The integrable structure of nonrational conformal field theory

A. Bytsko[*], *J. Teschner*[†]

[*] Steklov Mathematics Institute, Fontanka 27, 191023 St. Petersburg, Russia

[†] DESY Theory, Notkestr. 85, 22603 Hamburg, Germany

We dedicate this paper to L.D. Faddeev on the occasion of his 75th birthday.

1. Introduction

The main question which motivated this work is the following: How do conformal field theories look like if studied from the point of view of a possibly existing *integrable structure*? There are many quantum-field theoretical models of high interest for string theory and condensed matter physics which are expected to have conformal invariance, but not enough chiral symmetry to make a solution in terms of standard methods of conformal field theory look realistic. An interesting class of examples are nonlinear sigma models with targets being super-groups, which have recently attracted considerable interest both from string theory and condensed matter physics. Some of these theories are expected to be integrable. It therefore seems reasonable to expect that methods from the theory of integrable models can be used to understand the spectrum of these theories.

Such a program immediately faces an obstacle: Up to now it seemed that key features of conformal field theories like the factorization into left- and right-moving degrees of freedom are very hard to see with the help of the integrable structure. Using the traditional approaches based on the Bethe ansatz one usually has to go a rather long way until some of the features of conformal invariance become visible. We therefore looked for a simple, but prototypical example where we can improve on this state of affairs. The main point we want to illustrate with the example of Liouville theory is the following: The factorization into left- and right-movers can be made manifest in a very transparent way already on the level of an integrable lattice regularization of a conformal field theory.

The framework in which this turns out to be the case combines the use of Baxter's Q-operators with the Separation of Variables technique of Sklyanin [Sk85, Sk92, Sk95]. In the cases under consideration we will explicitly construct Q-operators $\mathsf{Q}^+(u)$ and $\mathsf{Q}^-(u)$ which contain the conserved charges of left- and right-moving degrees of freedom, respectively. Within the Separation of Variables framework one may then represent an eigenstate of $\mathsf{Q}^+(u)$ and $\mathsf{Q}^-(u)$ in terms of a wave-function constructed directly out of the corresponding eigenvalues $q^+(u)$ and $q^-(u)$. The combination of these two ingredients yields a quantum version of the Bäcklund transformation from Liouville theory to free field theory, making the factorization into left- and right-moving degrees of freedom transparent.

It also seems promising to view the integrable structure of conformal field theories as a useful starting point for the study of massive integrable models. One may expect that the integrable structure ”deforms smoothly” from the massless to the massive cases, but is simpler to study in the massless limits. This point of view was developed in particular in the beautiful series of works [BLZ1, BLZ2], where conformal field theories with central charge $c < 1$ were studied. One of our aims here is to study the counterpart of this theory for $c > 1$. The constructions from [BLZ1] no longer work in this case due to more severe ultraviolet problems. We will use an integrable lattice regularization to control such problems. This will also allow us study the Sinh-Gordon model, Liouville theory and quantum KdV theory in a uniform framework. We will observe that key objects of the integrable structure like the Baxter Q-operators are indeed related to each other by certain parametric limits.

The example chosen, Liouville theory, is of considerable interest in its own right. It has attracted a lot of attention for more than 25 years now due to its connections with noncritical string theory and two-dimensional quantum gravity (see [DGZ, GM] for reviews and references), as an example for interesting non-rational conformal field theories [T01, T08b], and due to its relations to the (quantized) Teichmüller spaces of Riemann surfaces [TT06, T07].

In the study of Liouville theory, the most popular approach so far was based on its conformal symmetry, leading to a complete solution in the sense of the Belavin-Polyakov-Zamolodchikov bootstrap approach [BPZ], see [CT82, GN84, DO92, ZZ96, PT99, T04] for some key steps in this program, and [T01] for a more complete list of references. Understanding Liouville theory from the point of view of its integrable structure has also attracted considerable interest in the past, going back to [FT86], and more recently being developed in [FKV, FK02]. This approach has also lead to beautiful results, see in particular [FK02].

What seemed somewhat unsatisfactory, however, was the lack of results that can be directly compared with the conformal field theory approach. It is the second main aim of this paper to re-derive the so-called reflection amplitude of Liouville theory with the help of its integrable structure. The formula for this quantity had been conjectured in [DO92, ZZ96]. A derivation for these conjectures was subsequently given in [T04]. Here we are going to re-derive this result in a completely different way, entirely based on the integrable structure of Liouville theory.

However, we feel that the interplay between conformal and integrable structures is still not completely understood. It seems particularly important to integrate the lattice Virasoro algebra [FV93] into the picture and to clarify the relations with the beautiful work [BMS] where closely related models of statistical mechanics were studied. What we do hope, however, is that this paper lays some useful groundwork which will ultimately lead to a better understanding of this important subject.

This paper is intended to give a reasonably concise overview over the main constructions, ideas and results of our work. It is not self-contained. In order to make the verification of our claims possible, we either give sketches of the proofs or indicate references where similar arguments can be found. A more detailed presentation is in preparation.

Note on notations: In order to distinguish objects associated to the three different models of interest, we shall sometimes use subscripts like $\mathsf{O}_{\rm ShG}$, $\mathsf{O}_{\rm Liou}$ or $\mathsf{O}_{\rm KdV}$. However, to unload the notation we shall omit these subscripts whenever it is clear from the context which model is considered.

Acknowledgements. We would like to thank V. Bazhanov, A. Bobenko, R. Kashaev for stimulating discussions. Many thanks also to L. Faddeev and V. Bazhanov for comments on the draft.

J.T. gratefully acknowledges support from the EC by the Marie Curie Excellence Grant MEXT-CT-2006-042695. A.B. was supported in part by the RFBR under grants 07-02-92166 and 08-01-00638.

2. Definition of the lattice models

The aim of this section is to define three lattice models, corresponding to the Sinh-Gordon model, Liouville theory and the scalar free field theory, respectively. Anticipating discussions of its integrable structure we will refer to the scalar free field theory as KdV theory below.

2.1 Lattice discretization

The classical counterparts of the models in question are dynamical systems whose degrees of freedom are described by the field $\phi(x,t)$ defined for $(x,t)\in[0,R]\times\mathbb{R}$ with periodic boundary conditions $\phi(x+R,t) = \phi(x,t)$. The dynamics of these models may be described in the Hamiltonian form in terms of variables $\phi(x,t)$, $\Pi(x,t)$, the Poisson brackets being

$$\{\,\Pi(x,t)\,,\,\phi(x',t)\,\}\;=\;2\pi\,\delta(x-x')\,.$$

The time-evolution of an arbitrary observable $O(t)$ is then given as

$$\partial_t O(t)\;=\;\{\,H\,,\,O(t)\,\}\,,$$

with Hamiltonian H being defined as

$$H = \int_0^R \frac{dx}{4\pi} h(x), \qquad \begin{aligned} h_{\rm ShG} &= \Pi^2 + (\partial_x\phi)^2 + 8\pi\mu\cosh(2b\phi)\,, \\ h_{\rm Liou} &= \Pi^2 + (\partial_x\phi)^2 + 4\pi\mu e^{-2b\phi}\,, \\ h_{\rm KdV} &= \Pi^2 + (\partial_x\phi)^2\,. \end{aligned} \tag{2.1}$$

In order to regularize the ultraviolet divergencies that arise in the quantization of these models we will pass to integrable lattice discretizations. First discretize the field variables according to the standard recipe

$$\phi_n \equiv \phi(n\Delta)\,, \quad \Pi_n \equiv \Delta\Pi(n\Delta)\,,$$

where $\Delta = R/\mathrm{N}$ is the lattice spacing. Quantization is then canonical: The variables Φ_n, Π_n, $n \in \mathbb{Z}/\mathrm{N}\mathbb{Z}$ are henceforth considered as operators with commutation relations

$$[\,\phi_n\,,\,\Pi_m\,] \,=\, 2\pi i\delta_{n,m}\,, \tag{2.2}$$

that can be realized in the usual way on the Hilbert space $\mathcal{H} \equiv (L^2(\mathbb{R}))^{\otimes \mathrm{N}}$. As another convenient set of variables let us introduce the operators f_k defined as

$$\mathsf{f}_{2n} \,\equiv\, e^{-2b\phi_n}\,, \qquad \mathsf{f}_{2n-1} \,\equiv\, e^{\frac{b}{2}(\Pi_n+\Pi_{n-1}-2\phi_n-2\phi_{n-1})}\,. \tag{2.3}$$

This change of variables is invertible for $\mathrm{N} \equiv 2L+1$ odd. We will therefore restrict our attention to this case in the following. The variables f_n satisfy the algebraic relations

$$\mathsf{f}_{2n\pm 1}\,\mathsf{f}_{2n} \,=\, q^2\,\mathsf{f}_{2n}\,\mathsf{f}_{2n\pm 1}\,, \quad q = e^{\pi i b^2}\,, \qquad \mathsf{f}_n\,\mathsf{f}_{n+m} \,=\, \mathsf{f}_{n+m}\,\mathsf{f}_n \;\; \text{for} \;\; m \geq 2\,. \tag{2.4}$$

These operators turn out to represent the initial data for time evolution in a particularly convenient way, as we are going to discuss next.

2.2 Lattice dynamics

A beautiful way to define a suitable dynamics in these lattice models was proposed by Faddeev and Volkov in [FV94]. This approach was adapted to the lattice Liouville model in [FKV]. Space-time is replaced by the cylindric lattice

$$\mathcal{L} \,\equiv\, \big\{\,(\nu,\tau)\,,\,\nu \in \mathbb{Z}/\mathrm{N}\mathbb{Z}\,,\,\tau \in \mathbb{Z}\,,\,\nu+\tau = \text{even}\,\big\}\,.$$

The condition that $\nu+\tau$ is even means that the lattice is rhombic: The lattice points closest to (ν,τ) are $(\nu\pm 1,\tau+1)$ and $(\nu\pm 1,\tau-1)$. We identify the variables f_n with the initial values of a discrete "field" $\mathsf{f}_{\nu,\tau}$ as

$$\mathsf{f}_{2r,0} \,\equiv\, \mathsf{f}_{2r}\,, \qquad \mathsf{f}_{2r-1,1} \,\equiv\, \mathsf{f}_{2r-1}\,.$$

One may then extend the definition recursively to all $(\nu,\tau)\in\mathcal{L}$ by

$$\mathsf{f}_{\nu,\tau+1}\;\equiv\;\mathsf{f}_{\nu,\tau-1}^{-\frac{1}{2}}\cdot g_\kappa\big(\mathsf{f}_{\nu-1,\tau}\big)g_\kappa\big(\mathsf{f}_{\nu+1,\tau}\big)\cdot\mathsf{f}_{\nu,\tau-1}^{-\frac{1}{2}}\,, \tag{2.5}$$

with functions g defined respectively by

$$\begin{aligned} g_\kappa(z) &= \frac{\kappa^2+z}{1+\kappa^2 z} && \text{for the Sinh-Gordon model,}\\ g_\kappa(z) &= \frac{z}{1+\kappa^2 z} && \text{for Liouville theory,}\\ g_\kappa(z) &= z && \text{for KdV theory.}\end{aligned} \tag{2.6}$$

where κ plays the role of a scale-parameter of the theory. In the massive case it can be identified with a certain function of the physical mass [T08a]. We refer to [FV94] for a nice discussion of the relation between the lattice evolution equation (2.5) and the classical Hirota equation, explaining in particular how to recover the Sinh-Gordon equation in the classical continuum limit.

In order to construct the unitary operators U that generate the time evolution above let us, following [FKV] closely, introduce the special functions $w_b(x)$ and $\varphi(x)$ which are defined as

$$w_b(x)\;=\;\frac{\zeta\,e^{\frac{\pi i}{2}x^2}}{\varphi(x)}\,,\qquad \varphi(x)\;=\;\exp\left(\int_{\mathbb{R}+i0}\frac{dt}{4t}\,\frac{e^{-2itx}}{\sinh(bt)\sinh(b^{-1}t)}\right)\,, \tag{2.7}$$

where $\zeta=e^{\frac{\pi i}{24}(b^2+b^{-2})}$. The special function $\varphi(x)$ has been introduced in a related context in [F95]. All the relevant properties (zeros, poles, asymptotic behavior, functional relations) can be found in [Vo05, BT06, BMS]. Out of these functions let us construct

$$\begin{aligned} G_v(e^{2\pi bx}) &= w_b(\tfrac{v}{2}+x)w_b(\tfrac{v}{2}-x) && \text{for the Sinh-Gordon model,}\\ G_v(e^{2\pi bx}) &= \zeta^{-1}\,e^{-i\frac{\pi}{2}(x+\frac{v}{2})^2}\,w_b\big(\tfrac{v}{2}-x\big) && \text{for Liouville theory,}\\ G_v(e^{2\pi bx}) &= \zeta^{-2}\,e^{-i\frac{\pi}{2}(x+\frac{v}{2})^2}\,e^{-i\frac{\pi}{2}(x-\frac{v}{2})^2} && \text{for KdV theory.}\end{aligned} \tag{2.8}$$

Let us then consider the operator U, defined as

$$\mathsf{U}\;=\;\prod_{n=1}^{\mathrm{N}}G_{2s}(\mathsf{f}_{2n})\cdot\mathsf{U}_0\cdot\prod_{r=1}^{\mathrm{N}}G_{2s}(\mathsf{f}_{2n-1})\,, \tag{2.9}$$

where U_0 is the parity operator that acts as $\mathsf{U}_0\cdot\mathsf{f}_k=\mathsf{f}_k^{-1}\cdot\mathsf{U}_0$. The functions $G_{2s}(z)$ satisfy the functional relations

$$G_{2s}(qz)\,/\,G_{2s}(q^{-1}z)\;=\;g_\kappa(z)\quad\text{ if }\;\kappa=e^{-\pi bs}\,, \tag{2.10}$$

where G_v and g_κ are chosen from (2.8) and (2.6) according to the case at hand. It easily follows from (2.10) that U is indeed the the generator of the time-evolution (2.5),

$$\mathsf{f}_{\nu,\tau+1}\;=\;\mathsf{U}^{-1}\cdot\mathsf{f}_{\nu,\tau-1}\cdot\mathsf{U}\,. \tag{2.11}$$

One of our tasks is to exhibit the integrability of this discrete time evolution.

2.3 Fock space representation

Classically the Hamiltonian density of KdV theory is the one of a free field theory. The correspondence with free field theory becomes manifest in the lattice model if we introduce lattice analogs of the fields $e^{b(\partial_t \pm \partial_x)\phi}$ as follows [Ge85, Vo92]

$$\begin{aligned} \mathsf{w}_n^+ &= q\,\mathsf{f}_{2n+1}\mathsf{f}_{2n+2}^{-1}\,, \qquad & \mathsf{w}_{\nu,\tau}^+ &\equiv q\,\mathsf{f}_{\nu,\tau}\,\mathsf{f}_{\nu+1,\tau-1}^{-1}\,, \\ \mathsf{w}_n^- &= q\,\mathsf{f}_{2n+1}\mathsf{f}_{2n}^{-1}\,, \qquad & \mathsf{w}_{\nu,\tau}^- &\equiv q\,\mathsf{f}_{\nu,\tau}\,\mathsf{f}_{\nu-1,\tau-1}^{-1}\,. \end{aligned} \tag{2.12}$$

Note that the operators w_n^+, w_n^- satisfy the following commutation relations:

$$\mathsf{w}_n^+\mathsf{w}_m^- = \mathsf{w}_m^-\mathsf{w}_n^+\,, \qquad \begin{aligned} \mathsf{w}_n^+\mathsf{w}_m^+ &= \omega_{nm}\mathsf{w}_m^+\mathsf{w}_n^+\,, \\ \mathsf{w}_n^-\mathsf{w}_m^- &= \omega_{nm}^{-1}\mathsf{w}_m^-\mathsf{w}_n^-\,, \end{aligned} \qquad \omega_{nm} \equiv \begin{cases} q^{2\,\mathrm{sgn}(m-n)} & \text{if } |n-m| = 1, \\ 1 & \text{if } |n-m| \neq 1. \end{cases} \tag{2.13}$$

The evolution generated by the operator $\mathsf{U}_{\mathrm{KdV}}$ is represented in these variables as

$$\mathsf{w}_{\nu,\tau+1}^+ = \mathsf{w}_{\nu-1,\tau}^+\,, \qquad \mathsf{w}_{\nu,\tau+1}^- = \mathsf{w}_{\nu+1,\tau}^-\,. \tag{2.14}$$

This means that that the variables w_n^+ and w_n^- represent the right and the left-moving degrees of freedom respectively.

We will sometimes use an alternative representation for the Hilbert space $\mathcal{H}$ which not only makes the chiral factorization into left- and right-moving degrees manifest for KdV-theory, but will also be used in the discussion of Liouville theory. Keeping in mind $\mathrm{N} = 2L+1$ let

$$\begin{aligned} \mathsf{p}_\circ &= \frac{1}{2\pi b\mathrm{N}} \sum_{n=-L}^{L} \log \mathsf{w}_n^\pm\,, \qquad \mathsf{q}_\circ = \frac{1}{2\pi}\sum_{n=1}^{\mathrm{N}} \phi_n\,, \\ \mathsf{a}_k^\pm &\equiv \frac{1}{2\pi b}\sum_{n=-L}^{L} e^{2i\frac{\pi}{\mathrm{N}}nk}\,\big(\log \mathsf{w}_n^\pm - 2\pi b \mathsf{p}_\circ\big)\,. \end{aligned} \tag{2.15}$$

We have the following commutation relations,

$$\begin{aligned} &[\,\mathsf{a}_n^+\,,\,\mathsf{a}_m^-\,] = 0\,, \qquad [\,\mathsf{a}_n^\pm\,,\,\mathsf{a}_m^\pm\,] = \pm\delta_{n+m,0}\,\frac{\sin 2\rho n}{\rho}\,, \qquad \rho \equiv \frac{\pi}{N} \\ &[\,\mathsf{p}_\circ\,,\,\mathsf{q}_\circ\,] = (2\pi i)^{-1}\,, \qquad [\,\mathsf{q}_\circ\,,\,\mathsf{a}_n^\pm\,] = 0\,, \qquad [\,\mathsf{p}_\circ\,,\,\mathsf{a}_n^\pm\,] = 0\,. \end{aligned} \tag{2.16}$$

Let $\mathcal{F}^\pm$ be the Fock spaces generated by the harmonic oscillators $(\mathsf{a}_n^\pm, \mathsf{a}_{-n}^\pm)$ for $n \neq 0$, respectively. There are representations for the Hilbert space $\mathcal{H}_{\mathrm{SG}}$ in which either $\mathsf{p}_\circ$ or $\mathsf{q}_\circ$ are represented as multiplication operators,

$$\begin{aligned} \mathcal{H}_{\mathrm{SG}} \simeq \mathcal{H}_{\mathrm{Fock}} &\equiv \int_{-\infty}^{\infty} dp\; \mathcal{F}_p^+ \otimes \mathcal{F}_p^-\,, \qquad & \mathsf{p}_\circ\,(\mathcal{F}_p^+ \otimes \mathcal{F}_p^-) &= p\,(\mathcal{F}_p^+ \otimes \mathcal{F}_p^-) \\ \simeq \mathcal{H}_{\mathrm{Schr}} &\equiv \int_{-\infty}^{\infty} d\phi_\circ\; \mathcal{F}_{\phi_\circ}^+ \otimes \mathcal{F}_{\phi_\circ}^-\,, \qquad & \mathsf{q}_\circ\,(\mathcal{F}_{\phi_\circ}^+ \otimes \mathcal{F}_{\phi_\circ}^-) &= \phi_0\,(\mathcal{F}_{\phi_\circ}^+ \otimes \mathcal{F}_{\phi_\circ}^-)\,. \end{aligned} \tag{2.17}$$

These representations $\mathcal{H}_{\mathrm{Fock}}$ and $\mathcal{H}_{\mathrm{Schr}}$ for $\mathcal{H}$ will be called the Fock and the (zero mode) Schrödinger representation, respectively.

3. Integrability

In order to exhibit the integrability of the discrete time evolutions introduced in the previous section one needs to construct mutually commutative families $\mathcal{Q}$ of self-adjoint operators T such that

$$\begin{aligned} &(\mathrm{A}) \quad [\,\mathsf{T}\,,\,\mathsf{T}'\,] \,=\, 0, \qquad \forall\,\mathsf{T},\mathsf{T}'\in\mathcal{Q}\,,\\ &(\mathrm{B}) \quad [\,\mathsf{T}\,,\,\mathsf{U}\,] \,=\, 0, \qquad \forall\,\mathsf{T}\in\mathcal{Q}\,,\\ &(\mathrm{C}) \quad \text{if } [\,\mathsf{T}\,,\,\mathsf{O}\,] \,=\, 0,\ \ \forall\,\mathsf{T}\in\mathcal{Q},\ \ \text{then}\ \ \mathsf{O}=\mathsf{O}(\mathcal{Q}). \end{aligned} \tag{3.1}$$

Within the framework of the quantum inverse scattering method one may conveniently define the family $\mathcal{Q}$ in terms of one-parameter families $\mathsf{T}(u)$ and $\mathsf{Q}(v)$ of operators that are mutually commuting for arbitrary values of the spectral parameters u and v, and which satisfy a functional relation of the form

$$\mathsf{T}(u)\mathsf{Q}(u) \,=\, a(u)\mathsf{Q}(u-ib)+d(u)\mathsf{Q}(u+ib)\,, \tag{3.2}$$

with $a(u)$ and $d(u)$ being certain model-dependent coefficient functions. The generator of lattice time evolution will be constructed from the specialization of the Q-operators to certain values of the spectral parameter u, making the integrability of the evolution manifest.

3.1 T-operators

The definition of T-operators for the models in question is standard. It is of the general form

$$\mathsf{T}(u) \,=\, \mathrm{tr}_{\mathbb{C}^2} M(u)\,, \qquad M(u)\,\equiv\, L_N(u)L_{N-1}(u)\ldots L_1(u)\,. \tag{3.3}$$

In the following subsection we will describe possible choices for the Lax-matrices $L_n(u)$ for the models of interest.

3.1.1 Sinh-Gordon model

For future use let us note that the L-operator of lattice Sinh-Gordon model [FST, IK82, Sk83] can be written as

$$L_n(u)\,\equiv\, L_n(\mu,\bar{\mu}) \,=\, \begin{pmatrix} \mathsf{u}_n+\mu\bar{\mu}^{-1}\mathsf{v}_n\mathsf{u}_n\mathsf{v}_n & \mu\mathsf{v}_n+\bar{\mu}^{-1}\mathsf{v}_n^{-1} \\ \mu\mathsf{v}_n^{-1}+\bar{\mu}^{-1}\mathsf{v}_n & \mathsf{u}_n^{-1}+\mu\bar{\mu}^{-1}\mathsf{v}_n^{-1}\mathsf{u}_n^{-1}\mathsf{v}_n^{-1}\end{pmatrix}\,, \tag{3.4}$$

where we have used the notations

$$\mathsf{u}_n \,=\, e^{\frac{b}{2}\Pi_n}\,, \quad \mathsf{v}_n \,=\, e^{-b\phi_n}\,, \quad \mu\equiv -ie^{\pi b(u-s)}\,, \quad \bar{\mu}\equiv -ie^{\pi b(u+s)}\,.$$

The key point about the definition (3.4) is the fact that the commutation relations for the matrix elements of $L_n(u)$ can be written in the Yang-Baxter form

$$R_{12}(u-v)L_{1n}(u)L_{2n}(v) \;=\; L_{2n}(u)L_{1n}(v)R_{12}(u-v)\,, \tag{3.5}$$

where the 4×4-matrix $R_{12}(u-v)$ is

$$R(u) \;=\; \begin{pmatrix} \sinh \pi b(u+ib) & & & \\ & \sinh \pi bu & i\sin \pi b^2 & \\ & i\sin\pi b^2 & \sinh \pi bu & \\ & & & \sinh\pi b(u+ib) \end{pmatrix}. \tag{3.6}$$

This implies as usual that the one-parameter family of operators $\mathsf{T}(u)$ is mutually commutative, $[\mathsf{T}(u),\mathsf{T}(v)]=0$.

3.1.2 Liouville theory

Faddeev-Tirkkonen [FT95] proposed the following L-matrix for the lattice Liouville model,

$$L^{+}_{\mathrm{Liou},n}(\mu,\bar\mu) \;=\; \begin{pmatrix} \mathsf{u}_n+\mu\bar\mu^{-1}\mathsf{v}_n\mathsf{u}_n\mathsf{v}_n & \mu\mathsf{v}_n \\ \mu\mathsf{v}_n^{-1}+\bar\mu^{-1}\mathsf{v}_n & \mathsf{u}_n^{-1} \end{pmatrix}. \tag{3.7}$$

This L-matrix can be obtained from $L_{\mathrm{ShG},n}(\mu,\bar\mu)$ in the limit

$$L^{+}_{\mathrm{Liou},n}(\mu,\bar\mu) \;\equiv\; \lim_{s\to\infty} e^{-\frac{\pi}{2}bs\sigma_3}\,\mathsf{u}_n^{\frac{s}{ib}}\cdot L_{\mathrm{ShG},n}(\mu,e^{+2\pi bs}\bar\mu)\cdot \mathsf{u}_n^{-\frac{s}{ib}}\,e^{+\frac{\pi}{2}bs\sigma_3}\,, \tag{3.8}$$

and it also satisfies (3.5). However, it is easy to see that the corresponding transfer matrix

$$\mathsf{T}^{+}(u) \;=\; \mathrm{tr}_{\mathbb{C}^2}\big(L^{+}_{\mathrm{N}}(u)\cdots L^{+}_{1}(u)\big) \tag{3.9}$$

generates only $L+1$ commuting operators if we have $N=2L+1$ degrees of freedom. $\mathsf{T}^{+}(u)$ alone will therefore *not* generate sufficiently many conserved quantities.

Fortunately there exist a second reasonable limit

$$L^{-}_{\mathrm{Liou},n}(\mu,\bar\mu) \;\equiv\; \lim_{s\to\infty} e^{+\frac{\pi}{2}bs\sigma_3}\,\mathsf{u}_n^{\frac{s}{ib}}\cdot L_{\mathrm{SG},n}(e^{-2\pi bs}\mu,\bar\mu)\cdot \mathsf{u}_n^{-\frac{s}{ib}}\,e^{-\frac{\pi}{2}bs\sigma_3}\,, \tag{3.10}$$

which leads to yet another solution to (3.5), namely

$$L^{-}_{\mathrm{Liou},n}(\mu,\bar\mu) \;=\; \begin{pmatrix} \mathsf{u}_n+\mu\bar\mu^{-1}\mathsf{v}_n\mathsf{u}_n\mathsf{v}_n & \mu\mathsf{v}_n+\bar\mu^{-1}\mathsf{v}_n^{-1} \\ \bar\mu^{-1}\mathsf{v}_n & \mathsf{u}_n^{-1} \end{pmatrix}. \tag{3.11}$$

The mutual commutativity of $\mathsf{T}^{+}(u)$ and $\mathsf{T}^{-}(v)$ for all u,v follows by standard arguments from the commutation relations

$$R'_{12}(u-v)\,L^{+}_{1}(u)\,L^{-}_{2}(v) \;=\; L^{-}_{2}(v)\,L^{+}_{1}(u)\,R'_{12}(u-v)\,, \tag{3.12}$$

where

$$R'_{12}(u) = \begin{pmatrix} e^{\pi b(u+ib)} & & & \\ & e^{\pi bu} & 0 & \\ & i\sin\pi b^2 & e^{\pi bu} & \\ & & & e^{\pi b(u+ib)} \end{pmatrix} . \tag{3.13}$$

We will later show that the splitting of the transfer matrix $\mathsf{T}(u)$ into $\mathsf{T}^+_{\rm Liou}(u)$ and $\mathsf{T}^-_{\rm Liou}(v)$ reflects the chiral factorization of Liouville theory into left- and right-moving degrees of freedom.

3.1.3 KdV theory

The operators $\mathsf{T}^{\pm}(u)$ for lattice KdV theory can finally be constructed from the Lax-matrices [Ge85, Vo92]

$$L^+_n(\mu) \equiv \begin{pmatrix} \mathsf{u}_n & \mu\,\mathsf{v}_n \\ \mu\,\mathsf{v}_n^{-1} & \mathsf{u}_n^{-1} \end{pmatrix} , \quad L^-_n(\bar\mu) \equiv \begin{pmatrix} \mathsf{u}_n & \bar\mu^{-1}\,\mathsf{v}_n^{-1} \\ \bar\mu^{-1}\,\mathsf{v}_n & \mathsf{u}_n^{-1} \end{pmatrix} .$$

These L-matrices also satisfy (3.5) and can be obtained [Vo92] from $L_{{\rm ShG},n}(u)$ and $L^{\pm}_{{\rm Liou},n}(u)$ by certain limiting procedures similar to (3.8),(3.10).

It was shown in Subsection 2.3 that the decoupling of the free field dynamics into right- and left-moving degrees of freedom becomes manifest in the lattice model in terms of the variables w^+_n and w^-_n. It is possible to show [Vo92] that the transfer matrices $\mathsf{T}^{\epsilon}(u)$, $\epsilon = \pm$, can be represented as a polynomial in the variables w^{ϵ}_n which is independent of $\mathsf{w}^{-\epsilon}_n$.

3.2 Construction of Q-operators

Algebraic constructions of Q-operators have previously been given in [Vo97] for the KdV model[1] and for the lattice Liouville theory [FKV, Ka01]. It has to be observed, however, that only the Q-operator related to the T-operator $\mathsf{T}^+_{\rm Liou}$ by means of a Baxter-type equation was considered in [FKV, Ka01]. We observed in the previous subsection that the T-operator $\mathsf{T}^+_{\rm Liou}$ does not generate sufficiently many conserved quantities. This suggests that we need a second Q-operator $\mathsf{Q}^-_{\rm Liou}$ related to $\mathsf{T}^-_{\rm Liou}$ by a Baxter-type relation in order to complete the proof of the integrability of the lattice Liouville model in the sense formulated above.

We will in the following give a uniform construction of Q-operators for all the models in question. For our purposes it will be most convenient to represent the Q-operators as integral operators with explicitly specified integral kernels. This facilitates the derivation of the analytic properties of the Q-operators, as first done in [BT06] for the Sinh-Gordon model, considerably.

[1] More precisely its chiral half, as will become clear later.

3.2.1 Representations as integral operators

In order to represent the Q-operators as integral operators it will be convenient to use the representation where the operators u_r and v_r are represented as

$$\mathsf{u}_n \,=\, e^{\pi b(2\mathsf{x}_n - \mathsf{p}_n)} \qquad \mathsf{v}_n \,=\, e^{\pi b \mathsf{p}_n}\,, \tag{3.14}$$

with x_n, p_n being realized on wave-functions $\Psi(\mathbf{x})$, $\mathbf{x} = (x_1, \dots, x_{\mathrm{N}})$ as

$$\mathsf{x}_n \cdot \Psi(\mathbf{x}) \,=\, x_n \Psi(\mathbf{x})\,, \qquad \mathsf{p}_n \cdot \Psi(\mathbf{x}) \,=\, \frac{1}{2\pi i}\frac{\partial}{\partial x_n}\Psi(\mathbf{x})\,.$$

Out of the special function $w_b(x)$ let us form a few useful combinations:

$$\begin{aligned}
\overline{W}_{v-i\eta}(x) \,&=\, \frac{w_b(x-\frac{v}{2})}{w_b(x+\frac{v}{2})}\,,\\
W^{+\mathrm{ShG}}_{i\eta+v}(x) \,&=\, \big(W^{-\mathrm{ShG}}_{i\eta-v}(x)\big)^{-1} \,=\, \frac{w_b(x+\frac{v}{2})}{w_b(x-\frac{v}{2})}\,,\\
W^{+\mathrm{Liou}}_{i\eta+v}(x) \,&=\, \big(W^{-\mathrm{Liou}}_{i\eta-v}(x)\big)^{-1} \,=\, \frac{\zeta^{-1}e^{-i\frac{\pi}{2}(x+\frac{v}{2})^2}}{w_b\big(x-\frac{v}{2}\big)}\,,\\
W^{+\mathrm{KdV}}_{i\eta+v}(x) \,&=\, \big(W^{-\mathrm{KdV}}_{i\eta-v}(x)\big)^{-1} \,=\, \frac{\zeta^{-1}e^{-i\frac{\pi}{2}(x+\frac{v}{2})^2}}{\zeta^{+1}e^{+i\frac{\pi}{2}(x-\frac{v}{2})^2}}\,,
\end{aligned} \qquad \eta \equiv \frac{1}{2}(b+b^{-1})\,. \tag{3.15}$$

From the known asymptotic properties of the function $w_b(x)$ it is easily found that $W_v^{\pm\mathrm{Liou}}$ and $W_v^{\pm\mathrm{KdV}}$ can be obtained from $W_v^{\pm\mathrm{ShG}}$ by taking suitable limits.

The Q-operators may then be constructed in the following general form:

$$\mathsf{Q}^+(u) \,=\, \mathsf{Y}_\infty^{-1}\cdot\mathsf{Y}^+(u)\,, \qquad \mathsf{Q}^-(u) \,=\, \mathsf{Y}^-(u)\cdot\mathsf{Y}_{-\infty}^{-1}\,, \tag{3.16}$$

where $\mathsf{Y}^\epsilon(u)$ can be represented as integral operators with kernels

$$\langle\,\mathbf{x}'\,|\,\mathsf{Y}^+(u)\,|\,\mathbf{x}\,\rangle \,=\, \prod_{n=1}^{\mathrm{N}} \overline{W}_{u-s}(x_n'-x_n)W^+_{u+s}(x'_{n-1}+x_n)\,, \tag{3.17}$$

$$\langle\,\mathbf{x}'\,|\,\mathsf{Y}^-(u)\,|\,\mathbf{x}\,\rangle \,=\, \prod_{n=1}^{\mathrm{N}} W^-_{u-s}(x'_{n-1}+x_n)\overline{W}_{u+s}(x_n'-x_n)\,, \tag{3.18}$$

whereas the operators $\mathsf{Y}_{\pm\infty}$ have the distributional kernels

$$\langle\,\mathbf{x}'\,|\,\mathsf{Y}_{\pm\infty}\,|\,\mathbf{x}\,\rangle \,=\, \prod_{n=1}^{\mathrm{N}} e^{\mp 2\pi i x_n'(x_n+x_{n+1})}\,. \tag{3.19}$$

The expressions for the kernel of the operators $\mathsf{Y}^\epsilon(u)$ are very similar to the remarkable factorized expressions for the matrix elements of Q-operators found in [BS90] for models with related quantum algebraic structures. We will present a systematic procedure to derive such factorized expressions for a certain class of models in [BT09].

The mutual commutativity of T- and Q-operators,

$$\big[\,\mathsf{Q}^{\epsilon}(u)\,,\,\mathsf{Q}^{\epsilon'}(v)\,\big]\;=\;0\,,\qquad \big[\,\mathsf{Q}^{\epsilon}(u)\,,\,\mathsf{T}^{\epsilon'}(v)\,\big]\;=\;0\,,\qquad \epsilon,\epsilon'=\pm\,, \tag{3.20}$$

can be shown either along the lines of [BS90, PG92, BT06] from the star-triangle relation satisfied by the function $W_u(x)$ [Ka00, Vo05, BT06, BMS] [2], or more elegantly by writing the Q-operators as traces of generalized monodromy matrices over q-oscillator type representations in auxilliary space [BT09], similar to the constructions of Q-operators in [BLZ1].

3.3 Proof of integrability

The key observation proving the integrability of the models is the fact that

$$\boxed{\mathsf{U}\;=\;\mathsf{U}^+\cdot\mathsf{U}^-\qquad \mathsf{U}^+\;=\;\mathsf{Q}^+(s_+)\qquad \mathsf{U}^-\;=\;(\mathsf{Q}^-(s_-))^{-1}} \tag{3.21}$$

where we have introduced the notations $s_+ = s - i\eta$, $s_- = -s - i\eta$ for convenience. The operators U^+ and U^- will be regarded as light cone evolution operators. Equation (3.21) is easily proven by noting that

$$\mathsf{Q}^+(s_+)\;=\;\mathsf{Y}_\infty^{-1}\cdot\prod_{n=1}^{\mathrm{N}}G_{2s}(\mathsf{f}_{2n-1})\,,\qquad (\mathsf{Q}^-(s_-))^{-1}\;=\;\mathsf{Y}_{-\infty}\cdot\prod_{n=1}^{\mathrm{N}}G_{2s}(\mathsf{f}_{2n-1})\,. \tag{3.22}$$

The operator Y_∞ satisfies $\mathsf{Y}_\infty^{-1}\cdot\mathsf{f}_{2n-1}\cdot\mathsf{Y}_\infty=\mathsf{f}_{2n}$. This implies

$$\mathsf{Q}^+(s_+)\cdot(\mathsf{Q}^-(s_-))^{-1}\;=\;\prod_{n=1}^{\mathrm{N}}G_{2s}(\mathsf{f}_{2n})\cdot\mathsf{Y}_\infty^{-1}\cdot\mathsf{Y}_{-\infty}\cdot\prod_{n=1}^{\mathrm{N}}G_{2s}(\mathsf{f}_{2n-1})\,.$$

It remains to notice that $\mathsf{Y}_\infty^{-1}\cdot\mathsf{Y}_{-\infty}=\mathsf{U}_0$ to conclude the proof of (3.21).

3.4 Chiral Q-operators in the lattice KdV model

Note that the Q-operators $\mathsf{Q}^+_{\rm KdV}$ and $\mathsf{Q}^-_{\rm KdV}$ are indeed the direct massless limits of $\mathsf{Q}^+_{\rm ShG}(s|u)\equiv\mathsf{Q}^+_{\rm ShG}(u)$ and $\mathsf{Q}^-_{\rm ShG}(s|u)\equiv\mathsf{Q}^-_{\rm ShG}(u)$, respectively,

$$\begin{aligned}\mathsf{Q}^+_{\rm KdV}(u)\;&=\;\lim_{\delta\to\infty}\;\mathsf{Q}^+_{\rm ShG}(s+\delta|u+\delta)\,,\\ \mathsf{Q}^-_{\rm KdV}(u)\;&=\;\lim_{\delta\to\infty}\;\mathsf{Q}^-_{\rm ShG}(s+\delta|u-\delta)\,.\end{aligned} \tag{3.23}$$

[2] The papers [Ka00, Vo05] derive integral identities which can be rewritten in the form of the star-triangle relation [BT06, BMS]. An elegant proof can be given by using arguments similar to [Ba08] from the Yang-Baxter equation satisfied by the corresponding R-matrix

The Baxter equations relate the Q-operators Q^ϵ with the T-operators T^ϵ. In the case of KdV theory we had seen that T^+ and T^- depend only on right- and left-moving degrees of freedom w_n^+ and w_n^-, respectively. This suggests that Q^+ and Q^- should have the same property. And indeed, it can be checked that

$$[\,\mathsf{Q}^+(u)\,,\,\mathsf{w}_n^-\,]\,=\,0\,,\qquad[\,\mathsf{Q}^-(u)\,,\,\mathsf{w}_n^+\,]\,=\,0\,,\tag{3.24}$$

making clear that $\mathsf{Q}^+(u)$ and $\mathsf{Q}^-(u)$ depend on the right- and left-moving degrees of freedom only. This property implies in particular that

$$[\,\mathsf{Q}^+(u)\,,\,\mathsf{p}_\circ\,]\,=\,0\,,\qquad[\,\mathsf{Q}^-(u)\,,\,\mathsf{p}_\circ\,]\,=\,0\,,\tag{3.25}$$

which means that $\mathsf{Q}^+(u)$ and $\mathsf{Q}^-(u)$ can be projected onto $\mathcal{F}_p^+$ and $\mathcal{F}_p^-$, respectively. We will use the notation $\mathsf{Q}_p^+(u)$ and $\mathsf{Q}_p^-(u)$ for the resulting operators acting within $\mathcal{F}_p^+$ and $\mathcal{F}_p^-$, respectively.

4. Analytic properties of Q-operators

It turns out that the operators $\mathsf{Q}^\epsilon(u)$ are hermitian up to a phase for $u\in\mathbb{R}$, see Subsection 4.3 below for the precise statement. It follows that the T- and the Q-operators can be diagonalized simultaneously. To each eigenstate of the evolution operator U, we therefore have a quadruple of functions $(t^+(u),q^+(u),t^-(u),q^-(u))$ related to each other by equations of Baxter type, as written out explicitly in (4.12) below.

Understanding the analytic properties of the Q-operators or (equivalently) of their eigenvalues $q^\epsilon(u)$, $\epsilon=\pm$ is a key step towards understanding the spectrum of the theories in question: It turns out that the analytic properties of the functions $q^\epsilon(u)$ following from their explicit constructions restrict the relevant class of solutions to the Baxter equations considerably. Let us call a pair of solutions of the Baxter equations (4.12) which has all these analytic properties *admissible*. Being an admissible pair of solutions to the Baxter equations is clearly *necessary* for functions $q^\epsilon(u)$, $\epsilon=\pm$ to represent eigenstates of U. The Separation of Variables Method of Sklyanin, developed for the models of interest in the following section, will then allow us to actually construct an eigenstate of U to each pair of admissible solutions to the Baxter equations. Being admissible is therefore not only necessary, but also sufficient for solutions to the Baxter equations q^ϵ, $\epsilon=\pm$ to represent eigenstates of U.

4.1 Analyticity

The functions $q^{\epsilon}(u)$ are meromorphic with poles contained in the sets

$$\begin{aligned} &\mathcal{S}_{\epsilon s} \cup (-\mathcal{S}_{\epsilon s}) && \text{for the Sinh-Gordon model,} \\ &\mathcal{S}_{\epsilon s} && \text{for Liouville and KdV theory,} \end{aligned} \tag{4.1}$$

where the set $\mathcal{S}_s$ is defined as

$$\mathcal{S}_s \;=\; s - i\big(\eta + b\mathbb{Z}^{\geq 0} + b^{-1}\mathbb{Z}^{\geq 0}\big)\,. \tag{4.2}$$

The proof is very similar to the one given in [BT06, Section 4] for the case of the Sinh-Gordon model.

In the case of KdV theory we may furthermore discuss the dependence of the operators $\mathsf{Q}_p^{\epsilon}(u)$ with respect to the parameters p. It is meromorphic and analytic in the strip

$$\mathbb{S}_p \;=\; \{\, p \in \mathbb{C};\; |\mathrm{Im}(p)| < N\tfrac{Q}{2}\,\}\,. \tag{4.3}$$

The proof becomes simple if one uses the alternative integral operator representation (A.6) for $\mathsf{Q}^{\epsilon}_{\mathrm{KdV}}(u)$ given in Appendix A.

4.2 Asymptotics

Probably the most important difference between the massive and the massless cases concern the asymptotic properties of the Q-operators. Whereas we can find exponential decay of the Q-operator at both ends of the u-axis in the case of the Sinh-Gordon model,

$$q_{\mathrm{ShG}}(u) \underset{\substack{|u|\to\infty \\ \mathrm{Im}(u)=\mathrm{const}}}{\sim} e^{\pi i \mathrm{N} s|u|} e^{-\pi \mathrm{N}\eta|u|}\,, \tag{4.4}$$

in the remaining cases we find exponential decay only at one and of the u-axis,

$$q^{\epsilon}(u) \underset{\substack{u\to\epsilon\infty \\ \mathrm{Im}(u)=\mathrm{const}}}{\sim} e^{\pi i \mathrm{N} s|u|} e^{-\pi \mathrm{N}\eta|u|}\,, \tag{4.5}$$

while we have oscillatory asymptotic behavior at the other end: There exists a real number p and constants N^{ϵ}, $C^{\epsilon}(p)$ and $D^{\epsilon}(p)$ such that

$$q^{\epsilon}(u) \underset{\substack{u\to-\epsilon\infty \\ \mathrm{Im}(u)=\mathrm{const}}}{\sim} N^{\epsilon}\, e^{-\frac{\pi i}{2}\mathrm{N}u^2}\, \big(C^{\epsilon}(p)\, e^{2\pi i p u} + D^{\epsilon}(p)\, e^{-2\pi i p u}\big)\,. \tag{4.6}$$

Most of the properties above can be proven by straightforward extensions of the arguments in [BT06]. This is not the case for the oscillatory asymptotics (4.6). We therefore give a sketch of the proof in Appendix A.

4.3 Hermiticity

Some of the properties of the Q-operators become most transparent in terms of the modified Q-operators $\hat{\mathsf{Q}}^{\epsilon}(u)$ which are defined as

$$\hat{\mathsf{Q}}^{\epsilon}(u) \;=\; \Xi^{\epsilon}(u)\,\mathsf{Q}^{\epsilon}(u)\,, \tag{4.7}$$

with normalization factors $\Xi^{\epsilon}(u)$ being chosen as

$$\begin{aligned} \Xi^{\epsilon}(u) \;&=\; \left(\frac{F(u+\epsilon s-i\eta)}{F(u-\epsilon s+i\eta)}\right)^{\mathrm{N}} \qquad \text{for the Sinh-Gordon model,}\\ \Xi^{\epsilon}(u) \;&=\; \left(\frac{F(u+\epsilon s-i\eta)}{F_0(u-\epsilon s+i\eta)}\right)^{\mathrm{N}} \qquad \text{for Liouville and KdV theory.} \end{aligned} \tag{4.8}$$

with $F(v)=(F_0(v))^{-1}\Phi(v)$, where $F_0(x)=\zeta^2 e^{\frac{\pi i}{4}(x^2+\frac{1}{2})}$, $\zeta=e^{\frac{\pi i}{24}(b^2+b^{-2})}$ and

$$\Phi(x) \;=\; \exp\left(\int_{\mathbb{R}+i0}\frac{dt}{8t}\,\frac{e^{-2itx}}{\sinh(bt)\sinh(b^{-1}t)\cosh((b+b^{-1})t)}\right)\,. \tag{4.9}$$

The function $\Phi(x)$ was introduced in [BMS] [3], where all properties relevant for us are listed in the appendix.

We then find that the operators $\hat{\mathsf{Q}}(u)$ are hermitian for all $u\in\mathbb{R}$,

$$\big(\hat{\mathsf{Q}}(u)\big)^{\dagger} \;=\; \hat{\mathsf{Q}}(u) \qquad \forall u\in\mathbb{R}\,. \tag{4.10}$$

This can be verified by using the integral identity (A.31) in [BT06], taking into account the functional relation $F(x+i\eta)F(x-i\eta)=(w_b(x))^{-1}$ [BMS].

This property implies in particular that the coefficients $C^{\epsilon}(p)$ and $D^{\epsilon}(p)$ that appear in (4.6) are complex conjugate to each other, $(C^{\epsilon}(p))^{*}=D^{\epsilon}(p)$. Of particular interest will be the so-called *reflection amplitude* defined by

$$R^{\epsilon}(p) \;=\; (C^{\epsilon}(p))^{*}\,/\,C^{\epsilon}(p)\,. \tag{4.11}$$

This quantity will play an important role later.

4.4 Functional relations

4.4.1 Baxter equations

The Q-operators all satisfy Baxter-type finite difference equations of the general form

$$\mathsf{T}(u)\mathsf{Q}(u) \;=\; A(u)\mathsf{Q}(u-ib)+D(u)\mathsf{Q}(u+ib)\,. \tag{4.12}$$

[3] A relative had previously appeared in [LZ97]

The coefficient functions $A(u)$ and $D(u)$ are model-dependent. In the massive case (Sinh-Gordon model) we find

$$\begin{aligned} A^+(u) \,&=\, A^-(u) \,=\, e^{-\pi b\mathrm{N}(u-\frac{i}{2}b)}\big(1+e^{-2\pi b(s-u+\frac{i}{2}b)}\big)^{\mathrm{N}}\,,\\ D^+(u) \,&=\, D^-(u) \,=\, e^{+\pi b\mathrm{N}(u+\frac{i}{2}b)}\big(1+e^{-2\pi b(s+u+\frac{i}{2}b)}\big)^{\mathrm{N}}\,, \end{aligned} \tag{4.13}$$

whereas we have for the massless cases (Liouville theory, KdV model) the expressions

$$\begin{aligned} &A^+(u) = e^{-\pi b\mathrm{N}(u-\frac{i}{2}b)}\big(1+e^{-2\pi b(s-u+\frac{i}{2}b)}\big)^{\mathrm{N}} && D^+(u) = e^{\pi b\mathrm{N}(u+\frac{i}{2}b)}\,,\\ &A^-(u) = e^{-\pi b\mathrm{N}(u-\frac{i}{2}b)} && D^-(u) = e^{\pi b\mathrm{N}(u+\frac{i}{2}b)}\big(1+e^{-2\pi b(s+u+\frac{i}{2}b)}\big)^{\mathrm{N}}\,. \end{aligned} \tag{4.14}$$

The proof of the Baxter equations given in [BT06] for the case of the Sinh-Gordon model which is similar to the methods of [Ba73, BS90, PG92, De99] can easily be extended to the other cases.

4.4.2 Quantum Wronskian relations

The following bilinear functional relation is particularly useful:

$$\hat{\mathsf{Q}}(v+i\delta_+)\,\hat{\mathsf{Q}}(v-i\delta_+) - \hat{\mathsf{Q}}(v+i\delta_-)\,\hat{\mathsf{Q}}(v-i\delta_-) \,=\, 1\,. \tag{4.15}$$

This relation is often called the quantum Wronskian relation. The proof of (4.15) in the case of the Sinh-Gordon model [BT06] can easily be extended to the other cases.

It is worth noting that the quantum Wronskian relation fixes the absolute value of the coefficient $C^\epsilon(p)$ which appears in (4.6) to be

$$|C^\epsilon(p)|^2 = (4\sinh(2\pi bp)\sinh(2\pi b^{-1}p))^{-1}. \tag{4.16}$$

The quantity $|C^\epsilon(p)|^{-2}$ will later be identified as a natural spectral measure.

4.5 Scale invariance

It is worth observing that the dependence of $\mathsf{Q}^\epsilon_{\text{Liou}}(s|u) \equiv \mathsf{Q}^\epsilon_{\text{Liou}}(u)$, $\epsilon = \pm$ w.r.t. the scale parameter s can (up to unitary equivalence) be absorbed into a shift of u,

$$\begin{aligned} \mathsf{Q}^+_{\text{Liou}}(s|u) \,&=\, \mathsf{G}^{-s}\cdot\mathsf{Q}^+_{\text{Liou}}(0|u-s)\cdot\mathsf{G}^{+s}\,,\\ \mathsf{Q}^-_{\text{Liou}}(s|u) \,&=\, \mathsf{G}^{-s}\cdot\mathsf{Q}^-_{\text{Liou}}(0|u+s)\cdot\mathsf{G}^{+s}\,, \end{aligned} \tag{4.17}$$

where G is the unitary operator $\mathsf{G} = \prod_{r=1}^{\mathrm{N}} \mathsf{u}_r^{-\frac{i}{b}}$. A similar (even simpler) property holds for $\mathsf{Q}^\epsilon_{\text{KdV}}(u)$. This reflects the scale invariance of these theories.

Equation (4.17) implies in particular that in the massless cases one may represent the eigenvalues of $\mathsf{Q}^+(u)$ and $\mathsf{Q}^-(u)$ by functions $q^+(u-s)$ and $q^-(u+s)$ which do not carry any dependence on s other than the one implied by the form of the arguments, respectively.

5. Separation of variables

The construction of the Q-operator allowed us to deduce a set of conditions that are necessary for functions $q^\epsilon(u)$ to represent an eigenvalue of $\mathsf{Q}^\epsilon(u)$. It remains to show that these conditions are also sufficient, i.e. that to each solution of these conditions there exists an eigenvector $\Psi_q \in \mathcal{H}$ such that $\mathsf{Q}^\epsilon(u)\Psi_q = q^\epsilon(u)\Psi_q$. We will now show how to construct such an eigenvector with the help of the separation of variables method [Sk85, Sk92, Sk95]. The upshot is to show existence of a representation $\mathcal{H}_{\rm SOV}$ for $\mathcal{H}$ in which states Ψ are represented by wave-functions $\Psi(\mathbf{y})$, $\mathbf{y} = (y_1, \dots, y_{\rm N})$ such that eigenstates of the $\mathsf{Q}^\epsilon(u)$ can be represented in a fully factorized form

$$\Psi(\mathbf{y}) \;=\; \prod_{k=1}^{\rm N} q^{\epsilon(k)}(y_k)\,, \tag{5.1}$$

for a certain choice of $\epsilon(k)$. The wave functions $\Psi(\mathbf{y})$ have to be normalizable w.r.t. to the measure $d\mu(\mathbf{y})$ which represents the scalar product in $\mathcal{H}_{\rm SOV}$. The main issue is to show that the conditions on $q^\epsilon(u)$ found above ensure the normalizability w.r.t. $d\mu(\mathbf{y})$.

In the case of the Sinh-Gordon model [BT06] the representation $\mathcal{H}_{\rm SOV}$ is simply the spectral representation for the commutative family of operators $\mathsf{B}(u)$ defined as the off-diagonal element of the monodromy matrix $M(u) = \big(\begin{smallmatrix} \mathsf{A}(u) & \mathsf{B}(u) \\ \mathsf{C}(u) & \mathsf{D}(u)\end{smallmatrix}\big)$. We will now briefly discuss how to adapt this method to the remaining cases.

5.1 Separation of variables for the Liouville and quantum KdV theories

The elements of the monodromy matrices $M^\epsilon(u)$, $\epsilon = \pm$, satisfy the relations

$$R_{12}(u-v)\,M_1^\epsilon(u)\,M_2^\epsilon(v) \;=\; M_2^\epsilon(v)\,M_1^\epsilon(u)\,R_{12}(u-v)\,, \tag{5.2}$$

$$R'_{12}(u-v)\,M_1^+(u)\,M_2^-(v) \;=\; M_2^-(v)\,M_1^+(u)\,R'_{12}(u-v)\,, \tag{5.3}$$

where $R'_{12}(u) = {\rm diag}(q,1,1,q)$ for KdV theory, while for Liouville theory

$$R'_{12}(u) = \begin{pmatrix} e^{\pi b(u+ib)} & & & \\ & e^{\pi bu} & 0 & \\ & i\sin\pi b^2 & e^{\pi bu} & \\ & & & e^{\pi b(u+ib)} \end{pmatrix}, \tag{5.4}$$

Let us use the notation $M^\epsilon(u) = \big(\begin{smallmatrix} \mathsf{A}^\epsilon(u) & \mathsf{B}^\epsilon(u) \\ \mathsf{C}^\epsilon(u) & \mathsf{D}^\epsilon(u)\end{smallmatrix}\big)$. The relations (5.2) imply in particular that

$$\begin{aligned} \mathsf{B}^\epsilon(u)\mathsf{B}^{\epsilon'}(v) \;&=\; \mathsf{B}^{\epsilon'}(v)\mathsf{B}^\epsilon(u)\,, \\ \mathsf{C}^\epsilon(u)\mathsf{C}^{\epsilon'}(v) \;&=\; \mathsf{C}^{\epsilon'}(v)\mathsf{C}^\epsilon(u)\,, \end{aligned} \qquad \epsilon,\epsilon' = \pm\,. \tag{5.5}$$

Note furthermore that $\mathsf{B}^\epsilon(u)$, $\mathsf{C}^{\epsilon'}(u)$ are positive self-adjoint for all $u \in \mathbb{R} + i/2b$. We may therefore simultaneously diagonalize either one of the the commutative families of operators $\mathsf{B}^\epsilon(u)$, $\epsilon = \pm$ or $\mathsf{C}^\epsilon(u)$, $\epsilon = \pm$. The main idea of the Separation of Variables method is to work within the spectral representation for one of these families.

Let us consider the spectral representation for the operators $\mathsf{B}^\epsilon(u)$, $\epsilon = \pm$. It will be called the B-representation. One may parameterize the corresponding eigenvalues as

$$\begin{aligned} b^+(u) \;&=\; -ie^{\pi bu} b_0 \prod_{a=1}^{L} \big(1 - e^{+2\pi b(u-y_a^+)}\big)\,, \\ b^-(u) \;&=\; -ie^{\pi bu} b_0 \prod_{a=0}^{L} \big(1 - e^{-2\pi b(u-y_a^-)}\big)\,, \end{aligned} \qquad b_0 = \prod_{a=1}^{L} e^{\pi b y_a^+} \prod_{a=0}^{L} e^{-\pi b y_a^-}\,. \tag{5.6}$$

The spectral representation for the operators $\mathsf{B}^\epsilon(u)$, $\epsilon = \pm$ is therefore equivalent to a representation in terms of wave-functions $\Psi(\mathbf{y})$, where $\mathbf{y} = (\, y_1^+, \dots, y_L^+\,;\, y_0^-, y_1^-, \dots, y_L^-)$. Let us define operators y_a^ϵ such that $\mathsf{y}_a^\epsilon \cdot \Psi(\mathbf{y}) = y_a^\epsilon \Psi(\mathbf{y})$.

Considering the operators $\mathsf{C}^\epsilon(u)$, $\epsilon = \pm$ instead yields what will be called the C-representation in terms of variables $\tilde{\mathbf{y}} = (\, \tilde{y}_1^-, \dots, \tilde{y}_L^-\,;\, \tilde{y}_0^+, \tilde{y}_1^+, \dots, \tilde{y}_L^+)$.

5.2 The Baxter equations

5.2.1 Liouville theory

Let us define operators $\mathsf{A}^\epsilon(\mathsf{y}_a^\epsilon)$, $\mathsf{D}^\epsilon(\mathsf{y}_a^\epsilon)$ by the prescription to order the operators y_a^ϵ to the left of the operators which appear in the expansion of $\mathsf{A}^\epsilon(u)$ in powers of $e^{\pi bu}$. It is an easy consequence of the algebraic relations (5.2) that these operators act on wave-functions $\Psi(\mathbf{y})$ as finite difference operators of the form

$$\mathsf{A}^\epsilon(\mathsf{y}_a^\epsilon) \cdot \Psi(\mathbf{y}) \;=\; A^\epsilon(y_a^\epsilon)\, \delta_{a-}^\epsilon \Psi(\mathbf{y})\,, \qquad \mathsf{D}^\epsilon(\mathsf{y}_a^\epsilon) \cdot \Psi(\mathbf{y}) \;=\; D^\epsilon(y_a^\epsilon)\, \delta_{a+}^\epsilon \Psi(\mathbf{y})\,, \tag{5.7}$$

where $\delta_{a\pm}^\epsilon$ are defined as

$$\delta_{a\pm}^\epsilon \Psi(\dots, y_a^\epsilon, \dots) \;=\; \Psi(\dots, y_a^\epsilon \pm ib, \dots)\,.$$

The coefficients $A^\epsilon(u)$, $D^\epsilon(u)$ are constrained by the quantum determinant condition

$$\Delta^\epsilon(u) \;\equiv\; \mathsf{A}^\epsilon(u)\mathsf{D}^\epsilon(u-ib) - \mathsf{B}^\epsilon(u)\mathsf{C}^\epsilon(u-ib) \;=\; \big(1 + e^{-2\pi b(s-\epsilon(u-\frac{i}{2}b))}\big)^{\mathrm{N}}\,. \tag{5.8}$$

As anticipated by the notation we shall adopt the choice (4.14) for the coefficients $A^\epsilon(u)$, $D^\epsilon(u)$.

The condition that $\Psi(\mathbf{y})$ represents an eigenstate of the transfer matrices $\mathsf{T}^\epsilon(u)$, $\epsilon = \pm$, with eigenvalues $t^\epsilon(u)$ becomes equivalent to the equations

$$t^\epsilon(y_a^\epsilon)\, \Psi(\mathbf{y}) \;=\; A^\epsilon(y_a^\epsilon)\, \delta_{a-}^\epsilon \Psi(\mathbf{y}) + D^\epsilon(y_a^\epsilon)\, \delta_{a+}^\epsilon \Psi(\mathbf{y})\,, \quad \epsilon = \pm\,. \tag{5.9}$$

The eigenfunctions for $\mathsf{T}^\epsilon(u)$ can therefore be constructed in the following form

$$\Psi_q(\mathbf{y}) \;=\; \prod_{a=1}^{L} q^-(y_a^+) \prod_{a=0}^{L} q^+(y_a^-)\,, \tag{5.10}$$

where $q_p^\epsilon(u)$, $\epsilon=\pm$ are solutions to the Baxter equations

$$t^\epsilon(u)\,q^\epsilon(u) \;=\; A^\epsilon(u)q^\epsilon(u-ib)+D^\epsilon(u)q^\epsilon(u+ib)\,. \tag{5.11}$$

Classifying eigenstates of $\mathsf{T}^\epsilon(u)$, $\epsilon=\pm$ thereby becomes equivalent to finding the proper set of solutions of the Baxter equations (5.11).

5.2.2 KdV theory

It is instructive to notice that the limit $s\to\infty$ which yields the lattice KdV model from Liouville theory forces one of the variables y_a^-, by convention chosen to be the variable $y_0^- \equiv y_\circ$, to diverge. The resulting parametrization for the eigenvalue $b^-(u)$ is

$$\begin{aligned} b^+(u) \;&=\; -ie^{\pi bu} b_\circ e^{-\pi b y_\circ}\prod_{a=1}^{L}\big(1-e^{+2\pi b(u-y_a^+)}\big)\,,\\ b^-(u) \;&=\; -ie^{\pi bu} b_\circ e^{+\pi b y_\circ}\prod_{a=1}^{L}\big(1-e^{-2\pi b(u-y_a^-)}\big)\,,\end{aligned} \qquad b_\circ=\prod_{a=1}^{L}e^{\pi b y_a^+}\prod_{a=1}^{L}e^{-\pi b y_a^-}\,. \tag{5.12}$$

The equations (5.7) degenerate for $a=\circ$ into

$$\mathsf{A}^\epsilon(\mathsf{y}_\circ)\Psi(\mathbf{y}) \;=\; A^\circ(\mathsf{y}_\circ)\delta_{\circ-}\Psi(\mathbf{y})\,,\qquad \mathsf{D}^\epsilon(\mathsf{y}_\circ)\Psi(\mathbf{y}) \;=\; D^\circ(\mathsf{y}_\circ)\delta_{\circ+}\Psi(\mathbf{y})\,,$$

where $A^\circ(u)=e^{-\pi b\mathrm{N}(u-\frac{i}{2}b)}$, $D^\circ(u)=e^{+\pi b\mathrm{N}(u+\frac{i}{2}b)}$, respectively, so that (5.9) for $a=\circ$ becomes

$$t_\circ\,\Psi(\mathbf{y}) \;=\; A^\circ(y_\circ)\delta_{\circ-}\Psi(\mathbf{y})+D^\circ(y_\circ)\delta_{\circ+}\Psi(\mathbf{y})\,, \tag{5.13}$$

where $t_\circ=t^+(-\infty)=t^-(\infty)$. We accordingly need to modify (5.10) as

$$\Psi_q(\mathbf{y}) \;=\; \prod_{a=1}^{L} q^-(y_a^+)\,q^\circ(y_\circ)\prod_{a=1}^{L} q^+(y_a^-)\,. \tag{5.14}$$

The equation (5.13) is solved by the exponential functions $q^\circ(y_\circ) \;=\; e^{-\frac{\pi i}{2}\mathrm{N}u^2}e^{2\pi i y_\circ p}$, with p being related to $t_\circ$ as $t_\circ=2\cosh(2\pi bp)$. We will see that p can take arbitrary real values.

5.3 The Sklyanin measure

Adopting the parametrization (5.6) for the eigenvalues of the operators $\mathsf{B}^\epsilon(u)$, $\epsilon=\pm$ one needs to find the set of all $\mathbf{y}\in\mathbb{C}^{\mathrm{N}}$ which parameterize a point in the spectrum of $\mathsf{B}^\epsilon(u)$ via (5.6). We shall adopt the following conjecture:

Conjecture 1. *All points in the spectrum of* $\mathsf{B}^\epsilon(u)$, $\epsilon = \pm$ *can be parameterized by real values of* $y_1^+, \dots, y_L^+$ *and* $y_0^-, \dots, y_L^-$.

Validity of the conjecture above is *not* crucial for the discussion below, we adopt it here to simplify the exposition. However, we are rather confident that it is correct. It can be checked in certain limits and special cases. The conjecture implies that the B-representation can be realized on a Hilbert space of the form

$$\mathcal{H}_{\text{SoV}}^{\mathsf{B}} \,=\, L^2\big((\mathbb{R}^L/S_L) \times (\mathbb{R}^{L+1}/S_{L+1})\,;\, d\mu_{\mathsf{B}}\big)\,.$$

Elements of $\mathcal{H}_{\text{SoV}}^{\mathsf{B}}$ are represented by wave-functions $\Psi(\mathbf{y})$ that are normalizable w.r.t. $d\mu_{\mathsf{B}}$ and totally symmetric under permutations among the sets of variables $\{y_a^+; a = 1, \dots L\}$ and $\{y_a^-; a = 0, \dots L\}$, respectively. The C-representation can similarly be realized on

$$\mathcal{H}_{\text{SOV}}^{\mathsf{C}} \,=\, L^2\big((\mathbb{R}^{L+1}/S_{L+1}) \times (\mathbb{R}^L/S_L)\,;\, d\mu_{\mathsf{C}}\big)\,,$$

Elements of $\mathcal{H}_{\text{SoV}}^{\mathsf{C}}$ are represented by wave-functions $\Psi(\tilde{\mathbf{y}})$ that are normalizable w.r.t. $d\mu_{\mathsf{C}}$ and totally symmetric under permutations among the sets of variables $\{\tilde{y}_a^+; a = 0, \dots L\}$ and $\{\tilde{y}_a^-; a = 1, \dots L\}$, respectively.

The Sklyanin measure $d\mu_{\mathsf{B}}$ can be found by the same method as used in [BT06] from the requirement that $\mathsf{A}^\epsilon(v)$ and $\mathsf{D}^\epsilon(v)$ are positive self-adjoint. We have

$$d\mu_{\mathsf{B}}(\mathbf{y}) \,=\, d\mu_{\mathsf{B}}^+(\mathbf{y}^+)\, d\mu_{\mathsf{B}}^-(\mathbf{y}^-)\,, \tag{5.15}$$

where

$$\begin{aligned} L!\, d\mu_{\mathsf{B}}^+(\mathbf{y}^+) \,&=\, \prod_{a=1}^{L} dy_a^+ \; e^{\pi Q(L+1)y_a^+} \prod_{b<a} 2\sinh \pi b(y_a^+ + y_b^+) 2\sinh \pi b^{-1}(y_a^+ - y_b^+)\,,\\ (L+1)!\, d\mu_{\mathsf{B}}^-(\mathbf{y}^-) \,&=\, \prod_{a=0}^{L} dy_a^- \; e^{\pi Q L y_a^-} \prod_{b<a} 2\sinh \pi b(y_a^- - y_b^-) 2\sinh \pi b^{-1}(y_a^- - y_b^-)\,. \end{aligned}$$

We have a very similar expression for $d\mu_{\mathsf{C}}(\mathbf{y})$.

In the case of the lattice KdV theory we get the following modifications:

$$d\mu_{\mathsf{B}}(\mathbf{y}) \,=\, d\mu_{\mathsf{B}}^+(\mathbf{y}^+)\, dy_\circ\, d\mu_{\mathsf{B}}^-(\mathbf{y}^-)\,, \tag{5.16}$$

where $d\mu_{\mathsf{B}}^+(\mathbf{y}^+)$ is unchanged, but $d\mu_{\mathsf{B}}^-(\mathbf{y}^-)$ is now given as

$$L!\, d\mu_{\mathsf{B}}^-(\mathbf{y}^-) \,=\, \prod_{a=1}^{L} dy_a^- \; e^{\pi Q(L+1)y_a^-} \prod_{b<a} 2\sinh \pi b(y_a^- - y_b^-) 2\sinh \pi b^{-1}(y_a^- - y_b^-)\,.$$

It is worth observing that the small asymmetry between the Liouville-variables y_a^+ and y_a^- disappears in the limit giving quantum KdV theory.

6. The spectra

6.1 The spectrum of quantum KdV theory

The fact that the dynamics generated by $\mathsf{U}_{\mathrm{KdV}}$ is "trivial" in the sense that it decouples into right- and left motions (2.14) of $\mathsf{w}_{\nu,t}^{+}$ and $\mathsf{w}_{\nu,t}^{-}$ respectively, does not mean that the lattice model characterized by the T-operators $\mathsf{T}^{\epsilon}_{\mathrm{KdV}}$, $\epsilon = \pm$, is trivial as an integrable model. As in classical (m)KdV theory one may define alternative and much less trivial evolutions from the families of operators $\mathsf{T}^{\epsilon}_{\mathrm{KdV}}$ or $\mathsf{Q}^{\epsilon}_{\mathrm{KdV}}$. The diagonalization of these operators is interesting in its own right.

6.1.1 The spectrum of the chiral free field

Let us first study the chiral free field theories with Hilbert space $\mathcal{F}_p^{\epsilon}$ and Q-operator $\mathsf{Q}_p^{\epsilon}(u)$ for fixed values of $\epsilon \in \{\pm\}$ and $p \in \mathbb{R}$. The spectral theorem for the commutative family of self-adjoint operators $\hat{\mathsf{Q}}_p^{\epsilon}(u)$ implies that the eigenstates $f_q^{\epsilon} \in \mathcal{F}_p^{\epsilon}$ of these operators form a *basis* for $\mathcal{F}_p^{\epsilon}$. This is the case for arbitrary real values of the variable p. Let $q_p^{\epsilon}(u)$ be the eigenvalue of the operator $\mathsf{Q}_p^{\epsilon}(u)$ on f_q^{ϵ}. It must be element of the set $\mathcal{Q}_p^{\epsilon}$, the set of all functions $q_p^{\epsilon}(u)$ that possess all the analytic and asymptotic properties implied by our explicit construction of the Q-operators as discussed in Section 4.

On the other hand let let us note that the SOV representation is realized on the Hilbert spaces

$$\mathcal{H}^{\epsilon}_{\mathrm{SoV}} \,=\, L^2(\mathbb{R}^L; d\mu^{\epsilon}_{\mathsf{B}})_{\mathrm{symm}}\,. \tag{6.1}$$

For a given element $q_p^{\epsilon}(u) \in \mathcal{Q}_p^{\epsilon}$ define

$$\Psi_q^{\epsilon}(\mathbf{y}^{\epsilon}) \,=\, \prod_{a=1}^{L} q_p^{\epsilon}(y_a^{\epsilon})\,. \tag{6.2}$$

It follows from the asymptotic properties of $q_p^{\epsilon}(u)$ that $\Psi_q^{\epsilon}(\mathbf{y}^{\epsilon})$ is normalizable w.r.t. $d\mu^{\epsilon}_{\mathsf{B}}$. There is a corresponding eigenstate $f_q^{\epsilon} \in \mathcal{F}_p^{\epsilon}$ of $\mathsf{Q}_p^{\epsilon}(u)$ which has as its eigenvalue the function $q_p^{\epsilon}(u)$ we had used in (6.2). We conclude that there is a one-to-one correspondence between the elements of $\mathcal{Q}_p^{\epsilon}$ and the eigenstates of $\mathsf{Q}_p^{\epsilon}(u)$ within $\mathcal{F}_p^{\epsilon}$. The fact that the wave-function Ψ_q^{ϵ} are all normalizable implies in particular that the spectrum of $\mathsf{Q}_p^{\epsilon}(u)$ is *purely discrete*.

6.1.2 The zero mode spectrum of quantum KdV theory

To each triple $q \equiv (q_p^{+}(u), q_p^{\circ}(u), q_p^{-}(v))$ of solutions to the Baxter equations (5.9) we may associate a wave-function of the form

$$\Psi_{q_p}(\mathbf{y}) \,=\, \prod_{a=1}^{L} q_p^{-}(y_a^{-})\, q_p^{\circ}(y_0)\, \prod_{a=1}^{L} q_p^{+}(y_a^{+})\,. \tag{6.3}$$

The asymptotic behavior (4.5), (4.6) ensures the (plane-wave) normalizability of $\Psi_q(\mathbf{y})$. We need to identify the set of solutions of the zero mode equation (5.13) which yields a *complete* set of $\mathsf{Q}^{\epsilon}_{\mathrm{KdV}}$-eigenstates in this way.

By means of induction it is easy to prove that T_0 has the following form:

$$\mathsf{T}_0 \;=\; 2\cosh \pi b \mathsf{p}_\circ\,, \tag{6.4}$$

where $e^{2\pi b\mathsf{p}_\circ} \;\equiv\; \prod_{n=1}^{\mathrm{N}} \mathsf{u}_n$. It easily follows from this observation that the vectors $\Psi_q(\mathbf{y})$ constructed from the choices $q_p^\circ(u) = e^{-\frac{\pi i}{2}u^2}e^{2\pi i p u}$, $p \in \mathbb{R}$, all represent linearly independent basis vectors for $\mathcal{H}$ in the sense of generalized functions.

6.2 The spectrum of Liouville theory

We are now going to analyze the spectrum of Liouville theory in a similar manner. To each eigenstate Ψ of the Q-operators $\mathsf{Q}^+(u)$ and $\mathsf{Q}^-(u)$ there exists a complex number p and a corresponding pair of elements $q_p = (q_p^+, q_p^-) \in \mathcal{Q}_p^+ \times \mathcal{Q}_p^-$, given by the eigenvalues of $\mathsf{Q}^\epsilon(u)$ on Ψ. Conversely, for a given value of p and each pair $q_p = (q_p^+, q_p^-) \in \mathcal{Q}_p^+ \times \mathcal{Q}_p^-$ of admissible solutions to the Baxter equations one may construct an eigenstate of the Q-operators $\mathsf{Q}^+(u)$ and $\mathsf{Q}^-(u)$ as

$$\Psi_{q_p}(\mathbf{y}) \;=\; \prod_{a=0}^{L} q_p^-(y_a^-)\,\prod_{b=1}^{L} q_p^+(y_b^+)\,. \tag{6.5}$$

With the help of our explicit formulae for the Sklyanin measure and the formulae (4.5), (4.6) for the asymptotic behavior of the functions $q_p^\epsilon(u)$ it is possible to check that the states (6.5) are plane-wave normalizable if $p \in \mathbb{R}$. More precisely one may show that

$$\big(\,\Psi_{q_p}\,,\,\Psi_{q_{p'}}\,\big) \;=\; \frac{\delta(p-p')}{4\sinh(2\pi b p)\sinh(2\pi b^{-1}p)}\,. \tag{6.6}$$

This means that $dp\, 4\sinh(2\pi b p)\sinh(2\pi b^{-1}p)$ is the natural spectral measure for the integration over p in the spectral representation.

One should note that the spectrum of the zero mode p is real and *purely continuous*. This follows from the works [Ka00, FK02], one of the main results of which can be stated as

$$\mathrm{Spec}(\mathsf{U}^+) \;=\; \big\{\, e^{-2\pi i(\Delta_p+m)/\mathrm{N}}\,;\, p\in\mathbb{R}_+\,,\; m\in\mathbb{Z}/\mathrm{N}\mathbb{Z}\,\big\}\,, \tag{6.7}$$

where

$$\Delta_s \;=\; \frac{c-1}{24} + s^2\,, \qquad c = 1 + 24\eta^2\,. \tag{6.8}$$

It is an important difference to the case of KdV theory that the eigenstates Ψ_{q_p} and $\Psi_{q_{-p}}$ are not independent. Indeed, it follows easily from (A.6) that the $q_p^\epsilon(u)$ are symmetric w.r.t. p, i.e. $q_p^\epsilon(u) = q_{-p}^\epsilon(u)$. It follows that

$$\Psi_{q_p}(\mathbf{y}) \;=\; \Psi_{q_{-p}}(\mathbf{y})\,. \tag{6.9}$$

We conclude that there is a one-to-one correspondence between triples $q = (p, q_p^+, q_p^-)$, $p \in \mathbb{R}^+$, $(q_p^+, q_p^-) \in \mathcal{Q}_p^+ \times \mathcal{Q}_p^-$ and the elements of a basis for $\mathcal{H}$ consisting of generalized eigenstates of the Q-operators.

7. The relation between quantum Liouville- and KdV-theory

7.1 The Bäcklund transformations

The key point for us to observe is the fact that the sets $\mathcal{Q}_p^\epsilon$ of admissible solutions of the Baxter equations are the same for Liouville theory and the quantum lattice KdV model. We may therefore construct operators W_χ which send the eigenstate Ψ_q of $\mathsf{Q}^\epsilon_{\rm Liou}(u)$, $\epsilon = \pm$ associated to a triple $q = (q; q_p^+, q_p^-)$ to the eigenstate Φ_q of $\mathsf{Q}^\epsilon_{\rm KdV}(u)$, $\epsilon = \pm$, which in the $\mathsf{B}_{\rm KdV}$-representation is represented by the wave-function

$$\Phi_q \;=\; W_{\chi_q} \prod_{a=1}^{L} q_p^-(y_a^-)\, q_p^\circ(y_\circ) \prod_{b=1}^{L} q_p^+(y_b^+)\,, \quad q_p^\circ(y_\circ) \;=\; e^{-\frac{\pi i}{2}{\rm N}y_\circ^2} e^{2\pi p y_\circ}\,. \tag{7.10}$$

The prefactor W_{χ_q} is required to satisfy $|W_{\chi_q}|^2 \;=\; 4\sinh(2\pi bp)\sinh(2\pi b^{-1}p)$ while its phase $e^{2i\chi_q} \equiv W_{\chi_q}/W^*_{\chi_q}$ is left arbitrary for the moment. The operators W_χ clearly satisfy

$$\boxed{\mathsf{W}_\chi \cdot \mathsf{Q}^\epsilon_{\rm Liou}(u) \;=\; \mathsf{Q}^\epsilon_{\rm KdV}(u) \cdot \mathsf{W}_\chi} \tag{7.11}$$

and they define unitary operators $\check{\mathsf{W}}_\chi$ from $\mathcal{H}$ to the subspace $\mathcal{H}_+$ of $\mathcal{H}$ on which the zero mode momentum $\mathsf{p}_\circ$ is positive. The operators W_χ can be seen as representatives for (generalizations of the) quantum Bäcklund transformations which map the interacting dynamics of Liouville theory to the free field dynamics. They make the decoupling of left- and right-moving degrees of freedom in Liouville theory manifest.

7.2 Relation with scattering theory

All what is nontrivial about Liouville theory is hidden in the way the decoupling between left- and right-movers is disguised when studying its dynamics in terms of the original degrees of freedom π_n, ϕ_n. The operators W_χ which trivialize the dynamics are rather nontrivial objects for which we do not have an explicit representation at the moment.[4] In the following we shall propose an interpretation of one of these operators related to the asymptotic behavior of the time evolution.

[4]Finding a more explicit representation would become possible once we had an explicit representation for the transformation from the original to the separated variables.

7.2.1 Wave- and scattering operators

One should note that the operators $\mathsf{Q}^{\epsilon}_{\rm Liou}(u)$ and $\mathsf{Q}^{\epsilon}_{\rm KdV}(u)$ coincide in the limit where the zero mode $\phi_\circ$ tends to infinity,

$$\lim_{\rho\to\infty} \langle\, \Psi_q \,,\, \mathsf{Q}^{\epsilon}_{\rm Liou}(u)\Phi_\rho \,\rangle \;=\; \lim_{\rho\to\infty} \langle\, \Psi_q \,,\, \mathsf{Q}^{\epsilon}_{\rm KdV}(u)\Phi_\rho \,\rangle \,, \tag{7.12}$$

for any wave-packet Φ_ρ that has support localized around $\phi_\circ = \rho$. We have, in particular, a similar statement for the evolution operator U. It then follows from standard arguments that wave-packets for time $\tau \to \pm\infty$ are always pushed into the asymptotic region $\phi_0 \to \infty$ where the dynamics becomes the free field dynamics. We may therefore define natural analogs of the wave operators from quantum mechanical scattering theory as

$$\mathsf{W}_{+\infty} \;=\; \lim_{\tau\to\infty} (\mathsf{U}_{\rm KdV})^{-\frac{\tau}{2}} \cdot (\mathsf{U}_{\rm Liou})^{+\frac{\tau}{2}} \,, \quad \mathsf{W}_{-\infty} \;=\; \lim_{\tau\to\infty} (\mathsf{U}_{\rm KdV})^{+\frac{\tau}{2}} \cdot (\mathsf{U}_{\rm Liou})^{-\frac{\tau}{2}} \,. \tag{7.13}$$

The operators $\mathsf{W}_{\pm\infty}$ are easily seen to represent a particular case of the Bäcklund transformations introduced in Subsection 7.1 above.

The scattering operator S which maps the asymptotic shape of a wave packet for $\tau \to -\infty$ to the one for $\tau \to \infty$ can then be defined as $\mathsf{S} \equiv \mathsf{W}_{+\infty} \cdot \mathsf{W}_{-\infty}^{-1}$. It can be described in terms of its eigenvalues S_{q_p} in the spectral representation.

7.2.2 Relation to space asymptotics of wave-functions

In quantum mechanical scattering theory there exist well-known results relating the scattering operator S to the (target-) space asymptotics of eigenfunctions of the corresponding Hamiltonian. It seems fairly clear that similar relations will hold in the present context, as now to be formulated more explicitly. We'd like to analyze the representation of eigenstates Ψ_q in the zero mode Schrödinger representation where they are represented by wave-functions $\Psi_q(\phi_\circ)$ taking values in $\mathcal{F}^+_{\phi_\circ} \otimes \mathcal{F}^-_{\phi_\circ}$. It follows from (7.12) that the asymptotic behavior for $\phi_\circ \to \infty$ of the wave-functions $\Psi_q(\phi_\circ)$ can be expanded into the eigenstates of $\mathsf{Q}^{\epsilon}_{\rm KdV}(u)$,

$$\Psi_{q_p}(\phi_\circ) \underset{\phi_0\to\infty}{\sim} N_p \left[\, e^{2\pi i p\phi_\circ} + S_{q_p} e^{-2\pi i p\phi_\circ} \,\right] (\, f^+_q \otimes f^-_q \,) \,, \tag{7.14}$$

where N_p is a normalization factor and $f^+_q \otimes f^-_q \in \mathcal{F}^+_p \otimes \mathcal{F}^-_p$ is an eigenstate of both $\mathsf{Q}^{+}_{\rm KdV}(u)$ and $\mathsf{Q}^{-}_{\rm KdV}(u)$ with eigenvalues $q^+_p(u)$ and $q^-_p(u)$, respectively. We claim that the so-called reflection amplitudes S_{q_p} which appear in the asymptotic behavior (7.14) are indeed the eigenvalues of the scattering operator S defined above.

7.3 Relation between the reflection amplitudes of Liouville and of KdV theory

Let us finally note that there is a remarkable relationship between the scattering amplitude S_{q_p} of Liouville theory and the reflection phases $R^\epsilon(p)$ of KdV-theory introduced in (4.11),

$$S_{q_p} \;=\; R_{q_p^+}\, R_{q_p^-} \quad \text{if} \quad q_p = (q_p^+(u), q_p^-(u))\,. \tag{7.15}$$

We have used the notation $R_{q_p^\epsilon}$, $\epsilon = \pm$ for the ratio $R^\epsilon(p) = (C^\epsilon(p))^*/C^\epsilon(p)$ of the coefficients which appear in the asymptotic behavior of $q_p^\epsilon(u)$ for $u \to -\epsilon\infty$ according to (4.6).

The relationship (7.15) allows one to calculate the scattering operator S from the asymptotics of the operators $\mathsf{Q}^\epsilon_{\rm KdV}(u)$ as determined in the Appendix. We do not go further into this direction for the case of the lattice models as we did not yet find a sufficiently nice formula for S. The situation becomes better in the continuum limit where (7.15) will be a key ingredient in our calculation of the Liouville reflection amplitude.

7.3.1 Derivation of equation (7.15)

Equation (7.15) can be verified by means of arguments which are similar to those in [T08a]. One may analyze the massless limit $s \to \infty$ in two different ways.

Let us, on the one hand, consider an eigenstate Ψ_q in the Sinh-Gordon model represented in the Schrödinger representation by a wave-function $\Psi_q(\phi_\circ) \in \mathcal{F}^+_{\phi_\circ} \otimes \mathcal{F}^-_{\phi_\circ}$. Note that the limit giving Liouville theory from the Sinh-Gordon model combines the limit $s \to \infty$ with $\phi_\circ \to -\infty$. It follows that the limit of the operator $\mathsf{Q}_{\rm ShG}(u)$ for $s \to \infty$ can also be regarded as the asymptotic behavior of $\mathsf{Q}^\epsilon_{\rm Liou}(u)$ for $\phi_\circ \to \infty$. Arguing as in Subsection 7.2.2 we conclude that the leading behavior of $\Psi_q(\phi_\circ)$ for $s \to \infty$ can be described in terms of eigenfunctions of $\mathsf{Q}^\epsilon_{\rm Liou}(u)$ as

$$\Psi_q(\phi_\circ) \;\simeq\; \big(\, C_{q_p}\, e^{2\pi i p\phi_\circ} + C^*_{q_p}\, e^{-2\pi i p\phi_\circ} \,\big)\,\big(\, f_q^+ \otimes f_q^- \,\big)\,, \tag{7.16}$$

where $f_q^+ \otimes f_q^- \;\in\; \mathcal{F}_p^+ \otimes \mathcal{F}_p^-$ is an eigenstate of both $\mathsf{Q}^+_{\rm KdV}(u)$ and $\mathsf{Q}^-_{\rm KdV}(u)$ with eigenvalues $q_p^+(u)$ and $q_p^-(u)$, respectively. The eigenstate Ψ_q is either even or odd under parity. In order to evaluate this condition note that $\arg S_{q_p} \;=\; -2\arg C_{q_p} \;=\; \rho_q(p) - 4\pi ps$, where $\rho_q(p)$ is independent of s. For $s \to \infty$ one gets the quantization condition to leading order as the condition that there exists an integer n such that allowed values p_n of the variable p satisfy

$$4\pi s\, p_n - \rho_q(p_n) \;=\; \pi n\,. \tag{7.17}$$

One may, on the other hand, note that the limit $s \to \infty$ of the Q-operators $\mathsf{Q}^\epsilon_{\rm ShG}(u)$ for $s \to \infty$ may according to (3.23) be described either as the asymptotics of the $\mathsf{Q}^+_{\rm KdV}(u)$ for $u \to -\infty$ or, equivalently as the asymptotics of $\mathsf{Q}^{-}_{\rm KdV}(u)$ for $u \to +\infty$. This implies for the eigenvalues of $\mathsf{Q}^{\epsilon}_{\rm ShG}(u)$ that we have, on the one hand

$$q^{\epsilon}(u) \simeq N_p \cos\big(2\pi p(u-s) + \theta_q^{+}(p)\big)\,, \tag{7.18}$$

where $N_p = (\sinh(2\pi bp)\sinh(2\pi b^{-1}p))^{-\frac{1}{2}}$, and on the other hand

$$q^{\epsilon}(u) \simeq N_p \cos\big(2\pi p(u+s) - \theta_q^{-}(p)\big)\,. \tag{7.19}$$

The compatibility between these two equations requires that there exists an integer n such that

$$4\pi p_n s - \theta_q^{+}(p_n) - \theta_q^{-}(p_n) = \pi n\,. \tag{7.20}$$

The equivalence of (7.17) and (7.20) yields our claim (7.15).

7.3.2 Interpretation of equation (7.15)

It seems natural to interpret (7.15) in the following way: In the same way as we used the evolution operator U to define the scattering operator S in Subsection 7.2.1 above, we may use the light-cone evolution operators U^{ϵ} to define light-cone scattering operators S^{ϵ} for $\epsilon = \pm$, respectively. It is clear that the eigenvalues of the operators S^{+} in a state defined by a pair $q_p = (q_p^{+}, q_p^{-})$ will not depend on q_p^{-}, and similarly for the eigenvalues of S^{-}. It seems natural to conjecture that the eigenvalues of S^{ϵ} are precisely the phases $R_{q_p^{\epsilon}}$ defined from the asymptotic behavior (4.6) of q_p^{ϵ}. This would mean that our relationship (7.15) is equivalent to $\mathsf{S} = \mathsf{S}^{+}\mathsf{S}^{-}$ which trivially follows from the factorization $\mathsf{U} = \mathsf{U}^{+}\mathsf{U}^{-}$ observed in (3.21) above.

8. Continuum limit

Following arguments which are very similar to those used in [T08a] we may now reformulate the conditions for the q-functions in terms of nonlinear integral equations which generalize the equations coming from the thermodynamic Bethe ansatz [YY, Za90, Za06] to arbitrary excited states. As shown in [T08a], one gets a characterization of the spectrum which is *completely* equivalent to the one derived above. On the level of the nonlinear integral equations it turns out to be straightforward to pass to the continuum limit. The limit is taken in such a way that $\mathrm{N} \to \infty$, $s \to \infty$ such that

$$mR = 4\sin\vartheta_0\,\mathrm{N}\,e^{-\pi bs}\,, \qquad \vartheta_0 \equiv \frac{\pi b^2}{1+b^2} \tag{8.1}$$

is kept constant. As the necessary arguments are very similar to those in [T08a] we will only briefly describe the resulting description of the q-functions for the continuum theories and some of the most important consequences for the spectrum of these theories.

8.1 Reformulation in terms of integral equations

As advertised earlier, one may express the eigenvalues of the Q-operators in terms of the solutions of certain nonlinear integral equations. These equations are best formulated in terms of the functions

$$Y_p^\epsilon\big(\tfrac{\pi}{2\eta}u\big) \;=\; q_p^\epsilon(u+i\delta)q_p^\epsilon(u-i\delta)\,, \tag{8.2}$$

where $2\delta = b^{-1}-b$. It suffices to consider the case that p is purely imaginary which is related the case of real p by means of analytic continuation. Assume that $q_p^\epsilon(u)$ has M^ϵ real zeros at positions ϑ_a^ϵ, $a=1,\dots,M$. The functions $q_p^\epsilon(u)$ can then be recovered from

$$\begin{aligned}\partial_\vartheta \log q_p^\epsilon\big(2\tfrac{\eta}{\pi}\vartheta\big) \;=\; -\epsilon\frac{mRe^{\epsilon\vartheta}}{2\sin\vartheta_0}+\sum_{a=1}^{M^\epsilon}\frac{1}{\sinh(\vartheta-\vartheta_a^\epsilon)}\\ +\int_{\mathbb{R}}\frac{d\vartheta'}{4\pi}\,\frac{1}{\cosh(\vartheta-\vartheta')}\,\partial_{\vartheta'}\log\big(1+Y_p^\epsilon(\vartheta')\big)\,,\end{aligned} \tag{8.3}$$

The nonlinear integral equations in question have an almost universal form,

$$\begin{aligned}\log Y_p^\epsilon(\vartheta) \;=-mRe^{\epsilon\vartheta}+\sum_{a=1}^{M^\epsilon}\log S(\vartheta-\vartheta_a^\epsilon-i\tfrac{\pi}{2})\\ +\int_{\mathbb{R}}\frac{d\vartheta'}{4\pi}\,\sigma(\vartheta-\vartheta')\log(1+Y_p^\epsilon(\vartheta'))\,,\end{aligned} \tag{8.4}$$

where

$$\sigma(\vartheta)\;=\;\frac{d}{d\vartheta}S(\vartheta)\;=\;\frac{4\sin\vartheta_0\cosh\vartheta}{\cosh 2\vartheta-\cos 2\vartheta_0}\,.$$

It is possible to prove that for arbitrary given input data $\mathbf{t}^\epsilon=(\vartheta_1^\epsilon\dots,\vartheta_{M^\epsilon}^\epsilon)$, $\vartheta_a^\epsilon\in\mathbb{R}$ the nonlinear integral equations (8.4) have a unique solution $Y_{p,\mathbf{t}}^\epsilon(\vartheta)$ which grows for $\vartheta\to-\epsilon\infty$ as $2\pi\epsilon\, ip\,\vartheta$.[5] The equations (8.4) have to be supplemented by the set of equations

$$\begin{aligned}2\pi\epsilon\, k_a^\epsilon \;=\; \epsilon\, mRe^{\epsilon\vartheta_a^\epsilon}+\sum_{\substack{b=1\\ b\neq a}}^{M^\epsilon}\arg S(\vartheta_a^\epsilon-\vartheta_b^\epsilon)\\ +\int_{\mathbb{R}}\frac{d\vartheta}{4\pi}\,\tau(\vartheta_a^\epsilon-\vartheta)\log(1+Y_{p,\mathbf{t}}^\epsilon(\vartheta))\,,\end{aligned} \tag{8.5}$$

where

$$\tau(\vartheta)\;\equiv\;\frac{4\sin\vartheta_0\sinh\vartheta}{\cosh 2\vartheta+\cos 2\vartheta_0}=i\sigma(\vartheta+i\tfrac{\pi}{2})\,. \tag{8.6}$$

The equations (8.5) represent strong constraints on the parameters $\mathbf{t}^\epsilon$. The fact that these parameters can only be real can be proven by means of an argument similar to the one of [YY, T08a] using the fact that the functions $Y_p^\epsilon(\vartheta)$ have to be real. This in turn follows from the hermiticity of the Q-operators observed above. In the following we shall adopt the basic conjecture that there exists a unique solution to the equations (8.5) for any given tuples $\mathbf{k}^\epsilon=(k_1^\epsilon,\dots,k_{M^\epsilon}^\epsilon)$. If so, we can conclude that eigenstates are uniquely labelled by p and the tuples $\mathbf{k}^\epsilon$.

[5] Bear in mind that we assume $p\in i\mathbb{R}$.

8.2 Analytic properties of the q-functions for the continuum theories

The integral equations characterizing the q-functions of the continuum theories are equivalent to either of the following two functional equations,

$$t^\epsilon(u)q^\epsilon(u) \;=\; q^\epsilon(u+ib)+q^\epsilon(u-ib)\,, \tag{8.7}$$

$$q^\epsilon(v+i\eta)q^\epsilon(v-i\eta)-q^\epsilon(v+i\delta)q^\epsilon(v-i\delta) \;=\; 1\,. \tag{8.8}$$

We observe no difference between the massive and the massless cases.

The analytic properties of the q-functions also simplify in the continuum limit. We find:

$$\begin{aligned}&\text{(i)}\;\; \text{The q-functions are entire analytic in } u \text{ for each of the cases considered.}\\ &\text{(ii)}\;\; \text{The q-functions } q_p^\epsilon(u) \text{ are entire analytic in } p \text{ for Liouville and KdV theory.}\end{aligned} \tag{8.9}$$

Important differences appear on the level of the asymptotic properties, as we shall now discuss. In the massive case we find [T08a] rapid decay of $q^\epsilon(u)$ at both ends of the real axis, more precisely,

$$\log q^\epsilon(u) \underset{\mathrm{Re}(u)\to\pm\infty}{\sim} -\frac{mR}{2\sin\vartheta_0}e^{\frac{\pi}{2\eta}|u|} \quad \text{for} \;\; |\mathrm{Im}(u)|<\eta\,. \tag{8.10}$$

The decay of $q^\epsilon(u)$ implies that the spectrum of the Sinh-Gordon field theory is purely discrete.

As in the case of the lattice theory, the main difference to the massless case is the appearance of oscillatory asymptotics at one end of the real axis, while it remains rapidly decaying at the other end,

$$\left.\begin{aligned} q_p^\epsilon(u) &\underset{\mathrm{Re}(u)\to-\epsilon\infty}{\sim} \frac{\cos(2\pi pu+\epsilon\theta_q(p))}{\sqrt{\sinh(2\pi bp)\sinh(2\pi b^{-1}p)}}\\ \log q_p^\epsilon(u) &\underset{\mathrm{Re}(u)\to\epsilon\infty}{\sim} -\frac{mR}{2\sin\vartheta_0}e^{\frac{\pi}{2\eta}|u|}\end{aligned}\right\} \quad \text{for} \;\; |\mathrm{Im}(u)|<\eta\,. \tag{8.11}$$

One may formulate the above statements about the asymptotics of the q-functions $q^\epsilon(u)$ for $u\to\epsilon\infty$ more precisely by saying that there exists an asymptotic expansion of the form

$$\log q_p^\epsilon(u) \;\sim\; -c_0\,e^{\frac{\pi}{2\eta}|u|}-\sum_{n=1}^{\infty}c_n\,\mathbb{I}_n^\epsilon\,e^{-\frac{\pi}{2\eta}(2n-1)|u|}\,. \tag{8.12}$$

For the classical continuum field theories it is well-known that the coefficients $\mathbb{I}_n^\epsilon$ represent the *local* conserved quantitites of the model in question. The coefficients $\mathbb{I}_1^\epsilon$ correspond to the light-cone Hamiltonians which are proportional to the generators L_0, $\bar{L}_0$ of the Virasoro algebra in the massless cases. For these cases it can be shown [T08a] that we have the following formula for the expectation values of $\mathbb{I}_n^\epsilon$ in a state characterized by $p\in\mathbb{R}$ and tuples $\mathbf{k}^\epsilon$:

$$\mathbb{I}_1^\epsilon \;=\; \frac{2\pi}{R}\bigg(P^2-\frac{1}{24}+\sum_{a\in\mathbb{K}}k_a^\epsilon\bigg)\,. \tag{8.13}$$

We clearly identify the zero-mode contribution $\propto p^2$ and integer-valued oscillator contributions k_a^ϵ. We therefore reproduced already a good part of the expected structure of the spectrum of the continuum Liouville theory [CT82].

8.3 Explicit calculation of the reflection amplitude

The reflection amplitude S_{q_p} introduced in Subsection 7.2.1 represents an important piece of data characterizing Liouville theory. We are now going to explain how to calculate this quantity for the class of states related to the primary states of the Liouville conformal field theory. The key observation underlying this calculation is equation (7.15) which relates the reflection amplitude to the asymptotics of the functions q_p^ϵ of KdV theory. These asymptotics were found in [T08a] based on [FL06]. To round off the picture, we will now briefly recall how this works.

Let us first observe, as can be seen e.g. from formula (8.13), that the states with $M^\epsilon = 0$, $\epsilon = \pm$, correspond to the Fock-vacua in the sectors labelled by p. According to (7.15), we may calculate $R_p \equiv S_{q_p}$ if we know the asymptotic behavior of the q-functions $q_p^\epsilon(u)$ corresponding to the Fock-vacua. These q-functions $q_p^\epsilon(u)$ can be characterized as the unique solutions of the functional equations (8.7), (8.8) which have the analytic properties (8.9), the asymptotic behavior (8.11), and the additional property to be non-vanishing within the strip $\mathbb{S}_u$. It was shown in [FL06] that a solution to this set of conditions is given by the Wronskian of certain solutions to the ordinary differential equation

$$\left[-\frac{d^2}{dx^2} - \frac{4}{b^2}p^2 + \kappa^2\big(e^{2x} + e^{-2x/b^2}\big)\right]\Psi = 0\,. \tag{8.14}$$

This generalizes similar results for other models which go back to [DT99, BLZ3]. In order to get $q_p^\epsilon(u)$, consider the solutions $\Psi_\pm$ to (8.14) which have the asymptotic behavior

$$\begin{aligned} \Psi_+ \;&\sim\; \frac{1}{\sqrt{2\kappa}}\exp\Big(\frac{x}{2b^2} - \kappa b^2 e^{-x/b^2}\Big) \quad \text{for} \quad x\to-\infty\,,\\ \Psi_- \;&\sim\; \frac{1}{\sqrt{2\kappa}}\exp\Big(-\frac{x}{2} - \kappa e^{x}\Big) \quad \text{for} \quad x\to+\infty\,, \end{aligned} \tag{8.15}$$

respectively. The functions $q_p^\epsilon(\vartheta)$ are then simply given as

$$q_p^+(u) \;\equiv\; q_p^-(-u) \;\equiv\; \Psi_+\frac{d}{dx}\Psi_- - \Psi_-\frac{d}{dx}\Psi_+\,, \tag{8.16}$$

provided that we identify the respective parameters as follows,[6]

$$\kappa \;=\; -\frac{\kappa_0}{2\sin\vartheta_0}\,\frac{mR}{2}\,e^{\frac{\pi}{2\eta}u}, \quad \kappa_0 \;=\; -\frac{2\sqrt{\pi}}{\Gamma\big(-\frac{1}{2(1+b^2)}\big)\Gamma\big(1-\frac{b^2}{2(1+b^2)}\big)}\,. \tag{8.17}$$

[6]Concerning the comparison with [FL06] note that the parameter n used there is related to b^2 via $n = 2/b^2$.

The characterization (8.16) of $q_p^\epsilon(u)$ in terms of the ODE (8.14) allowed the authors of [FL06] to determine the asymptotics of $q_p^\epsilon(u)$. The explicit expression for $S_p = e^{2i\theta(p)}$ which follows from formula (177) in [FL06] is

$$S_p \,=\, -\rho^{-8i\delta p}\,\frac{\Gamma(1+2ibp)\Gamma(1+2ib^{-1}p)}{\Gamma(1-2ibp)\Gamma(1-2ib^{-1}p)}\,, \tag{8.18}$$

in which we have used the abbreviation

$$\rho \,\equiv\, \frac{R}{2\pi}\,\frac{m}{4\sqrt{\pi}}\,\Gamma\bigg(\frac{1}{2+2b^2}\bigg)\Gamma\bigg(1+\frac{b^2}{2+2b^2}\bigg)\,. \tag{8.19}$$

We recover the expression proposed in [ZZ96], for which a full derivation was given in [T04]. We'd like to stress how different the present derivation of the reflection amplitude – based on the integrable structure of Liouville theory – is compared to the one in [ZZ96, T04], which was based on the conformal symmetry. It would be very interesting further elucidate the interplay between the integrable and the conformal structure of Liouville theory.

A. Asymptotic behavior of Q-operators

Let us first note that the Q-operators for Liouville theory and for the KdV model have the same asymptotic behavior. To this aim let us consider the eigenvalue equation in the form

$$\langle\, q\,|\,\mathsf{Q}^\epsilon_{\rm Liou}(u)\,|\,t\,\rangle \,=\, q^\epsilon(u)\,\langle\, q\,|\,t\,\rangle\,, \tag{A.1}$$

where $\langle\, q\,|$ is a generalized eigenstate of $\mathsf{Q}^\epsilon_{\rm Liou}(u)$ with eigenvalue $q^\epsilon(u)$, and $|\,t\,\rangle$ is a test function from a suitable dense subspace $\mathcal{T}$ of $\mathcal{H}$ like those defined in [BT06]. The left hand side of (A.1) can be represented as

$$\int d\mathbf{x}'d\mathbf{x}\;\langle\, q'\,|\,\mathbf{x}'\,\rangle\,\langle\,\mathbf{x}'\,|\,\mathsf{Y}^\epsilon_{\rm Liou}(u)\,|\,\mathbf{x}\,\rangle\,, \tag{A.2}$$

where $\langle\, q'\,| \equiv \langle\, q\,|\,\mathsf{Y}_\infty^{-1}$. Following [BT06, Section 4.2.] it is not hard to see that the bulk of the domain of integration over $\mathbf{x}'$, $\mathbf{x}$ gives contributions which decay exponentially when $|u|\to\infty$. One may observe, however, that the integration over $\mathbf{x}'$ may receive contributions from the region in the integration over $\mathbf{x}'$ where $x_r = y_r - \delta$, $\delta\to\infty$. This is due to the fact that the wave-function $\langle\, q'\,|\,\mathbf{x}'\,\rangle$ has plane-wave like behavior w.r.t. the zero mode $x_0 = \sum_{n=1}^{\rm N} x_n$ in this limit. A look at the formula (A.4) for the kernel $\langle\,\mathbf{x}'\,|\,\mathsf{Y}^\epsilon_{\rm Liou}(u)\,|\,\mathbf{x}\,\rangle$ then reveals that it becomes equal to the kernel $\langle\,\mathbf{x}'\,|\,\mathsf{Y}^\epsilon_{\rm KdV}(u)\,|\,\mathbf{x}\,\rangle$ for large δ. This observation reduces the problem to find the asymptotic behavior of $\mathsf{Q}^\epsilon_{\rm Liou}(u)$ to the corresponding problem for $\mathsf{Q}^\epsilon_{\rm KdV}(u)$.

To solve this problem, an alternative integral operator representation will be useful. In order to find it, let us consider a variant of the Q-operators defined as

$$\tilde{\mathsf{Q}}^+(u) \,=\, (\mathsf{Q}^+(s_+))^{-1}\cdot\mathsf{Q}^+(u)\,,\qquad \tilde{\mathsf{Q}}^-(u) \,=\, (\mathsf{Q}^-(s_-))^{-1}\cdot\mathsf{Q}^-(u)\,. \tag{A.3}$$

One advantage of the Q-operators $\tilde{\mathsf{Q}}^+(u)$ and $\tilde{\mathsf{Q}}^-(u)$ is the fact that the kernels representing these operators can be written in an even more explicit form,

$$\langle\,\mathbf{x}'\,|\,\tilde{\mathsf{Q}}^+(u)\,|\,\mathbf{x}\,\rangle \;=\; \prod_{n=1}^{\mathrm{N}} W^-_{2s+i\eta}(x'_n+x'_{n+1})\overline{W}_{u-s}(x'_n-x_r)W^+_{u+s}(x'_{n-1}+x_n)\,, \tag{A.4}$$

$$\langle\,\mathbf{x}'\,|\,\tilde{\mathsf{Q}}^-(u)\,|\,\mathbf{x}\,\rangle \;=\; \prod_{n=1}^{\mathrm{N}} W^-_{u-s}(x'_{n-1}+x_n)\overline{W}_{u+s}(x'_n-x_n)W^+_{i\eta-2s}(x_n+x_{n+1})\,, \tag{A.5}$$

Let $\langle\,\mathbf{t}\,|$, $\mathbf{t}\;=\;(t_1,\ldots,t_{\mathrm{N}})$ now be the generalized eigenstates of the operators u_n such that $\langle\,\mathbf{t}\,|\,\mathsf{u}_n\;=\;\langle\,\mathbf{t}\,|\,e^{\pi b t_n}$. By means of straightforward computations it is possible to show that

$$\begin{aligned}\langle\,\mathbf{t}'\,|\,\tilde{\mathsf{Q}}^+_{\mathrm{KdV}}(u)\,|\,\mathbf{t}\,\rangle\; &= \delta(p-p')\,E_s\,e^{-\frac{\pi i}{2}\mathrm{N}u^2}\,e^{-2\pi i\tau_r t_r} \\ &\times\int_{\mathbb{R}} dx\;e^{4\pi i p x}\;\prod_{n=1}^{N}\varphi(w+x+\tau_n)\varphi(w-x-\tau_n)\,,\end{aligned} \tag{A.6}$$

where E_s is a constant, and we have used the notation $2p\equiv\sum_{s=1}^{\mathrm{N}}t_s$ and $\tau_r\equiv\sum_{s=1}^{r-1}(t'_s-t_s)$. We are now in the position to prove that

$$\tilde{\mathsf{Q}}^+_{\mathrm{KdV}}(u)\underset{\substack{u\to-\infty\\ \mathrm{Im}(u)=\mathrm{const}}}{\sim} E_s\,e^{-\frac{\pi i}{2}\mathrm{N}u^2}\left(e^{2\pi i\mathsf{p}_\circ(u-s)}\mathsf{A}^+_+ + e^{-2\pi i\mathsf{p}_\circ(u-s)}\mathsf{A}^+_-\right), \tag{A.7}$$

where $\mathsf{A}^+_\pm$ are operators represented by the kernels

$$\langle\,\mathbf{t}'\,|\,\mathsf{A}^+_\pm\,|\,\mathbf{t}\,\rangle\; = \delta(p-p')\,e^{-2\pi i\tau_r t_r}\int_{\mathbb{R}} dy\;e^{\mp 4\pi i p y}\;\prod_{r=1}^{N}\varphi\big(y\mp\tau_r+\tfrac{i}{2}\eta\big)\,, \tag{A.8}$$

respectively. Indeed, it is easy to see that the dominant contributions to the asymptotics $u\to\infty$ come from the region in the integration over x where $|x|\;\sim\;u$. In order to isolate the contributions from $x\pm u\;=\;\mathcal{O}(1)$, respectively, let us change the variable of integration to $y^\epsilon\;=\;\frac{u-s}{2}\mp x$. Taking into account that $\varphi(x)\;\sim\;1$ for $x\;\to\;\infty$ it becomes easy to verify our claim.

References

[Ba73] R. Baxter, *Eight-vertex model in lattice statistics and one-dimensional anisotropic heisenberg chain. I. Some fundamental eigenvectors* Annals of Physics **76** (1973) 1-24

[BLZ1] V.V. Bazhanov, S.L. Lukyanov, A.B.Zamolodchikov, *Integrable Structure of Conformal Field Theory*, I: Commun.Math.Phys. **177** (1996) 381-398, II: Commun.Math.Phys. **190** (1997) 247-278, III: Commun.Math.Phys. **200** (1999) 297-324.

[BLZ2] V.V. Bazhanov, S.L. Lukyanov, A.B.Zamolodchikov, *Integrable quantum field theories in finite volume: Excited state energies* Nucl. Phys. **B489** (1997) 487-531

[BLZ3] V.V.Bazhanov, S.L.Lukyanov, A.B.Zamolodchikov, *Spectral determinants for Schroedinger equation and Q-operators of Conformal Field Theory*, J. Statist. Phys. **102** (2001) 567-576

[BMS] V.V. Bazhanov, V.V. Mangazeev, S.M. Sergeev, *Faddeev-Volkov solution of the Yang-Baxter equation and discrete conformal symmetry*, Nucl. Phys. **B784** (2007) 234-258

[Ba08] V.V. Bazhanov, *Chiral Potts model and the discrete Sine-Gordon model at roots of unity*, Preprint arXiv:hep-th/0809.2351

[BS90] V.V. Bazhanov, Yu. G. Stroganov, *Chiral Potts model as a descendant of the six-vertex model*, Journal of Statistical Physics **59** (1990) 799-817

[BPZ] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, *Infinite conformal symmetry in 2D quantum field theory*, Nucl. Phys. **B241** (1984) 333-380

[BT06] A. Bytsko, J. Teschner, *Quantization of models with non-compact quantum group symmetry. Modular XXZ magnet and lattice sinh-Gordon model*, J. Phys. **A39** (2006) 12927-12981

[BT09] In preparation

[CT82] T. Curtright, C. Thorn, *Conformally Invariant Quantization of the Liouville Theory*, Phys. Rev. Lett. **48** (1982) 1309-1313

[De99] S.E. Derkachov, *Baxter's Q-operator for the homogeneous XXX spin chain*, J.Phys. **A32** (1999) 5299-5316

[DGZ] P. Di Francesco, P. Ginsparg, J. Zinn-Justin, 2*D gravity and random matrices.* Phys. Rep. **254** (1995), no. 1-2, 133 pp

[DT99] P. Dorey, R. Tateo *Anharmonic oscillators, the thermodynamic Bethe ansatz, and non-linear integral equations*, J. Phys. **A32** (1999) L419-L425

[DO92] H. Dorn, H.-J. Otto, *On Correlation Functions for Non-critical Strings with* $c < 1$ *but* $d > 1$, Phys.Lett. **B291** (1992) 39-43, and: *Two and three-point functions in Liouville theory*, Nucl.Phys. **B429** (1994) 375-388

[FT86] L.D. Faddeev, L.A. Takhtajan, *Liouville model on the lattice.* Field theory, quantum gravity and strings (Meudon/Paris, 1984/1985), 166–179, Lecture Notes in Phys., 246, Springer, Berlin, 1986

[FV93] L.D. Faddeev, A. Yu. Volkov, *Abelian current algebra and the Virasoro algebra on the lattice*, Phys. Lett. **B315** (1993) 311-318

[FV94] L.D. Faddeev, A. Yu. Volkov, *Hirota Equation as an Example of an Integrable Symplectic Map*, Letters in Mathematical Physics **32** (1994) 125-135

[F95] L.D. Faddeev, *Discrete Heisenberg-Weil group and modular group* Lett. Math. Phys. **34** (1995) 249-254

[FKV] L.D. Faddeev, R.M. Kashaev, A. Yu. Volkov, *Strongly coupled quantum discrete Liouville theory. I: Algebraic approach and duality*, Comm. Math. Phys. **219** (2001) 199-219

[FK02] L.D. Faddeev, R.M. Kashaev, *Strongly Coupled Quantum Discrete Liouville Theory. II: Geometric Interpretation of the Evolution Operator*, J. Phys. **A35** (2002) 4043-4048

[FST] L.D. Faddeev, E.K. Sklyanin, L.A. Takhtajan, *Quantum inverse problem method: I* Theor. Math. Phys. **57** (1980) 688-706

[FT95] L.D. Faddeev, O. Tirkkonen, *Connections of the Liouville model and XXZ spin chain*, Nucl. Phys. **B453** (1995) 647-669

[FL06] V.A. Fateev, S.L. Lukyanov, *Boundary RG Flow Associated with the AKNS Soliton Hierarchy*, J. Phys. **A39** (2006) 12889-12926

[GM] G. Moore, P. Ginsparg, *Lectures on 2D gravity and 2D string theory*, Proceedings of the TASI Summer School 1992, eds. J. Harvey and J. Polchinski, World Scientific, Singapore

[GN84] J.-L. Gervais, A. Neveu, *Novel triangle relation and absence of tachyons in Liouville string field theory* Nuclear Physics **B238** (1984) 125-141

[Ge85] J.-L. Gervais, *Transport matrices associated with the Virasoro algebra*, Phys. Lett. **B160** (1985) 279-282

[IK82] A.G. Izergin, V.E. Korepin, *Lattice versions of quantum field theory models in two dimensions* Nuclear Physics **B205** (1982) 401-413

[Ka00] R.M. Kashaev: *The quantum dilogarithm and Dehn twists in quantum Teichmüller theory*, In: Integrable structures of exactly solvable two–dimensional models of quantum field theory (Kiev, 2000), 211–221 (NATO Sci.Ser.II Math.Phys.Chem., **35**, Kluwer Acad. Publ., Dordrecht, 2001)

[Ka01] R.M. Kashaev, *The Non-Compact Quantum Dilogarithm and the Baxter Equations*, Journal of Statistical Physics **102** (2001) 923-936

[LZ97] S. Lukyanov, A.B. Zamolodchikov, *Exakt expectation values of exponential fields in the Sine-Gordon model*, Nucl. Phys. **B 493** (1997) 571-587

[PG92] V. Pasquier, M. Gaudin, *The periodic Toda chain and a matrix generalization of the Bessel function*, J. Phys. **A25** (1992) 5243-5252

[PT99] B. Ponsot, J. Teschner, *Liouville bootstrap via harmonic analysis on a noncompact quantum group*, Preprint arXiv:hep-th/9911110

[Sk83] E.K. Sklyanin, *Some algebraic structures connected with the Yang-Baxter equation. Representations of quantum algebras* Functional Analysis and Its Applications **17** (1983) 273-284

[Sk85] E.K. Sklyanin, *The quantum Toda chain*, Lect. Notes Phys. **226** (1985) 196–233

[Sk92] E.K. Sklyanin, *Quantum inverse scattering method. Selected topics*. In: *Quantum groups and quantum integrable systems* (World Scientific, 1992) 63–97

[Sk95] E.K. Sklyanin, *Separation of variables – new trends*, Prog. Theor. Phys. Suppl. **118** (1995) 35–60

[TT06] L.A. Takhtajan, L.-P. Teo, *Quantum Liouville theory in the background field formalism I. Compact Riemann surfaces*, Commun. Math. Phys. **268** (2006) 135-197

[T01] J. Teschner, *Liouville theory revisited*, Class. Quant. Grav. **18** (2001) R153-R222

[T04] J. Teschner, *A lecture on the Liouville vertex operators*, Int. J. Mod. Phys. **A19S2** (2004) 436-458

[T07] J. Teschner, *From Liouville theory to the quantum geometry of Riemann surfaces.* Prospects in mathematical physics, 231–246, Contemp. Math., 437, Amer. Math. Soc., Providence, RI, 2007

[T08a] J. Teschner, *On the spectrum of the Sinh-Gordon model in finite volume*, Nucl.Phys.**B799** (2008) 403-429

[T08b] J. Teschner, *Nonrational conformal field theory*, Preprint arXiv:hep-th/0803.0919

[Vo92] A. Volkov, *Quantum Volterra model*, Phys. Lett. **A167** (1992) 345-355

[Vo97] A. Volkov, *Quantum Lattice KdV equation*, Lett. Math. Phys. **39** (1997) 313-329

[Vo05] A.Yu. Volkov, *Noncommutative hypergeometry*, Comm. Math. Phys. **258** (2005) 257–273

[YY] C.N. Yang, C.P. Yang, *Thermodynamics of a One-Dimensional System of Bosons with Repulsive Delta-Function Interaction* J. Math. Phys. **10** (1969) 1115-1122

[Za90] Al.B.Zamolodchikov, *Thermodynamic Bethe ansatz in relativistic models: Scaling 3-state potts and Lee-Yang models* Nucl. Phys. **B342** (1990) 695-720

[Za06] Al.B.Zamolodchikov, *On the Thermodynamic Bethe Ansatz Equation in Sinh-Gordon Model*, J.Phys. **A39** (2006) 12863-12887

[ZZ96] A.B.Zamolodchikov, Al.B.Zamolodchikov, *Structure Constants and Conformal Bootstrap in Liouville Field Theory*, Nucl. Phys. **B477** (1996) 577-605