Solutions of $D = 11$ supergravity

We consider the ansatz for the bosonic fields of $D = 11$ supergravity given by

$$\begin{aligned} ds^2 &= \Phi^{-2/3}\left[2dx^+dx^- + h\left(dx^+\right)^2 + 2dx^+C + dx_1^2\right] + \Phi^{1/3}ds^2\left(CY_4\right) \\ G &= dx^+ \wedge dx^- \wedge dx_1 \wedge d\Phi^{-1} + dx^+ \wedge V + dx^+ \wedge dx_1 \wedge d\left(\Phi^{-1}C\right) \end{aligned} \tag{3.1}$$

where Φ, h are functions, C is a one-form and V is a three-form all defined on[3] the Calabi-Yau four-fold, CY_4. Our conventions for $D = 11$ supergravity [18] are as in [19]. One finds that all the equations of motion are satisfied provided that

$$\begin{aligned} \nabla^2_{CY}\Phi &= 0 \\ d *_{CY} dC &= 0 \\ dV = d *_{CY} V &= 0 \\ \nabla^2_{CY}h &= -|V|^2_{CY} \end{aligned} \tag{3.2}$$

where $|V|^2_{CY} \equiv (1/3!)V^{ijk}V_{ijk}$ with indices raised with respect to the CY metric. When $C = h = V = 0$ we have the standard M2-brane class of solutions with a transverse CY_4 space.

If we choose the three-form V to only have $(2,1)$ plus $(1,2)$ pieces and be primitive on the CY_4 then the solutions generically preserve 2 supersymmetries[4], which is enhanced to 4 supersymmetries if the CY_4 us flat. More specifically, we introduce the orthonormal frame $e^+ = \Phi^{-1/6}dx^+, e^- = \Phi^{-1/6}(dx^- + C + \frac{h}{2}dx^+), e^2 = \Phi^{-1/6}dx^1$,

[3]It is straightforward to also consider other eight-dimensional special holonomy manifolds, but for simplicity we shall restrict our attention to CY_4.

[4]As an aside, we note that we can also add a closed, primitive $(2,2)$-form F on CY_4 to the four-form flux while still preserving the same amount of supersymmetry. This changes two of the equations to $\nabla^2_{CY}\Phi = -(1/2)|F|^2_{CY}$ and $d *_{CY} dC = V \wedge F$.

etc. and choose positive orientation to be given by $e^{+-2}\wedge \mathrm{Vol}_{\mathrm{CY}}$, where $\mathrm{Vol}_{\mathrm{CY}}$ is the volume element on CY_4. Consider first the special case that $C = h = V = 0$. Then, as usual, a non-flat CY_4 breaks 1/8 of the supersymmetry, and the harmonic function Φ can be added "for free" (the projection on the Killing spinors arising from the CY_4 automatically imply the projection $\Gamma^{+-2}\epsilon = -\epsilon$). Switching on C, h, V we find that generically we need to also impose $\Gamma^{+}\epsilon = 0$ and $\Gamma^{ijk}V_{ijk}\epsilon = 0$. Note as an aside that we can "skew-whiff" by changing the sign of the four-form flux and obtain solutions that generically don't preserve any supersymmetry (apart from the special case when $SE_7 = S^7$).

We now specialise to the case that the CY_4 is a metric cone over a seven-dimensional Sasaki-Einstein manifold SE_7, $ds^2(CY_4) = dr^2 + r^2 ds^2(SE_7)$. In order to get solutions with $\mathrm{Sch}(z)$ symmetry we now set

$$\begin{aligned}
\Phi &= r^{-6} \\
C &= r^{\lambda_1}\beta \\
h &= r^{\lambda_2} q \\
V &= d\left(r^{\lambda_3}\tau\right)
\end{aligned} \tag{3.3}$$

where q is a function, β and τ are, respectively, a one-form and a two-form on SE_7, and λ_i are constants which we will take to be positive. The full solution now reads

$$\begin{aligned}
ds^2 &= r^4\left[2dx^+dx^- + r^{\lambda_2}q\left(dx^+\right)^2 + 2r^{\lambda_1}\,dx^+\beta + dx_1^2\right] + \frac{dr^2}{r^2} + ds^2\left(SE_7\right) \\
G &= 6r^5\,dx^+\wedge dx^-\wedge dx_1\wedge dr + dx^+\wedge d\left(r^{\lambda_3}\tau\right) + dx^+\wedge dx_1\wedge d\left(r^{6+\lambda_1}\beta\right)
\end{aligned} \tag{3.4}$$

Generically, when $C, h, V \neq 0$, solutions with $2 + \lambda_1/2 = 1 + \lambda_2/4 = \lambda_3/2 \equiv z$ will be $\mathrm{Sch}(z)$ invariant. In particular, the scaling now acts as $(x^+, x^-, x_1, r) \to (\mu^z x^+, \mu^{2-z}x^-, \mu x_1, \mu^{-1/2}r)$. Note that if we set $C = h = V = 0$ then we have the standard $AdS_4\times SE_7$ solution. Generically, when $C, h, V \neq 0$, we still need to impose the projections mentioned above in order to preserve supersymmetry. Thus, generically, half of the Poincaré supersymmetries of the $AdS_4\times SE_7$ solution are preserved, while none of the special conformal supersymmetries are. It would be interesting to explore special subclasses of solutions with enhanced supersymmetry, which occur, for example, when the CY_4 is flat.

In [8], supersymmetric solutions with $C = V = 0$, $h \neq 0$ were constructed with

$$\nabla^2_{SE}q + \lambda_2(6+\lambda_2)q = 0 \tag{3.5}$$

and give rise to solutions with $z = 1+\lambda_2/4 \geq 5/4$, with the bound only achievable for $SE_7 = S^7$. In particular supersymmetric solutions with $z = 2$ were found, but they suffer from a similar instability to that found for the analogous type IIB solutions in [6]. In [8], supersymmetric solutions with $h = V = 0$, $C \neq 0$ were constructed with

$$\triangle_{SE}\beta = \lambda_1 (\lambda_1 + 4)\beta, \qquad d^\dagger \beta = 0 \tag{3.6}$$

and give rise to solutions with $z = 2 + \lambda_1/2 \geq 3$, with the bound achievable for any SE_7 space. More specifically, the bound is achieved when β is a one-form dual to a Killing vector on the SE_5 space; and one can always choose the one-form dual to the Reeb vector on the SE_7 space. It was also shown in [8] that one can combine these classes of solutions with $C, h \neq 0$, (still with $V = 0$), and providing that one can choose $4+2\lambda_1 = \lambda_2$ then they have dynamical exponent again with $z = 2+\lambda_1/2 \geq 3$.

We now consider $V \neq 0$. This implies $h \neq 0$ and we need to set $\lambda_2 = 2(\lambda_3 - 2)$. In addition to (2.6) we also need to solve

$$\begin{aligned} \triangle_{SE}\tau = \lambda_3 (\lambda_3 + 2)\tau, \qquad d^\dagger \tau = 0 \\ \nabla^2_{SE} q + 4(\lambda_3 - 2)(\lambda_3 + 1)q = -\lambda_3^2 |\tau|^2_{SE} - |d\tau|^2_{SE} \; . \end{aligned} \tag{3.7}$$

The solutions for which $\lambda_3 = 4 + \lambda_1$ are invariant under $\mathrm{Sch}(z)$ with $z = 2 + \lambda_1/2 = \lambda_3/2$. If $C \neq 0$ then necessarily we have $\lambda_1 \geq 2$ and hence $z \geq 3$.

If we set $C = 0$ then we just need to solve (3.7). Let us illustrate with some simple solutions when $SE_7 = S^7$. In fact it is easiest to directly solve (3.2). For example, if we let z^a be standard complex coordinates on $\mathbb{R}^8$, with Kähler form $\omega = (i/2) dz^a \wedge d\bar{z}^a$ we can take $V = c\, dz^1 d\bar{z}^2 d\bar{z}^3 + c.c.$, where c is constant, which obviously has only $(1,2)$ and $(2,1)$ pieces and is primitive, and $h = -c^2 r^2$ (setting a possible solution of the homogeneous equation in (3.2) to zero). This gives a supersymmetric solution with $\lambda_3 = 3$ and hence $z = 3/2$. In particular we note that the metric component g_{++} is always negative. A simple solution with $z = 2$ is obtained by splitting $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4$ and considering a sum of terms which are $(1,1)$ and primitive on one factor with a factor dx^i on the other:

$$\begin{aligned} V &= c\Big\{ \big[x^1 (dx^1 \wedge dx^2 - dx^3 \wedge dx^4) + x^3 (dx^2 \wedge dx^3 - dx^1 \wedge dx^4)\big] \wedge dx^5 \\ &+ \quad \big[x^2 (dx^1 \wedge dx^2 - dx^3 \wedge dx^4) + x^4 (dx^2 \wedge dx^3 - dx^1 \wedge dx^4)\big] \wedge dx^6 \\ &+ \quad \big[x^5 (dx^5 \wedge dx^6 - dx^7 \wedge dx^8) + x^7 (dx^6 \wedge dx^7 - dx^5 \wedge dx^8)\big] \wedge dx^1 \\ &+ \quad \big[x^6 (dx^5 \wedge dx^6 - dx^7 \wedge dx^8) + x^8 (dx^6 \wedge dx^7 - dx^5 \wedge dx^8)\big] \wedge dx^2 \Big\} \; . \end{aligned} \tag{3.8}$$

Solving for h (and setting to zero a solution of the homogeneous equation in (3.2)) we get

$$h = -\frac{c^2}{20} r^4 \ .$$

For this solution, the metric component g_{++} is again always negative. Clearly there are many additional simple constructions for the S^7 case that could be explored as well as for the more general class of other explicit SE_7 metrics.

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A Comments on solving (2.7)

Here we make a few further comments concerning solving (2.7) (which also have obvious analogues for solving (3.7)). To solve (2.7), we first solve the first line for σ and then substitute into the second. It is illuminating to expand out the source term in the right hand side of the equation in the second line using a complete set of scalar harmonics on the SE_5 space:

$$-\lambda_3^2|\sigma|^2_{SE} - |d\sigma|^2_{SE} = \sum_{I_l} a_{I_l} Y^{I_l} \tag{A.1}$$

where $\nabla^2_{SE} Y^{I_l} = -l(l+4)$, corresponding to the harmonic function $P^{I_l} = r^l Y^{I_l}$ on the CY_3 cone. We then find

$$q = \sum_{I_l} \frac{a_{I_l}}{4\lambda_3^2 - (l+2)^2} Y^{I_l} + q_0 \ . \tag{A.2}$$

In this expression we have allowed for the possibility of an arbitrary solution to the homogeneous equation, q_0, assuming it exists. The point is that the relevant putative eigenvalue for q_0 is fixed by the eigenspectrum of the Laplacian acting on one-forms. For the special case when $SE_5 = S^5$, for example, there is always such a possibility of adding a solution to the homogeneous equation. Another point to notice about (A.2) is that it only makes sense providing that the coefficient $a_{I_l} = 0$ whenever $2\lambda_3 = l+2$.

For the special case when $SE_5 = S^5$, not only is this coefficient zero but the sum appearing in (A.2) is a finite sum terminating at $l = 2\lambda_3 - 4$. To see this we observe that

$$a_{I_l} \propto \int_{S^5} Y^{I_l} (\lambda_3^2 |\sigma|^2_{SE} + |d\sigma|^2_{SE}) \tag{A.3}$$

which can be recast as an integral on the flat cone $\mathbb{R}^6$

$$a_{I_l} \propto \int_{\mathbb{R}^6} e^{-r^2} P^{I_l} W^{ij} W_{ij} \tag{A.4}$$

where for S^5, $r^2 = \sum_i x^i x^i$ and

$$P^{I_l} = C^I_{i_1 \dots i_l} x^{i_1} \cdots x^{i_l} \tag{A.5}$$

with $C^I_{i_1 \dots i_l}$ defining the scalar harmonics on S^5. To proceed we write W as

$$W = C^J_{j;ki_1 \dots i_{\lambda_3 - 1}} x^{i_1} \cdots x^{i_{\lambda_3 - 2}} \, dx^j \wedge dx^k \tag{A.6}$$

where $C^J_{j;i_1 \dots i_{\lambda_3-1}}$ define the vector spherical harmonics on S^5. In carrying out the integral (A.4) we will get all possible contractions of the l indices of the scalar spherical harmonic $C^I_{i_1 \dots i_l}$ with some of the $2\lambda_3 - 4$ indices

$$C^J_{[j;k] i_1 \dots i_{\lambda_3-2}} C^J_{[j;k] i'_1 \dots i'_{\lambda_3-2}} \tag{A.7}$$

In particular, since the tensor defining the scalar harmonic is traceless, we conclude that the a_{I_l} are zero for all I_l with $l > 2\lambda_3 - 4$.

Let us now consider this issue for a general SE_5 space, but in the special case when σ is a one-form dual to a Killing vector on SE_5 corresponding to $\lambda_3 = 2$ and hence $z = 2$. As above, we have (A.4). Write

$$W = d(r^2 \sigma) \equiv dT \tag{A.8}$$

and observe that on the CY_3 cone $\nabla_i T_j = \nabla_{[i} T_{j]}$ and that $\nabla^2_{CY} T_i = 0$. We then compute

$$\begin{aligned}
a_{I_l} &\propto \int_{CY} e^{-r^2} P^{I_l} W^{ij} W_{ij} \\
&= 4 \int_{CY} e^{-r^2} P^{I_l} \left(\nabla^i T^j\right) \left(\nabla_i T_j\right) \\
&= 2 \int_{CY} \nabla^2_{CY} \left(e^{-r^2} P^{I_l}\right) T^2 \\
&= 2 \int_{CY} e^{-r^2} \left(-4r\partial_r P^{I_l} - 12 P^{I_l} + 4r^2 P^{I_l}\right) T^2 \\
&= 4 \int_{CY} e^{-r^2} \left(-l + 2\right) P^{I_l} T^2 \ .
\end{aligned} \tag{A.9}$$

In getting to the last line one needs to take into account the r^5 factor in the measure and use

$$\int_0^\infty r^{n+2} e^{-r^2} dr = \frac{n+1}{2} \int_0^\infty r^n e^{-r^2} dr \ . \tag{A.10}$$

We thus conclude from (A.9) that the problematic coefficient a_{I_l} in (A.2) when $l = 2\lambda_3 - 2 = 2$ again vanishes for this class of solutions.

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