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Functional equations for one-loop master integrals for heavy-quark production and Bhabha scattering

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Abstract

The method for obtaining functional equations, recently proposed by one of the authors [1], is applied to one-loop box integrals needed in calculations of radiative corrections to heavy-quark production and Bhabha scattering. We present relationships between these integrals with different arguments and box integrals with all propagators being massless. It turns out that functional equations are rather useful for finding imaginary parts and performing analytic continuations of Feynman integrals. For the box master integral needed in Bhabha scattering, a new representation in terms of hypergeometric functions admitting one-fold integral representation is derived. The hypergeometric representation of a master integral for heavy-quark production follows from the functional equation.

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1 Introduction

The production of heavy quarks at hadron colliders has become a very important field of research. The very large production rates for both top and bottom quarks at the CERN Large Hadron Collider (LHC) will allow for studies of heavy quarks with high precision. The full next-to-leading-order (NLO) radiative corrections to the hadroproduction of heavy flavors was completed in Ref. [2]. The theoretical NLO predictions suffer from the usual uncertainty resulting from the freedom in the choice of renormalization and factorization scales of perturbative QCD. To reduce such uncertainties, next-to-next-to-leading order (NNLO) calculations are needed. The computation of NNLO corrections is complicated due to great technical difficulties, mainly related to the evaluation of two-loop Feynman integrals. To overcome such difficulties, it is of great importance to develop new methods and approaches for calculating Feynman integrals. The main complications of these calculations are related to the fact that the integrals depend on several kinematical variables. As was noted in Ref. [3], the most appropriate methods for calculating such integrals may be those based on a different kind of recurrence relations. Such methods can be based on the solution of recurrence relations with respect to the exponent of a propagator in the integral [4] or on the solution of dimensional recurrences [5, 6].

A significant simplification of the computation of Feynman integrals depending on several kinematical variables may be achieved by using a new type of relationships between Feynman integrals through functional equations with respect to kinematical variables as proposed in Ref. [1]. As was shown in Ref. [1], Feynman integrals with several kinematical variables can be expressed in terms of integrals with a lesser number of variables, which significantly simplifies their evaluation.

It is the purpose of the present paper to apply the general method for finding functional equations [1] to integrals required in calculations of radiative corrections to important physical processes and to use those relations for the analytic computation of these integrals.

Our paper is organized as follows. In Section 2, we give definitions and notations. In Section 3, we present functional equations for the on-shell master integrals from heavy-quark production and Bhabha scattering. In Section 4, a new hypergeometric representation in terms of the Appell functions F_1 and F_3 and the Gauss hypergeometric function ${}_2F_1$ for the one-loop box integral from Bhabha scattering is presented. In Section 5, we describe how to use functional equations to find imaginary parts of the considered integrals. New analytic results for the imaginary parts are presented. Using dispersion relations, we write also one-fold integral representations for real parts of integrals. In Section 6, we present functional equations for master integrals from heavy-quark production with one quark leg off shell.

2 Definitions and notations

As was shown in Ref. [1], functional equations for one-loop integrals corresponding to diagrams with four external legs can be derived, for example, from the following equation obtained in

Refs. [5, 7]:

$$
\begin{aligned}
&G_4\,\mathbf{j}^+ I_5^{(d+2)}(m_1^2,m_2^2,m_3^2,m_4^2,m_5^2;\{s_{kr}\}) - (\partial_j\Delta_5) I_5^{(d)}(m_1^2,m_2^2,m_3^2,m_4^2,m_5^2;\{s_{kr}\})\\
&= (\partial_j\partial_1\Delta_5)\; I_4^{(d)}(m_2^2,m_3^2,m_4^2,m_5^2; s_{23},s_{34},s_{45},s_{25};\; s_{35},s_{24})\\
&\quad + (\partial_j\partial_2\Delta_5)\; I_4^{(d)}(m_1^2,m_3^2,m_4^2,m_5^2; s_{13},s_{34},s_{45},s_{15};\; s_{35},s_{14})\\
&\quad + (\partial_j\partial_3\Delta_5)\; I_4^{(d)}(m_1^2,m_2^2,m_4^2,m_5^2; s_{12},s_{24},s_{45},s_{15};\; s_{25},s_{14})\\
&\quad + (\partial_j\partial_4\Delta_5)\; I_4^{(d)}(m_1^2,m_2^2,m_3^2,m_5^2; s_{12},s_{23},s_{35},s_{15};\; s_{25},s_{13})\\
&\quad + (\partial_j\partial_5\Delta_5)\; I_4^{(d)}(m_1^2,m_2^2,m_3^2,m_4^2; s_{12},s_{23},s_{34},s_{14};\; s_{24},s_{13}), \qquad (1)
\end{aligned}
$$

where the operator $\mathbf{j}^+$ shifts the indices $\nu_j \to \nu_j + 1$, G_4 is the Gram determinant, and Δ_5 is the modified Cayley determinant, defined as

$$
G_4 = -16 \begin{vmatrix} p_{15}p_{15} & p_{15}p_{25} & p_{15}p_{35} & p_{15}p_{45}\\ p_{15}p_{25} & p_{25}p_{25} & p_{25}p_{35} & p_{25}p_{45}\\ p_{15}p_{35} & p_{25}p_{35} & p_{35}p_{35} & p_{35}p_{45}\\ p_{15}p_{45} & p_{25}p_{45} & p_{35}p_{45} & p_{45}p_{45}\end{vmatrix}, \qquad \Delta_5 = \begin{vmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15}\\ Y_{12} & Y_{22} & Y_{23} & Y_{24} & Y_{25}\\ Y_{13} & Y_{23} & Y_{33} & Y_{34} & Y_{35}\\ Y_{14} & Y_{24} & Y_{34} & Y_{44} & Y_{45}\\ Y_{15} & Y_{25} & Y_{35} & Y_{45} & Y_{55}\end{vmatrix}, \qquad (2)
$$

$$
Y_{ij} = m_i^2 + m_j^2 - s_{ij}, \qquad s_{ij} = p_{ij}^2, \qquad p_{ij} = p_i - p_j, \qquad \partial_j = \frac{\partial}{\partial m_j^2}. \qquad (3)
$$

The integral $I_5^{(d)}(\{m_j^2\},\{s_{kr}\})$ corresponds to a diagram with five external legs and the integrals $I_4^{(d)}(\{m_j^2\},\{s_{kr}\})$ in Eq. (1) are defined as

$$
\begin{aligned}
&I_4^{(d)}(m_n^2,m_j^2,m_k^2,m_l^2;\; s_{nj},s_{jk},s_{kl},s_{nl};\; s_{jl},s_{nk})\\
&= \int \frac{d^dq}{i\pi^{d/2}}\, \frac{1}{[(q-p_n)^2-m_n^2][(q-p_j)^2-m_j^2][(q-p_k)^2-m_k^2][(q-p_l)^2-m_l^2]}. \qquad (4)
\end{aligned}
$$

The diagram corresponding to the integral $I_4^{(d)}(m_1^2,m_2^2,m_3^2,m_4^2;\;\{s_{ij}\})$ is presented in Fig. 1.

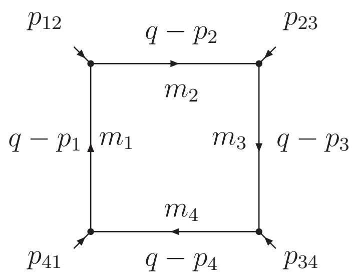

Figure 1: *Diagram corresponding to the integral* $I_4^{(d)}(m_1^1,m_2^2,m_3^2,m_4^2;\; s_{12},s_{23},s_{34},s_{14};\; s_{24},s_{13})$.

In what follows, we will use the following short-hand notation for integrals needed in the calculation of the one-loop radiative corrections to the process $e^+e^- \to e^+e^-$, the so-called Bhabha scattering [8], and heavy-quark production:

$$
\begin{aligned}
B(s,t) &= I_4^{(d)}(0,m^2,0,m^2;\; m^2,m^2,m^2,m^2;\; s,t),\\
D_2(s,t) &= I_4^{(d)}(0,0,0,m^2;\; 0,0,m^2,m^2;\; t,s). \qquad (5)
\end{aligned}
$$

The diagrams corresponding to these integrals are depicted in Fig. 2.

$$s = (p_{12} + p_{41})^2$$
$$t = (p_{12} + p_{23})^2$$

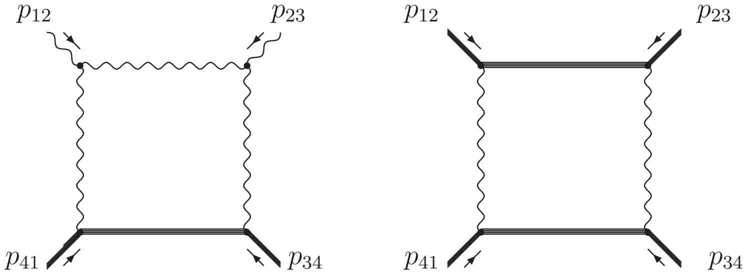

Figure 2: *Diagrams corresponding to the integrals $D_2(t,s)$ and $B(s,t)$. Lines with a zero internal mass $m_i = 0$ (for internal lines) or a zero virtuality $s_{ij} = 0$ (for external lines) are shown wavy. Solid lines have a non-zero internal mass or a non-zero virtuality.*

3 Functional equations for the integrals $B(s,t)$ and $D_2(t,s)$

In this section, we present in detail the derivation of functional equations for the scalar integrals $B(s,t)$ and $D_2(t,s)$. As was proposed in Ref. [1], one can obtain a functional equation for the integral $I_4^{(d)}(\{m_j^2\},\{s_{kr}\})$ by eliminating terms with $I_5^{(d)}(\{m_j^2\},\{s_{kr}\})$ from Eq. (1) through an appropriate choice of kinematical variables. The integral $I_5^{(d)}(\{m_j^2\},\{s_{kr}\})$ depends on 15 kinematical variables, while the integral $I_4^{(d)}(\{m_j^2\},\{s_{kr}\})$ depends on 10 variables. Therefore, to obtain a functional equation for the integral $I_4^{(d)}(\{m_j^2\},\{s_{kr}\})$ with all 10 kinematical variables arbitrary, we can impose conditions on some 5 variables. To eliminate terms with $I_5^{(d)}(\{m_j^2\},\{s_{kr}\})$ from Eq. (1), two equations must be fulfilled:

$$G_4 = 0, \qquad \partial_j \Delta_5 = 0, \tag{6}$$

thus fixing two kinematical variables. There are several options to use the remaining three variables. One option is to set these variables to some particular values and obtain a functional equation connecting the integral of interest with integrals which are easy to evaluate and/or integrals similar to the original one but with different kinematics. Another option to choose variables is to reduce the number of terms in the functional equation by requiring some derivatives $\partial_i \partial_j \Delta_5$ to be zero. Also, one can use some combination of these two options. The derivation of the numerous functional equations corresponding to the different options and their combinations described above was done on a computer. We describe the most useful functional equations below.

To be definite, let us assume that our integral of interest is the last integral on the right-hand side of Eq. (1). If we set in Eq. (1) $m_1 = m_3 = 0$, $m_2^2 = m_4^2 = m^2$, $s_{12} = s_{23} = s_{34} = s_{14} = m^2$, and $s_{24} = s$, $s_{13} = t$, then the last integral on the right-hand side of this equation corresponds to our integral $B(s,t)$.

Setting $m_5^2 = 0$ and choosing different particular values of the remaining two kinematical variables and/or requiring some second derivatives $\partial_i \partial_j \Delta_5$ to be zero, one can get from Eq. (1) rather different functional equations for the integral $B(s,t)$. For some specific choice of kinematical variables, also the integral $D_2(s,t)$ appears in the functional equation. By imposing different conditions, we try to find equations connecting the integrals $B(s,t)$ and $D_2(s,t)$ with simpler integrals, for example, with integrals having more massless propagators and simpler external kinematics. We would like to note that, in the present investigation, we always set $m_5^2=0$ in order to avoid the appearance of integrals with three propagators having nonzero mass in the functional equations.

Substituting $j=2$, $m_2^2=m_3^2=m_5^2=0$, $s_{12}=s_{23}=s_{34}=s_{14}=m^2$, $s_{24}=s$, and $s_{13}=t$ into Eq. (1) and choosing s_{15}, s_{25}, s_{35}, and s_{45} from the conditions

$$G_4=0, \qquad \partial_2\Delta_5=0, \qquad \partial_1\partial_2\Delta_5=0, \qquad \partial_2\partial_3\Delta_5=0, \tag{7}$$

we arrive at the following equation:

$$B(s,t)=\frac{m^2}{s}(1+\alpha_+)D_2(t,m^2\alpha_+)+\frac{m^2}{s}(1+\alpha_-)D_2(t,m^2\alpha_-), \tag{8}$$

where

$$\alpha_\pm=\frac{1\pm\beta_s}{1\mp\beta_s}, \qquad \beta_s=\sqrt{1-\frac{4m^2}{s}}. \tag{9}$$

Thus, we have a relation connecting the integral $B(s,t)$ with an integral having only one massive propagator, i.e. with the integral $D_2(s,t)$. It turns out that the integral $D_2(s,t)$ in Eq. (8) satisfies the following functional equation:

$$D_2(t,s)=\frac{m^2}{s}\;D_2\left(t,\frac{m^4}{s}\right)+\frac{s-m^2}{s}\;I_4^{(d)}\left(0,0,0,0;\;0,0,0,0;\;\frac{(s-m^2)^2}{s},t\right), \tag{10}$$

which can be obtained from Eq. (1) by setting $j=5$, $m_1^2=m_2^2=m_3^2=m_5^2=s_{12}=s_{23}=0$, $s_{34}=s_{14}=m_4^2=m^2$, and $s_{24}=s$, $s_{13}=t$ and imposing the conditions

$$G_4=\partial_5\Delta_5=\partial_1\partial_5\Delta_5=\partial_3\partial_5\Delta_5=0. \tag{11}$$

The last integral in Eq. (10) corresponds to the box integral with all propagators massless and the squares of all external momenta equal to zero. By using Eq. (10) and taking into account the relation $\alpha_+\alpha_-=1$, one can write the integral in Eq. (8) with argument α_+ as

$$D_2(t,m^2\alpha_+)=\alpha_-\;D_2(t,m^2\alpha_-)+(1-\alpha_-)\;I_4^{(d)}\left(0,0,0,0;\;0,0,0,0;\;s-4m^2,t\right). \tag{12}$$

Substituting this relation into Eq. (8), gives

$$B(s,t)=(1-\beta_s)D_2(t,m^2\alpha_-)+\beta_s\;I_4^{(d)}(0,0,0,0;\;0,0,0,0;\;s-4m^2,t). \tag{13}$$

We illustrate this relation in Fig. 3. From Eq. (13), the master integral from heavy-quark production is found to be

$$D_2(s,t)=\frac{t+m^2}{2t}\;B\left(\frac{(t+m^2)^2}{t},s\right)+\frac{t-m^2}{2t}\;I_4^{(d)}\left(0,0,0,0;\;0,0,0,0;\;\frac{(t-m^2)^2}{t},s\right). \tag{14}$$

Here, we would like to remark that, for $\varepsilon=(4-d)/2\to 0$, the integral D_2 has a pole proportional to $1/\varepsilon^2$, while the leading singularity of the integral $B(s,t)$ is $1/\varepsilon$. The leading $1/\varepsilon^2$ singularity on the right-hand side comes from the massless integral I_4.

Analytic formulae for both integrals on the right-hand side of Eq. (14) are given in the next sections.

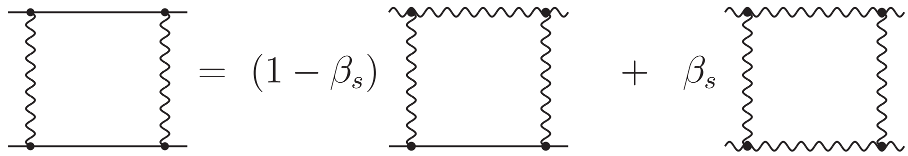

Figure 3: A schematic depiction of Eq. (13). Wavy lines correspond to massless scalar propagators and solid lines to massive propagators.

4 Analytic result for the integral $B(s,t)$

Using the method of dimensional recurrences [5, 6], the following hypergeometric representation for the integral $B(s,t)$ was obtained in Ref. [9]:

$$\begin{aligned} B(s,t) = & \frac{(-2)}{mt(s-4m^2)} I_2^{(d)}(0,0;\ t)\ F_1\left(\frac{d-3}{2},1,\frac{1}{2};\frac{d-1}{2};\frac{tz}{4},-\frac{t\theta}{4m^2}\right) \\ & + \frac{2(2-d)}{t(s-4m^2)} I_2^{(d)}(0,m^2;\ 0) \\ & \times \left[F_2\left(\frac{d-3}{2},1,1,\frac{3}{2},\frac{d-2}{2};\frac{s}{s-4m^2},-m^2 z\right) - \frac{1}{d-3}\phi\left(-m^2 z,\theta\right)\right], \end{aligned} \tag{15}$$

where

$$z = \frac{4u}{t(4m^2-s)}, \qquad \theta = 1 - \frac{4m^2}{t}, \qquad u = 4m^2 - s - t, \tag{16}$$

and $I_2^{(d)}$ are the one-loop propagator type integrals

$$\begin{aligned} I_2^{(d)}(0,m^2;\ 0) &= \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{k_1^2(k_1^2-m^2)} = -\Gamma\left(1-\frac{d}{2}\right) m^{d-4}, \\ I_2^{(d)}(0,0;\ p^2) &= \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{k_1^2(k_1-p)^2} = \frac{-\pi^{\frac{3}{2}}\left(-p^2\right)^{\frac{d}{2}-2}}{2^{d-3}\Gamma\left(\frac{d-1}{2}\right)\sin\frac{\pi d}{2}}. \end{aligned} \tag{17}$$

Here, the Appell hypergeometric functions are

$$\begin{aligned} F_1\left(\frac{d-3}{2},1,\frac{1}{2},\frac{d-1}{2};x,y\right) &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{d-1}{2}\right)_{r+s}} \frac{\left(\frac{1}{2}\right)_s}{(1)_s} x^r y^s, \\ F_2\left(\frac{d-3}{2},1,1,\frac{3}{2},\frac{d-2}{2};x,y\right) &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{3}{2}\right)_r \left(\frac{d-2}{2}\right)_s} x^r y^s. \end{aligned} \tag{18}$$

The function $\phi(x,y)$ is

$$\phi(x,y) = F_{1;1;0}^{1;2;1}\left[\begin{matrix} \frac{d-3}{2}: & \frac{d-3}{2}, 1; & 1; \\ \frac{d-1}{2}: & \frac{d-2}{2}; & -; \end{matrix}\ x,y\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{d-1}{2}\right)_{r+s}} \frac{\left(\frac{d-3}{2}\right)_r}{\left(\frac{d-2}{2}\right)_r} x^r y^s, \tag{19}$$

where $F_{1;1;0}^{1;2;1}$ is the Kampé de Fériet function [10]. The Appell function F_1 admits a one-fold integral representation (see Appendix). The functions F_2 and ϕ admit two-fold integral representations, and this is the reason why their ε expansions are problematic.

We discovered that both functions can be represented in terms of the Gauss hypergeometric function ${}_2F_1$ and Appell function F_3, defined as [10]:

$$F_3\left(\alpha,\alpha',\beta,\beta',\gamma,x,y\right) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \qquad |x|<1, \quad |y|<1. \tag{20}$$

We found two methods to obtain such a representation for the ϕ function. The first method is as follows. We write the function $\phi(x,y)$ as

$$\phi(x,y) = \sum_{r=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_r \left(\frac{d-3}{2}\right)_r}{\left(\frac{d-2}{2}\right)_r \left(\frac{d-1}{2}\right)_r} x^r \, {}_2F_1\left[\begin{matrix} 1, \frac{d-3}{2}+r\, ; \\ \frac{d-1}{2}+r\, ; \end{matrix} y\right], \tag{21}$$

and then perform an analytic continuation of ${}_2F_1$ transforming it to two functions ${}_2F_1$ with argument $1/y$. Thus, we obtain two terms. One of these terms is just the Gauss function ${}_2F_1$ and another one is the Horn function

$$H_2\left(\frac{d-5}{2}, 1, 1, 1, \frac{d-2}{2}, x, \frac{1}{y-1}\right), \tag{22}$$

defined as [11]

$$H_2\left(\alpha,\beta,\gamma,\gamma',\delta,x,y\right) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\gamma')_n}{(\delta)_m} \frac{x^m}{m!}\frac{y^n}{n!}, \qquad \frac{1}{|y|}-|x|>1, \ \ |x|,|y|<1. \tag{23}$$

The Horn function (22) can be expressed in terms of ${}_2F_1$ functions and the Appell function F_3 using Eq. (65) on p. 295 in Ref. [12]. Combining all terms, we arrive at the following result:

$$\begin{aligned}\phi(x,y) = &-\frac{(d-3)(d-4)}{(d-5)(d-7)}\frac{1}{x(1-y)} F_3\left(1,1,3-\frac{d}{2},1,\frac{9-d}{2};\frac{1}{x},\frac{1}{1-y}\right) \\ &+ \frac{(3-d)\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{5-d}{2}\right)}{\sqrt{\pi}\,(1-y)(-x)^{\frac{(d-5)}{2}}}\sqrt{1-\frac{1}{x}}\ \ {}_2F_1\left[\begin{matrix}1,1\,;\\ \frac{3}{2}\,;\end{matrix}\frac{1-x}{1-y}\right] \\ &+ \frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{5-d}{2}\right)}{(-y)^{\frac{d-3}{2}}}\ \ {}_2F_1\left[\begin{matrix}1,\frac{d-3}{2}\,;\\ \frac{d-2}{2}\,;\end{matrix}\frac{x}{y}\right].\end{aligned} \tag{24}$$

The Appell function F_3 for this specific set of parameters can be written as a one-fold integral:

$$F_3\left(1,1,3-\frac{d}{2},1,\frac{9-d}{2};\frac{1}{x},\frac{1}{1-y}\right) = \frac{\Gamma\left(\frac{9-d}{2}\right)x(y-1)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(3-\frac{d}{2}\right)}\int_0^1 \frac{(1-v)^{2-\frac{d}{2}}\arcsin\sqrt{\frac{v}{1-y}}}{(1-x-v)\sqrt{1-y-v}}\,dv. \tag{25}$$

This formula can be obtained from the integral representation of the F_3 function [11] (see also Appendix). The function $\phi(-m^2z,\theta)$ from Eq. (15) reads:

$$\begin{aligned}\phi(-m^2z,\theta) = (d-3)\Gamma\left(\frac{d-5}{2}\right)\Bigg\{&\frac{-\phi_1(s,t)}{\sqrt{\pi}\Gamma\left(2-\frac{d}{2}\right)} \\ &+\frac{1}{2}\Gamma\left(\frac{d-3}{2}\right)(-\theta)^{\frac{3-d}{2}}\,{}_2F_1\left[\begin{matrix}1,\frac{d-3}{2}\,;\\ \frac{d-2}{2}\,;\end{matrix}\frac{4m^2u}{(t+u)(s+u)}\right] \\ &-\frac{t(u+t)\Gamma\left(\frac{d}{2}-1\right)}{2\sqrt{m^2stu}}\left[\frac{t(t+u)}{4m^2u}\right]^{\frac{d-5}{2}}\arcsin\sqrt{\frac{w}{4m^2(t+u)}}\Bigg\},\end{aligned} \tag{26}$$

where

$$\phi_1(s,t) = \int_0^1 \frac{(1-v)^{2-\frac{d}{2}} \arcsin\sqrt{\frac{vt}{4m^2}}}{\left[\frac{w}{4(u+t)} - v\right]\sqrt{1-\theta-v}}\, dv, \tag{27}$$

and

$$w = 16m^4 - 4m^2 s - ts. \tag{28}$$

From this expression, it follows that any coefficient in the ε expansion of the ϕ function can be expressed in terms of one-fold integrals. Several terms of the ε expansion of the function ϕ are given in Ref. [13].

We present also another method to represent the function $\phi(x,y)$ in terms of the Appell function F_3 and the Gauss functions ${}_2F_1$. For the function $\phi(x,y)$, one can write the following integral representation [13]:

$$\phi(x,y) = \frac{d-3}{2} \int\limits_0^1 \frac{v^{\frac{d-5}{2}}}{1-yv}\, {}_2F_1\left[\begin{matrix}1, \frac{d-3}{2}\,; \\ \frac{d-2}{2}\,;\end{matrix} xv\right] dv\ . \tag{29}$$

From this integral representation, one can derive a differential equation. Differentiating both sides of Eq. (29) w.r.t. x, using the following formula for the derivative of the Gauss function,

$$\frac{d}{dz}\, {}_2F_1\left[\begin{matrix}1, b\,; \\ c\,;\end{matrix} z\right] = \frac{(bz-c+1)}{z(1-z)}\, {}_2F_1\left[\begin{matrix}1, b\,; \\ c\,;\end{matrix} z\right] + \frac{(c-1)}{z(1-z)}, \tag{30}$$

and after some simplification of the resulting integrand, one obtains the following equation [9]:

$$\begin{aligned} 2(y-x)x\frac{\partial\phi(x,y)}{\partial x} &= [y-(d-3)(y-x)]\phi(x,y) \\ &- x(d-3)\,{}_2F_1\left(1, \frac{d-3}{2}, \frac{d-2}{2}, x\right) + (d-4)y\ {}_2F_1\left(1, \frac{d-3}{2}, \frac{d-1}{2}, y\right). \end{aligned} \tag{31}$$

This first-order differential equation can be easily solved yielding:

$$\begin{aligned} \phi(x,y) &= \frac{(d-3)}{(d-2)}\, \frac{x}{x-y}\ F_3\left(\frac{1}{2}, 1, 1, \frac{d-3}{2}, \frac{d}{2};\ \frac{x}{x-y}, x\right) \\ &+ \left(\frac{y}{y-x}\right)^{1/2}\ {}_2F_1\left[\begin{matrix}1, \frac{d-3}{2}\,; \\ \frac{d-1}{2}\,;\end{matrix} y\right]\ {}_2F_1\left[\begin{matrix}1, \frac{d-3}{2}\,; \\ \frac{d-2}{2}\,;\end{matrix} \frac{x}{y}\right]. \end{aligned} \tag{32}$$

As it happens in the previous case, instead of the Kampé de Fériet function, we obtain the more familiar Appell function F_3, which, for the above parameters, admits the one-fold integral representation:

$$F_3\left(\frac{1}{2}, 1, 1, \frac{d-3}{2}, \frac{d}{2};\ \frac{x}{x-y}, x\right) = \frac{\Gamma\left(\frac{d}{2}\right)(x-y)}{\sqrt{\pi}\,\Gamma\left(\frac{d-3}{2}\right)x(1-x)} \int_0^1 \frac{(1-v)^{\frac{d-5}{2}}}{1-\frac{vx}{x-1}} \ln\frac{1+\sqrt{\frac{xv}{x-y}}}{1-\sqrt{\frac{xv}{x-y}}}\, dv. \tag{33}$$

This relation is obtained from the integral representation given in the Appendix. By using Eqs. (32) and (33), we obtain the following expression for the function $\phi(-m^2 z,\theta)$ from Eq. (15):

$$\begin{aligned} \phi(-m^2 z,\theta) &= \frac{(d-3)\Gamma\left(\frac{d}{2}-1\right)(4m^2-s)t}{2\sqrt{\pi}\,\Gamma\left(\frac{d-3}{2}\right)w}\phi_2(s,t) \\ &+ \left[\left(1-\frac{4m^2}{s}\right)\theta\right]^{\frac{1}{2}}\ {}_2F_1\left[\begin{matrix}1, \frac{d-3}{2}\,; \\ \frac{d-1}{2}\,;\end{matrix} \theta\right]\ {}_2F_1\left[\begin{matrix}1, \frac{d-3}{2}\,; \\ \frac{d-2}{2}\,;\end{matrix} \frac{4m^2u}{(s-4m^2)(t-4m^2)}\right], \end{aligned} \tag{34}$$

where

$$\phi_2(s,t) = \int_0^1 \frac{(1-v)^{\frac{d-5}{2}}}{1-\frac{4m^2uv}{w}} \ln \frac{1+\sqrt{-\frac{4m^2uv}{ts}}}{1-\sqrt{-\frac{4m^2uv}{ts}}} dv, \tag{35}$$

θ is defined in Eq. (16) and w in Eq. (28).

Thus, we obtained two rather different hypergeometric representations for the ϕ function. In both cases, the hypergeometric functions admit one-fold integral representations, so that all the coefficients in the ε expansion may be expressed only in terms of one-fold integrals.

The Appell function F_2 from Eq. (15) can also be expressed in terms of the Appell function F_3 and the Gauss function ${}_2F_1$. To obtain such a relation, we use the formula for the analytic continuation of the Appell function F_3 from Ref. [11] and obtain:

$$\begin{aligned} F_2\left(\frac{d-3}{2},1,1,\frac{d-2}{2},\frac{3}{2};x,y\right) &= \frac{(d-4)}{(d-5)(d-7)xy}\, F_3\left(1,1,3-\frac{d}{2},\frac{1}{2},\frac{9-d}{2};\frac{1}{x},\frac{1}{y}\right) \\ &- \frac{\sqrt{\pi}\,\Gamma\left(\frac{5-d}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{2\sqrt{-y}(-x)^{\frac{d}{2}-2}} + \frac{\Gamma\left(\frac{5-d}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{\sqrt{\pi}\,(-x)^{\frac{d}{2}-2}\sqrt{1-x}}\; {}_2F_1\left[\begin{matrix}1,\frac{1}{2};\\ \frac{3}{2};\end{matrix}\; \frac{y}{1-x}\right] \\ &+ \frac{\sqrt{\pi}\,\Gamma\left(\frac{5-d}{2}\right)}{2\Gamma\left(\frac{6-d}{2}\right)\sqrt{-y}\,(1-y)^{\frac{d}{2}-2}}\; {}_2F_1\left[\begin{matrix}1,\frac{d}{2}-2;\\ \frac{d}{2}-1;\end{matrix}\; \frac{x}{1-y}\right]. \end{aligned} \tag{36}$$

The Appell function F_3 with this particular set of parameters also can be expressed in terms of the one-fold integral

$$F_3\left(1,1,3-\frac{d}{2},\frac{1}{2},\frac{9-d}{2};\frac{1}{x},\frac{1}{y}\right) = \frac{-\Gamma\left(\frac{9-d}{2}\right)}{\sqrt{\pi}\,\Gamma\left(3-\frac{d}{2}\right)}\, \frac{x\sqrt{y}}{1-x} \int_0^1 \frac{(1-v)^{\frac{4-d}{2}}}{1+\frac{v}{x-1}} \ln \frac{1+\sqrt{\frac{v}{y}}}{1-\sqrt{\frac{v}{y}}}\, dv. \tag{37}$$

Therefore, the Appell function F_2 from Eq. (15) reads:

$$\begin{aligned} &F_2\left(\frac{d-3}{2},1,1,\frac{3}{2},\frac{d-2}{2};\frac{s}{s-4m^2},-m^2z\right) \\ &= \frac{\Gamma\left(\frac{5-d}{2}\right)}{16\sqrt{\pi}\Gamma\left(2-\frac{d}{2}\right)} \frac{(4m^2-s)}{m^2} \left[t(s-4m^2)m^2u\right]^{\frac{1}{2}} \phi_3(s,t) \\ &+ \Gamma\left(\frac{5-d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)\left[\frac{\pi st}{16um^2}\right]^{\frac{1}{2}}\left[\frac{4m^2-s}{s}\right]^{\frac{d-3}{2}}\left[1+\frac{i}{\pi}\ln\frac{1+\sqrt{-\frac{u}{t}}}{1-\sqrt{-\frac{u}{t}}}\right] \\ &+ \frac{\sqrt{\pi}\,\Gamma\left(\frac{5-d}{2}\right)}{4\Gamma\left(3-\frac{d}{2}\right)}\left[\frac{t(4m^2-s)}{um^2}\right]^{\frac{1}{2}}\left[\frac{t(4m^2-s)}{w}\right]^{\frac{d-4}{2}} {}_2F_1\left[\begin{matrix}1,\frac{d-4}{2};\\ \frac{d-2}{2};\end{matrix}\; -\frac{ts}{w}\right], \end{aligned} \tag{38}$$

where

$$\phi_3(s,t) = \int_0^1 \frac{(1-v)^{\frac{4-d}{2}}}{1+\frac{(s-4m^2)v}{4m^2}} \ln \frac{1+\sqrt{\frac{vt(s-4m^2)}{4m^2u}}}{1-\sqrt{\frac{vt(s-4m^2)}{4m^2u}}}\, dv. \tag{39}$$

Thus, we found that the Appell function F_2, the function ϕ and, therefore, also the integral $B(s,t)$ are expressible in terms of hypergeometric functions admitting one-fold integral representations. Such a representation of $B(s,t)$ would be convenient for obtaining higher-order terms in the ε expansion of this integral. The first terms in the ε expansion of the integral $B(s,t)$ were obtained in Ref. [14].

In Ref. [9], a one-fold integral representation for a scalar box integral with arbitrary masses, external momenta and space-time dimension d was presented. To obtain the formula for $B(s,t)$ directly from such a representation by setting masses and scalar invariants to their specific values would be not so easy because the appropriate analytic formulae are rather lengthy, and also the analytic continuations needed for the hypergeometric functions are rather nontrivial. Our result can be considered as a confirmation that the one-fold integral representations for box integrals with physically relevant kinematics do exist.

5 Imaginary parts and spectral representation for the integral D_2

The integral $I_4^{(d)}$ with all internal lines massless and external legs on shell can be calculated analytically. Using the method of dimensional recurrences [6], we obtain the following relation for the last integral in Eq. (13), assuming $|s+t| \le |s|$ and $|s+t| \le |t|$:

$$I_4^{(d)}(0,0,0,0;\ 0,0,0,0;\ s,t) = \frac{-4(d-3)}{st(d-4)} \times \left\{ I_2^{(d)}(0,0;\ t)\, {}_2F_1\left[\begin{matrix}1, \frac{d}{2}-2\,; \\ \frac{d}{2}-1\,;\end{matrix} \frac{s+t}{s}\right] + I_2^{(d)}(0,0;\ s)\, {}_2F_1\left[\begin{matrix}1, \frac{d}{2}-2\,; \\ \frac{d}{2}-1\,;\end{matrix} \frac{s+t}{t}\right]\right\}. \tag{40}$$

This formula is in agreement with the result obtained in Ref. [15] (see also Refs. [16, 17]). Thus, Eqs. (14), (15), and (40) provide us with a hypergeometric representation for the integral D_2. The ε expansions of the real and imaginary parts of this integral through order ε^2 were given in Ref. [18]. We expect that, with our hypergeometric representation, one can derive a shorter result for the ε^2 term in the expansion of the integral D_2 than that given in Ref. [18].

The functional equations in Eqs. (10) and (13) can be used for finding imaginary parts of the integrals in some kinematical regions. As one can see from Eq. (10), if $s > m^2$ and $t < 0$, the integral $I_4^{(d)}(0,0,0,m^2;\ 0,0,m^2,m^2;\ s,t)$ has an imaginary part that arises only from the integral $I_4^{(d)}$ with all propagators massless, which can be easily found from Eq. (40), so that

$$\mathrm{Im}\ D_2(t,s) = \frac{4(d-3)}{(d-4)} \frac{\sin\frac{\pi d}{2}}{t(m^2-s)} I_2^{(d)}\left(0,0;\ \frac{(s-m^2)^2}{-s}\right) {}_2F_1\left[\begin{matrix}1, \frac{d}{2}-2\,; \\ \frac{d}{2}-1\,;\end{matrix} 1+\frac{(s-m^2)^2}{st}\right], \qquad s > m^2, \quad t < 0. \tag{41}$$

In a similar fashion, if $s > 4m^2$ and $t < 0$, the imaginary part of the integral $B(s,t)$ can be obtained from Eq. (13). In this case, the imaginary part originates only from the second integral on the right-hand side of Eq. (13). Again one can use Eq. (40) to find:

$$\mathrm{Im}\ B(s,t) = \frac{4(d-3)}{(d-4)} \frac{\beta_s\ \sin\frac{\pi d}{2}}{t(4m^2-s)} I_2^{(d)}\left(0,0;\ 4m^2-s\right) {}_2F_1\left[\begin{matrix}1, \frac{d}{2}-2\,; \\ \frac{d}{2}-1\,;\end{matrix} 1+\frac{s-4m^2}{t}\right], \qquad s > 4m^2, \quad t < 0. \tag{42}$$

One can write $D_2(s,t)$ in the fixed-t spectral representation

$$D_2(t,s) = \frac{1}{\pi}\int\limits_{m^2}^{\infty} \frac{dx\,\mathrm{Im}\,D_2(t,x)}{s-x}. \tag{43}$$

Substituting Eq. (41) into Eq. (43) leads to the following integral representation for $D_2(t,s)$:

$$D_2(t,s) = \frac{4(d-3)\sin\frac{\pi d}{2}}{(d-4)t\pi}\int\limits_{m^2}^{\infty} \frac{dx}{(s-x)(x-m^2)} \times\, I_2^{(d)}\left(0,0;\,\frac{(x-m^2)^2}{-x}\right)\,{}_2F_1\left[\begin{matrix}1,\frac{d}{2}-2\,;\\ \frac{d}{2}-1\,;\end{matrix}\,1+\frac{(x-m^2)^2}{xt}\right]. \tag{44}$$

To use $D_2(t,s)$ in calculations of heavy-quark production, one needs to know it at $t>0$ and $s<0$. To perform the analytic continuation of $D_2(t,s)$ into the region $t>0$, one can use Eq. (44). Using a formula for the analytic continuation of the Gauss hypergeometric function [11] and introducing the new integration variable

$$v = \frac{xt}{(m^2-x)^2+xt}, \tag{45}$$

leads to the following expression:

$$D_2(t,s) = \frac{2(d-3)\sin\frac{\pi d}{2}}{\pi s t^{4-\frac{d}{2}}}\int\limits_0^1 \frac{dv}{1-v\sigma}\, I_2^{(d)}\left(0,0;\,\frac{v-1}{v}\right)\left\{\frac{m^2-s}{1-v} - \frac{(s+m^2)}{\sqrt{(1-v)(1-v\theta)}}\right\} \times\left\{\frac{v}{6-d}\;{}_2F_1\left[\begin{matrix}1,3-\frac{d}{2}\,;\\ 4-\frac{d}{2}\,;\end{matrix}\,v\right] + \frac{\pi}{2}v^{\frac{d}{2}-2}\cot\frac{\pi d}{2} + i\frac{\pi}{2}v^{\frac{d}{2}-2}\right\}, \tag{46}$$

where

$$\sigma = 1+\frac{(s-m^2)^2}{st}, \tag{47}$$

and θ is defined in Eq. (16). Substituting the expression for $I_2^{(d)}$ in this formula, we obtain for the imaginary part:

$$\mathrm{Im}\,D_2(t,s) = \frac{-\pi^{\frac{3}{2}}(d-3)t^{\frac{d}{2}-4}}{s2^{d-4}\Gamma\left(\frac{d-1}{2}\right)}\left\{\frac{m^2-s}{d-4}\;{}_2F_1\left[\begin{matrix}1,1\,;\\ \frac{d-2}{2}\,;\end{matrix}\,\sigma\right] - \frac{s+m^2}{d-3}F_1\left(1,1,\frac{1}{2};\frac{d-1}{2};\sigma,\theta\right)\right\}, \quad s<0, \quad t>0. \tag{48}$$

Applying the Euler transformation to the hypergeometric functions ${}_2F_1$ and F_1 (see, for example, Ref. [11]), we find agreement with the expression for the imaginary part obtained in this region from the analytic result for the integral $B(s,t)$ given in Eq. (15).

For the real part, one can write the following representation:

$$\mathrm{Re}\,D_2(t,s) = \frac{2(d-3)(1-\sigma)^{\frac{d}{2}-2}}{(d-4)\;t(s-m^2)} I_2^{(d)}(0,0;-t)\;{}_2F_1\left[\begin{matrix}1,\frac{d-4}{2}\,;\\ \frac{d-2}{2}\,;\end{matrix}\,\sigma\right] + \cot\frac{\pi d}{2}\;\mathrm{Im}\,D_2(t,s) + \Phi(t,s), \quad s<0, \quad t>0, \tag{49}$$

where

$$\Phi(t,s) = \frac{2^{4-d}\,\sqrt{\pi}\,(d-3)(s+m^2)}{(6-d)\Gamma\left(\frac{d-1}{2}\right)\,s\,t^{4-\frac{d}{2}}} \int\limits_0^1 \frac{dv}{1-v\sigma}\,\frac{v^{3-\frac{d}{2}}(1-v)^{\frac{d-5}{2}}}{\sqrt{(1-v\theta)}}\,{}_2F_1\left[\begin{matrix}1, 3-\frac{d}{2};\\ 4-\frac{d}{2};\end{matrix} v\right]. \tag{50}$$

Equations (48) and (49) are convenient for the ε expansion. Appropriate formulae for the ε expansion of the Gauss hypergeometric function can be taken from Refs. [19, 20]. To obtain the ε expansion of the Appell hypergeometric function F_1, one can use a one-fold integral representation (see Appendix). The first two terms in Eq. (49) give singular contributions proportional to $1/\varepsilon^2$ and $1/\varepsilon$. The last term in Eq. (49) is regular in ε. Its ε expansion can be derived from integral representation of Eq. (50).

In the case when $t > 0$ and $s < 0$, the imaginary part of $B(s,t)$ can be found from Eq. (15) and reads:

$$\mathrm{Im}\; B(s,t) = \frac{\pi^{\frac{3}{2}}\;2^{4-d}\;t^{\frac{d-5}{2}}}{m(s-4m^2)\Gamma\left(\frac{d-1}{2}\right)} F_1\left(\frac{d-3}{2}, 1, \frac{1}{2}; \frac{d-1}{2}; 1-\frac{t}{4m^2-s}, 1-\frac{t}{4m^2}\right). \tag{51}$$

The imaginary part of the integral $D_2(s,t)$ for $s > 0$ and $t < 0$ reads:

$$\begin{aligned}\mathrm{Im}\; D_2(s,t) = \sin\frac{\pi d}{2}\; I_2^{(d)}(0,0;\; -s)\; &\left\{\frac{2(d-3)}{(d-4)s(m^2-t)}\;{}_2F_1\left[\begin{matrix}1, \frac{d}{2}-2;\\ \frac{d}{2}-1;\end{matrix} 1+\frac{st}{(t-m^2)^2}\right]\right. \\ &\left. - \frac{(t+m^2)}{m\sqrt{s}(t-m^2)^2}\; F_1\left(\frac{d-3}{2}, 1, \frac{1}{2}, \frac{d-1}{2}; 1+\frac{st}{(t-m^2)^2}, 1-\frac{s}{4m^2}\right)\right\}. \end{aligned}\tag{52}$$

The first few terms of the ε expansion of the Appell function F_1 can be found in Refs. [9, 21].

6 Functional equations for D_2 with one quark leg off shell

For the computation of the radiative corrections to heavy-quark production in the NNLO approximation, one needs to know the integral D_2 with one quark leg off shell. This is a rather nontrivial task, which will be considered in a forthcoming publication. In this section, we just want to outline the strategy of such calculations. Similar to the case of D_2 with all legs on shell, one can derive the functional equations:

$$\begin{aligned} &I_4^{(d)}(0,0,0,m^2;0,0,s_{34},m^2;\; s,t) \\ &= \frac{m^2}{s} I_4^{(d)}\left(0,0,0,m^2;0,\frac{(m^2-s_{34})(m^2-s)}{s}, s_{34}, m^2, \frac{m^4}{s}, t\right) \\ &\quad + \frac{s-m^2}{s} I_4^{(d)}\left(0,0,0,0;\; \frac{(m^2-s_{34})(m^2-s)}{s}, 0,0,0;\; \frac{(m^2-s)^2}{s}, t\right), \end{aligned}\tag{53}$$

$$\begin{aligned} &I_4^{(d)}(0,0,0,m^2;\; 0,0,s_{34},m^2;\; s,t) \\ &= \frac{m^2}{s_{34}} I_4^{(d)}\left(0,0,0,m^2;\; \frac{(m^2-s_{34})(m^2-s)}{s_{34}}, 0, m^2, \frac{m^4}{s_{34}};\; s, \frac{tm^2}{s_{34}}\right) \\ &\quad + \frac{s_{34}-m^2}{s_{34}} I_4^{(d)}\left(0,0,0,0;\; \frac{tm^2}{s_{34}}, \frac{(m^2-s_{34})^2}{s_{34}}, 0,0;\; \frac{(m^2-s_{34})(m^2-s)}{s_{34}}, t\right). \end{aligned}\tag{54}$$

Equations (53) and (54) were obtained from Eq. (1) by setting

$$m_1^2 = m_2^2 = m_3^2 = m_5^2 = s_{12} = s_{23} = 0, \qquad m_4^2 = s_{14} = m^2,$$
$$G_4 = \partial_5\Delta_5 = \partial_1\partial_5\Delta_5 = 0, \tag{55}$$

in both cases and additionally

$$\partial_3\partial_5\Delta_5 = 0, \tag{56}$$

in Eq. (53) and

$$\partial_2\partial_5\Delta_5 = 0, \tag{57}$$

in Eq. (54). The above equations are rather similar to the functional equations for the integral D_2 with all legs on shell. In some kinematical regions, the imaginary part of the integral $D_2(s_{34}, s, t)$ arises from integrals with all propagators massless. In Eq. (53), the integral with massless propagators can be expressed in terms of three Gauss hypergeometric functions [15]. The analytic expression for the massless integral in Eq. (54) for arbitrary d is not known at present. The first terms in the ε expansion of both integrals may be found in Refs. [15, 16]. We expect that, similar to the D_2 integral with all legs on shell, the integral D_2 with one leg off shell can be represented in terms of hypergeometric functions admitting a one-fold integral representation.

7 Conclusions

The usefulness of functional equations turns out to be threefold. First, we obtain a hypergeometric representation of the integral needed for NLO calculations of heavy-quark production from the result for the integral from Bhabha scattering. Second, since the ε^2 term in the expansion of the integral D_2 is known [18], one can use it to obtain the ε^2 term for Bhabha scattering. Third, functional equations provide a simple method to obtain imaginary parts of integrals. For some kinematic regions, the imaginary parts of integrals with nonzero internal masses can be related to the integrals with all lines massless.

It is also important that the functional equations provide a tool for performing analytic continuations of the considered integrals. As was already observed in Ref. [1] and now also in this paper, functional equations suitable for analytic continuation express the considered integral in terms of the same integral with transformed arguments that has no imaginary part plus simpler integrals (usually with massless lines) giving the imaginary part of the integral. Such analytic continuation is achieved by solving algebraic equations, so that the explicit analytic form of the integral is not needed.

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9 Appendix

In this appendix, we present formulae which were used in the derivation of some equations in the main text.

The integral representation of the Appell hypergeometric function F_1 reads:

$$F_1(\alpha,\beta,\beta',\gamma;\ x,y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int\limits_0^1 \frac{u^{\alpha-1}(1-u)^{\gamma-\alpha-1}}{(1-ux)^{\beta}(1-uy)^{\beta'}} du,$$

$$\text{Re}\ \alpha > 0, \quad \text{Re}\ (\gamma-\alpha) > 0. \tag{58}$$

The integral representation of the Appell function F_3 used in the derivation of the one-fold integral representations of the ϕ_i functions reads:

$$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma(\beta)} \int_0^1 \frac{u^{\gamma-\beta-1}(1-u)^{\beta-1}}{(1-x+ux)^{\alpha}}\ {}_2F_1\left[\begin{matrix}\alpha',\beta'\,; \\ \gamma-\beta\,;\end{matrix} uy\right]\ du. \tag{59}$$

This integral representation follows from Eq. (20) in Ref. [22].

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