

A Toy Model For Single Field Open Inflation

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Inflation in an open universe produced by Coleman-De Luccia (CDL) tunneling induces a friction term that is strong enough to allow for successful small-field inflation in models that would otherwise suffer from a severe overshoot problem. In this paper, we present a polynomial scalar potential which allows for a full analysis. This provides a simple model of single-field open inflation on a small-field inflection point after tunneling. We present numerical results and compare them with analytic approximations.

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1 Introduction

Recently, inflation in open universes has gained significant interest in the context of the string theory landscape [1, 2, 3, 4, 5]. Transitions between its plethora of different vacua most likely happen via tunneling processes [6, 7, 8]. The resulting bubble looks like an open Friedmann-Robertson-Walker universe for observers in the interior. If we assume that at early times, the universe was trapped in a false vacuum at high energy, it is conceivable that we are now sitting inside a bubble of (hopefully) true vacuum. Even though cosmological observations give rather stringent limits on the curvature of the universe, $-0.0133 < \Omega_k < 0.0084$ at 95%CL, having an open universe is not excluded [9]. Quite contrary to this, it appears to us that there is a strong theoretical prior for $\Omega_k \lesssim 0$ if one accepts the landscape paradigm of string theory. There simply is no other way known to traverse the vacuum energy landscape but by tunneling, leading to an open, inflating universe inside the low energy bubble. In this case, the curvature of the universe will be redshifted to (probably) unobservably small values by exponential expansion.

It is quite easy to devise potentials with successful large-field inflation – that is, inflation lasts long enough to solve the well-known cosmological homogeneity, flatness and isotropy problem. However, these models typically involve field motion over a distance of several Planck masses. This necessitates protection of the potential by a spontaneously broken shift symmetry. Such a shift symmetry can be derived from a fundamental description such as string theory, using constructions such as N-flation or axion monodromy.

In contrast to this, small-field inflection point inflation models do not need a symmetry. Instead, they derive their slow-roll flatness from fine-tuning dimension-6 operators which may be computable in a given string theory embedding (for a recent review on large-field and small-field inflation models from string theory, see e.g. [10]). Small-field inflation models potentially suffer from a severe overshoot problem [11].

Open inflation after CDL tunneling was discussed before in the context of single-field chaotic inflation in a quadratic potential [12]. In this model, the barrier and the false vacuum were supplied by adding a non-polynomial term to the standard $m^2\phi^2$ term that has the form of a Lorentz-resonance line. Here, the negative curvature inside the CDL bubble is not needed to prevent an overshoot problem: the quadratic potential after exit from the barrier generates large-field inflation which has a dynamical slow-roll attractor with a wide

field range. In [2], a two-field chaotic open inflation model was constructed. Here, tunneling proceeds from a high-mass high-vacuum energy de Sitter minimum predominantly in the direction of a 2nd non-inflaton scalar field, and exits into a shallow quadratic large-field inflation potential valley.

In [13], it was shown that in inflation after tunneling, the resulting friction term – coming from the curvature contribution – in the evolution equation for the inflaton can alleviate this problem.¹ For monomial potentials of sufficiently high order, the negative spatial curvature inside the CDL bubble is strong enough to allow for successful inflation in small-field models that would otherwise suffer from a severe overshoot problem. In this paper, we analyze a simple polynomial single-field scalar potential, encompassing tunnelling from the false to the inflationary vacuum, as well as a subsequent period of open inflation.

In Section 2, we give a brief review of how to compute the tunneling process, based on pioneering work by [6]. In Section 3 we review inflation in an open universe. Section 4 contains the description of tunneling and inflation in a toy model, where we present numerical results and compare them with analytic approximations developed in [16, 17]. Finally we conclude in Section 5.

2 Tunneling

The analysis of tunneling from false to true vacuum in scalar field was pioneered by [6, 7]. The probability per unit volume for the transition from the false vacuum at ϕ_+ to the true vacuum located at ϕ_- in a potential $V(\phi)$ which has a barrier at $\phi_+ < \phi_T < \phi_-$ is given by

$$\frac{\Gamma}{V} = Ae^{-B}, \quad (2.1)$$

where B is given by

$$B = 2\pi^2 \int dr r^3 \left(\frac{1}{2} \phi_B'^2 + V(\phi) \right) - 2\pi^2 \int dr r^3 V(\phi_+), \quad (2.2)$$

r the Euclidean radius, and ϕ_B is the $O(4)$ symmetric, so-called bounce solution to the Euclidean equation of motion

$$\phi_B''(r) + \frac{3}{r} \phi_B'(r) - \partial_\phi V(\phi) \Big|_{\phi=\phi_B} = 0. \quad (2.3)$$

¹The slowing effect of the negative curvature inside a CDL bubble on a scalar field was discussed before for a piecewise linear potential in [14], and noted in general already in [15].

We work in units $M_p^2 = \frac{1}{8\pi G} = 1$. In general, the field ϕ does not materialize exactly in the true vacuum at ϕ_- , but rather some distance away from it² at a point that we denote by ϕ_0 , see Figure 2(c). In the center of the nucleated bubble $r = 0$, the field is located at $\phi(0) = \phi_0 > \phi_T$ with $\phi'(0) = 0$. Crossing the bubble wall located approximately at R_T (with $\phi(R_T) = \phi_T$), the bounce interpolates between ϕ_0 and the false vacuum at $\phi(R_+) = \phi_+^3$.

3 Open Inflation

In this section, we give a brief overview of open inflation driven by a scalar field. Scalar field dynamics in an open FRW background

$$ds^2 = dt^2 - a(t)^2 \left(\frac{1}{1+r^2} dr^2 + r^2 d\Omega \right), \quad (3.4)$$

where $a(t)$ is the scale factor, is governed by the following equations of motion

$$\ddot{\phi} = -3\frac{\dot{a}}{a}\dot{\phi} - \partial_\phi V(\phi), \quad (3.5)$$

$$\frac{\ddot{a}}{a} = \frac{1}{3} \left(-\dot{\phi}^2 + V(\phi) \right), \quad (3.6)$$

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) + \frac{1}{a^2}. \quad (3.7)$$

At early times, the friction term in (3.5) is dominated by the curvature contribution in (3.7) $\frac{1}{a^2}$, while $a \propto t$. Inflation is defined as accelerating expansion

$$\frac{\ddot{a}}{a} \equiv \dot{H} + H^2 = H^2(1 - \epsilon) > 0, \quad (3.8)$$

where the Hubble parameter $H = \frac{\dot{a}}{a}$. We define $\epsilon = -\frac{\dot{H}}{H^2}$ with $0 < \epsilon < 1$ for inflation to take place. Hence we see that for a scalar field in an open FRW universe with $a \propto t$ at early times, $\epsilon \lesssim 1$ due to the contribution of the potential energy to H in the denominator.

To successfully solve the cosmological flatness and horizon problem, inflation should last sufficiently long, for approximately $N = 60$ e-folds or so⁴, where the number of e-folds is $N \equiv \ln \frac{a_0}{a}$, a_0 is the scale factor at the end of inflation, and N is measured backwards from the end of inflation. In other words, $N = 0$ when $\epsilon = 1$, and $N > 0$ for earlier times.

²Only in the thin-wall limit does the field materialize infinitesimally close to the true vacuum.

³Strictly speaking, $R_+ \rightarrow \infty$.

⁴The exact number of required e-folds depends on the energy scale of both inflation and reheating.

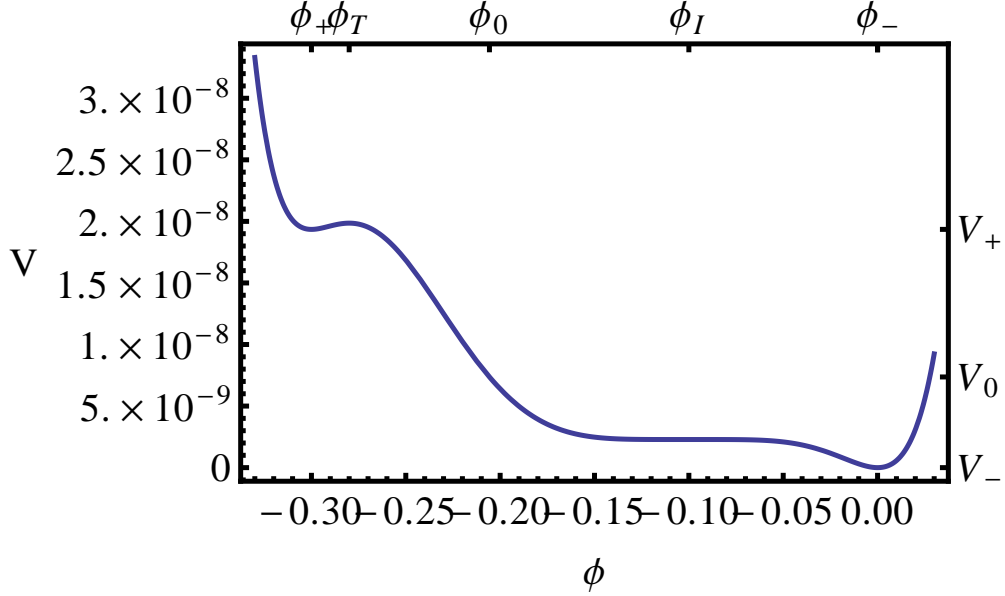


Figure 1: Plot of the potential. $\phi_+ = -\frac{30}{100}$ is the location of the false vacuum. $\phi_T = -\frac{28}{100}$ the location of the top of the barrier. $\phi_0 = -0.205$ is the location the field tunnels to. $\phi_I = -\frac{10}{100}$ is the inflection point.

4 Toy Model

We arrange for the potential to have an inflection point at $\phi = -\frac{1}{10}$ and further critical points at $-\frac{28}{100}, -\frac{30}{100}, 0$ by writing down the derivative of V

$$V'(\phi) = \phi\left(\phi + \frac{10}{100}\right)^4\left(\phi + \frac{30}{100}\right)\left(\phi + \frac{28}{100}\right), \quad (4.9)$$

and integrating the potential subject to the condition $V(0) = 0$. This gives

$$V(\phi) = \sum_{i=2}^8 c_i \phi^i \quad \text{with : } c_2 = 42 \times 10^{-7}, c_3 = \frac{3940}{3} \times 10^{-7}, c_4 = 1865 \times 10^{-6},$$

$$c_5 = 1448 \times 10^{-5}, c_6 = \frac{188}{3} \times 10^{-3}, c_7 = 14 \times 10^{-2},$$

$$c_8 = 1/8, \quad (4.10)$$

see Figure 1. Note that such an 8th order polynomial potential can be generated quite naturally, e.g. by higher-dimension operators present in explicit string theory constructions

of warped D3-brane inflation [18]. The false minimum is located at $\phi_+ = -\frac{3}{10}$, the potential barrier at $\phi_T = -\frac{28}{100}$, the inflection point at $\phi_I = -\frac{1}{10}$ and the true minimum at $\phi_- = 0$. The field tunnels from ϕ_+ to some position ϕ_0 located between ϕ_T and ϕ_I . Then it rolls down the slope towards ϕ_- , driving inflation.

4.1 Tunneling process

While tunneling can in principle proceed either via Coleman de-Luccia (CdL) [7] or Hawking-Moss tunneling [8], a quick check reveals that due to $|V''(\phi_T)|/H^2(\phi_T) \approx 900$, the tunneling process is described by the CdL bounce. This solution to the tunneling problem can be easily found numerically, see Figure 2(c). In Euclidean space, the bounce of the full gravitational CdL instanton interpolates between a point extremely close to the false vacuum $\phi_+ = -\frac{3}{10}$ far outside of the bubble $r \geq R_+$ and $\phi_0 \approx -0.2057$ for $r = 0$ inside of the bubble. The Euclidean action for the CdL instanton gives

$$\Gamma = e^{-B} = e^{-S_E(\phi_B)+S_E(\phi_+)} \approx e^{-7 \times 10^4} \quad .$$

We can apply the results of [17] to analytically estimate the tunneling location ϕ_0 without gravity. For this purpose, we note in jumping ahead, that close to the inflection point ϕ_I we have $V(\phi) \sim (\phi - \phi_I)^5$. However, an analytic expression for the bounce in a quintic potential is not known. Therefore we choose to approximate the potential by a piecewise quartic-quartic potential with the same values for ϕ_+, ϕ_T, ϕ_I and V_+, V_T, V_I as used for the original scalar potential (4.10). Equations (6) and (8) in [17] give the value of ϕ_0 for tunneling in such a piecewise quartic-quartic potential excluding the effects of gravity as

$$\phi_0 = \phi_T + \left(\frac{1 - \sqrt{1 + z^2}}{z} \right)^2 (\tilde{\phi}_- - \phi_T) \approx -0.26, \quad (4.11)$$

where

$$z = \frac{(1 + 2\alpha)\sqrt{\Delta} + \sqrt{4\alpha(1 + \alpha) + \Delta}}{1 - \Delta}, \quad \Delta = \frac{V(\phi_T) - V(\phi_+)}{V(\phi_T) - V(\tilde{\phi}_-)}, \quad \alpha = \frac{\phi_+ - \phi_T}{\phi_T - \tilde{\phi}_-}. \quad (4.12)$$

The additional terms ϕ_T appear because in [17], the coordinate system was chosen in such a way as to have $\phi_T = 0$. While the true minimum of the potential (4.10) is located at ϕ_- , it is clear that the more appropriate “true” minimum to use is $\tilde{\phi}_- = \phi_I$. Hence we obtain $\phi_0 \approx -0.26$ which is in agreement with the numerical results.

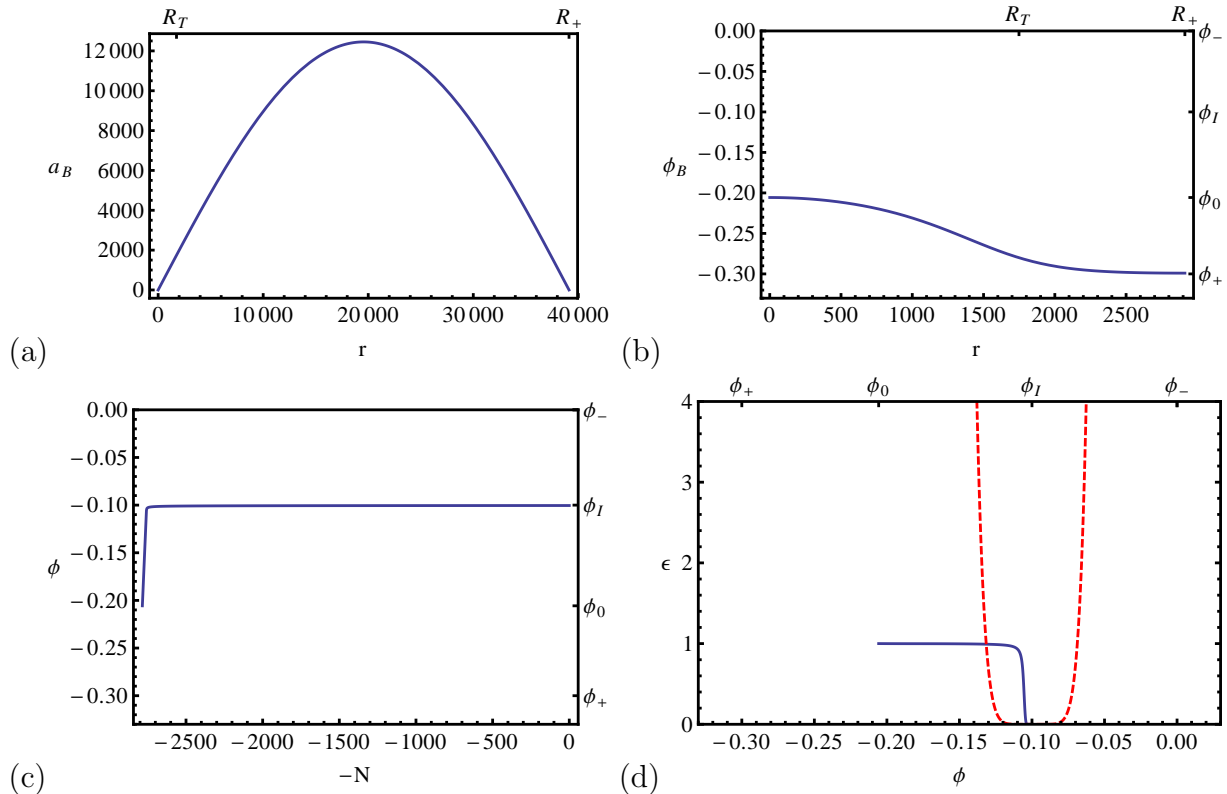


Figure 2: For the potential in (4.10): (a) Scale factor a for the bounce solution. (b) Scalar field ϕ for the bounce solution, interpolating between ϕ_0 and ϕ_+ . (c) Inflation trajectory. The field rolls down from ϕ_0 and its classical motion gets stuck at ϕ_I , so we are just displaying the first ≈ 2500 of infinitely many e-folds of pure classical slow-roll. The field undergoes slow-roll eternal inflation close to the inflection point. (d) ϵ as a function of the inflaton ϕ , both the strict definition $\epsilon = \epsilon_H = -\dot{H}/H^2$ (solid blue) as well as in terms of the scalar potential $\epsilon_V = \frac{1}{2}(V'(\phi)/V(\phi))^2$ (dashed red). While ϵ_V has two crossings of 1, indicating a possible end of inflation, this estimate does not correctly capture the real dynamics. As can be seen in panel (c), the classical motion of the inflaton comes to a complete halt at ϕ_I . For an analytical explanation, see the main text. Note, that diffusion due to quantum fluctuations of the inflaton in dS space will have the field crossing ϕ_I in finite time eventually, and thus lead to a finite maximum of e-folds observed at every given comoving point [19].

4.2 Inflationary trajectory

Numerically solving the field dynamics, Equations (3.5) and (3.6), for the inflationary period, we find that the field stops at the inflection point. Consequently, inflation never ends, see

Figure 2(c), which shows the evolution of ϵ (3.8) as a function of the number of e-folds. Initially, ϵ is only a little smaller than unity. As outlined in Section 3, this can be understood as a consequence of the fact that for open inflation, $a \propto t$ at early times. In fact, the friction term in Equation (3.5) dominates the potential contribution and causes a slow down of the scalar field, see Figure 2(d). Together with the constant contribution to H from the potential plateau, this is responsible for the complete stop of the scalar field at the inflection point. Ignoring the curvature term, the inflaton overshoots the inflationary plateau within less than an efold. This kind of behavior was outlined in [13]. There, we found that for open inflation in monomial potentials of order ≥ 4 without uplifts, the inflaton comes to a stop at the inflection point. Writing the potential (4.10) as expansion around the inflection point $\phi_I = -\frac{1}{10}$, we find⁵

$$V(\phi_I + \delta\phi) = 0.023 - 7200 \delta\phi^5 + \mathcal{O}(\delta\phi^6), \quad (4.13)$$

which shows that the potential is sufficiently flat around ϕ_I for the results of [13] to apply. Strictly speaking, the analysis was performed for potentials with $V(\phi_I) = 0$ and additionally assuming curvature domination throughout. While the eventual failure of curvature domination leads to a non-zero velocity at ϕ_I . However, the presence of a non-zero uplift $V(\phi_I) > 0$ more than compensates for this such that the induced friction term can easily make the inflaton come to a complete stop.

4.3 Tilted model

As we strive to develop a model where inflation eventually ends, we simply add a small negative slope and a corresponding uplift (to ensure $V(\phi_-) = 0$) to the potential

$$V(\phi) = 1.5 \times 10^{-15} - 5 \times 10^{-7} \phi + \sum_{i=2}^8 c_i \phi^i, \quad (4.14)$$

resulting in a finite duration of inflation, see Figure 3. Introducing the extra tilt does not change position of ϕ_+ , ϕ_T , ϕ_0 , ϕ_I , ϕ_- by much. It does however alter the expansion of the potential around the inflection point

$$V(\phi_I + \delta\phi) = 0.023 - 5 \times 10^{-7} \delta\phi - 7200 \delta\phi^5 + \mathcal{O}(\delta\phi^6), \quad (4.15)$$

⁵Note that this expression is exact as we are dealing with a finite power series.

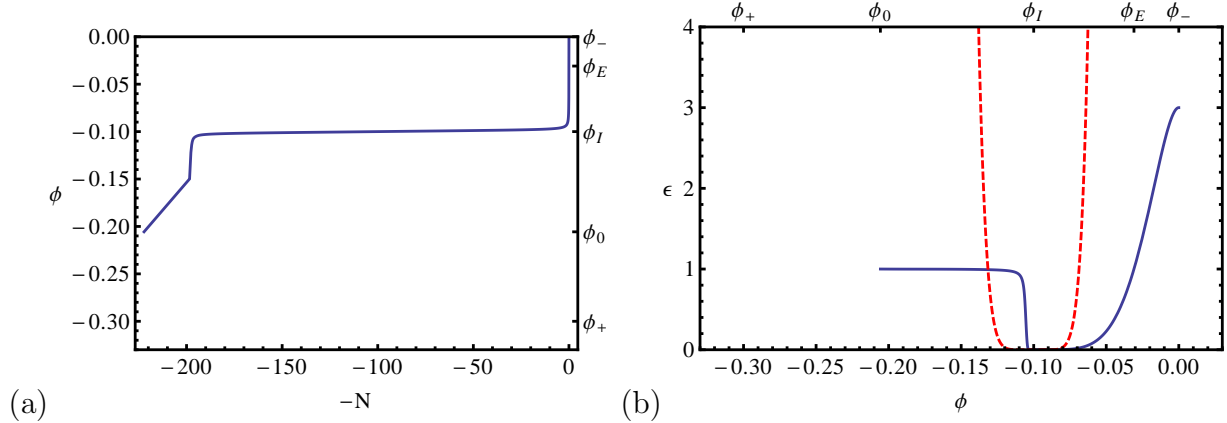


Figure 3: For the potential in (4.14): (a) The inflationary trajectory in the inflationary universe on the inside of the bubble. While the field rolls down from ϕ_0 , ϵ is just a little smaller than 1, i.e. inflation is taking place. Then, the field enters a slow roll phase around the inflection point ϕ_I (without slow-roll eternal inflation) due to the combined effect of the curvature and the potential energy $V(\phi_I) > 0$. Finally, the field rolls off and accelerated expansion stops at ϕ_E . (b) ϵ as a function of the number of e-folds, both the strict definition $\epsilon = \epsilon_H = -\dot{H}/H^2$ (solid blue) as well as in terms of the scalar potential $\epsilon_V = \frac{1}{2}(V'(\phi)/V(\phi))^2$ (dashed red). At the end of inflation at ϕ_E , we have $\epsilon_H = 1$.

introducing a linear term there as well which will be responsible for the end of inflation. As pointed out in [13], the field can enter slow-roll already to the left of the inflection point at ϕ_I due to vacuum energy domination from the substantial uplift $V(\phi_I) = 0.023$. Writing eq. (4.15) as

$$V(\phi_I + \delta\phi) = V_0 \left[1 - \lambda_1 \delta\phi - \frac{\lambda_5}{5} \delta\phi^5 + \mathcal{O}(\delta\phi^6) \right], \quad (4.16)$$

we can determine the total amount of e-folds $N_{tot.}$ to be

$$N_{tot.} \approx \int_{\phi_0}^{\phi_1} d\phi \frac{1}{\sqrt{2\epsilon_V}} \approx \frac{\pi}{\sqrt{2} \sqrt[4]{\lambda_1^3 \lambda_5}} \approx 196 \quad . \quad (4.17)$$

This agrees with the numerical results, see Figure 3. Using this result, we can determine the slow-roll parameters ϵ_V , η_V in a similar analytical approximation, and use them to write the spectral index n_s in the limit $N_{tot.} \gg N_e$ as

$$n_s \approx 1 - \frac{8}{3N_e} \approx 0.97, \quad (4.18)$$

with $N_e \approx 60$. This is in good agreement with current observations $n_s = 0.967 \pm 0.014$ [9].

5 Conclusions

We presented a simple toy model for open inflation after tunneling. Within a simple polynomial single field potential, the scalar field tunnels from a false vacuum towards the true vacuum, subsequently undergoing an extended phase of inflation. While at first glance, the numerical factors appearing in the potential appear gigantic, it should be kept in mind that they can be compensated by rescaling ϕ with a number of order 10 – apart from the tiny linear tilt in the potential. However, we only introduced this tilt to make inflation come to an end. Without it, the field would stay at the inflection point forever (or until another tunneling event takes it to the true minimum). Let us stress this point: without introducing the linear tilt, we would be in a situation where small field inflation after tunneling suffers from the opposite of an overshoot problem!

We numerically computed the CdL bounce for tunneling from the false minimum towards the inflection point, and compared it to an analytical estimate. We then computed the inflationary trajectory numerically and found that only through introducing a small linear tilt inflation can come to an end. For the numerical values chosen here, we obtain about 200 e-folds of inflation. The scalar spectral index $n_s \approx 1$ is in agreement with current observations. We like to point out that we do not advertise this model as a realistic candidate “theory”, but merely as a toy model which shows that it is quite possible (and seemingly easy) to obtain a simple model of open small-field inflation without any overshoot problem after tunneling.

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References

- [1] L. Susskind, *The Anthropic landscape of string theory*, [hep-th/0302219](#).
- [2] K. Sugimura, D. Yamauchi, and M. Sasaki, *Multi-field open inflation model and multi-field dynamics in tunneling*, *JCAP* **1201** (2012) 027, [[arXiv:1110.4773](#)].
- [3] D. Yamauchi, A. Linde, A. Naruko, M. Sasaki, and T. Tanaka, *Open inflation in the landscape*, *Phys.Rev.* **D84** (2011) 043513, [[arXiv:1105.2674](#)].
- [4] M. Kleban and M. Schillo, *Spatial Curvature Falsifies Eternal Inflation*, [arXiv:1202.5037](#).
- [5] A. H. Guth and Y. Nomura, *What can the observation of nonzero curvature tell us?*, [arXiv:1203.6876](#).
- [6] S. R. Coleman, *The Fate of the False Vacuum. 1. Semiclassical Theory*, *Phys. Rev.* **D15** (1977) 2929–2936.
- [7] S. R. Coleman and F. De Luccia, *Gravitational Effects on and of Vacuum Decay*, *Phys. Rev.* **D21** (1980) 3305.
- [8] S. Hawking and I. Moss, *Supercooled Phase Transitions in the Very Early Universe*, *Phys.Lett.* **B110** (1982) 35.
- [9] **WMAP Collaboration** Collaboration, E. Komatsu *et. al.*, *Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation*, *Astrophys.J.Suppl.* **192** (2011) 18, [[arXiv:1001.4538](#)].
- [10] D. Baumann and L. McAllister, *Advances in Inflation in String Theory*, *Ann.Rev.Nucl.Part.Sci.* **59** (2009) 67–94, [[arXiv:0901.0265](#)].
- [11] R. Brustein and P. J. Steinhardt, *Challenges for superstring cosmology*, *Phys.Lett.* **B302** (1993) 196–201, [[hep-th/9212049](#)].
- [12] A. D. Linde, *A Toy model for open inflation*, *Phys.Rev.* **D59** (1999) 023503, [[hep-ph/9807493](#)].

- [13] K. Dutta, P. M. Vaudrevange, and A. Westphal, *The Overshoot Problem in Inflation after Tunneling*, *JCAP* **1201** (2012) 026, [[arXiv:1109.5182](#)].
- [14] B. Freivogel, M. Kleban, M. Rodriguez Martinez, and L. Susskind, *Observational consequences of a landscape*, *JHEP* **0603** (2006) 039, [[hep-th/0505232](#)].
- [15] G. Dvali and S. Kachru, *New old inflation*, [hep-th/0309095](#).
- [16] K. Dutta, P. M. Vaudrevange, and A. Westphal, *An Exact Tunneling Solution in a Simple Realistic Landscape*, [arXiv:1102.4742](#). * Temporary entry *.
- [17] K. Dutta, C. Hector, P. M. Vaudrevange, and A. Westphal, *More Exact Tunneling Solutions in Scalar Field Theory*, [arXiv:1110.2380](#).
- [18] N. Agarwal, R. Bean, L. McAllister, and G. Xu, *Universality in D-brane Inflation*, *JCAP* **1109** (2011) 002, [[arXiv:1103.2775](#)].
- [19] A. D. Linde and A. Westphal, *Accidental Inflation in String Theory*, *JCAP* **0803** (2008) 005, [[arXiv:0712.1610](#)].