

Threshold resummation in SCET vs. perturbative QCD: an analytic comparison

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Abstract:

We compare threshold resummation in QCD, as performed using soft-collinear effective theory (SCET) in the Becher-Neubert approach, to the standard perturbative QCD formalism based on factorization and resummation of Mellin moments of partonic cross-sections. We consider various forms of the SCET result, which correspond to different choices of the soft scale μ_s that characterizes this approach. We derive a master formula that relates the SCET resummation to the QCD result for any choice of μ_s . We then use it first, to show that if SCET resummation is performed in N -Mellin moment space by suitable choice of μ_s it is equivalent to the standard perturbative approach. Next, we show that if SCET resummation is performed by choosing for μ_s a partonic momentum variable, the perturbative result for partonic resummed cross-sections is again reproduced, but like its standard perturbative counterpart it is beset by divergent behaviour at the endpoint. Finally, using the master formula we show that when μ_s is chosen as a hadronic momentum variable the SCET and standard approach are related through a multiplicative (convolutive) factor, which contains the dependence on the Landau pole and associated divergence. This factor depends on the luminosity in a non-universal way; it lowers by one power of log the accuracy of the resummed result, but it is otherwise subleading if one assumes the luminosity not to contain logarithmically enhanced terms. Therefore, the SCET approach can be turned into a prescription to remove the Landau pole from the perturbative result, but the price to pay for this is the reduction by one logarithmic power of the accuracy at each order and the need to make assumptions on the parton luminosity.

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1 Threshold resummation and the Landau pole

The interest in the resummation of logarithmically enhanced contributions due to soft gluon radiation in perturbative QCD (threshold resummation, henceforth) has been recently revived due to its relevance for many LHC processes, such as Higgs [1] or top [2] production. Threshold resummation was originally performed (to next-to-leading logarithmic accuracy) by factorizing the hadronic cross-section in Mellin space in terms of a luminosity and a partonic cross-section, and then exponentiating logarithmically enhanced corrections to the latter to all orders through eikonal [3] or factorization [4] techniques. Subsequent derivations and generalizations to all logarithmic orders were obtained, among others, from a suitable two-scale generalized factorization theorem [5] and through renormalization-group improvement of the kinematics of the gluon radiation phase-space [6], with an additional hypothesis of factorization of virtual corrections.

In all these approaches, resummation is performed after Mellin transformation of the hadronic cross-section, which factorizes it into the product of a parton luminosity and a partonic cross-section. More importantly, in Mellin space the partonic cross-section in the soft limit can be obtained by exponentiating single-particle emission cross-sections, thanks to the fact that in Mellin space the n -particle longitudinal phase space factorizes. The large logs which are resummed are then logarithms of N , the variable which is Mellin conjugate to τ (a dimensionless ratio which equals one at threshold), rather than the original $\ln(1 - \tau)$.

More recently, factorization and exponentiation were directly performed at the level of Feynman diagrams, without the need for a Mellin transformation, using path-integral methods to separate off soft gluon modes [7, 8]. In the latter approach the standard resummed results are readily recovered, but the way the terms which dominate in the eikonal limit emerge order-by-order in perturbation theory (and the next-to-eikonal corrections to them) is particularly transparent. Indeed, an important use of resummed results is to provide predictions for higher order terms which can even be used to construct approximate expressions for unknown fixed-order corrections (see e.g. Ref. [9]).

However, regardless of how resummation is proven, resummed expressions for partonic cross-sections with a fixed logarithmic accuracy in momentum space (i.e. next ^{k} -to-leading $\ln(1 - z)$, where z is a partonic scaling variable) turn out to be ill-defined: they lead to divergent hadronic cross-sections upon convolution with a parton luminosity [6, 10]. This behaviour is already present at the fixed-coupling level [10], and it persists when the coupling runs [6]. It can be traced [10] to the fact that the truncation of resummed results to any finite logarithmic accuracy in momentum space induces terms which violate longitudinal momentum conservation, thereby leading to factorial divergence of the perturbative expansion: the result is well-defined provided only the truncation to finite logarithmic accuracy is performed in Mellin space (i.e., next ^{k} -to-leading $\ln N$, rather than next ^{k} -to-leading $\ln(1 - z)$), and the Mellin transform is inverted exactly to power accuracy, i.e. retaining terms to all logarithmic orders in $1 - z$ and only neglecting terms which are down by powers of $1 - z$ [6]. As a consequence, perturbative QCD resummation, even if derived using a momentum-space argument, must be performed in Mellin space (to finite logarithmic accuracy) if it is to respect momentum conservation, and to lead to finite physical (hadronic) cross-sections.

At the running-coupling level, however, a new difficulty arises: namely, it turns out

that the next^k-to-leading $\ln N$ series of contributions to the partonic cross-section at any finite logarithmic accuracy, viewed as a series in the strong coupling α_s , corresponds, upon inverse Mellin transformation, to a divergent series of contributions to the partonic cross-section. This divergence can be traced to the Landau pole in the strong coupling: as long known [11], resummed results correspond to effectively replacing the hard scale M^2 at which the strong coupling is evaluated with a scale $M^2(1-z)^a$ related to the soft-gluon radiation process (with a a process-dependent exponent, e.g. $a = 2$ for Drell-Yan). Because the hadronic observable is found by convoluting the partonic cross-section with a luminosity, the integration over parton momenta always intercepts the region $z \rightarrow 1$ where the strong coupling blows up, and this manifests itself as a divergence of the expansion in powers of $\alpha_s(M^2)$. This divergence, which is of non-perturbative origin, can be removed by addition of subleading terms: within the commonly used “minimal prescription” of Ref. [10] this is done by choosing a particular integration path to perform the Mellin inversion integral, which corresponds to adding a term which is more suppressed than any power of $1/M^2$, while with the more recent “Borel prescription” [12, 13] this is done by adding a higher twist term to make the divergent series Borel summable.

An alternative approach to resummation can be based on the soft-collinear effective field theory (SCET) [14–17], which provides [18] an alternative derivation of QCD factorization: threshold resummation based on SCET was performed in Refs. [19–23]. This approach provides a powerful alternative way of determining resummed results for hadronic observables, which can then be used for phenomenology through the standard Mellin-space formalism of Ref. [10]. However, it was pointed out in Ref. [23] that, thanks to the fact that the effective theory deals with the hadronic degrees of freedom, in a SCET approach resummed expressions can be directly derived in terms of the hadronic kinematic variable, i.e., in practice, SCET allows one to perform the resummation of $\ln(1-\tau)$, where τ is a measurable dimensionless kinematic ratio. The advantage is that the divergences related to the need to integrate over the parton kinematics are no longer present: hence, in particular, the difficulties related to the Landau pole of the strong coupling disappear. The approach of Ref. [23] has been subsequently developed for phenomenology, and applied to various physical processes, such as deep-inelastic scattering [24], Drell-Yan [25] and Higgs [26] production.

Henceforth, for brevity, we will refer to the approach of Ref. [23] as SCET approach, and that of Refs. [3–6] as QCD approach. It should be observed, however, that whereas the QCD implementation of resummation is unique to any finite perturbative order, the aforementioned [19–23] alternative implementations of threshold resummation in SCET, to the best of our understanding, lead to results which differ (possibly by subleading terms) even when truncated to finite perturbative order. Here we will concentrate on the SCET approach of Ref. [23], which has been widely used in particular for phenomenological applications.

However, results obtained in the approach of Ref. [23] are not easily compared to those obtained using the standard approach of Refs. [3–6], because the direct connection to factorization and resummation at the level of partonic cross-sections is lost. Indeed, as mentioned, the presence of the Landau pole implies that the expansion in powers of $\alpha_s(M^2)$ of the resummed partonic cross-section diverges. Hence, if the resummed SCET result of Ref. [23] is free of divergences, its expansion to fixed order must necessarily differ from that of the standard Mellin-space resummation.

This difference has never been determined so far: its computation is the goal of this paper. Clearly, its knowledge is crucial in order to determine the theoretical and phenomenological viability of the SCET resummation of Ref. [23]. Some phenomenological comparisons of resummed predictions for relevant physical processes obtained using SCET to standard perturbative results have been performed in Refs. [24–26]. Differences are found to be reasonably small: however, this does not shed light on their analytic form. But knowledge of this analytic form is necessary if we wish to know, first, whether up to the stated accuracy the SCET and QCD approaches are equivalent, and second, even if they are, what is the kind of subleading suppression of the terms introduced in the SCET approach to tame the perturbative divergence, i.e. what is the accuracy of the SCET approach (be it power or logarithmic).

The answer to these questions is presented here in several steps. In Section 2, after summarizing the known form of resummed results both in the perturbative QCD and SCET approach, we recall that the definition of next^k-to-leading log accuracy in the SCET approach of Ref. [23] and in the standard perturbative QCD approach are different, and only agree at the leading log level. Beyond the leading log, SCET results are always less accurate by one power of log than the perturbative ones: so N^kLL in the perturbative case always include terms which only appear in the N^{k+1}LL SCET result, and so forth. In order to proceed to a comparison, it is necessary to discuss the dependence of the SCET resummation on the soft scale: in Section 3 we summarize how SCET results in Mellin space, or in momentum space, at either the partonic or hadronic level can be obtained by different choices of soft scale. A comparison is then made possible through the derivation of a general relation between the SCET result and the standard result, by expressing the latter in terms of the convolution of the former with a function C_r which depends on the soft scale. We establish this result at next-to-next-to-leading logarithmic order (but we conjecture it to hold to all orders): it provides a master formula which enables a full comparison of the QCD and SCET results, both from an analytic and a numerical point of view.

As a preliminary step, this master formula can be used to prove the fact that if SCET resummation is performed in Mellin space it is completely equivalent to the standard approach, and in particular it has the same logarithmic accuracy at each order. This result was established already in Refs. [24, 25], but with the aforementioned lower log accuracy of the SCET results. This is done in Section 4, where we also digress to show that if SCET resummation is performed in momentum space by choosing a partonic scaling variable z , it coincides with the perturbative result up to power suppressed corrections, but, like the perturbative result, it diverges at the partonic endpoint $z = 1$. We can then (in Section 5) tackle the computation of the function C_r which relates the SCET and perturbative resummation when the soft scale is chosen as a measurable hadronic scale. In this case, the SCET result is free of Landau pole, and thus the divergence is entirely contained in the C_r function. This function depends on the PDF luminosity in a non-universal way, and thus whether or not it is subleading depends on the form of the PDF. In particular, if one assumes that the luminosity does not contain logarithmically enhanced terms, then we can show that this function is always logarithmically subleading, provided only the less accurate SCET definition of logarithmic accuracy is used. However, any logarithmically enhanced contribution to the parton luminosity $\mathcal{L}(x)$ proportional to $\ln^k(1-x)$, with $k \geq 1$, will lead to contributions to C_r which are of the same order as

those induced by perturbative resummation.

Therefore, we conclude that it is only for a particular class of luminosities that SCET with a hadronic choice of soft scale reproduces the perturbative result, and can thus be considered to be equivalent to the standard approach and to provide an alternative prescription to remove the divergence of the perturbative expansion. Even when this is the case, the momentum-space SCET resummation prescription of Ref. [23] requires lowering by one order (one power of log) the accuracy of the resummed result at each logarithmic order. Furthermore, in the SCET prescription, terms which are introduced in order to remove the perturbative divergence are only logarithmically subleading, rather than being power suppressed (as in the Borel prescription) or exponentially suppressed (as in the minimal prescription), along with power suppressed terms. Finally, subleading terms which are induced by SCET resummation are suppressed by powers or logs of the hadronic scale: this feature of SCET resummation may also be a limitation, because the partonic and hadronic scales, though related, do not coincide, and in fact it may well be that the former is close to threshold while the latter is not [27].

2 Threshold resummation at fixed logarithmic accuracy

For definiteness, we will concentrate on the production of Drell-Yan pairs at hadron colliders. This choice does not entail loss of generality, and the extension to other processes is straightforward. We will consider in particular the invariant mass distribution $\frac{d\sigma_{\text{DY}}}{dM^2}$, with M the invariant mass of the pair. We define the hadronic scaling variable

$$\tau = \frac{M^2}{s} \quad (2.1)$$

where s is the hadronic center-of-mass energy squared, so the threshold limit is $\tau \rightarrow 1$. Perturbative QCD factorization takes the form

$$\sigma(\tau, M^2) = \int_{\tau}^1 \frac{dz}{z} C(z, M^2) \mathcal{L}\left(\frac{\tau}{z}\right), \quad (2.2)$$

where \mathcal{L} is the parton luminosity, and $\sigma(\tau, M^2)$ is a dimensionless cross-section

$$\sigma(\tau, M^2) = \frac{1}{\tau\sigma_0} \frac{d\sigma_{\text{DY}}}{dM^2} \quad (2.3)$$

defined by requiring that at the Born level (i.e. at order α_s^0) $C(z, M^2) = \delta(1-z)$. Note that Eq. (2.2) is a schematic expression: in general, a sum over different parton subprocesses must be included. In the sequel, without significant loss of generality, we shall always choose the renormalization and factorization scales equal to each other and to the physical hard scale $\mu_F^2 = \mu_R^2 = M^2$.

2.1 Perturbative QCD: resummation in N space

As discussed in Section 1, standard QCD resummation is more conveniently performed by taking a Mellin transform

$$\sigma(N, M^2) = \int_0^1 d\tau \tau^{N-1} \sigma(\tau, M^2) = C(N, M^2) \mathcal{L}(N) \quad (2.4)$$

which factorizes both the convolution Eq. (2.2) and the gluon radiation phase space. In Eq. (2.4) by slight abuse of notation we denote with $C(N, M^2)$ and $\mathcal{L}(N)$ the Mellin transforms of $C(z, M^2)$ and $\mathcal{L}(z)$ respectively.

The N -space resummed coefficient function has the form

$$C_{\text{QCD}}(N, M^2) = \bar{g}_0(\alpha_s) \exp \bar{\mathcal{S}} \left(M^2, \frac{M^2}{N^2} \right) \quad (2.5)$$

where

$$\bar{\mathcal{S}} \left(M^2, \frac{M^2}{N^2} \right) = \int_0^1 dz z^{N-1} \left[\frac{1}{1-z} \int_{M^2}^{M^2(1-z)^2} \frac{d\mu^2}{\mu^2} 2A(\alpha_s(\mu^2)) + D(\alpha_s([1-z]^2 M^2)) \right]_+ \quad (2.6)$$

The functions $\bar{g}_0(\alpha_s)$, $A(\alpha_s)$ and $D(\alpha_s)$ are given as power series in α_s , with $\bar{g}_0(0) = 1$ and $A(0) = D(0) = 0$; $A(\alpha_s)$ is order by order the coefficient of the soft singularity in the Altarelli-Parisi splitting function for the relevant partonic subprocess, while the functions $D(\alpha_s)$ and $\bar{g}_0(\alpha_s)$ are process-dependent. Specifically, in the case of Drell-Yan production initiated by quark-antiquark collisions, the relevant Altarelli-Parisi splitting function is

$$P_{qq}(\alpha_s, x) = \frac{A(\alpha_s)}{(1-x)_+} [1 + O(1-x)]. \quad (2.7)$$

As a result, the resummed coefficient function takes the form (using the notation of Ref. [10])

$$C_{\text{QCD}}(N, M^2) = g_0(\alpha_s) \exp \mathcal{S}(\bar{\alpha}L, \bar{\alpha}), \quad (2.8)$$

$$\mathcal{S}(\bar{\alpha}L, \bar{\alpha}) = \frac{1}{\bar{\alpha}} g_1(\bar{\alpha}L) + g_2(\bar{\alpha}L) + \bar{\alpha} g_3(\bar{\alpha}L) + \bar{\alpha}^2 g_4(\bar{\alpha}L) + \dots, \quad (2.9)$$

$$\bar{\alpha} \equiv 2\alpha_s(M^2)\beta_0, \quad L \equiv \ln \frac{1}{N}, \quad (2.10)$$

where β_0 is the first coefficient of the QCD β function, defined as

$$\mu^2 \frac{d\alpha_s(\mu^2)}{d\mu^2} = -\beta_0 \alpha_s^2(\mu^2) + \mathcal{O}(\alpha_s^3); \quad \beta_0 = \frac{11C_A - 2n_f}{12\pi} \quad (2.11)$$

and the functions g_i are of order $g_1(\bar{\alpha}L) = \mathcal{O}(\alpha_s^2)$ and $g_i(\bar{\alpha}L) = \mathcal{O}(\alpha_s)$ for $i > 1$, and are straightforwardly obtained performing the integrals in Eq. (2.6), and thus each determined by a finite number of coefficients in the expansion of the functions A and D . Note that the functions g_0 and \mathcal{S} do not coincide with \bar{g}_0 and $\bar{\mathcal{S}}$ of Eq. (2.5), because, by definition, \mathcal{S} unlike $\bar{\mathcal{S}}$ does not contain terms which are not logarithmically enhanced, while g_0 includes non-logarithmic contributions both from \bar{g}_0 itself, and from the integral Eq. (2.6).

The standard perturbative QCD resummation predicts correctly all contributions to $\ln C_{\text{QCD}}(N, M^2)$ up to a given order: in other words, if contributions up to g_n are included in $\mathcal{S}(\bar{\alpha}L, \bar{\alpha})$ Eq. (2.9) then $\ln C_{\text{QCD}}(N, M^2)$ is determined up to subleading corrections of order $\mathcal{O}(\alpha_s^{k+(n-1)} L^k)$. This is standardly called $N^{n-1}\text{LL}$ resummation. However, once the exponential is expanded out in order to obtain the coefficient function $C_{\text{QCD}}(N, M^2)$, at each order in α_s , only a restricted number of logarithmically enhanced terms is predicted correctly, and furthermore, inclusion of the prefactor $g_0(\alpha_s)$ Eq. (2.8) (which is not

logarithmically enhanced) is mandatory in order to improve the accuracy beyond NLL. In fact its inclusion already at the NLL level increases the number of contributions to the coefficient function which are predicted correctly. In Tab. 1 we summarize the order up to which the expansion of the functions Eq. (2.8,2.9) should be included to achieve a given logarithmic accuracy, and, in the last column, the order of the contributions to the resummed coefficient function $C_{\text{QCD}}(N, M^2)$ which, as a consequence, are predicted correctly.

log approx.	g_i up to	g_0 up to order	accuracy: $\alpha_s^n L^k$
LL	$i = 1$	$(\alpha_s)^0$	$k = 2n$
NLL	$i = 2$	$(\alpha_s)^1$	$2n - 2 \leq k \leq 2n$
NNLL	$i = 3$	$(\alpha_s)^2$	$2n - 4 \leq k \leq 2n$

Table 1: Orders of logarithmic approximations and accuracy of the predicted logarithms in perturbative QCD.

2.2 The SCET approach

Resummation in SCET in the approach of Ref. [23], which henceforth we will refer to as SCET resummation for short, is directly given in the physical space of momentum fractions. The relevant expression for Drell-Yan pair production has been computed in Ref. [25], and it is given by

$$C_{\text{SCET}}(z, M^2, \mu_s^2) = H(M^2)U(M^2, \mu_s^2)S(z, M^2, \mu_s^2) \quad (2.12)$$

where $H(M^2)$, the so-called hard function, has an expansion in powers of α_s computed at the hard scale M^2 ;

$$S(z, M^2, \mu_s^2) = \tilde{s}_{\text{DY}} \left(\ln \frac{M^2}{\mu_s^2} + \frac{\partial}{\partial \eta}, \mu_s \right) \frac{1}{1-z} \left(\frac{1-z}{\sqrt{z}} \right)^{2\eta} \frac{e^{-2\gamma\eta}}{\Gamma(2\eta)}, \quad (2.13)$$

where

$$\eta = \int_{M^2}^{\mu_s^2} \frac{d\mu^2}{\mu^2} \Gamma_{\text{cusp}}(\alpha_s(\mu^2)); \quad \Gamma_{\text{cusp}}(\alpha_s) = A(\alpha_s) \quad (2.14)$$

and $\tilde{s}_{\text{DY}}(L, \mu)$ has a perturbative expansion in powers of $\alpha_s(\mu^2)$. Note that the function $\Gamma_{\text{cusp}}(\alpha_s)$ coincides with the function $A(\alpha_s)$ of Eq. (2.6). Finally,

$$U(M^2, \mu_s^2) = \exp \left\{ - \int_{M^2}^{\mu_s^2} \frac{d\mu^2}{\mu^2} \left[\Gamma_{\text{cusp}}(\alpha_s(\mu^2)) \ln \frac{\mu^2}{M^2} - \gamma_W(\alpha_s(\mu^2)) \right] \right\} \quad (2.15)$$

where $\gamma_W(\alpha_s)$ has a power expansion in α_s . The resummed expression as given in Ref. [25] actually depends on several energy scales, which here for simplicity are all taken to be equal to the hard scale M^2 .

Two important formal aspects characterize the SCET resummed result. The first is that it depends on a “soft scale” μ_s , and in fact the logs which are being resummed in

SCET are $\ln \frac{\mu_s}{M}$. Hence, different choices of soft scale lead to different forms of the SCET resummation, as we shall discuss in greater detail in the next Section.

The second is related to the well-known fact that at the endpoint $z = 1$ the coefficient function $C_{\text{SCET}}(z, M^2, \mu_s^2)$ is a distribution, rather than an ordinary function. This distribution is usually expressed in terms of the so-called plus distribution $\frac{1}{(1-z)_+}$. The distributional nature of the SCET result emerges in the following way. The convolution product of $C_{\text{SCET}}(z, M^2, \mu_s^2)$ with any well-behaved test function of z is well defined as long as η is a fixed, positive number: the factor $(1-z)^{2\eta}$ acts as a regulator of the soft singularity at $z = 1$. The result can then be analytically continued to negative values of η (which is typically the case in DY-like processes) by means of the identity

$$\int_0^1 dz (1-z)^{2\eta-1} f(z) = \int_0^1 dz (1-z)^{2\eta-1} [f(z) - f(1)] + \frac{1}{2\eta} f(1). \quad (2.16)$$

Eq. (2.16) defines a distribution on a space of test functions $f(z)$, regular in the range $0 \leq z \leq 1$, which is usually written

$$(1-z)^{2\eta-1} = [(1-z)^{2\eta-1}]_+ + \frac{1}{2\eta} \delta(1-z). \quad (2.17)$$

It is important to note that η is of order α_s : therefore, the term proportional to $\delta(1-z)$ in Eq. (2.17) combines with the factor $1/\Gamma(2\eta) = 2\eta + \mathcal{O}(\eta^2)$ in Eq. (2.12) to form an order- α_s^0 contribution (with the correct kinematic structure).

As in the perturbative case, a given logarithmic accuracy is obtained by including a finite number of terms in the perturbative expansion of the functions which determine the resummed result, namely Γ_{cusp} , γ_W , H and \bar{s}_{DY} . The accuracy which, according to Ref. [25], is obtained by including in the SCET expression Eq. (2.12-2.15) coefficients up to a given order, as well as the corresponding nomenclature, are summarized in Tab. 2. As in the case of Tab. 1, the last column gives the order of the contributions to C_{SCET} which are predicted exactly. As mentioned in Section 1 and as is apparent comparing Tab. 1 to Tab. 2, beyond LL the SCET results are always less accurate than the QCD results of the same name: the QCD NLL includes terms of order $\alpha_s^n \ln^k \frac{\mu_s}{M}$ with $k \geq 2n - 2$, but the SCET NLL only includes terms with $k \geq 2n - 1$. This was already observed in Ref. [28].

RG-impr. PT	log. approx.	Γ_{cusp}	γ_W	H, \bar{s}_{DY}	accuracy: $\alpha_s^n \ln^k \frac{\mu_s}{M}$
—	LL	1-loop	tree-level	tree-level	$k = 2n$
LO	NLL	2-loop	1-loop	tree-level	$2n - 1 \leq k \leq 2n$
NLO	NNLL	3-loop	2-loop	1-loop	$2n - 3 \leq k \leq 2n$
NNLO	NNNLL	4-loop	3-loop	2-loop	$2n - 5 \leq k \leq 2n$

Table 2: Different approximation schemes for the evaluation of the resummed cross-section formulae in the SCET approach.

When comparing the two different definitions of logarithmic accuracy, Tab. 1 and Tab. 2, one should distinguish a purely terminological issue and an issue of substance. The terminological issue is how each given accuracy is called: this is clearly immaterial.

The issue of substance is whether at (say) NLL the SCET expression Eq. (2.12-2.15) may be upgraded to the higher accuracy of the NLL QCD expression Eq. (2.5-2.6) (without having to resort to the yet more accurate NNLL SCET expression), and likewise at all subsequent logarithmic orders. We will show that the answer to this question depends on the choice of soft scale μ_s .¹

3 Choice of the soft scale and SCET-QCD comparison

In the standard perturbative QCD approach to soft resummation, the energy scale which characterizes soft gluon emission is of the order of $M(1-z)$: when the observed final state carries away a fraction z of the available partonic energy, the energy available for unobserved radiation is $M(1-z)$, which is much smaller than M if z is close to 1. The fact that the scale involved is partonic has phenomenological implications: because the partonic center-of-mass energy is always smaller than the hadronic one, threshold resummation may be relevant even for processes which are relatively far from hadronic threshold, provided the parton luminosity is peaked for small values of the momentum fraction [27]. This indeed is known to happen for Higgs production in gluon fusion at the LHC. [30, 31]

In SCET resummation, however, one resums logs of the large ratio M/μ_s of the hard scale M to the soft scale μ_s , and various choices for the soft scale μ_s are possible: in particular, the choice which has been advocated in Refs. [23–26], and which removes the problem of the Landau pole, consists of choosing for μ_s a scale which characterizes the (hadronic) physical process.

If μ_s is chosen as a function of the partonic scaling variable z , then the resummed SCET partonic cross-section $C_{\text{SCET}}(z, M^2, \mu_s^2)$ Eq. (2.12) can be directly compared to the momentum-space perturbative QCD expression, which may be obtained by determining the inverse Mellin transform $C_{\text{QCD}}(z, M^2)$ of the resummed N -space expression Eq. (2.5). We will study this case in detail in the next Section. However, if μ_s is chosen as a function of the hadronic scaling variable τ , the SCET and perturbative QCD resummed results must be compared at the level of physical cross-sections $\sigma_{\text{QCD}}(\tau, M^2)$ and $\sigma_{\text{SCET}}(\tau, M^2)$, which are respectively obtained substituting $C_{\text{QCD}}(z, M^2)$ or $C_{\text{SCET}}(z, M^2, \mu_s^2)$ in the factorized expression Eq. (2.2), with some particular choice of soft scale μ_s .

It is important to understand that these different choices of soft scale lead to resummed predictions with different analytic structure. To see this, note that if the soft scale only depends on the parton momentum fraction z , then Eq. (2.2) is a convolution, in the sense that upon Mellin transformation it factorizes according to Eq. (2.4). This factorization is of course a necessary and sufficient condition for parton radiation to respect longitudinal momentum conservation. But if in Eq. (2.2) the coefficient function depends on τ through the soft scale, then the convolution structure is destroyed. This means that with this particular choice of soft scale, upon Mellin transformation the cross-section no longer factorizes, thereby violating longitudinal momentum conservation. This also violates the QCD factorization theorem, because the short-distance partonic cross section depends on long-distance physics through the hadronic variable τ . The possibility of making this

¹In other contexts, such as for example the resummation of jet veto logs [29], SCET results which correspond either of two different accuracies, respectively akin to Tab. 2 or Tab. 1, may be achieved by suitable choices of terms to be included in the resummed expression.

choice of soft scale, and indeed the very possibility of making alternative choices of soft scale, some of which preserve factorization and some of which do not, appears puzzling in a standard perturbative QCD approach. We will not attempt to address the issue of principle of understanding this apparent structural discrepancy between SCET and QCD results. Rather, we will take the SCET and QCD expressions at face value: our aim will be to determine how they are related to each other.

We will now derive a master formula which relates the SCET resummed expression for generic choice of the soft scale to the standard perturbative QCD expression. For definiteness, we specialize to the next-to-next-to-leading log case, but all relevant structures are already present at this order so generalization to higher logarithmic orders is straightforward. First, we give the explicit expression of the QCD result Eq. (2.5) to this order. Then, we give the SCET expression Eq. (2.12-2.15) to the same order, and we perform its (exact) Mellin transform in order to allow for a comparison with the QCD expression, which is given in N space. Finally, by comparing the two expressions we derive a master formula which relates them, as a function of the soft scale μ_s , through a suitable factor (in Mellin space) or a convolutive function (in momentum space).

3.1 Perturbative QCD resummation to NNLL

The NNLL resummed expression in perturbative QCD is given by Eq. (2.5) with [9, 32] (see also Ref. [27])

$$A(\alpha_s) = \frac{A_1}{4}\alpha_s + \frac{A_2}{16}\alpha_s^2 + \frac{A_3}{64}\alpha_s^3 + \mathcal{O}(\alpha_s^4), \quad (3.1)$$

$$A_1 = \frac{4C_F}{\pi}; \quad (3.2)$$

$$D(\alpha_s) = D_1\alpha_s + D_2\alpha_s^2 + \mathcal{O}(\alpha_s^3), \quad (3.3)$$

$$D_1 = 0, \quad D_2 = \frac{C_F}{16\pi^2} \left[C_A \left(-\frac{1616}{27} + \frac{88}{9}\pi^2 + 56\zeta_3 \right) + \left(\frac{224}{27} - \frac{16}{9}\pi^2 \right) n_f \right]. \quad (3.4)$$

We can perform the z integral in Eq. (2.5) using Eq. (A.4):

$$\begin{aligned} \bar{\mathcal{S}}_{\text{QCD}} \left(M^2, \frac{M^2}{\bar{N}^2} \right) &= \int_{M^2}^{M^2/\bar{N}^2} \frac{d\mu^2}{\mu^2} \left[A(\alpha_s(\mu^2)) \left(\ln \frac{1}{\bar{N}^2} - \ln \frac{\mu^2}{M^2} \right) + \frac{1}{2} D(\alpha_s(\mu^2)) \right] \\ &\quad + \frac{\pi^2}{12} \frac{d^2}{dL^2} \int_{M^2}^{M^2/\bar{N}^2} \frac{d\mu^2}{\mu^2} \left[A(\alpha_s(\mu^2)) \left(\ln \frac{1}{\bar{N}^2} - \ln \frac{\mu^2}{M^2} \right) + \frac{1}{2} D(\alpha_s(\mu^2)) \right] \\ &= \int_{M^2}^{M^2/\bar{N}^2} \frac{d\mu^2}{\mu^2} \left[A(\alpha_s(\mu^2)) \left(\ln \frac{1}{\bar{N}^2} - \ln \frac{\mu^2}{M^2} \right) + \frac{1}{2} D(\alpha_s(\mu^2)) \right] \\ &\quad + \frac{C_F\pi}{3} \alpha_s \left(\frac{M^2}{\bar{N}^2} \right), \end{aligned} \quad (3.5)$$

where (as per Eq. (A.5)) $\bar{N} = Ne^\gamma$. We have neglected subleading ($N^3\text{LL}$) terms (including the replacement $N \rightarrow \bar{N}$ in the argument of α_s in the last term) and we have brought all integrals to a common form using

$$\int_0^{1-1/N} \frac{dz}{1-z} 2 \int_{M^2}^{M^2(1-z)^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) = - \int_{M^2}^{M^2/N^2} \frac{d\nu^2}{\nu^2} \int_{M^2}^{\nu^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2))$$

$$= - \int_{M^2}^{M^2/N^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) \left(\ln \frac{1}{N^2} - \ln \frac{\mu^2}{M^2} \right) \quad (3.6)$$

$$\int_0^{1-1/N} \frac{dz}{1-z} D(\alpha_s(M^2(1-z)^2)) = -\frac{1}{2} \int_{M^2}^{M^2/N^2} \frac{d\mu^2}{\mu^2} D(\alpha_s(\mu^2)). \quad (3.7)$$

In order to ease the subsequent comparison to the SCET result, we separate off the non-logarithmic constant from the last term in Eq. (3.5):

$$\begin{aligned} \bar{\mathcal{S}}_{\text{QCD}} \left(M^2, \frac{M^2}{N^2} \right) &= \int_{M^2}^{M^2/N^2} \frac{d\mu^2}{\mu^2} \left[A(\alpha_s(\mu^2)) \left(\ln \frac{1}{N^2} - \ln \frac{\mu^2}{M^2} \right) + \hat{D}_2 \alpha_s^2(\mu^2) \right] \\ &\quad + \frac{C_F \pi}{3} \alpha_s(M^2) \end{aligned} \quad (3.8)$$

where

$$\hat{D}_2 = \frac{D_2}{2} - \frac{C_F \pi}{3} \beta_0 = \frac{C_F}{16\pi^2} \left[C_A \left(-\frac{808}{27} + 28\zeta_3 \right) + \frac{112}{27} n_f \right]. \quad (3.9)$$

We can thus write

$$C_{\text{QCD}}(N, M^2) = \hat{g}_0(\alpha_s(M^2)) \exp \hat{\mathcal{S}}_{\text{QCD}} \left(M^2, \frac{M^2}{N^2} \right), \quad (3.10)$$

where

$$\hat{g}_0(\alpha_s) = 1 + \hat{g}_{01} \alpha_s + \mathcal{O}(\alpha_s^2); \quad (3.11)$$

$$\hat{\mathcal{S}}_{\text{QCD}} \left(M^2, \frac{M^2}{N^2} \right) = \int_{M^2}^{M^2/N^2} \frac{d\mu^2}{\mu^2} \left[A(\alpha_s(\mu^2)) \left(\ln \frac{1}{N^2} - \ln \frac{\mu^2}{M^2} \right) + \hat{D}_2 \alpha_s^2(\mu^2) \right]. \quad (3.12)$$

Note that \hat{g}_0 and $\hat{\mathcal{S}}$ cannot be identified with g_0 and \mathcal{S} in Eq. (2.5), because the integral in Eq. (3.12) does contain some terms which are not logarithmically enhanced:

$$\frac{A_1}{4} \int_{M^2}^{M^2/N^2} \frac{d\mu^2}{\mu^2} \alpha_s(\mu^2) \left(\ln \frac{1}{N^2} - \ln \frac{\mu^2}{M^2} \right) = \frac{2C_F}{\pi} \gamma^2 \alpha_s(M^2) + \text{logarithms} + \mathcal{O}(\alpha_s^2), \quad (3.13)$$

so that

$$\hat{g}_{01} = g_{01} - \frac{2C_F}{\pi} \gamma^2. \quad (3.14)$$

However, the form Eq. (3.12) of the exponent in the QCD result is especially suited for comparison to the SCET result, as we now show.

3.2 SCET resummation to NNLL

We turn to the SCET expression, which is given by Eq. (2.12) with, to NNLL

$$\gamma_W(\alpha_s) = \gamma_W^{(2)} \frac{\alpha_s^2}{16\pi^2} + \mathcal{O}(\alpha_s^3), \quad (3.15)$$

$$\gamma_W^{(2)} = C_F C_A \left(-\frac{808}{27} + \frac{11\pi^2}{9} + 28\zeta_3 \right) + C_F T_F n_f \left(\frac{224}{27} - \frac{4\pi^2}{9} \right). \quad (3.16)$$

In order to compare it to the perturbative QCD result, we perform a Mellin transform with respect to z . This is easy to do, because the z dependence is all contained in the soft function $S(z, M^2, \mu_s^2)$, whose Mellin transform is

$$\begin{aligned} \mathcal{M} [S(z, M^2, \mu_s^2)] &= \bar{s}_{\text{DY}} \left(\ln \frac{M^2}{\mu_s^2} + \frac{\partial}{\partial \eta}, \mu_s \right) \frac{\Gamma(N - \eta) \Gamma(2\eta) e^{-2\gamma\eta}}{\Gamma(N + \eta) \Gamma(2\eta)} \\ &= \left[1 + \frac{C_F}{2\pi} \alpha_s(\mu_s^2) \left(\ln^2 \frac{M^2}{\mu_s^2 \bar{N}^2} + \frac{\pi^2}{6} \right) \right] \bar{N}^{-2\eta} + \mathcal{O} \left(\frac{1}{N} \right). \end{aligned} \quad (3.17)$$

It follows that the Mellin transform of the coefficient function Eq. (2.12) is

$$\begin{aligned} C_{\text{SCET}}(N, M^2, \mu_s^2) &= H(M^2) \left[1 + \frac{C_F \pi}{12} \alpha_s(\mu_s^2) + \frac{C_F}{2\pi} \alpha_s(\mu_s^2) \ln^2 \frac{M^2}{\mu_s^2 \bar{N}^2} \right] \\ &\times \exp \int_{M^2}^{\mu_s^2} \frac{d\mu^2}{\mu^2} \left[\Gamma_{\text{cusp}}(\alpha_s(\mu^2)) \left(\ln \frac{1}{\bar{N}^2} - \ln \frac{\mu^2}{M^2} \right) + \frac{\gamma_W^{(2)}}{16\pi^2} \alpha_s^2(\mu^2) \right] + \mathcal{O} \left(\frac{1}{N} \right). \end{aligned} \quad (3.18)$$

It is very important to observe that the Mellin transform has been computed *at fixed* μ_s . This means that firstly, Eq. (3.18) is not the Mellin transform of the SCET expression when μ_s depends on z (which we will discuss in the next Section): in that case the Mellin transform would also act on the z dependence through μ_s . And second, that if μ_s depends on τ the cross-section $\sigma_{\text{SCET}}(\tau, M^2)$ computed using Eq. (2.2) does not factorize into the product of $C_{\text{SCET}}(N, M^2, \mu_s^2)$ Eq. (3.18) times a parton luminosity $\mathcal{L}(N)$ upon Mellin transformation: the Mellin integral over τ would also act on the τ dependence through μ_s which, as already noted, does not have the form of a convolution integral.

Equation (3.18) can be brought in a form which is suitable for comparison to the QCD expression by separating off the constant as in Eq. (3.8), thus leading to

$$C_{\text{SCET}}(N, M^2, \mu_s^2) = \hat{H}(M^2) E(N, M^2, \mu_s^2) \exp \hat{\mathcal{S}}_{\text{SCET}}(M^2, \mu_s^2), \quad (3.19)$$

with

$$\hat{H}(M^2) = H(M^2) \exp \left[\frac{C_F \pi}{12} \alpha_s(M^2) \right] = 1 + \alpha_s(M^2) \left(H_1 + \frac{C_F \pi}{12} \right) + \mathcal{O}(\alpha_s^2); \quad (3.20)$$

$$\hat{\mathcal{S}}_{\text{SCET}}(M^2, \mu_s^2) = \int_{M^2}^{\mu_s^2} \frac{d\mu^2}{\mu^2} \left[\Gamma_{\text{cusp}}(\alpha_s(\mu^2)) \left(\ln \frac{1}{\bar{N}^2} - \ln \frac{\mu^2}{M^2} \right) + \hat{\gamma}_W^{(2)} \alpha_s^2(\mu^2) \right], \quad (3.21)$$

$$\hat{\gamma}_W^{(2)} = \frac{\gamma_W^{(2)}}{16\pi^2} - \frac{C_F \pi}{12} \beta_0 = \frac{C_F}{16\pi^2} \left[C_A \left(-\frac{808}{27} + 28\zeta_3 \right) + \frac{112}{27} n_f \right]; \quad (3.22)$$

$$E(N, M^2, \mu_s^2) = 1 + \frac{C_F}{2\pi} \alpha_s(\mu_s^2) \left(\ln \frac{1}{\bar{N}^2} - \ln \frac{\mu_s^2}{M^2} \right)^2. \quad (3.23)$$

3.3 The master formula

The QCD expression Eqs. (3.10-3.12) and the SCET expression Eqs. (3.19-3.23) are easily related, by noting that, because $\Gamma_{\text{cusp}}(\alpha_s) = A(\alpha_s)$ and $\hat{D}_2 = \hat{\gamma}_W^{(2)}$, the integrands in Eqs. (3.12) and (3.21) coincide, so

$$\hat{\mathcal{S}}_{\text{SCET}}(M^2, Q^2) = \hat{\mathcal{S}}_{\text{QCD}}(M^2, Q^2) \equiv \hat{\mathcal{S}}(M^2, Q^2). \quad (3.24)$$

It follows that, splitting the integral as $\int_{M^2}^{M^2/\bar{N}^2} \frac{d\mu^2}{\mu^2} = \int_{M^2}^{\mu_s^2} \frac{d\mu^2}{\mu^2} + \int_{\mu_s^2}^{M^2/\bar{N}^2} \frac{d\mu^2}{\mu^2}$, we get

$$C_{\text{QCD}}(N, M^2) = C_r(N, M^2, \mu_s^2) C_{\text{SCET}}(N, M^2, \mu_s^2) \quad (3.25)$$

where

$$C_r(N, M^2, \mu_s^2) = \frac{\hat{g}_0(\alpha_s(M^2)) \exp \hat{\mathcal{S}}\left(\mu_s^2, \frac{M^2}{\bar{N}^2}\right)}{\hat{H}(M^2) E(N, M^2, \mu_s^2)}. \quad (3.26)$$

The non-logarithmic terms in fact cancel to the accuracy of our computation. Indeed, by substituting the value [32]

$$g_{01} = \frac{C_F}{\pi} (4\zeta_2 - 4 + 2\gamma^2) \quad (3.27)$$

in Eq. (3.14), and the value [25]

$$H_1 = \frac{C_F}{\pi} \left(\frac{7}{2} \zeta_2 - 4 \right) \quad (3.28)$$

in Eq. (3.20) we get

$$\frac{\hat{g}_0(\alpha_s(M^2))}{\hat{H}(M^2)} = 1 + \mathcal{O}(\alpha_s^2), \quad (3.29)$$

so deviations from unity are of the same order as the first contribution which, at NNLL accuracy, is not included in $H(M^2)$ (according to Tab. 2). The expression of C_r can be further simplified by including the function $E(N, M^2, \mu_s^2)$ Eq. (3.23) in the function $\hat{\mathcal{S}}$: indeed

$$\begin{aligned} E(N, M^2, \mu_s^2) &= \exp \left[\frac{A_1}{8} \alpha_s(\mu_s^2) \left(\ln \frac{1}{\bar{N}^2} - \ln \frac{\mu_s^2}{M^2} \right)^2 \right] + \mathcal{O}(\alpha_s^2) \\ &= \exp \int_{\mu_s^2}^{M^2/\bar{N}^2} \frac{d\mu^2}{\mu^2} \left[\frac{A_1}{4} \alpha_s(\mu^2) \left(\ln \frac{1}{\bar{N}^2} - \ln \frac{\mu^2}{M^2} \right) - \frac{A_1}{8} \beta(\alpha_s(\mu^2)) \left(\ln \frac{1}{\bar{N}^2} - \ln \frac{\mu^2}{M^2} \right)^2 \right]. \end{aligned} \quad (3.30)$$

Using Eq. (3.30) in the definition of $C_r(N, M^2, \mu_s^2)$ we obtain our final expression

$$C_r(N, M^2, \mu_s^2) = \exp \hat{\mathcal{S}}_r \left(\mu_s^2, \frac{M^2}{\bar{N}^2} \right), \quad (3.31)$$

with

$$\begin{aligned} \hat{\mathcal{S}}_r \left(\mu_s^2, \frac{M^2}{\bar{N}^2} \right) &= \int_{\mu_s^2}^{M^2/\bar{N}^2} \frac{d\mu^2}{\mu^2} \left[\left(A(\alpha_s(\mu^2)) - \frac{A_1 \alpha_s(\mu^2)}{4} \right) \left(\ln \frac{1}{\bar{N}^2} - \ln \frac{\mu^2}{M^2} \right) \right. \\ &\quad \left. + \frac{A_1}{8} \beta(\alpha_s(\mu^2)) \left(\ln \frac{1}{\bar{N}^2} - \ln \frac{\mu^2}{M^2} \right)^2 + \hat{D}_2 \alpha_s^2(\mu^2) \right]. \end{aligned} \quad (3.32)$$

Equation (3.25) together with the explicit expression Eqs. (3.31-3.32) of the function C_r provides the master formula which relates SCET and perturbative QCD resummation. It

is the main result of this paper. We note that no term of order α_s appears in the integrand of Eq. (3.32): indeed, the inclusion of the term $E(N, M^2, \mu_s^2)$ has the effect of removing the term proportional to $A_1 \alpha_s$ (see Eq. (3.1)). The remaining contributions to the integrand in Eq. (3.32) start at $\mathcal{O}(\alpha_s^2)$.

It is important to observe that while $C_{\text{QCD}}(N, M^2)$ does not admit a Mellin inverse, because it has a cut in the complex N plane starting at the value N_L at which the strong coupling blows up, $C_{\text{SCET}}(N, M^2, \mu_s^2)$ does admit a Mellin inverse as long as μ_s is kept fixed, because the argument of the strong coupling in the SCET expression does not depend on N . This means that if Eq. (3.25) is expanded in powers of $\alpha_s(M^2)$, and then the expansion is Mellin-inverted term by term, the expansion of the left-hand side is divergent, while on the right-side the Mellin inverse of the expansion of $C_{\text{SCET}}(N, M^2, \mu_s^2)$ converges to $C_{\text{SCET}}(z, M^2, \mu_s^2)$ Eq. (2.12). Therefore, the divergence has been isolated in the Mellin inverse of the expansion of the function $C_r(N, M^2, \mu_s^2)$ Eq. (3.26).

If the perturbative expansion of both sides of Eq. (3.25) in powers of $\alpha_s(M^2)$ is truncated to any finite order, then the Mellin inverse of both sides exists, and one gets the momentum-space relation

$$C_{\text{QCD}}(z, M^2) = \int_z^1 \frac{dy}{y} C_r\left(\frac{y}{z}, M^2, \mu_s^2\right) C_{\text{SCET}}(y, M^2, \mu_s^2), \quad (3.33)$$

where $C_{\text{SCET}}(z, M^2, \mu_s^2)$ is given by Eq. (2.12) (expanded out to the given order), while both $C_{\text{QCD}}(z, M^2)$ and $C_r(z, M^2, \mu_s^2)$ should be understood as the truncation to the given order of the Mellin inverse of the expansion of the corresponding N -space quantities. Equation (3.33) is then the momentum-space version of the master formula.

The master formula Eqs. (3.25,3.33) has been established at next-to-next-to-leading logarithmic order, defined according to Tab. 2. Note, however, that the accuracy is upgraded to the higher one of Tab. 1 if non-logarithmic terms cancel to $\mathcal{O}(\alpha_s^3)$, i.e. if the function $\hat{H}(M^2)$ in the SCET coefficient function Eq. (3.19) is replaced by a function $\bar{H}(M^2)$ such that

$$\frac{\hat{g}_0(\alpha_s(M^2))}{\bar{H}(M^2)} = 1 + \mathcal{O}(\alpha_s^3). \quad (3.34)$$

Of course, this can always be achieved by letting $\bar{H}(M^2) = \hat{H}(M^2) + \bar{H}_2 \alpha_s^2(M^2)$ and suitably choosing the value of \bar{H}_2 , while including the $\mathcal{O}(\alpha_s^2)$ to $g_0(\alpha_s(M^2))$, as per Tab. 1. (Whether $\bar{H}(M^2)$ coincides with the $\mathcal{O}(\alpha_s^2)$ expression of $\hat{H}(M^2)$ as obtained using SCET is an issue that we will not address here). We conclude that the master formula holds up to NNLL accuracy, defined as in Tab. 1. It is easy to convince oneself that this argument should hold to all logarithmic orders.

4 Perturbative QCD vs. SCET: partonic cross-sections

The master formula Eqs. (3.25-3.33) shows how SCET resummation can be used to reproduce standard results. Indeed, it immediately implies that if we fix the soft scale in terms of the Mellin-space variable,

$$\mu_s = \frac{M}{N}, \quad (4.1)$$

then $C_r(N, M^2, \mu_s^2) = 1$, i.e.

$$C_{\text{QCD}}(N, M^2) = C_{\text{SCET}}\left(N, M^2, \frac{M^2}{N^2}\right), \quad (4.2)$$

so the standard QCD result is reproduced: with this choice, SCET resummation is performed at the level of Mellin-space partonic cross-sections. Notice that because with this choice the SCET and QCD expressions coincide, they also have the same accuracy. So with this choice the SCET results actually has the accuracy of Tab. 1, rather than the lower accuracy of Tab. 2. The equivalence of Mellin-space SCET resummation to the QCD expressions was already established in Ref. [33]; it was also pointed out in Ref. [25], but with the lower accuracy of Tab. 2.

Alternatively, one may try to use SCET resummation for partonic cross-sections, but using the momentum-space SCET formula Eq. (2.12), with μ_s fixed as a momentum-space partonic scale, namely

$$\mu_s = M(1 - z). \quad (4.3)$$

This choice for instance was adopted recently in Ref. [34] to perform threshold resummation for top production. This choice also provides another way of re-deriving the standard perturbative resummation from SCET. Indeed, it can be shown that, away from the endpoint $z = 1$, all logarithmically enhanced terms $\frac{\ln^p(1-z)}{1-z}$ in the partonic cross-section are reproduced order by order with this choice.

This is very easily seen at the leading-log, fixed-coupling level. Indeed, in this limit one has

$$\eta = \frac{\alpha_s A_1}{2} \ln(1 - z), \quad (1 - z)^{2\eta} = \exp[\alpha_s A_1 \ln^2(1 - z)], \quad (4.4)$$

so that

$$\begin{aligned} C_{\text{SCET}}(z, M^2, M^2(1 - z)^2) &= \exp\left[-\frac{A_1 \alpha_s}{4} \int_{M^2}^{M^2(1-z)^2} \frac{d\mu^2}{\mu^2} \ln \frac{\mu^2}{M^2}\right] \frac{(1 - z)^{2\eta}}{1 - z} \frac{1}{\Gamma(2\eta)} \\ &= \exp\left[-\frac{A_1 \alpha_s}{2} \ln^2(1 - z)\right] \frac{(1 - z)^{2\eta}}{1 - z} \frac{1}{\Gamma(2\eta)}. \end{aligned} \quad (4.5)$$

But to leading log order one may expand $1/\Gamma(2\eta)$ to first order in α_s , so

$$C_{\text{SCET}}(z, M^2, M^2(1 - z)^2) = \alpha_s A_1 \frac{\ln(1 - z)}{1 - z} \exp\left[\frac{A_1 \alpha_s}{2} \ln^2(1 - z)\right] + \text{NLL}; \quad z \neq 1. \quad (4.6)$$

On the other hand, the perturbative result in the same approximation is the inverse Mellin transform of

$$C_{\text{QCD}}(N, M^2) = \exp\left[\frac{\alpha_s A_1}{2} \ln^2 \frac{1}{N}\right] + \text{NLL}, \quad (4.7)$$

i.e., using the results of Appendix B,

$$C_{\text{QCD}}(z, M^2) = \frac{1}{1 - z} \exp\left(\frac{\alpha_s A_1}{2} \frac{\partial^2}{\partial \xi^2}\right) \frac{(1 - z)^\xi}{\Gamma(\xi)} \Big|_{\xi=0} + \text{NLL}; \quad z \neq 1. \quad (4.8)$$

Expanding the exponential and keeping only leading log terms this is seen to coincide with Eq. (4.6).

However, as pointed out in Ref. [10] and discussed in Section 1, $C_{\text{QCD}}(z, M^2)$ (defined as the leading-log truncation of the inverse Mellin of Eq. (4.7)) is ill-defined at the endpoint $z = 1$: it behaves as a distribution which leads to a divergent integral upon convolution with any reasonably behaved luminosity, and, if expanded order by order in α_s , it diverges factorially. The SCET expression, either in the form of Eq. (4.5) or of Eq. (4.6), is also ill-defined as $z \rightarrow 1$. Indeed, because now η depends on z (see Eq. (4.4)), it is no longer possible to use Eq. (2.17) to regulate the behaviour of $C_{\text{SCET}}(z, M^2, \mu_s)$. Note that Eq. (2.17) also had the effect of generating the required $\mathcal{O}(\alpha_s^0)$ contribution to $C_{\text{SCET}}(z, M^2, M^2(1-z)^2)$ proportional to $\delta(1-z)$. Furthermore, as $z \rightarrow 1$ the coefficient function Eq. (4.5) oscillates with a factorially-growing amplitude, because of the factor $\frac{1}{\Gamma(2\eta)}$. The fact that the SCET resummed expression diverges at the partonic endpoint was already noticed in Ref. [34]. Because of these difficulties, we will not pursue further the choice Eq. (4.3) of soft scale.

5 Perturbative QCD vs. SCET in momentum space: hadronic cross-sections

We now turn finally to the choice of soft scale which is recommended in Refs. [23–26], specifically as a solution to the problem of the Landau pole, namely, a soft scale fixed in terms of the hadronic momentum scale²

$$\mu_s = M(1 - \tau). \quad (5.1)$$

With this choice of soft scale, the SCET and perturbative QCD results can only be compared at the level of hadronic cross-sections

$$\sigma_{\text{QCD}}(\tau, M^2) = \int_{\tau}^1 \frac{dz}{z} C_{\text{QCD}}(z, M^2) \mathcal{L}\left(\frac{\tau}{z}\right), \quad (5.2)$$

$$\sigma_{\text{SCET}}(\tau, M^2) = \int_{\tau}^1 \frac{dz}{z} C_{\text{SCET}}(z, M^2, \mu_s^2) \mathcal{L}\left(\frac{\tau}{z}\right). \quad (5.3)$$

Indeed, with the choice of soft scale Eq. (5.1) the resummed SCET cross-section Eq. (5.3) is no longer in the form of a convolution product, because the integrand depends on τ explicitly in the lower integration bound and in the argument of \mathcal{L} , but also implicitly through μ_s^2 . As a consequence, upon Mellin transformation with respect to τ , $\sigma_{\text{SCET}}(\tau, M^2)$, unlike the standard QCD result, does not factorize into a parton luminosity and a partonic cross-section.

Therefore, the comparison must be carried out directly at the level of hadronic cross-sections Eqs. (5.2-5.3), using the momentum-space form Eq. (3.33) of the master formula (always understood as a truncation to arbitrary but finite order in α_s , as discussed in the end of Section 3.3). This is somewhat problematic, because the power counting of Tabs. 1-2 was defined at the level of coefficient functions and thus necessarily at the level

²In Refs. [23–26] a slightly more general choice of soft scale is considered: namely, the soft scale Eq. (5.1) is generally rescaled by a function of τ which does not vanish at $\tau = 1$, and is chosen in such a way that the finite-order perturbative expansion of \tilde{s}_{DY} is reliable. Because this modification does not introduce any extra logarithmic enhancement, it does not affect our discussion, and we will not consider it.

of a partonic cross-section. Of course, it is possible to define a given logarithmic order at the level of SCET coefficient functions, then use this expression to compute the cross-section $\sigma_{\text{SCET}}(\tau, M^2)$ using Eq. (5.3). However, because this expression is not factorized, the question whether $\sigma_{\text{SCET}}(\tau, M^2)$ and $\sigma_{\text{QCD}}(\tau, M^2)$ agree at any given order can only be answered by comparing them directly, and counting logs of the hadronic scale $1 - \tau$. The result will then inevitably depend on the choice of parton distributions. The only alternative is to simply conclude that the SCET result with this choice cannot be compared to the perturbative one, and cannot be endowed with a perturbative meaning [35].

We will perform this comparison by computing the difference between $\sigma_{\text{SCET}}(\tau, M^2)$ and $\sigma_{\text{QCD}}(\tau, M^2)$ up to $\mathcal{O}(\alpha_s^2(M^2))$ and using the master formula to relate results. We will then discuss the structure of the result to all orders.

5.1 Fixed-order comparisons

We start by computing the function $C_r(N, M^2, \mu_s^2)$ Eq. (3.31) explicitly. Up to order α_s^2 we find

$$C_r(N, M^2, \mu_s^2) = 1 + \alpha_s^2(M^2) \left(-\frac{A_1}{3} \beta_0 \ln^3 \frac{c}{N} + \frac{A_2}{8} \ln^2 \frac{c}{N} + 2\hat{D}_2 \ln \frac{c}{N} \right) + \mathcal{O}(\alpha_s^3) \quad (5.4)$$

where

$$c = \frac{M e^{-\gamma}}{\mu_s}. \quad (5.5)$$

The corresponding momentum-space expression is readily obtained by performing the inverse Mellin transform of Eq. (5.4) with the help of Eq. (A.9):

$$C_r(z, M^2, \mu_s^2) = \delta(1 - z) + \alpha_s^2(M^2) \left(-\frac{A_1}{3} \beta_0 \frac{\partial^3}{\partial \xi^3} + \frac{A_2}{8} \frac{\partial^2}{\partial \xi^2} + 2\hat{D}_2 \frac{\partial}{\partial \xi} \right) c^\xi K(z, \xi) \Big|_{\xi=0} + \mathcal{O}(\alpha_s^3), \quad (5.6)$$

where the function

$$K(z, \xi) = \Delta(\xi) \ln^{\xi-1} \frac{1}{z} \quad (5.7)$$

plays the role of a generating function.

The difference between the resummed physical cross-sections in the QCD and SCET formalisms is now found substituting the explicit expression of C_r Eq. (5.6) in the master formula Eq. (3.33):

$$\sigma_{\text{QCD}}(\tau, M^2) = \sigma_{\text{SCET}}(\tau, M^2) + \alpha_s^2(M^2) \left(-\frac{A_1}{3} \beta_0 \frac{\partial^3}{\partial \xi^3} + \frac{A_2}{8} \frac{\partial^2}{\partial \xi^2} + 2\hat{D}_2 \frac{\partial}{\partial \xi} \right) c^\xi \Sigma(\tau, \xi) \Big|_{\xi=0} \quad (5.8)$$

where

$$\begin{aligned} \Sigma(\tau, \xi) &= \int_\tau^1 \frac{dz}{z} K(z, \xi) \sigma_{\text{SCET}}\left(\frac{\tau}{z}, M^2\right) \\ &= (1 - \tau)^\xi \Delta(\xi) \sum_{n=0}^{\infty} \frac{1}{n + \xi} \frac{1}{n!} (1 - \tau)^n \sigma_{\text{SCET}}^{(n)}(\tau, M^2); \end{aligned} \quad (5.9)$$

$$\sigma_{\text{SCET}}^{(n)}(\tau, M^2) = \frac{\partial^n}{\partial \tau^n} \sigma_{\text{SCET}}(\tau, M^2), \quad (5.10)$$

up to corrections suppressed by powers of $1 - \tau$, as shown in Appendix B.

Equation (5.8) provides the sought-for explicit comparison of the QCD and SCET results at the level of hadronic cross-sections. Note that the non-convolutive nature of the SCET result implies that the generating function for the correction term is now given by the function $\Sigma(\tau, \xi)$, which depends on the parton luminosity, rather than by the universal function $K(z, \xi)$ Eq. (5.7).

In order to understand the correction term in Eq. (5.8), we note that, with the choice of μ_s Eq. (5.1), we get

$$c^\xi \Sigma(\tau, \xi) = e^{-\gamma \xi} \Delta(\xi) \sum_{n=0}^{\infty} \frac{1}{n + \xi} \frac{1}{n!} (1 - \tau)^n \sigma_{\text{SCET}}^{(n)}(\tau, M^2), \quad (5.11)$$

so the dependence on $(1 - \tau)^\xi$ cancels. It follows that ξ derivatives acting on $c^\xi \Sigma(\tau, \xi)$ do not induce any extra logarithmic enhancement, other than that of $\sigma_{\text{SCET}}(\tau, M^2)$ itself:

$$\sigma_{\text{QCD}}(\tau, M^2) = \sigma_{\text{SCET}}(\tau, M^2) + \alpha_s^2(M^2) \sum_{n=0}^{\infty} \frac{C_n}{n!} (1 - \tau)^n \sigma_{\text{SCET}}^{(n)}(\tau, M^2), \quad (5.12)$$

where the constants C_n are τ -independent:

$$C_n = \left(-\frac{A_1}{3} \beta_0 \frac{\partial^3}{\partial \xi^3} + \frac{A_2}{8} \frac{\partial^2}{\partial \xi^2} + 2\hat{D}_2 \frac{\partial}{\partial \xi} \right) \frac{e^{-\gamma \xi} \Delta(\xi)}{n + \xi} \Big|_{\xi=0}; \quad (5.13)$$

$$C_0 = -\frac{2}{3} \zeta_3 A_1 \beta_0 - \frac{\pi^2}{48} A_2, \quad (5.14)$$

$$C_n = \frac{A_1 \beta_0}{n} \left(\frac{\pi^2}{6} - \frac{2}{n^2} \right) - \frac{A_2}{4n^2} + \frac{2\hat{D}_2}{n}, \quad n > 0. \quad (5.15)$$

Therefore, up to order α_s^2 , the correction term is just $\alpha_s^2(M^2)$ times a linear combination of derivatives of σ_{SCET} with respect to $\ln(1 - \tau)$. It follows that the correction term is at most of order

$$\alpha_s^2 \times \alpha_s^k \ln^{2k}(1 - \tau) \times \ln^p(1 - \tau) = \alpha_s^h \ln^{2h+p-4}(1 - \tau); \quad h \equiv k + 2, \quad (5.16)$$

where terms of order $\alpha_s^k \ln^{2k}(1 - \tau)$ are due to the coefficient functions, while terms of order $\ln^p(1 - \tau)$ are due to the parton luminosity.

In other words, at order α_s^n , terms $\ln^k(1 - \tau)$ in the SCET and QCD result coincide if $2n - 3 + p \leq k \leq 2n$. There are now various possibilities. If we simply neglect all logarithmic enhancements from the parton luminosity, i.e. if we set $p = 0$, then we conclude that the SCET and QCD results differ by terms which are NNLL according to the QCD counting Tab. 1, but N³LL correction according to the SCET counting Tab. 2. Hence we conclude that, neglecting logarithmic enhancements from the luminosity, the SCET result does reproduce the QCD result to NNLL accuracy, albeit with the less accurate SCET definition of what is meant by NNLL. However, if a logarithmic enhancement from the luminosity is present, this is no longer the case, and the discrepancy can become arbitrarily

large (i.e. even at the leading log level) by just increasing the value of p . Note that this in particular means that with this choice of soft scale it is not possible to upgrade the accuracy of the SCET expression, as given in Tab. 2, to that of the QCD expression, as given in Tab. 1, because at each logarithmic order the SCET expression differs from the perturbative QCD results by terms which, though consistent with the accuracy of Tab. 2, spoil the higher logarithmic accuracy of Tab. 1.

One may then ask whether logarithmic enhancements due to the PDF are expected to be present, and whether they should be counted. Because the P_{qq} and P_{gg} splitting functions behave as $P \sim \frac{1}{(1-x)_+}$ as $x \rightarrow 1$, contributions to all parton distributions $f(x)$ which are enhanced by $\ln(1-x)$ terms will always be induced by perturbative evolution. In a parton distribution evaluated at some reference scale Q_0 these terms will be accompanied by powers of $\alpha_s(Q_0^2)$, which is not small if the reference scale is taken as some low “initial” scale. Hence, in general, one does expect logarithmically enhanced contributions to PDFs, unless one wishes to make some fine-tuned assumption about the PDF itself, which can only hold at one single scale. The second question is whether these terms should be included or not in the power counting Eq. (5.16). This is a question which cannot be answered on the basis of first principles. Two relevant observations here are the following. First, once one substitutes any explicit expression of the parton luminosity in the expression Eq. (5.8) for the difference between the SCET and QCD result, there is no way to separate what comes from the luminosity and what comes from the coefficient function, because the SCET expression is not factorizable. Hence, in order to discard the luminosity logs from the power counting ones has to invoke the explicit SCET expression Eq. (2.12): in other words, one must argue that the SCET expression contains more information than that which is contained in the order-by-order perturbative result. The second observation is that in practice these correction terms may be parametrically large in realistic situations, and they may lead to significant discrepancies between the predictions obtained using $\sigma_{\text{SCET}}(\tau, M^2)$ or $\sigma_{\text{QCD}}(\tau, M^2)$.

5.2 All orders

The fixed $\mathcal{O}(\alpha_s^2)$ computation of Section 5.1 can be easily generalized to all orders. First of all, we note that the argument is based on the observation that to $\mathcal{O}(\alpha_s^2)$ the correction term Eq. (5.8) can be expressed as a series of derivatives of the function $c^\xi \Sigma(\tau, \xi)$ with respect to ξ , but these do not lead to an extra logarithmic enhancement beyond that which is already present in $\Sigma(\tau, \xi)$. The argument of Section 5.1 would thus hold, to NNLL but to all orders in α_s , provided only the correction term in Eq. (5.8) was a series of series of derivatives of the function $c^\xi \Sigma(\tau, \xi)$ with respect to ξ to all orders in α_s . This is true if and only if $\hat{\mathcal{S}}_r$ depends on μ_s only through powers of $\ln \frac{c}{N}$, with c given by Eq. (5.5).

Now, we observe that the generic term in $\hat{\mathcal{S}}_r$, Eq. (3.32), has the form

$$\int_{\mu_s^2}^{M^2/N^2} \frac{d\mu^2}{\mu^2} \alpha_s^n(\mu^2) \left(\ln \frac{1}{N^2} - \ln \frac{\mu^2}{M^2} \right)^m = \int_1^{c^2/N^2} \frac{dt}{t} \alpha_s^n(t\mu_s^2) \left(\ln \frac{c^2}{N^2} - \ln t \right)^m, \quad (5.17)$$

with $n \geq 2$ and $m = 0, 1, 2$. This is not a function of $\ln \frac{c}{N}$ only, because of the dependence of α_s on μ_s . In order to generalize the argument to all orders we must thus study this

dependence. Note that because

$$\alpha_s^n(t\mu_s^2) = \alpha_s^n(M^2) - n\beta_0\alpha_s^{n+1}(M^2)\ln\frac{t\mu_s^2}{M^2} + \mathcal{O}(\alpha_s^{n+2}) \quad (5.18)$$

terms in $\hat{\mathcal{S}}_r$ which are not a function of $\ln\frac{c}{N}$ only first appear at order $\alpha_s^3(M^2)$.

Furthermore,

$$\alpha_s^{n+1}(M^2)\ln\frac{\mu_s^2}{M^2}\int_1^{c^2/N^2}\frac{dt}{t}\left(\ln\frac{c^2}{N^2}-\ln t\right)^m = \frac{1}{m+1}\alpha_s^{n+1}(M^2)\ln\frac{\mu_s^2}{M^2}\ln^{m+1}\frac{c^2}{N^2}. \quad (5.19)$$

For $n = 2$, this term contributes to Eq. (5.8) an order- α_s^3 correction. This gives a series of extra contributions to the correction term, on top of those whose order was given in Eq. (5.16), which are at most of order

$$\alpha_s^3\ln(1-\tau)\times\alpha_s^k\ln^{2k}(1-\tau)\times\ln^p(1-\tau) = \alpha_s^h\ln^{2h-5+p}(1-\tau); \quad h \equiv k+3 \quad (5.20)$$

while higher-order terms are even more suppressed.

The power counting which ensues from Eq. (5.20) is the same as that of Section 5.1: neglecting logarithmic enhancements from the luminosity (i.e. if $p = 0$), the SCET result does reproduce the QCD result to NNLL accuracy, but with the less accurate SCET definition of what is meant by NNLL. If a logarithmic enhancement from the luminosity is included, the QCD and SCET results differ, with the discrepancy appearing at any desired logarithmic order (including leading log) if the enhancement of the luminosity is sufficiently strong.

It is interesting to observe that a different power counting might be appropriate in the phenomenologically relevant case in which the hadronic τ is actually far from threshold, yet threshold resummation effects are non-negligible because the partonic center-of-mass energy is lower than the hadronic one, as discussed in the beginning of Section 3. In this case, it might be appropriate to simply take μ_s as some numerical constant, and compare directly the SCET and QCD expressions of the coefficient function C_{SCET} and C_{QCD} through the master formula Eq. (3.33). But if τ is far from threshold, then μ_s Eq. (5.1) is of the same order as the hard scale M . As a consequence, in this case $C_r(N, M^2, \mu_s^2)$ Eq. (3.31) manifestly starts at NLL order, as confirmed by inspection of Eq. (5.4), which gives

$$C_r(N, M^2, \mu_s^2) = 1 + \mathcal{O}\left(\alpha_s^2(M^2)\ln^3\frac{1}{N}\right). \quad (5.21)$$

One must therefore conclude that this class of NLL terms are resummed, through the leading-log function g_1 in Eq. (2.9), by the QCD result $C_{\text{QCD}}(z, M^2)$ but are not resummed at all in $C_{\text{SCET}}(z, M^2, \mu_s^2)$. This problem may be alleviated through generalizations of the choice Eq. (5.1) such as those proposed in Ref. [25] (and mentioned above at the beginning of Section 5), whereby one rescales μ_s Eq. (5.1) by a factor (determined for instance using scale-optimization methods), because they lead to smaller values of μ_s . These choices, however, do not affect the counting of logs, and it is therefore impossible to assess their impact in a comparison with standard QCD results, other than by numerical methods.

6 Summary

We have analyzed in detail the relation between the approach to threshold resummation based on perturbative factorization of Refs. [3–6], and the SCET approach of Refs. [23–26], with the main goal of exploring the viability, both theoretical and phenomenological, of the SCET prescription to treat the divergent nature of the perturbative QCD expansion in the soft limit. By deriving a master formula which connects resummed results in these two approaches, we have shown that the way they are related depends on the choice of soft scale in the SCET expression. We have explicitly performed calculations up to next-to-next-to-leading logarithmic accuracy, though it is easy to convince oneself that the structure of our master formula holds to any logarithmic order.

We have shown that if SCET resummation is performed in Mellin space, then it coincides with the standard perturbative result. The SCET and QCD results then have the same accuracy, and are both beset by the problem of the divergence of the perturbative expansion. With this (partonic) choice of soft scale SCET and QCD provide alternative ways of deriving the same resummed result.

If SCET resummation is performed in momentum space, as advocated in Ref. [23], the SCET and QCD results differ by a non-universal term, which depends on the parton luminosity (explicitly given up to $\mathcal{O}(\alpha_s^2)$ in Eq. (5.4)): the SCET approach separates off the series of divergent contributions which is then contained in this term, with the SCET resummed result now given by a convergent perturbative expansion. The price to pay for this is fourfold. First, because the difference term is non-universal, it may spoil the logarithmic accuracy of the resummed result depending on the parton luminosity. In particular, if the parton luminosity contains logarithmically enhanced contributions (as it generally will, based on its behaviour upon QCD evolution), the difference term may enter at any logarithmic accuracy (including at the leading-log level), *unless* one decides that logarithms coming from the luminosity should not be included in the power counting. Note however that, because this correction term is not factorized, there is no way of actually isolating the logs that come from the luminosity, other than to assume that the luminosity does not contain any.

If this problem of non-universality is neglected, the SCET and QCD results are equivalent, however only by redefining the logarithmic accuracy to be always by one power lower, according to the counting of Tab. 2 rather than the more accurate perturbative QCD counting Tab. 1. Hence the second price to pay is that the logarithmic accuracy of the SCET result in this case is always lower by one power of log, to all orders in α_s . Third, while perturbative QCD resummation prescriptions such as the minimal [10] or Borel [12, 13] prescription introduce corrections to the perturbative result which are power suppressed or more, the SCET prescription introduces a deviation which is only logarithmically suppressed. And finally, the power counting and suppression in the SCET result must be done at the level of the hadronic scale $1 - \tau$, while in QCD it is done at the level of the partonic scale $1 - z$. In many cases of physical interest [27] it may turn out that the latter is small even when the former isn't: in these cases the QCD counting will be more accurate.

It will be interesting to investigate the phenomenological implications of this state of affairs. Our result enables such an investigation, by providing a closed-form expression for the difference between the SCET and QCD results.

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A Mellin transforms

We collect here some useful results on Mellin transforms, while referring to Section 2 of Ref. [6] and the appendices of Refs. [13, 27] for a fuller treatment.

The Mellin transform which are necessary for the computation of resummed terms, such as the exponent of Eq. (2.5), can be performed using

$$\int_0^1 dz z^{N-1} \left[\frac{\ln^p(1-z)}{1-z} \right]_+ = - \sum_{k=0}^{p+1} \frac{\Gamma^{(k)}(1)}{k!} \frac{d^k}{dL^k} \int_0^{1-1/N} dz \frac{\ln^p(1-z)}{1-z} + \mathcal{O}\left(\frac{1}{N}\right) \quad (\text{A.1})$$

where $L = \ln \frac{1}{N}$ Eq. (2.10).

At NNLL

$$\begin{aligned} & \int_0^1 dz z^{N-1} \left[\frac{F(\ln(1-z))}{1-z} \right]_+ \\ &= - \left[1 - \gamma \frac{d}{dL} + \frac{1}{2} \left(\gamma^2 + \frac{\pi^2}{6} \right) \frac{d^2}{dL^2} \right] \int_0^{1-1/N} dz \frac{F(\ln(1-z))}{1-z} + \text{N}^3\text{LL} + \mathcal{O}\left(\frac{1}{N}\right), \end{aligned} \quad (\text{A.2})$$

(where $\gamma = -\Gamma'(1)$ is the Euler constant) for any function $F(\ell)$ which admits a Taylor expansion around $\ell = 0$. Now,

$$\begin{aligned} \left(1 - \gamma \frac{d}{dL} + \frac{\gamma^2}{2} \frac{d^2}{dL^2} \right) L^p &= L^p - \gamma p L^{p-1} + \frac{p(p-1)}{2} \gamma^2 L^{p-2} \\ &= (L - \gamma)^p + \mathcal{O}(L^{p-3}). \end{aligned} \quad (\text{A.3})$$

Hence, to NNLL accuracy, Eq. (A.1) can be written in the equivalent form

$$\begin{aligned} & \int_0^1 dz z^{N-1} \left[\frac{F(\ln(1-z))}{1-z} \right]_+ \\ &= - \int_0^{1-1/\bar{N}} dz \frac{F(\ln(1-z))}{1-z} - \frac{\pi^2}{12} \frac{d^2}{dL^2} \int_0^{1-1/N} dz \frac{F(\ln(1-z))}{1-z} \end{aligned} \quad (\text{A.4})$$

where

$$\bar{N} = N e^\gamma; \quad \ln \frac{1}{\bar{N}} = L - \gamma. \quad (\text{A.5})$$

An essential ingredient in the discussion of Section 5 is the inverse Mellin transform of

$$\ln^n \frac{c}{N} \quad (\text{A.6})$$

where c is a constant. In order to compute it, we start from the identity

$$\ln^n \frac{1}{N} = \frac{d^n}{d\xi^n} \Delta(\xi) \int_0^1 dz z^{N-1} \ln^{\xi-1} \frac{1}{z} \Big|_{\xi=0}, \quad (\text{A.7})$$

where $\Delta(\xi) = \frac{1}{\Gamma(\xi)}$. Eq. (A.7) should be (and usually is) written in the form

$$\ln^n \frac{1}{N} = \frac{d^n}{d\xi^n} \Delta(\xi) \int_0^1 dz z^{N-1} \left[\ln^{\xi-1} \frac{1}{z} \right]_+ \Big|_{\xi=0} + \delta_{n0}, \quad (\text{A.8})$$

so that the integral is well defined even when the derivatives and the limit $\xi \rightarrow 0$ are taken under the integral sign. This is not necessary for our present purposes. Rewriting Eq. (A.7) with N replaced by N/c we obtain

$$\begin{aligned} \ln^n \frac{c}{N} &= \frac{d^n}{d\xi^n} \Delta(\xi) \int_0^1 dz z^{\frac{N}{c}-1} \ln^{\xi-1} \frac{1}{z} \Big|_{\xi=0} \\ &= \frac{d^n}{d\xi^n} c^\xi \Delta(\xi) \int_0^1 dz z^{N-1} \ln^{\xi-1} \frac{1}{z} \Big|_{\xi=0} \end{aligned} \quad (\text{A.9})$$

after rescaling the integration variable $z \rightarrow z^c$. The inverse Mellin transform of $\ln^n \frac{c}{N}$ can now be immediately read off Eq. (A.9).

B Convolutions

In this Appendix we compute the integral

$$\Sigma(\tau, \xi) = \Delta(\xi) \int_\tau^1 \frac{dz}{z} \sigma\left(\frac{\tau}{z}\right) \ln^{\xi-1} \frac{1}{z}, \quad (\text{B.1})$$

where $\sigma(\tau) \equiv \sigma_{\text{SCET}}(\tau, M^2)$ for simplicity, up to terms suppressed by powers of $1 - \tau$. Using

$$\ln \frac{1}{z} = 1 - z + \mathcal{O}((1-z)^2) \quad (\text{B.2})$$

we find

$$\begin{aligned} \Sigma(\tau, \xi) &= \Delta(\xi) \int_\tau^1 \frac{dz}{z} (1-z)^{\xi-1} \sigma\left(\frac{\tau}{z}\right) \\ &= \Delta(\xi) \sum_{n=0}^{\infty} \frac{\sigma^{(n)}(\tau)}{n!} \tau^n \int_\tau^1 dz (1-z)^{\xi-1} \frac{(1-z)^n}{z^{n+1}}. \end{aligned} \quad (\text{B.3})$$

Expanding $1/z^{n+1}$ in powers of $1-z$ we see that the integral is a sum of terms proportional to

$$(1-\tau)^{\xi+m}, \quad m \geq n. \quad (\text{B.4})$$

Hence, the derivatives $\sigma^{(n)}(\tau)$ appear in $\Sigma(\tau, \xi)$ multiplied by $(1-\tau)^m$, with $m \geq n$. Now

$$\begin{aligned} (1-\tau)\sigma^{(1)}(\tau) &= -\frac{d\sigma(\tau)}{d\ln(1-\tau)} \\ (1-\tau)^2\sigma^{(2)}(\tau) &= -\frac{d\sigma(\tau)}{d\ln(1-\tau)} + \frac{d^2\sigma(\tau)}{d\ln^2(1-\tau)} \\ &\dots \end{aligned} \quad (\text{B.5})$$

which means that $(1-\tau)^m \sigma^{(n)}(\tau)$ with $m > n$ is power-suppressed, and can be neglected in Eq. (B.3) since we are only interested in logarithmically-enhanced contributions to $\Sigma(\tau, \xi)$. Hence

$$\Sigma(\tau, \xi) = \Delta(\xi) (1-\tau)^\xi \sum_{n=0}^{\infty} \frac{(1-\tau)^n \sigma^{(n)}(\tau)}{n!(n+\xi)} + \text{power-suppressed terms}. \quad (\text{B.6})$$

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