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Counting master integrals: integration-by-parts procedure with effective mass

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Abstract

We show that the new relation between master integrals recently obtained in Ref. [1] can be reproduced using the integration-by-parts technique implemented with an effective mass. In fact, this relation is recovered as a special case of a whole family of new relations between master integrals.

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Recently, one of us in collaboration with Mikhail Kalmykov found [1] a new relation between some specific Feynman integrals, which is actually absent in modern computer programs based on the integration-by-parts (IBP) technique [2] (for a recent review, see Ref. [3]). The new relation arises in the framework of the so-called differential reduction (see Refs. [1,4] and references cited therein) developed by these authors during last several years. This decreases the number of master integrals and, thus, leads to a simplification of calculations.

In this short note, we recover this relation directly in the framework of the IBP technique by introducing an effective mass originating from the reduction of one-loop integrals to simple propagators (see Refs. [5,6] and Eq. (2) below). In fact, this relation is found to be a special case of a whole family of new relations between master integrals.

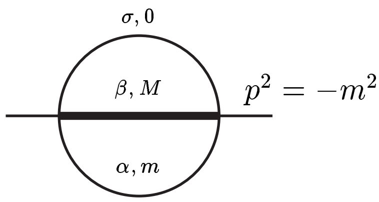

Figure 1: Two-loop sunset diagram $J_{012}(\sigma,\beta,\alpha)$ involving propagators with masses 0, M, and m raised to the powers σ, β, and α, respectively, taken on the mass shell $p^2 = -m^2$.

Following Ref. [1], let us consider the two-loop self-energy sunset-type diagram J_{012} with on-shell kinematics, defined as

$$J_{012}(\sigma,\beta,\alpha) = \frac{1}{\pi^n} \int \frac{d^n k_1 d^n k_2}{[(p-k_1)^2]^\sigma [(k_1-k_2)^2 + M^2]^\beta [k_2^2 + m^2]^\alpha} \bigg|_{p^2=-m^2}, \qquad (1)$$

where $n = 4 - 2\varepsilon$ is the dimensionality of space time. It is depicted in Fig. 1.

Considering the standard Feynman representation of the following one-loop diagram as a one-fold integral

$$\begin{aligned} I(\alpha_1,\alpha_2) &\equiv \frac{1}{\pi^{n/2}} \int \frac{d^n k}{[k^2 + M_1^2]^{\alpha_1} [(p-k)^2 + M_2^2]^{\alpha_2}} \\ &= \frac{\Gamma(\alpha_1+\alpha_2-n/2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 \frac{ds\, s^{n/2-1-\alpha_1} \overline{s}^{n/2-1-\alpha_2}}{[p^2 + M_1^2/s + M_2^2/\overline{s}]^{\alpha_1+\alpha_2-n/2}}, \end{aligned} \qquad (2)$$

with $\overline{s} = 1 - s$, we can interpret this as an integral over a new propagator with the effective mass $M_1^2/s + M_2^2/\overline{s}$. In previous papers [5,6], this procedure was used to decrease the numbers of loops in analyses of different types of master integrals and, thus, to simplify calculations.

Using Eq. (2) with $M_1 = M$ and $M_2 = 0$, we can represent the considered two-loop diagram $J_{012}(\sigma,\beta,\alpha)$ as the one-fold integral

$$J_{012}(\sigma,\beta,\alpha) = \frac{\Gamma(\sigma+\beta-n/2)}{\Gamma(\sigma)\Gamma(\beta)} \int_0^1 \frac{ds}{s^{\beta+1-n/2} \overline{s}^{\sigma+1-n/2}} I_{12}(\alpha, \beta+\sigma-n/2), \qquad (3)$$

where

$$I_{12}(\alpha_1, \alpha_2) = \frac{1}{\pi^{n/2}} \int \frac{d^n k}{[(p-k)^2 + m^2]^{\alpha_1} [k^2 + M^2/s]^{\alpha_2}} \bigg|_{p^2 = -m^2} \tag{4}$$

is a one-loop on-shell diagram.

Applying the IBP procedure to the one-loop integral $I(\alpha_1, \alpha_2)$ considered in Eq. (2), with the distinguished line carrying the index α_1 (see, for example, Ref. [5]),[1] we have the general relation

$$(n - 2\alpha_1 - \alpha_2) I(\alpha_1, \alpha_2) = \alpha_2 \left[I(\alpha_1 - 1, \alpha_2 + 1) - \left(p^2 + M_1^2 + M_2^2 \right) I(\alpha_1, \alpha_2 + 1) \right] - 2\alpha_1 M_1^2 I(\alpha_1 + 1, \alpha_2). \tag{5}$$

Thus, for $I_{12}(1, \alpha_2)$ considered in Eq. (4), we can apply Eq. (5) with $M_1 = m$, $M_2 = M/\sqrt{s}$, $p^2 = -m^2$, and $\alpha_1 = 1$. The result is

$$(n - 2 - \alpha_2) I_{12}(1, \alpha_2) = \alpha_2 I_{12}(0, 1 + \alpha_2) - \alpha_2 \frac{M^2}{s} I_{12}(1, 1 + \alpha_2) - 2m^2 I_{12}(2, \alpha_2), \tag{6}$$

where the tadpole $I_{12}(0, 1 + \alpha_2)$ has the form

$$I_{12}(0, 1 + \alpha_2) = \frac{\Gamma(\alpha_2 + \varepsilon - 1)}{\Gamma(\alpha_2 + 1)} \left(\frac{s}{M^2} \right)^{\alpha_2 + \varepsilon - 1}. \tag{7}$$

Integrating Eq. (6) with $\alpha_2 = \sigma + \beta - n/2$, multiplied by the factor

$$\frac{\Gamma(\sigma + \beta - n/2)}{\Gamma(\sigma)\Gamma(\beta)} \frac{1}{s^{\beta + 1 - n/2} \bar{s}^{\sigma + 1 - n/2}}$$

as on the r.h.s. of Eq. (3), over s, we obtain

$$(4 - 3\varepsilon - \sigma - \beta) J_{012}(\sigma, \beta, 1) = \frac{\pi}{\sin[\pi(2 - \varepsilon - \sigma)]} \frac{\Gamma(\sigma + \beta - 3 + 2\varepsilon)}{\Gamma(\sigma)\Gamma(\beta)} (M^2)^{3 - 2\varepsilon - \sigma - \beta} - M^2 \beta J_{012}(\sigma, 1 + \beta, 1) - 2m^2 J_{012}(\sigma, \beta, 2). \tag{8}$$

Putting $\sigma = \beta = 1$, we recover the new relation discovered in Ref. [1] as a special case of a more general class of relations between IBP master integrals.

In conclusion, applying the IBP procedure to a one-loop integral with an effective mass in one of its propagators, we produced the new relation (8) between ordinary IBP master integrals. This relation coincides for $\sigma = \beta = 1$ with the one recently discovered in Ref. [1], but it is more general and obtained in a more straightforward way. We intend to extend this analysis to the case of the off-shell sunset diagrams in a future work. The effective-mass procedure applied here to reduce the number of master integrals with respect to the one achieved by the ordinary IBP procedure may in principle be applied whenever the considered topology contains a bubble subdiagram. We expect that relations between ordinary IBP master integrals thus obtained may be usefully implemented in modern computer packages based on the IBP procedure.

[1] I.e. the factor coming in IBP procedure has the form $n = d(k - k_1)^\mu / dk^\mu$, where $k - k_1$ is the momentum of the propagator carrying the index α_1.

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