Applied Mathematics

Helmut Schmidt Universität
Universität der Bundeswehr Hamburg

Angewandte Mathematik und Optimierung Schriftenreihe
Applied Mathematics and Optimization Series
AMOS # 60(2017)

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Angewandte Mathematik und Optimierung Schriftenreihe (AMOS), ISSN-Print 2199-1928
Angewandte Mathematik und Optimierung Schriftenreihe (AMOS), ISSN-Internet 2199-1936

Mixed Integer PDE Constrained Optimization for the Control of a Wildfire Hazard

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Abstract. We derive an optimization problem for a mission planning problem of a firefighting department challenged by a wildfire. Here the fire is modeled using partial differential equations (PDEs), and the response from the firefighters is modeled as a dynamic network flow. The firefighters influence the spread of the wildfire, and vice versa, the fire restricts the movement options of the firefighters. These mutual interactions have to be incorporated into the model. The presented approach to formulate this problem mathematically is to replace the infinite dimensional constraints imposed by the PDE by a finite dimensional system. These systems however tend to be very large even for a moderate resolution of the approximation. This causes a direct approach using a finite difference method to be outperformed by a new method, in which the PDE is solved in a pre-optimization step. We demonstrate the superiority of this approach in a computational study, where both methods are compared for various approximation resolutions.

Keywords: Mixed Integer Programming, PDE Constrained Optimization.

1 Introduction

In a forest close to inhabited regions is an ongoing wildfire spread. Leaving it burning uncontrolled might endanger the local population and their properties, hence the firefighters are trying to plan their response in an optimal way, without endangering themselves. A road network is passing through the forest that can now be used for firefighting operations. The forest itself cannot be crossed; all movements are restricted to the said road network. In order to prevent endangering the firefighters, no movement should take place on roads leading through or too close to burning territory. The resources necessary to control the fire (water, equipment, and manpower) are limited, therefore an optimal resource allocation and proper scheduling might make the difference between getting the fire under control or a major disaster.

In this situation an optimal planning has to take two different types of dynamics into account: Firstly, the physics of the fire, which allows to predict the spread direction and velocity, and secondly, the movement of the firefighters and their extinguishing agents (water). Those two systems cannot be considered separately. The ultimate goal of any firefighter mission is to influence the spread of the fire, but during this mission, the fire might temporarily prevent the firefighters from reaching certain areas.

For the modeling of the fire a time dependent PDE is used, and a dynamic network flow is used to model the movements of the firefighters, or more precisely, the water that they use. In order to express the interdependencies, the flow variables of the network are used as control variables for the PDE and additional constraints are imposed on the network flow which include the state of the PDE. The inclusion of these interdependencies make our model unique in comparison to other recent work. For example, Göttlich et al. [3] studied an evacuation planning problem in response to a gas hazard, where the latter is modeled by a PDE, which is independent of the network dynamics (i.e., flows of evacuating people). Frank et al. [2] consider the coolest path problem, where an object traverses a network graph while being heated or cooled on the arcs. Here the heat PDE gives rise to objective function coefficients for a shortest path problem, but does not constrain the combinatorial decisions.

2 The Mathematical Model

In order to solve the planning problem of the response to the wildfire, an integrated model for the spread of the fire and for the movement of the firefighters is formulated. We define the sets and variables of this model, and then the constraints and the objective function.

For the dynamic flow of the water used by the firefighters we assume that the road network is given in form of a graph $G := (V, A)$ with capacities $c_{i,j}$ and traversing times $\delta_{i,j}$ for all arcs $(i, j) \in A$. Graph G is embedded in the plane by endowing each vertex $i \in V$ with a coordinate $x_i \in \Omega$, where $\Omega := [0, L]^2$ is a square area of interest. The arcs $(i, j) \in A$ are associated with a straight lines between the coordinates of their respective incident vertices. The flow can start in source nodes, denoted by $S \subset V$, and ends in demand nodes $D \subset V$, which are nodes suitable for extinguishing the fire. We introduce a discretization of the time horizon $[0, T]$ by the set of time $\mathbb{T} := \{0, \Delta t, \dots, n_t \Delta t = T\}$.

The variables $v_{i,j,t} \in \mathbb{R}_+$ represent the flow (of water) on arc (i, j), starting in i at time $t \in \mathbb{T}$. In nodes $i \in S$ the flow can enter the network, and the intensity at time $t \in \mathbb{T}$ is specified by the variables $w_{i,t} \in \mathbb{R}_-$. Vice versa, the flow leaves the network in nodes $i \in D$ at time $t \in \mathbb{T}$, with an intensity given by $w_{i,t} \in \mathbb{R}_+$. The temperature in the forest at location $x \in \Omega$ at time $t \in \mathbb{T}$ is given by $u(x, t)$. Finally, binary decision variables $z_{i,t} \in \{0, 1\}$ for $i \in V$ and $t \in \mathbb{T}$ are introduced to link the temperature to the flow, with $z_{i,t} = 0$ if and only if the temperature at x_i at time t exceeds a certain threshold U_B, at which further firefighter operations have to be terminated for safety reasons, that is, the area is burning.

The following constraints are now used to ensure the desired behavior of the firefighter operations (i.e., the flow of water), where we use a dynamic maximum flow formulation (see [6] for a survey):

$$v_{i,j,0} = 0 \qquad \forall (i,j) \in A, \quad \text{(1a)}$$

$$\sum_{i \in V:(i,k) \in A, \delta_{i,k} \leq t} v_{i,k,t-\delta_{i,k}} = \sum_{j \in V:(k,j) \in A} v_{k,j,t} + w_{k,t} \qquad \forall k \in V, t \in \mathbb{T}, \quad \text{(1b)}$$

$$u(x_i,t) - (1 - z_{i,t})M \leq U_B \qquad \forall i \in V, t \in \mathbb{T}, \quad \text{(1c)}$$

$$\sum_{s \in \mathbb{T}: s \leq \min(\delta_{i,j}, T-t)} v_{i,j,t+s} \leq c_{i,j} z_{j,t} \qquad \forall (i,j) \in A, t \in \mathbb{T}. \quad \text{(1d)}$$

The initial condition (1a) guarantees that no flow is inside the network at $t = 0$. The flow conservation is ensured by (1b). Constraints (1c) and (1d) prevent flow from passing through a burning area, where the first sets the binary switch variable $z_{i,t}$ to zero if the threshold temperature is reached, and the second only allow for flow (w.r.t. to the capacity restriction) as long as $z_{j,t} = 1$.

The dynamics of the fire is modeled by the following PDE system:

$$u_t(x,t) - c \cdot \nabla u(x,t) - d\Delta u(x,t) = y(x,t,w) \qquad \forall\ (x,t) \in \Omega \times (0,T), \quad \text{(2a)}$$

$$\frac{\partial}{\partial n} u(x,t) = h_R(u_R - u(x,t)) \qquad \forall\ (x,t) \in \partial\Omega \times (0,T), \quad \text{(2b)}$$

$$u(x,0) = f(x) \qquad \forall x \in \Omega, \quad \text{(2c)}$$

$$u(x,t) \geq 0, \qquad \forall (x,t) \in \Omega \times (0,T). \quad \text{(2d)}$$

This is a convection-diffusion equation with Robin type boundary conditions on the spacial domain Ω and the time domain $[0,T]$. The fire model is able to express the effect of the wind and the diffusive behavior of fire, while still being a linear PDE (which we need later for computational reasons[1]). Condition (2b) imposes that the normal derivative at the boundary is directly proportional to the difference of the temperature on the boundary and the temperature U_R. Parameter d is the coefficient of the diffusion term, it determines the speed of the fire spread. Parameter c is the velocity-vector of the wind. Furthermore condition (2d) ensures that, when the fire is exitinguished (at temperature zero), the control function cannot push the temperature any lower thereafter.

The term in this PDE that represents an outer influence is $y(x,t,w)$, which depends on the outflow of water $w_{i,t}$ for $i \in D$ of a nearby node ($x \approx x_i$) as follows: The controls at the different vertices and different points in time are independent of each other, hence y is the sum of several individual control functions. It is further assumed that each outflow variable $w_{i,t}$ ($i \in D$) has only a local effect with a peak at the coordinate of its vertex and acts only for a certain duration T_E. We assume that the spatial effect follows a Gaussian distribution

[1] We remark that there are more complex fire models known, for example [4], where a further nonlinear term expresses the consumption of fuel (here: wooden trees), but on such models our presented computational techniques do not work. Their adaptation is a direction for future research.

with the coordinates of the vertex x_i at its center:

$$y(x,t,w) = \lambda \sum_{\tau \in \mathbb{T}} \sum_{i \in D} -w_{i,\tau} \chi_{[t,t+T_E)}(t) \exp\left(-\frac{\|x - x_i\|_2^2}{\sigma^2}\right), \tag{3}$$

where λ and σ are parameters that represent the spatial influence of the outflow of water on the surrounding fire (more precisely, its temperature), and χ_I is the characteristic function (i.e., $\chi_I(t) = 1$ for $t \in I$ and 0 otherwise) that restricts the duration of the influence to a time interval of size T_E.

The objective is to minimize the damage caused by the fire. We assume that the damage is proportional to a weighted integral of the temperature $u(x,t)$ in Ω over a time horizon $[0,T]$.

$$\begin{aligned} &\min \int_0^T \int_\Omega \omega(x) u(x,t) \, \mathrm{d}x \, \mathrm{d}t, \\ &\text{s.t.} (1a) - (1d), (2a) - (2d). \end{aligned} \tag{4}$$

We present two approaches to solve (4). Neither of them solves this model directly. Instead, we derive suitable finite dimensional systems that approximate (4), which turn out to be linear mixed-integer problems (MILP) and thus can be solved using a state-of-the-art MILP solver.

Finite Differences. The first approach uses a one-to-one replacement of the constraints and objective with a discrete counterpart. The PDE is replaced by a linear system obtained from a convergent finite difference method [5] and the integral is replaced by a quadrature formula. The domain is discretized by replacing Ω with an equidistant grid of length $\Delta x = \frac{L}{n_x}$ with $n_x \in \mathbb{N}$. The interval $[0,T]$ is replaced by the discrete time set $\mathbb{T}$, which was already used for setting up to the network flow. Then for each point $(i\Delta x, j\Delta x, t)$ of the grid a variable $u_{i,j,t}$ is added. The function $u(x,t)$ is approximated at each gridpoint, i.e., $u((i\Delta x, j\Delta x), t) \approx u_{i,j,t}$. All constraints that depend on u have to be adjusted for those discrete variables. The PDE and its initial and boundary conditions (2a)-(2c) are replaced by a linear system

$$\begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = b, \tag{5}$$

where the coefficients in the matrices A_1, A_2 and the vector b are derived according to a finite difference scheme. Condition (2d) is converted by enforcing it for the discrete variables:

$$u_{i,j,t} \geq 0, \qquad \forall i,j \in \{0, \dots n_x\}, t \in \mathbb{T}. \tag{6}$$

From the network conditions only (1c) has to be adjusted as

$$u_{j,k,t} - (1 - z_{i,t})M \leq U_B, \qquad \forall i \in V, x_i = (j\Delta x, k\Delta x), t \in \mathbb{T}. \tag{7}$$

Note that we assume here for simplicity that the coordinates x_i of the nodes $i \in V$ are aligned to the grid. More generally, then one can take the weighted sum of the neighboring grid points according to their distance to the position of the vertex x_i, which still gives a linear constraint. The objective function can be approximated by the trapezoidal rule applied at the grid points. Then the first linear mixed-integer approximation of (4) is:

$$\begin{aligned} &\min\ (\Delta x)^2 \Delta t \sum_{t\in\mathbb{T}} \sum_{i,j=0}^{n_x} \lambda_t \mu_i \nu_j \omega(i\Delta x, j\Delta x) u_{i,j,t}, \\ &\text{s.t. } (1a) - (1d), (5), (6), (7), \end{aligned} \tag{8}$$

where: $\lambda_0 = 0.5$, and $\lambda_t = 1$ if $t > 0$; $\mu_0, \mu_{n_x}, \nu_0, \nu_{n_x} = 0.5$, and $\mu_i, \nu_j = 1$ otherwise.

Finite Elements. The second approach is based on the observation that because of the principle of superposition for linear PDEs the continuous state u can be defined as

$$u = u_{inh} + \sum_{t\in\mathbb{T}} \sum_{i\in V} w_{i,t} \hat{u}_{i,t}, \tag{9}$$

where u_{inh} is the solution of (2a) for $w = \mathbf{0}$, and $\hat{u}_{i,t}$ are the solutions of (2a) for each individual summand of u and homogeneous boundary and initial conditions. Since the summands of the control functions for a fixed vertex i can be obtained by shifting $\hat{u}_{i,0}(t)$ to the right it holds for all $\tau \in \mathbb{T}$

$$\hat{u}_{i,\tau}(t) = \begin{cases} 0, & 0 \le t \le \tau, \\ \hat{u}_{i,0}(t-\tau), & \tau < t \le T. \end{cases}$$

Therefore only $|V| + 1$ PDEs have to be solved in order to obtain u, and (9) can be used to replace (2a)-(2c) in the continuous model. This also makes it possible to separate the solution of the PDE from the optimization process, which opens up the possibility to use adaptive finite element methods instead of finite differences. Finite element methods in contrast to finite differences define a linear combination of base functions and thus can be used to derive values anywhere in Ω and not only on a grid. So independent on the meshes of the finite element method, it is possible to define the discrete variables as:

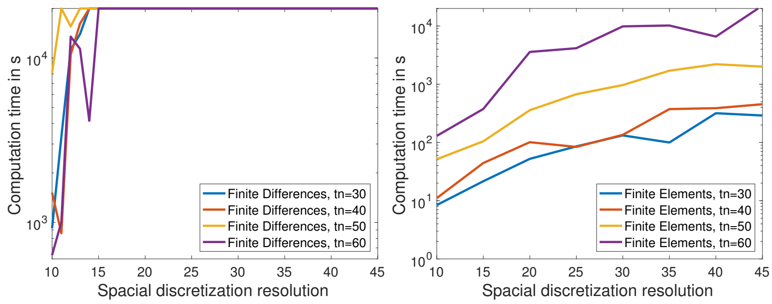
$$u_{i,j,t} = u_{inh}(i\Delta x, j\Delta x, t) + \sum_{\tau\in\mathbb{T}} \sum_{k\in V} w_{k,\tau} \hat{u}_{k,\tau}(i\Delta x, j\Delta x, t). \tag{10}$$

With this we define the MILP for the second approach:

$$\begin{aligned} &\min\ (\Delta x)^2 \Delta t \sum_{t\in\mathbb{T}} \sum_{i,j=0}^{n_x} \lambda_t \mu_i \nu_j \omega(i\Delta x, j\Delta x) u_{i,j,t}, \\ &\text{s.t. } (1a) - (1d), (6), (7), (10). \end{aligned} \tag{11}$$

3 Computational Results and Conclusion

Two different MILP were derived that approximate the continuous problem (4). The second model (11) has much less constraints and variables compared to model (8) based on finite differences. Yet it remains to be shown that the second model indeed outperforms the first one. For solving the required PDEs, the object oriented software package oFEM [1] has been employed. The computational results for a problem formulated for the two models are included in the Figures 1a and 1b. The different graphs show the runtimes for different degrees of time and space discretizations. The figures illustrate that the finite difference method was only able to solve problems with only a 10x10 spacial grid and up to 60 timesteps within a time limit of $20,000s$. In contrast, the second method still solves problems with a 45x45 spacial grid and 50 timesteps within the same timeframe, using IBM ILOG CPLEX 12.6.3.0 on a 2014 Mac mini with a 2.6 GHz Intel Core i7 CPU and 16 GB RAM.



(a) *Finite Differences Approach* (b) *Finite Elements Approach* (c) *Video*

Fig. 1: Computational Results.

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