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On quantum effects in spontaneous emission by a relativistic electron beam in an undulator

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Abstract

Robb and Bonifacio (2011) claimed that a previously neglected quantum effect results in noticeable changes in the evolution of the energy distribution associated with spontaneous emission in long undulators. They revisited theoretical models used to describe the emission of radiation by relativistic electrons as a continuous diffusive process, and claimed that in the asymptotic limit for a large number of undulator periods the evolution of the electron energy distribution occurs as discrete energy groups according to Poisson distribution. We show that these novel results have no physical sense, because they are based on a one-dimensional model of spontaneous emission and assume that electrons are sheets of charge. However, electrons are point-like particles and, as is well-known, the bandwidth of the angular-integrated spectrum of undulator radiation is independent of the number of undulator periods. If we determine the evolution of the energy distribution using a three-dimensional theory we find the well-known results consistent with a continuous diffusive process. The additional pedagogical purpose of this paper is to review how quantum diffusion of electron energy in an undulator with small undulator parameter can be simply analyzed using the Thomson cross-section expression, unlike the conventional treatment based on the expression for the Lienard-Wiechert fields.

1 Introduction

In a recent article [1] it is stated that quantum effects in spontaneous emission by a relativistic electron beam in an undulator can be described by a drift-diffusion equation only when the parameter

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$$\epsilon = \frac{N_w \hbar \omega}{\gamma m c^2}, \quad (1)$$

is much smaller than unity, where N_w is the number of undulator periods, \hbar is the reduced Planck constant, ω is the photon frequency, γ the relativistic Lorentz factor, m the electron rest mass and c the speed of light. In that work it is argued that when $\epsilon \geq 1$, a drift-diffusion equation is no more sufficient to describe the the evolution of the distribution of electron momenta, which "occurs as discrete momentum groups according to a Poisson distribution".

In this paper we will show that results in [1] are incorrect, because they are based on a one-dimensional model of the spontaneous radiation emission. This model does not account for the angular distribution of the radiation, but only for the emission on axis, which is characterized by an overall relative bandwidth $\sim 1/N_w$. In contrast to this, the electron recoil related with the quantized nature of photons depends on the entire angular distribution of the radiation, which is fundamentally linked to the Thomson scattering phenomenon in the case for a small undulator parameter $K \ll 1$, [2, 3]. When the angular distribution of radiation is properly accounted for, the overall, angle-integrated relative bandwidth is independent on the number of undulator periods. As a result, it turns out that a three-dimensional drift-diffusion model is valid when the parameter

$$\zeta = \frac{\hbar \omega}{\gamma m c^2}, \quad (2)$$

is much smaller than unity. This means that a Fokker-Planck approach is always valid in all cases of practical interest.

In this work we will first review the spectral-angular characteristics of undulator radiation. For reasons of simplicity, from the very beginning we will consider the limit for $N_w \gg 1$ and $K \ll 1$. Our considerations can easily be applied to arbitrary values of N_w and K , but the choice of $N_w \gg 1$ and $K \ll 1$ easily allows one to underline the fundamental point that the angle-integrated spectrum of radiation does not depend on the number of undulator periods N_w . Using a Fokker-Planck equation we will derive the diffusion coefficient in agreement with [4]. Finally, the diffusion coefficient will also be derived by exploiting the relation between undulator radiation and Thomson scattering, which stresses once more the intrinsic three-dimensional nature of the radiation pattern.

2 Spontaneous emission process and associated quantum effects

2.1 Spectral-angular distribution of radiation

As is well-known, spontaneous radiation emission from an ultrarelativistic electron in an undulator can be modeled fully classically as long as the energy of the emitted photons $\hbar\omega$ is much smaller than the electron energy γmc^2 . In this case, the knowledge of the classical characteristics of radiation can easily be used to discuss quantum effects on the electron motion integrated along the trajectory.

Characteristics of spontaneous radiation have been studied long time ago in [5, 6]. In this section, we briefly review them, focusing on the particular case of a planar undulator, and following notations introduced in previous works of us [7]. In order to do so, we first call with $\vec{E}(\omega)$ the transverse component of the electric field generated by an electron in the space-frequency domain². Based on the ultrarelativistic approximation $\gamma^2 \gg 1$ and on the consequent paraxial approximation, we introduce the slowly varying electric field envelope $\vec{E} = \vec{E} \exp[-i\omega z/c]$, which does not vary much along the longitudinal coordinate z on the scale of the reduced wavelength $\lambda = \lambda/(2\pi)$. We can specify "how near" ω is to the resonant frequency of the undulator, $\omega_{r0} = 2k_w c \bar{\gamma}_z^2$, by introducing a detuning parameter C , defined as $C = \omega/(2\bar{\gamma}_z^2 c) - k_w = (\Delta\omega/\omega_{r0})k_w$, where $\omega = \omega_{r0} + \Delta\omega$. Here $k_w = 2\pi/\lambda_w$, λ_w is the undulator period, K is the undulator parameter, which is related to the undulator magnetic field B by

$$B = \frac{Kmc^2 k_w}{e}, \quad (3)$$

and $\bar{\gamma}_z = \gamma/\sqrt{1+K^2/2}$. We further simplify our considerations by considering from the beginning the case for $K^2 \ll 1$ and $N_w \gg 1$. These two assumptions do not change the nature of our considerations, and are introduced for simplicity only. One obtains

$$\vec{E} = -\frac{\omega K e L_w}{2c^2 z \gamma} \exp\left[i\frac{\omega\theta^2 z}{2c}\right] \left\{ \left[1 - \frac{\theta_x^2 \omega}{k_w c}\right] \vec{e}_x + \left[\frac{\theta_x \theta_y \omega}{k_w c}\right] \vec{e}_y \right\}$$

² By this, $\vec{E}(\omega)$ is defined as the Fourier transform of the electric field in the time domain, $\vec{E}(t)$, according to $\vec{E}(\omega) = \int_{-\infty}^{\infty} \vec{E}(t) \exp[i\omega t] dt$, and has a dimension of an electric field multiplied by a time.

$$\times \text{sinc} \left[\frac{L_w}{4} \left(C + \frac{\omega \theta^2}{2c} \right) \right], \quad (4)$$

where θ_x and θ_y are horizontal and vertical angles identifying the angular position of an observer and $\theta^2 = \theta_x^2 + \theta_y^2$. Here and everywhere in this paper we will be using Gaussian units. The total energy emitted per unit spectral interval per unit solid angle turns out to be

$$\frac{dW}{d\omega d\Omega} = \frac{\omega^2 K^2 L_w^2 e^2}{16\pi^2 c^3 \gamma^2} \left\{ \left[1 - \frac{\theta_x^2 \omega}{k_w c} \right]^2 + \left[\frac{\theta_x \theta_y \omega}{k_w c} \right]^2 \right\} \text{sinc}^2 \left[\frac{L_w}{4} \left(C + \frac{\omega \theta^2}{2c} \right) \right], \quad (5)$$

in agreement with [6].

2.2 Angle-integrated spectral distribution of radiation

We now integrate Eq. (5) over all angles by using the fact that $N_w \gg 1$. When this is the case, the bandwidth of the radiation spectrum does not depend on the number of undulator periods. This is represented, mathematically, by the fact that the sinc function in Eq. (5) can be substituted with a Dirac- δ function according to $\text{sinc}^2[x/a]/(\pi a) \rightarrow \delta(x)$ for $a \rightarrow 0$. Integrating over the solid angle we obtain

$$\frac{dW}{d\omega} = \frac{e^2 \omega K^2 L_w}{4c^2 \gamma^2} \left[1 + \left(\frac{\omega}{ck_w \gamma^2} - 1 \right)^2 \right], \quad (6)$$

for $\omega < 2c\gamma^2 k_w$, and zero otherwise. Note that here we already set $\bar{\gamma}_z \simeq \gamma$ in the limit for $K \ll 1$. Eq. (6) is in agreement with expressions in literature, e.g. [5] (where the energy spectrum was first calculated) and [8]³.

For us, the important point to be underlined by inspection of Eq. (6) is the fact that the radiation spectrum depends on the number of undulator periods only through a scaling factor. In other words, the bandwidth is independent of N_w . The reason for this is that we are now considering the spectrum integrated over angles. At variance, the on-axis spectral bandwidth exhibits a dependence on the number of undulator periods, and scales as $1/N_w$. The authors of [1] consider from the very beginning a one-dimensional model and explicitly state that "the linewidth of wiggler radiation is $\Delta\omega/\omega \sim 1/N_w$ ". This is correct if one considers the on-axis spectrum only, for example analyzing the undulator output through a pinhole.

³ A typing error is present in Eq. (2.11) of [8].

In our case of interest, however, we want to discuss the effect of the electron recoil due to the quantized nature of radiation, and the electron does not distinguish radiation emitted on axis from radiation emitted at an angle. The one-dimensional model in [1] cannot be applied, and the linewidth of the radiation is independent of N_w .

2.3 Drift-diffusion model

The previous derivations and observations should convince the reader that the parameter ϵ defined in Eq. (1) is unphysical, and that a Fokker-Planck equation can properly describe the evolution of the electron density, as long as $\zeta = \hbar\omega/(\gamma mc^2) \ll 1$. The coefficient of quantum diffusion in a bending magnet was calculated for the first time in [9]. This expression is valid for calculations of energy diffusion in the undulator at large values of the undulator parameter. At arbitrary values of K the quantum diffusion coefficient was calculated in [4].

Let us write the evolution equation for a particular projection of the electron phase space as a function the energy-time variables. Calling with $f = f(\mathcal{E}, t)$ this projection of the electron density phase space, and with $\psi(\mathcal{E}, \Delta\mathcal{E})d\Delta\mathcal{E}$ the probability to find an electron with energy between \mathcal{E} and $\mathcal{E} + \Delta\mathcal{E}$ in the time interval Δt , we write the evolution equation as

$$\frac{\partial f}{\partial t} = -C_1 \frac{\partial f}{\partial \mathcal{E}} + \frac{1}{2} C_2 \frac{\partial^2 f}{\partial \mathcal{E}^2} \quad (7)$$

where

$$C_1 = \frac{1}{\Delta t} \int d\Delta\mathcal{E} \psi(\mathcal{E}, \Delta\mathcal{E}) \Delta\mathcal{E} \quad (8)$$

and

$$C_2 = \frac{1}{\Delta t} \int d\Delta\mathcal{E} \psi(\mathcal{E}, \Delta\mathcal{E}) \Delta\mathcal{E}^2 \quad (9)$$

Eq. (8) is just the rate of mean energy lost of an electron, Eq. (9) gives the diffusion coefficient we are after. We impose energy conservation by setting $\Delta\mathcal{E} = \hbar\omega$. By noting that

$$\frac{1}{\hbar\omega} \frac{dW}{d(\hbar\omega)} = \psi(\mathcal{E}, \Delta\mathcal{E}) \quad (10)$$

and using Eq. (6) we obtain

$$\frac{C_2}{m^2 c^4} = \frac{d\langle(\Delta\gamma)^2\rangle}{dt} = \frac{c}{L_w} \frac{1}{m^2 c^4} \int_0^\infty d\omega \hbar\omega \frac{dW}{d\omega} = \frac{7}{15} r_e c \lambda_c K^2 k_w^3 \gamma^4 . \quad (11)$$

Not surprisingly, Eq. (11) is in agreement with the result obtained in [4] in the limit for $K \ll 1$. Note that despite the use of a one-dimensional model, authors of [1] find parametric agreement with Eq. (11) in the case for $\epsilon \ll 1$. The reason for this is that for $\epsilon \ll 1$ they use a drift-diffusion equation. They cannot recover the exact numerical result, since they miss the contribution to the diffusion coefficient coming from radiation emitted at angles different from zero, due to the incorrect choice of a one-dimensional model, but the right parameters are nevertheless present in this asymptote. However, the use of the one-dimensional model leads to the introduction of the unphysical parameter ϵ , and to the consequent introduction of an artificial quantum effect that does not exist in reality for $\epsilon \geq 1$.

2.4 Relation with Thomson scattering

It is straightforward to underline the well-known equivalence between the previously obtained results and Thomson scattering of radiation. In fact, in the limit for $K^2 \ll 1$ and $N_w \gg 1$ and in the reference system of the electron, the undulator magnetic field is seen as a plane wave interacting with the electron with frequency

$$\omega_R = \gamma c k_w . \quad (12)$$

This simple observation includes the essence of the Weizsäcker-Williams method of virtual quanta [10], and allows to calculate the quantum diffusion coefficient, Eq. (11), following an alternative derivation in the rest frame.

Due to the presence of the electron, the plane wave scatters radiation as a function of the rest frame angle. Under the approximation $\hbar\omega_R \ll m_e c^2$ the process differential cross-section for horizontally polarized incident radiation is just the Thomson cross-section for polarized radiation:

$$\frac{d\sigma}{d\Omega_R} = r_e^2 \left[\cos^2(\theta_R) \cos^2(\phi_R) + \sin^2(\phi_R) \right] , \quad (13)$$

where θ_R and ϕ_R are spherical coordinate angles in the rest frame⁴ of the

⁴ We will label the rest frame with R , and the lab frame with L .

electron and $r_e = e^2/(mc^2)$ is the classical electron radius.

In the language of photons we can say that Eq. (13) is related to the probability of scattering a photon in the solid angle $d\Omega_R = \sin\theta_R d\theta_R d\phi_R$. In fact, remembering that the radiation pulse in the rest frame has a duration given by $L_w/(\gamma c)$, the number of photons scattered in $d\Omega_R$ can be written as

$$\frac{dN_{phR}}{d\Omega_R} = \frac{d\sigma}{d\Omega_R} \frac{L_w}{\gamma c} \frac{1}{\hbar\omega_R} \bar{S}_R, \quad (14)$$

where \bar{S}_R is the time-averaged Poynting vector of the radiation incident on the electron in the rest frame. Note that only elastic scattering takes place under the over-mentioned assumption $\hbar\omega_R \ll m_e c^2$. Therefore, there is no change of photon frequency in the scattering process. Since the wave packet incident on the electron includes $N_w \gg 1$ period we can assume, with accuracy $1/N_w \ll 1$, that the incoming wave packet is composed of a single frequency. This explain why Eq. (14) is not analyzed in frequency.

The magnitude of the time-averaged Poynting vector of the radiation incident on the electron in the rest frame can be found remembering that⁵

$$\begin{aligned} E_R &\simeq \gamma B_L \\ B_R &= \gamma B_L \end{aligned} \quad (15)$$

where B_L is the undulator field in the laboratory frame. With the help of Eq. (3), the time-averaged Poynting vector of the radiation incident on the electron in the rest frame can be written as

$$\bar{S}_R = \frac{c}{8\pi} \left(\frac{\gamma K m c^2 k_w}{e} \right)^2. \quad (16)$$

The relation between frequencies in the laboratory frame and in the rest frame obey the following Lorentz transformation

$$\omega_L(\theta_R) = \gamma \omega_R (1 + \cos\theta_R). \quad (17)$$

By energy conservation we can identify the change in the electron energy in the laboratory frame with the photon energy in the laboratory frame. Eq. (17) allows us to calculate this quantity by averaging over the number of photons scattered at angles θ_R in the rest frame, Eq. (14). We can write the rate of change in the spread of $\Delta\gamma$ as

⁵ Note that $E_L = \gamma \vec{\beta} \times \vec{B}_R$, which presents a correction of order $1/\gamma^2$ with respect to Eq. (15). In our case, this correction can be omitted and $E_R \simeq \gamma B_L$.

$$\frac{d\langle(\Delta\gamma)^2\rangle}{dt} = \frac{c}{L_w} \int d\Omega_R \left(\frac{\hbar\omega_L(\theta_R)}{mc^2} \right)^2 \frac{dN_{phR}}{d\Omega_R}. \quad (18)$$

With the help of Eqs. (12)-(14), Eq. (16) and Eq. (17) we find

$$\frac{d\langle(\Delta\gamma)^2\rangle}{dt} = \frac{7}{15} r_e c \lambda_c K^2 k_w^3 \gamma^4, \quad (19)$$

in perfect agreement with Eq. (11).

3 Conclusions

In this paper we showed that quantum effects in spontaneous radiation emission can be satisfactorily modeled via a drift-diffusion model. It is of fundamental importance to treat spontaneous radiation within a three-dimensional model. This is explained by the fact that an electron feeling photon recoil does not filter photons along a privileged direction, but reacts to photons emitted at all angles. In this case, contrarily to what has been argued in [1], the linewidth of spontaneous radiation must be integrated over all angles, and is independent of the number of undulator periods N_w . It follows from our analysis that if one enforces a three-dimensional model for the spontaneous emission, a drift-diffusion model remains valid up to photon energies smaller than the electron energy, which practically means always. This conclusion is also in contrast with [1], where the assumption of a linewidth scaling with $1/N_w$ leads to the identification of an unphysical parameter scaling as N_w , and to the rise of artificial quantum effects when this parameters becomes comparable with unity.

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