

Mellin-Barnes representations of Feynman diagrams, linear systems of differential equations, and polynomial solutions

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Abstract

We argue that the Mellin-Barnes representations of Feynman diagrams can be used for obtaining linear systems of homogeneous differential equations for the original Feynman diagrams with arbitrary powers of propagators without recourse to the integration-by-parts technique. These systems of differential equations can be used (i) for the differential reductions to sets of basic functions and (ii) for counting the numbers of master integrals.

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1 Introduction

The importance of the differential-equation approach to the description of the analytical properties of Feynman diagrams has been recognized a long time ago [1]. Within dimension regularization [2], the differential-equation technique [3], based on the integration-by-part (IBP) relations [4], has been one of the most popular tools for the analytical evaluation of Feynman diagrams during the last decade [5]. Here we argue that linear systems of homogeneous differential equations may be derived for Feynman diagrams starting from their Mellin-Barnes representations without resorting to IBP relations [6].

Our starting point is the multiple Mellin-Barnes representation of Feynman diagrams [7,8,9], which may be written in the following form:

$$\begin{aligned}\Phi(\mathbf{A}, \vec{B}; \mathbf{C}, \vec{D}; \vec{z}) &= \int_{-i\infty}^{+i\infty} d\vec{t} \phi(\vec{t}) \vec{z}^{\vec{t}} \\ &= \text{Const} \int_{-i\infty}^{+i\infty} \prod_{a,b,c,r} dt_c \frac{\Gamma(\sum_{i=1}^m A_{ai}t_i + B_a)}{\Gamma(\sum_{j=1}^r C_{bj}t_j + D_b)} z_k^{\sum_l \alpha_{kl}t_l},\end{aligned}\quad (1)$$

where z_k are ratios of Mandelstam variables and $\mathbf{A}, \vec{B}, \mathbf{C}, \vec{D}, \alpha$ are matrices and vectors depending linearly on the dimension n of space-time and the powers of the propagators. An important property of Feynman diagrams is that the matrices \mathbf{A} and \mathbf{C} only include integers. Let us define the polynomials P_i and Q_i as

$$\frac{P_i(\vec{t})}{Q_i(\vec{t})} = \frac{\phi(\vec{t} + \vec{e}_i)}{\phi(\vec{t})}, \quad (2)$$

where \vec{e}_i is a unit vector with nonzero element at the i -th place. Then, the integral in Eq. (1) satisfies the following linear system of homogeneous differential equations:

$$Q_i(\vec{t})|_{t_j \rightarrow \theta_j} \frac{1}{z_i} \Phi(\mathbf{A}, \vec{B}; \mathbf{C}, \vec{D}; \vec{z}) = P_i(\vec{t})|_{t_j \rightarrow \theta_j} \Phi(\mathbf{A}, \vec{B}; \mathbf{C}, \vec{D}; \vec{z}), \quad (3)$$

where $\theta_i = z_i d/dz_i$, we assume that, in Eq. (1), the number of variables z_i is equal to number of integration variables t_i , and, for simplicity, we put $\alpha_{ij} = \delta_{ij}$. Moreover, the function Φ defined in Eq. (1) satisfies the differential contiguous relations:

$$\begin{aligned}\Phi(\mathbf{A}, \vec{B} + \vec{e}_a; \mathbf{C}, \vec{D}; \vec{z}) &= \left(\sum_{i=1}^m A_{ai} \theta_i + B_a \right) \Phi(\mathbf{A}, \vec{B}; \mathbf{C}, \vec{D}; \vec{z}), \\ \Phi(\mathbf{A}, \vec{B}; \mathbf{C}, \vec{D} - \vec{e}_b; \vec{z}) &= \left(\sum_{j=1}^r C_{bj} \theta_j + D_b \right) \Phi(\mathbf{A}, \vec{B}; \mathbf{C}, \vec{D}; \vec{z}).\end{aligned}\quad (4)$$

This system of differential equations may be analyzed using the Gröbner basis technique [10]. Specifically, (i) the holonomic rank r of the system in Eq. (3) can be evaluated,¹ and (ii) starting from the direct differential operators in Eq. (4) and

¹This can also be done using a finite number of prolongations and converting the original system into a Pfaffian system [11].

the system of differential equations in Eq. (3), the inverse differential operators can be constructed [12]. By the action of such differential operators on the function Φ , the value of any parameter B_j, D_k may be shifted by an arbitrary integer. This algorithm is working for holonomic functions if the number of linearly independent solutions is finite. As was shown in Ref. [13], within analytical regularization [14], Feynman diagrams satisfy holonomic systems of linear differential equations under the condition that all particles have different masses. In fact, this statement is the basis of the algorithm proposed in Ref. [15]. To our understanding, a rigorous proof for the case of on-shell diagrams or diagrams with zero internal masses does not yet exist, albeit very interesting work has been done in this direction [16]. For our analysis, it is sufficient to assume that there is a set of independent variables z_i for each Mellin-Barnes integral² in Eq. (1). Moreover, it has been shown [18] recently that Mellin-Barnes integrals satisfy systems of differential equations corresponding to Gelfand-Kapranov-Zelevinsky hypergeometric equations [19]. Another necessary condition is that two contours differing by a translation by one unit along the real axis are equivalent. From the analysis performed in Refs. [8,20], we expect that this statement is valid for all Feynman diagrams before constructing ε expansions.

The aim of the present paper is to illustrate this approach and to outline how it can be used to count master integrals, considering as examples one-fold Mellin-Barnes integrals and the corresponding Feynman diagrams. In fact, the evaluation of master integrals is often the bottleneck of multi-loop calculations of both Feynman diagrams and scattering amplitudes, especially if several different mass scales are involved, and any opportunity to reduce their number below the achievements of the standard techniques of solving IBP relations, as implemented in various publicly available computer codes, should be highly welcome. The technique advocated here may also allow one to gain deeper insights into the mathematical structures of multi-scale Feynman diagrams.

2 One-fold Mellin-Barnes integral

Let us consider the function

$$\Phi(\vec{A}, \vec{B}; \vec{C}, \vec{D}; z; r) = C_\Phi \int_{\gamma+iR} dt \frac{\prod_{i=0}^K \Gamma(A_i + t) \prod_{j=0}^L \Gamma(C_j - t)}{\prod_{k=0}^R \Gamma(B_k + t) \prod_{l=0}^J \Gamma(D_l - t)} \Gamma(-t) z^{t+r}, \quad (5)$$

where C_Φ is some z -independent constant depending on ratios of Γ functions with arguments being linear combinations of powers of propagators and the space-time dimension n . In the remainder of this letter, we assume that $r = 0$. In order to restore a nonzero value of r , it is sufficient to substitute $\theta \rightarrow \theta - r$. Let us assume that the differences between any two parameters A_i, B_j, C_k, D_l are not integers.

² The application of the summation technique to Mellin-Barnes integrals with $z_i = 1$ was discussed in Ref. [17].

Then, this function satisfies the following homogeneous differential equation:

$$\begin{aligned} & (-1)^{L+1} \theta \prod_{i=1}^R (\theta + B_i - 1) \prod_{j=1}^L (\theta - C_j) \Phi(\vec{A}, \vec{B}; \vec{C}, \vec{D}; z) \\ &= (-1)^J z \prod_{i=1}^K (\theta + A_i) \prod_{j=1}^J (\theta - D_j + 1) \Phi(\vec{A}, \vec{B}; \vec{C}, \vec{D}; z), \end{aligned} \quad (6)$$

where $\theta = zd/dz$. Let us consider the case of the non-confluent function, for which the orders of the differential equations on the l.h.s. and r.h.s. of Eq. (6) are equal to each other, viz.

$$K + J = 1 + L + R \equiv p, \quad (7)$$

so that the function Φ satisfies a differential equation of order p . In this case, there are p linearly independent solutions of the differential equation. In accordance with Takayama's algorithm [12], the differential operators inverse to the operators defined by Eq. (4) can be constructed, and the result of the differential reduction applied to the function Φ has the following form:

$$P_0 \Phi(\vec{A} + \vec{m}_1, \vec{B} + \vec{m}_2; \vec{C} + \vec{m}_3, \vec{D} + \vec{m}_4; z) = \sum_{i=0}^{p-1} R_i \theta^i \Phi(\vec{A}, \vec{B}; \vec{C}, \vec{D}; z), \quad (8)$$

where P_0, R_i are some polynomials and \vec{m}_i are sets of integers.

Theorem 1:

Any Feynman diagram associated with the function Φ defined by the Mellin-Barnes integral in Eq. (5), under the conditions that all parameters as well the differences between any two parameters are not integer and Eq. (7) is valid, has p master integrals (including all integrals following from the original one by contracting one or more lines), where p is defined by Eq. (7).

This theorem follows from Takayama's analysis of the differential equations in Eq. (6) with irreducible monodromy groups.

Under the conditions of **Theorem 1** and the additional condition $C_i \neq C_j$, it is possible to close the contour of integration [8] and to write the function Φ in terms of linear combinations of hypergeometric functions whose series representations are well defined in the vicinity of the point $z = 0$:

$$\begin{aligned} \frac{z^{-r}}{C_\Phi} \Phi(\vec{A}, \vec{B}; \vec{C}, \vec{D}; z) &= \frac{\prod_{i=1}^K \Gamma(A_i) \prod_{j=1}^L \Gamma(C_j)}{\prod_{k=1}^R \Gamma(B_k) \prod_{l=1}^J \Gamma(D_l)} {}_pF_{p-1} \left(\begin{matrix} \vec{A}, \vec{1} - \vec{D} \\ \vec{B}, \vec{1} - \vec{C} \end{matrix} \middle| (-1)^{1+L+J} z \right) \\ &+ \sum_{m=1}^L z^{C_m} \frac{\prod_{i=1}^K \Gamma(A_i + C_m) \prod_{j=1, j \neq m}^L \Gamma(C_j - C_m)}{\prod_{k=1}^R \Gamma(B_k + C_m) \prod_{l=1}^J \Gamma(D_l - C_m)} \\ &\times {}_pF_{p-1} \left(\begin{matrix} \vec{A} + C_m, \vec{1} + C_m - \vec{D} \\ \vec{B} + C_m, \vec{1} + C_m - \vec{C} \end{matrix} \middle| (-1)^{L+J} z \right), \end{aligned} \quad (9)$$

where \vec{C} denotes the set of parameters \vec{C} excluding C_m .

Corollary 1:

The number of nontrivial master integrals of a Feynman diagram satisfying the conditions of **Theorem 1** and the additional condition that $C_i \neq C_j$ is equal to the number of basic functions for any hypergeometric function on the r.h.s. of Eq. (9).

Comment 1:

The application of the relation

$$\Gamma(nx) = n^{nx-1/2} (2\pi)^{\frac{1-n}{2}} \Gamma(x) \Gamma\left(x + \frac{1}{n}\right) \cdots \Gamma\left(x + \frac{n-1}{n}\right), \quad (10)$$

where n is integer and $x = t + p/q$, allows us to express any one-fold Mellin-Barnes integral as an integral of the form defined by Eq. (1).

A special consideration is necessary when some parameters or some differences between parameters are integer. In this case, the final expression for the differential reduction of Horn-type hypergeometric functions has a simpler form (for details, see Ref. [21]). To evaluate the dimension of the solution space of the solutions of the differential equation in Eq. (6), a classical technique can be applied [22]. However, for practical applications to Feynman diagrams, we only need two particular cases, namely case (i) when the differences between upper (A_i, D_j) and lower (B_k, C_l) parameters are positive integers and case (ii) when some of the parameters are positive integers.

In case (i), the original Mellin-Barnes integral in Eq. (5) may be simplified as

$$\int dt z^t \frac{\Gamma(A+m+t)}{\Gamma(A+t)} F(t) = (A+m-1+\theta)(A+m-2+\theta) \cdots (A+\theta) \int dt z^t F(t). \quad (11)$$

In this case, the order of the differential equation in Eq. (6) is $p-1$, and there are $p-1$ nontrivial master-integrals. After the application of step-up/step-down operators to the r.h.s. of Eq. (9), we obtain the hypergeometric functions ${}_{p-1}F_{p-2}$, and **Corollary 1** is valid.

In case (ii), the differential equation in Eq. (6) has a common differential factor θ , since $z(\theta+1)f(z) = \theta(zf(z))$. In this case, the result of the differential reduction of Eq. (8) has the following form:

$$P_0 \Phi(\vec{A} + \vec{m}_1, \vec{B} + \vec{m}_2; \vec{C} + \vec{m}_3, \vec{D} + \vec{m}_4; z) = \sum_{i=0}^{p-2} R_i \theta^i \Phi(\vec{A}, \vec{B}; \vec{C}, \vec{D}; z) + R_{p-1}(z), \quad (12)$$

where P_0, R_i are some polynomials and \vec{m}_i are sets of integers. The same is true for the r.h.s. of Eq. (9), so that **Corollary 1** is again valid.

Corollary 2:

The number of nontrivial master-integrals of the Feynman diagram associated with the function Φ defined by Eq. (5), under the condition that $C_i \neq C_j$, is equal to number of basic functions for any hypergeometric function on the r.h.s. of Eq. (9).

Comment 2:

The differential equation in Eq. (6) can be factorized due to the relation

$$z(\theta+1+a)f(z) = (\theta+a)(zf(z)), \quad (13)$$

where a is an arbitrary parameter. This equation corresponds to the factor $\Gamma(a - t)\Gamma(1 - a + t)$ in the numerator or denominator of the Mellin-Barnes integral in Eq. (5).

Comment 3:

Also Tarasov [15] proposed that, in the framework of Ref. [13], there is a one-to-one correspondence between the number of master integrals obtained from the IBP relations and the dimension of the solution space of a corresponding system of differential equations.

In this way, we showed that our conjecture presented in Ref. [21] is correct. Below, we present its multivariable generalization:

Proposition 1:

When a multivariable Mellin-Barnes integral can be presented as a linear combination of multivariable Horn-type hypergeometric functions with rational coefficients³ about some points $z_i = z_i^0$, the holonomic rank of the corresponding system of linear differential equations is equal to the holonomic rank of any hypergeometric function in its hypergeometric representation.

The proof of this proposition is based on the same technology, namely the comparison of the holonomic rank of the system of differential equations in Eq. (3) with the holonomic rank of each term of its hypergeometric representation. A two-variable example of this statement was presented in Ref. [23].

Conjecture 1:

Any polynomial (rational) solution of a multivariable linear system of differential equations related to a Feynman diagram can be written as a product of one-loop bubble integrals and massless propagator or vertex integrals.

3 Feynman diagrams

Several examples of Feynman diagrams corresponding to the function Φ in Eq. (5) were presented in Refs. [21,23,24]. The results of the analysis performed in Refs. [21,23,24] are in agreement with **Corollary 2** of the present paper. Nevertheless, for the illustration of the advocated technique, we consider here the diagrams depicted in Fig. 1. In all examples, we put $r_i = 0$, since non-zero values of r_i do not affect the order of the differential equations and may be easily restored by the redefinitions $\theta_i \rightarrow \theta_i - r_i$.

3.1 One-loop vertex QED₁

Let us consider the one-loop QED vertex diagram with one massive internal line and two external lines being on mass shell, which is labeled QED₁ in Fig. 1. It is

³We called such a representation hypergeometric.

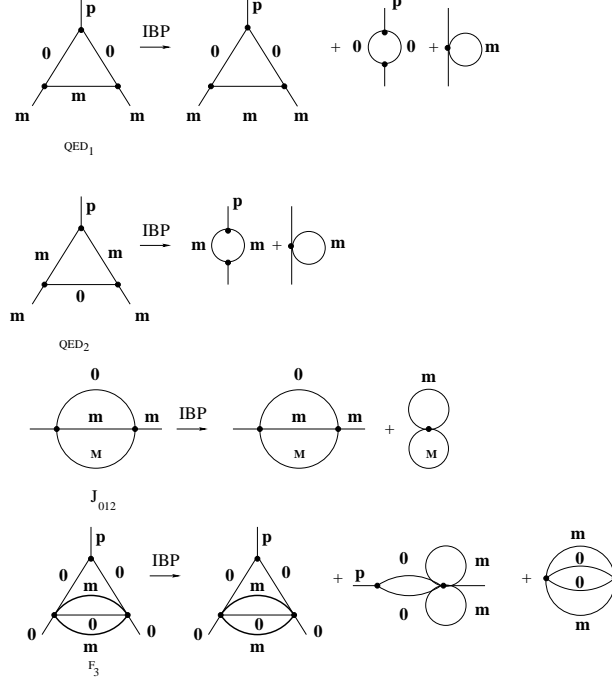


Figure 1: The diagrams on the r.h.s. emerge from the original one on the l.h.s. with arbitrary positive powers of propagators by the systematic contraction of one line.

given by

$$\begin{aligned}
C_{\text{QED}_1}(\sigma_1, \sigma_2, j_3) &\equiv \int \frac{d^n k}{[(k-p_1)^2]^{\sigma_1} [(k+p_2)^2]^{\sigma_2} [k^2-m^2]^{j_3}} \Big|_{p_1^2=p_2^2=m^2} \\
&= \frac{i^{1-n} \pi^{n/2} (-m^2)^{\frac{n}{2}-\sigma_{12}-j_3}}{\Gamma(n-\sigma_{12}-j_3) \Gamma(j_3) \Gamma(\sigma_1) \Gamma(\sigma_2)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dt \left(-\frac{k^2}{m^2}\right)^t \Gamma(-t) \\
&\quad \times \Gamma(\sigma_1+t) \Gamma(\sigma_2+t) \Gamma\left(\sigma_{12}+j_3-\frac{n}{2}+t\right) \Gamma(n-j_3-2\sigma_{12}-2t) ,
\end{aligned} \tag{14}$$

where $\sigma_{12} = \sigma_1 + \sigma_2$ and $k^2 = (p_1 - p_2)^2$. This diagram corresponds to a Φ function with six parameters,

$$C_{\text{QED}_1}(\sigma_1, \sigma_2, j_3) = \Phi_{\text{QED}_1} \left(\sigma_1, \sigma_2, \sigma_{12}+j_3-\frac{n}{2}; -; \frac{n-j_3}{2}-\sigma_{12}, \frac{n-j_3+1}{2}-\sigma_{12}, 0; -; z \right) , \tag{15}$$

and satisfies a third-order differential equation,

$$\begin{aligned}
&[\theta(\theta-C_1)(\theta-C_2) + z(\theta+A_1)(\theta+A_2)(\theta+A_3)] \Phi_{\text{QED}_1}(A_1, A_2, A_3; -; C_1, C_2, 0; -; z) \\
&= 0 .
\end{aligned} \tag{16}$$

With the help of differential operators,

$$\Phi_{\text{QED}_1}(A_i+1) = (\theta+A_i) \Phi_{\text{QED}_1}(A_i) , \tag{17}$$

it is easy to reduce the value of A_1 (or A_2) to unity, and the one of A_3 to $1 - C_1$ or $1 - C_2$, depending on whether j_3 is even or odd. In this case, there is a double factorization of the differential equation in Eq. (16) due to Eq. (13),

$$\theta(\theta - C_1) [(\theta - C_2) + z(\theta + A_2)] \Phi_{\text{QED}_1}(1, A_2, 1 - C_1; -, C_1, C_2, 0; -, z) = 0. \quad (18)$$

As a consequence, there are two polynomial solutions. We conclude from our analysis that there are three master integrals corresponding to the order of the differential equation in Eq. (16): one is a nontrivial function and two are polynomials.

3.2 One-loop vertex QED₂

Let us now consider the one-loop QED vertex diagram with two massive internal lines and two external lines being on mass shell, which is labeled QED₂ in Fig. 1. It is given by

$$\begin{aligned} C_{\text{QED}_2}(j_1, j_2, \sigma) &\equiv \int \frac{d^n k}{[(k - p_1)^2 - m^2]^{j_1} [(k + p_2)^2 - m^2]^{j_2} (k^2)^\sigma} \Big|_{p_1^2 = p_2^2 = m^2} \\ &= \frac{i^{1-n} \pi^{n/2} (-m^2)^{\frac{n}{2} - j_{12} - \sigma} \Gamma(n - j_{12} - 2\sigma)}{\Gamma(j_1) \Gamma(j_2) \Gamma(n - j_{12} - \sigma)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dt \left(-\frac{k^2}{m^2}\right)^t \\ &\quad \times \frac{\Gamma(-t) \Gamma(j_1 + t) \Gamma(j_2 + t) \Gamma(j_{12} + \sigma - \frac{n}{2} + t)}{\Gamma(j_{12} + 2t)}. \end{aligned} \quad (19)$$

This diagram corresponds to a Φ function with six parameters,

$$C_{\text{QED}_2}(j_1, j_2, \sigma) = \Phi_{\text{QED}_2}\left(j_1, j_2, j_{12} + \sigma - \frac{n}{2}, \frac{j_{12}}{2}, \frac{j_{12} + 1}{2}, 0; -, -, z\right). \quad (20)$$

Repeatedly applying differential operators, $\Phi_{\text{QED}_2}(A_i + 1) = (\theta + A_i) \Phi_{\text{QED}_2}(A_i)$, and $\Phi_{\text{QED}_2}(B_i - 1) = (\theta + B_i - 1) \Phi_{\text{QED}_2}(B_i)$, we reduce A_1 , A_2 , and B_1 to unity, so that the six-parameter Φ function reduces to a four-parameter one,

$$\Phi_{\text{QED}_2}(1, 1, A; 1, B; 0; -, z) = \Phi_{\text{QED}_2}(1, A; B; 0; -, z), \quad (21)$$

which satisfies a second-order differential equation,

$$\theta[(\theta + B - 1) + z(\theta + A)] \Phi_{\text{QED}_2}(1, A; B; 0; -, z) = 0. \quad (22)$$

Consequently, there are two master integrals, one of which is polynomial. This result does not depend on whether σ is integer or not.

3.3 Two-loop sunset diagram J_{012}

Let us now consider the two-loop sunset diagram from Ref. [25], which is given by

$$\begin{aligned} J_{012}(\sigma, \alpha, \beta) &\equiv \int \frac{d^n(k_1 k_2)}{[(k_1 - p)^2]^\sigma [k_1^2 - M^2]^\alpha [(k_1 - k_2)^2 - m^2]^\beta} \Big|_{p^2 = m^2} \\ &= \frac{[i^{1-n} \pi^{n/2}]^2 (-m^2)^{n - \alpha - \sigma - \beta} \Gamma(\frac{n}{2} - \sigma)}{\Gamma(\sigma) \Gamma(\alpha) \Gamma(\beta)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dt \left(\frac{M^2}{m^2}\right)^t \\ &\quad \times \frac{\Gamma(\sigma + \alpha - \frac{n}{2} + t) \Gamma(\alpha + \beta + \sigma - n + t) \Gamma(-t) \Gamma(\frac{n}{2} - \alpha - t) \Gamma(2n - \alpha - 2\sigma - 2\beta - 2t)}{\Gamma(n - \sigma - \alpha - t) \Gamma(\frac{3n}{2} - \sigma - \alpha - \beta - t)}. \end{aligned} \quad (23)$$

This integral corresponds to a Φ function with eight parameters and satisfies a fourth-order differential equation,

$$\begin{aligned} & \theta \left(\theta - \frac{n}{2} + \alpha \right) (\theta - n + \frac{\alpha}{2} + \sigma + \beta) (\theta - n + \frac{\alpha-1}{2} + \sigma + \beta) \Phi_{J_{012}} = \\ & z \left(\theta - \frac{n}{2} + \sigma + \alpha \right) (\theta - n + \alpha + \beta + \sigma) (\theta - n + \sigma + \alpha + 1) \left(\theta - \frac{3n}{2} + \sigma + \alpha + \beta + 1 \right) \Phi_{J_{012}} . \end{aligned} \quad (24)$$

With the help of step-up/step-down operators, this equation can be written as follows:

$$\left(\theta - \frac{n}{2} + I_1 \right) (\theta - n + I_2) \left[\theta \left(\theta - n + \frac{1}{2} + I_3 \right) - z \left(\theta - \frac{3n}{2} + I_4 \right) \right] \Phi_{J_{012}} = 0 . \quad (25)$$

Consequently, there are four master integrals, two of which are polynomial. All topologically possible integrals for this diagram are depicted in Fig. 1. The right-most one is polynomial in this approach. In order to have four master integrals, it is necessary that the diagram with the original topology has three master integrals, which we may take to have the propagator powers $(1, 1, 1)$, $(1, 1, 2)$, and $(1, 2, 1)$. We note that this is in accordance with the results of Ref. [26]. However, one of these three master integrals should be polynomial. This polynomial solution was first derived in Ref. [27]. An alternative derivation has recently be presented in Ref. [28].

3.4 Three-loop vertex diagram F

Let us consider the vertex diagram denoted as F_3 in Fig. 1, which is given by

$$\begin{aligned} & F(\vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3, \alpha_1, \alpha_2) \\ &= \int \frac{d^n(k_1 k_2 k_3)}{[(k-p_1)^2]^{\sigma_1} [(k+p_2)^2]^{\sigma_2} [k_2^2 - m^2]^{\alpha_1} [k_3^2 - m^2]^{\alpha_2} [(k_1 + k_2 + k_3)^2]^{\sigma_3}} \Big|_{p_1^2 = p_2^2 = 0} , \end{aligned} \quad (26)$$

where we have introduced a “dressed” massless propagator, as in Eq. (45) of Ref. [21]. Instead of three massless lines, we consider q_1 , q_2 , and q_3 massless lines corresponding to propagators with powers σ_1 , σ_2 , and σ_3 , respectively. In the present case, this is equivalent to the redefinition $\sigma_j \rightarrow \sigma_j - \frac{n}{2}(q_j - 1)$. The Mellin-Barnes integral for the vertex diagram of Eq. (26) has the following form:

$$\begin{aligned} & F(\vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3, \alpha_1, \alpha_2) \\ &= \text{Const} \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dt \left(-\frac{p^2}{m^2} \right)^t \frac{\Gamma(\alpha_1 + t) \Gamma(\alpha_2 + t) \Gamma(\alpha_{12} - \frac{n}{2} + t) \Gamma(\frac{n}{2} + t)}{\Gamma(\alpha_{12} + 2t)} \\ & \times \frac{\Gamma(\vec{\sigma}_3 - \frac{n}{2} q_3 - t) \Gamma(\frac{n}{2} q_{13} - \vec{\sigma}_{13} + t) \Gamma(\frac{n}{2} q_{23} - \vec{\sigma}_{23} + t) \Gamma(\vec{\sigma}_{123} - \frac{n}{2}(q_{123} - 1) - t)}{\Gamma(\frac{n}{2}(q_3 + 1) - \sigma_3 + t) \Gamma(\frac{n}{2} q_{123} - \vec{\sigma}_{123} + t)} . \end{aligned} \quad (27)$$

where Const is some constant irrelevant for the differential equation.

This diagram corresponds to a Φ function with twelve parameters. The first simplification arises from the step

$$\frac{\Gamma(\alpha_1 + t) \Gamma(\alpha_2 + t)}{\Gamma(\alpha_{12} + 2t)} \rightarrow \frac{\Gamma(I_1 + t)}{\Gamma(\frac{1}{2} + I_2 + t)} , \quad (28)$$

where I_a are integers. The further procedure strongly depends on whether the values of q_1 and q_2 are even or odd. For simplicity, we put $q_1 = q_2 = q_3 = 1$ and denote this integral as F_3 . Then the following step is valid:

$$\left. \frac{\Gamma\left(\frac{n}{2}q_{13} - \vec{\sigma}_{13} + t\right) \Gamma\left(\frac{n}{2}q_{23} - \vec{\sigma}_{23} + t\right)}{\Gamma\left(\frac{n}{2}(q_3 + 1) - \sigma_3 + t\right)} \right|_{q_1=q_2=q_3=1} \rightarrow \Gamma(n + I_3 + t) , \quad (29)$$

so that F_3 satisfies a homogeneous differential equation of fourth order that may be written in the following form:

$$\begin{aligned} & \left(\theta + \frac{1}{2} + I_1\right) \left(\theta + \frac{3}{2}n + I_2\right) \left(\theta + \frac{n}{2} + I_3\right) (\theta + n + I_4) \Phi_{F_3} \\ & = z (\theta + I_5) \left(\theta - \frac{n}{2} + I_7\right) \left(\theta + \frac{n}{2} + I_6\right) (\theta + n + I_8) \Phi_{F_3} . \end{aligned} \quad (30)$$

Applying step-up/step-down operators, this equation can be converted to the form

$$\left(\theta + \frac{n}{2}\right) (\theta + n) \left[\left(\theta + \frac{1}{2} + I_1\right) \left(\theta + \frac{3}{2}n + I_2\right) - z (\theta + I_3) \left(\theta - \frac{n}{2} + I_4\right) \right] \Phi_{F_3} = 0 . \quad (31)$$

Consequently, for the vertex diagram F_3 there are four master integrals, two of which are polynomial. All topologically possible integrals for this case are shown in Fig. 1. In order to have four master integrals, it is necessary that the diagram with the original topology has two master integrals.

4 Discussion and conclusions

We proposed a novel way of finding linear systems of homogeneous differential equations for Feynman diagrams with arbitrary powers of propagators. It is based on the Mellin-Barnes representation and does not rely on the IBP relations [4]. Systems of equations such as Eq. (3) are the left ideals in the Weyl algebra of linear differential operators with polynomial coefficients. Exploiting the Gröbner basis technique [10], the original diagrams may be explicitly reduced to sets of basis functions, and their holonomic ranks, i.e. the numbers of their linearly independent solutions, may be evaluated.

For the one-variable case, we presented a very simple algorithm for counting polynomial (rational) solutions of differential equations arising from the Mellin-Barnes representations associated with Feynman diagrams. This reduces the problem to the one of factorizing the homogeneous differential equation over the field of polynomials. We presented a few simple examples to illustrate our technique. The generalization of this algorithm to the multivariable case is nontrivial.

With the help of the new technology presented here, we proved the conjecture formulated in Ref. [21] regarding the counting of the numbers of master integrals via hypergeometric representations. This result may be useful for searching polynomial (rational) solutions of multivariable Feynman diagrams.

We suggest that any polynomial (rational) solution corresponds to a product of one-loop bubbles and massless single-scale diagrams with coefficients that are

products of Gamma functions (see **Conjecture 1**). Even in the one-variable case, such a correspondence does not emerge from the application of standard IBP relations, as was pointed out in Refs. [27,28] for the case of the two-loop sunset diagram J_{012} . With help of the technique presented here, all such algebraic relations between master integrals of the type studied in Ref. [21,23,24] may be easily derived.

Hypergeometric functions provide us with a remarkable tool for deepening our understanding of the mathematical structures underlying Feynman diagrams, and the present analysis allows us to draw the following picture. From the fact that any Feynman diagram with arbitrary powers of propagators is reducible to a set of master integrals, including bubble diagrams and massless propagators, it follows that a given Feynman diagram corresponds to a special function with a reducible monodromy group (see also Ref. [29,30]). The dimension of the irreducible part of the monodromy group, which is equal to the dimension of the solution space of the Pfaff system of differential equations, is equal to the number of master integrals generated via IBP relations, provided the full set of the latter is exploited. It is interesting to note that the simplest way of avoiding the reducibility of the monodromy group is to introduce different non-integer parameters for each propagator. This may be regarded as a generalization of the analytical regularization.

The proposed method to derive differential equations from the Mellin-Barnes representations of Feynman diagrams with non-unit values of z_i can be directly applied to study massless propagator diagrams at higher loop orders. In this case, we have to tackle with multiple (threefold and higher) Mellin-Barnes integrals. We shall return to this issue in a future publication.

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