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Coverage Maximization for Multistatic Sonar Location under Budget Restrictions

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Abstract

A multistatic sonar system consists of one or more sources that are able to emit underwater sound, and receivers that listen to the reflected sound waves. Knowing the speed of sound in water, the time when the sound was sent from a source, and the arrival time of the sound at one (or more) receiver(s), it is possible to determine the location of surrounding objects. The propagation of underwater sound is a complex phenomenon that depends on various attributes of the water (density, pressure, temperature, and salinity) and the emitted sound (pulse length and volume), as well as the reflection properties of the water's surface. These effects can be approximated by nonlinear equations. Furthermore, natural obstacles in the water, such as the coastline, need to be taken into consideration. Given a fixed number of sources and receivers and area of the ocean that should be endowed with a sonar system for surveillance, we consider the problem of determining the best locations for the sources and receivers in order to maximize the covered area. We give an integer nonlinear formulation for this problem, and we discuss several ways to derive an integer linear formulation from it. We then compare these formulations numerically using a test bed from coastlines around the world and a state-of-the-art mixed-integer program (MIP) solver (IBM ILOG CPLEX).

1. Introduction to Sonar

Sonar is a technique to detect objects that are under water or at the surface using sound propagation. Active sonar has been in use for nearly 100 years and has become a key component of undersea detection. The basic operating principle of active sonar is that acoustic energy is emitted from a source and its echoes are detected by a receiver; these echoes reveal information about surrounding objects. In a monostatic system, the source and the receiver are collocated Bistatic sonar uses a source and a receiver pair in different locations. Multistatic sonar uses several sources and receivers simultaneously as a network. For the surveillance of a large area of the ocean, a number of both types of devices must be deployed. This leads to an optimization problem to find a maximal coverage of a desired area with a yet-to-be-designed multistatic network that consists of a given and fixed number of sources and receivers. No algorithm currently in the literature provides an optimal simultaneous placement of a given number of sources and receivers. In a discretized setting, we describe mathematical models designed to determine the sensor layout that will best cover a portion of the ocean (a tile) by sonar surveillance, with adequate detection probability throughout the tile. Using a definite range ("cookie-cutter") detection model, we perform preprocessing calculations to determine whether

a given part of the tile can be surveilled by a given source-receiver pair as a binary yes/no outcome. We model the physical properties of sound traveling between sources, target, and receivers; the properties of the ocean (temperature, density, salinity); as well as geometrical considerations (obstacles such as islands or coastlines). Section 2 contains the details of our model. We formulate an integer nonlinear program for the multistatic sonar source-receiver location problem and discuss several linearizations in Section 3. We compare these formulations empirically using topological data from coastal areas around the world and a state-of-the-art solver MIP solver in Section 4. Section 5 contains concluding remarks.

2. Input Data

We obtain ocean topography data from Ryan et al. (2009). At present, we do not use sea level information and only distinguish in a binary fashion between the ocean (negative elevation value) and the dry land (positive elevation value). A desired part of the ocean and shoreline (a tile) is taken from the database. Since the resolution of the data is too fine to let each data pixel become a possible target/source/receiver location, we aggregate the raw input data into larger rectangular areas called grid cells. We then average the elevation data from all pixels within a cell and apply the resulting elevation to the entire cell. Denote the set of rectangles with negative elevation (i.e., those that are underwater) by G (for grid) and the number of elements in G by $n := |G|$.

The sonar signal is characterized by the range of the day ϱ_0, which indicates how quickly the signal diminishes as the target, source, and receiver become farther apart. In a definite range ("cookie-cutter") sensor model, a target in a cell $k \in G$ is detected by a source placed in cell $i \in G$ and a receiver placed in cell $j \in G$ with binary probability $p_{i,j,k} \in \{0, 1\}$. Denote by $d_{i,j}$ the Euclidean distance between (the centers of) cell i and j. Necessary for detection ($p_{i,j,k} = 1$) is that the target k is inside the Cassini oval defined by the equation $d_{i,k} \cdot d_{k,j} \leq \varrho_0^2$, c.f. Karatas et al. (2016); Craparo et al. (2017); Karatas et al. (2014); Karatas and Craparo (2015). If the target is too close to the line of sight from source to receiver, then the original signal and its reflection off the target become indistinguishable at the receiver. This phenomenon is known as the direct blast effect. The pulse length κ_b determines the severity of this effect, since longer pulses are more prone to overlapping with the reflected signal. The direct blast zone is defined by the ellipsoid $d_{i,k} + d_{k,j} \leq d_{i,j} + 2\kappa_b$, c.f. Karatas and Craparo (2015). To account for the direct blast effect, we say that $p_{i,j,k} = 0$, if the target lies within the direct blast zone. Additionally, if an obstacle lies on either straight-line path of source to target, target to receiver, or source to receiver, then $p_{i,j,k} = 0$.

Let P_i denote the value (reward) for covering each grid cell $i \in G$. This parameter can be used to reflect the fact that some parts of the tile are more valuable than others and should get a higher priority if not all of the grid can be covered. For simplicity, we use $P_i := \frac{1}{n} \cdot 100\%$ for all $i \in G$. Our objective function thus yields the percentage of the ocean that is covered. Denote the number of available sources and receivers by $c_s, c_r \in \mathbb{N}$, respectively.

3. Model Formulations

All of our model formulations utilize the binary decision variables $s_i, r_i \in \{0, 1\}$ for each $i \in G$, which model the decision whether to place a source ($s_i = 1$) or a receiver ($r_i = 1$) in cell i, and $x_i \in \{0, 1\}$ for $i \in G$, which models the coverage outcome of grid cell i

($x_i = 1$ if cell i is covered). The objective (in all formulations) is to maximize the total value from all covered cells, which we calculate as follows:

$$\sum_{i \in G} P_i x_i. \tag{3.1}$$

Constraints that are common to all models are the two deployment constraints that require that all sources are used:

$$\sum_{i \in G} s_i = c_s, \tag{3.2}$$

and all receivers are used:

$$\sum_{j \in G} r_j = c_r. \tag{3.3}$$

3.1. *An Integer Nonlinear Model.*

In the first nonlinear formulation the binary variables s_i and r_j are multiplied in order to represent the joint decision of placing a source in cell i **and** a receiver in cell j. A grid cell k can only be covered if there is at least one feasible combination of a source in i and a receiver in j that allows a detection in k. An over-covering (i.e., covering with more than one source-receiver combination) is allowed, but it does not give any additional benefit:

$$\sum_{i \in G} \sum_{j \in G} p_{i,j,k} s_i r_j \geq x_k, \quad \forall k \in G. \tag{3.4}$$

In general, for any given $k \in G$ the non-negative matrix $(p_{i,j,k})_{i,j}$ is indefinite. Only recently, some solvers have the ability to solve optimization problems with this type of restriction to proven global optimality. For instance, the solver CPLEX is able to process constraints of this form since version 12.6 (Bliek et al. 2014). Thus, the base run for comparison with the other reformulation approaches is to solve the model:

$$\min\{(3.1) | (3.2), (3.3), (3.4); s, r, x \in \{0,1\}^G\}. \tag{3.5}$$

3.2. *The Oldest Linearization Technique*

The first documented linearization of a product of binaries $s_i r_j$ introduces a new binary variable $h_{i,j} \in \{0,1\}$ with $h_{i,j} = 1$ if and only if $s_i = 1$ and $r_j = 1$. In this method, independently described in Fortet (1959); Balas (1964); Zangwill (1965); Watters (1967), the constraints $2h_{i,j} \leq s_i + r_j$ and $s_i + r_j \leq 1 + h_{i,j}$ (for all $i, j \in G$) are a linear description of this relationship. In our case, because of the non-negativity of all $p_{i,j,k}$, only the first constraint is necessary. Thus the first linear version of (3.5) is

$$\min\ (3.1),\ \text{s.t.} \sum_{i \in G} \sum_{j \in G} p_{i,j,k} h_{i,j} \geq x_k, \quad \forall k \in G, \tag{3.6a}$$

$$2h_{i,j} \leq s_i + r_j, \quad \forall i, j \in G, \tag{3.6b}$$

$$(3.2), (3.3), s, r, x \in \{0,1\}^G, h \in \{0,1\}^{G \times G}. \tag{3.6c}$$

Compared to the nonlinear integer formulation (3.5), this binary linear model has an additional n^2 binary variables and n^2 constraints.

3.3. *Standard Linearization of the Model*

A linearization for $s_i r_j$ similar to the previous one from Glover and Woolsey (1974) introduces continuous auxiliary variables $h_{i,j} \in [0,1]$ together with the constraints $h_{i,j} \leq s_i, h_{i,j} \leq r_j$ and $s_i + r_j \leq 1 + h_{i,j}$. This is perhaps the first and most natural formulation to

come to mind (and for good reason: Padberg (1989) showed that these three constraints are facet defining), and is hence often called "standard linearization." As before, the third constraint is not required in our case. Then, the second linear version of (3.5) is

$$\min\ (3.1),\ \text{s.t.} \sum_{i\in G}\sum_{j\in G} p_{i,j,k}h_{i,j} \geq x_k, \quad \forall k \in G, \tag{3.7a}$$

$$\left.\begin{array}{rcl} h_{i,j} & \leq & s_i \\ h_{i,j} & \leq & r_j \end{array}\right\}, \quad \forall i,j \in G, \tag{3.7b}$$

$$(3.2), (3.3), s, r, x \in \{0,1\}^G, h \in [0,1]^{G\times G}. \tag{3.7c}$$

Compared to the nonlinear binary formulation (3.5), this mixed-integer linear model has an additional n^2 continuous variables and $2n^2$ constraints.

3.4. *Glover's Linearization*

To adapt a linearization technique from Glover (1975), we set $L_{j,k} := \sum_{i\in G} p_{i,j,k}$ for all $j,k \in G$, and the model reads:

$$\min\ (3.1),\ \text{s.t.} \sum_{j\in G} z_{j,k} \geq x_k, \quad \forall k \in G, \tag{3.8a}$$

$$\left.\begin{array}{rcl} \sum_{i\in G} p_{i,j,k}s_i & \geq & z_{j,k} \\ L_{j,k}r_j & \geq & z_{j,k}, \end{array}\right\}\ \forall j,k \in G, \tag{3.8b}$$

$$(3.2), (3.3), s, r, x \in \{0,1\}^G, z \in \mathbb{R}_+^{G\times G}. \tag{3.8c}$$

In order to understand that this formulation of the problem is equivalent to the previous ones, fix k and think of the auxiliary variables $z_{j,k}$ as the row sum of the matrix $(p_{i,j,k}s_ir_j)_{i,j}$ for the j-th row. Now also fix j. In case $r_j = 0$, the row sum is equal to 0. This is ensured by the second inequality in (3.8b). Otherwise, if $r_j = 1$, this very inequality allows $z_{j,k}$ to hold any value between zero and the upper bound of the row sum, which is $L_{j,k}$. This upper bound may be tight, which happens if and only if $s_i = 1$ for all $i \in G$. Other than that, it is bounded from above by the first inequality in (3.8b), which is where the actual $s_i = 1$ values are taken into account. There is no need to bound $z_{j,k}$ from below here, since this is taken care of by the inequality of (3.8a). By the objective function (3.1), it is beneficial to set as many x_k variables to 1 as possible; hence, there is no need for a further lower bound on z_{j_k}.

This model introduces n^2 additional continuous variables and $2n^2$ additional constraints (compared to (3.5)). We remark that because of the non-negativity of $p_{i,j,k}$ we could remove some (in this case) redundant constraints from this formulation, similar to the models (3.6) and (3.7); this was noted by Adams and Forrester (2005). In this case, the remaining formulation equals a linearization formulation given by Chaovalitwongse et al. (2004); Pardalos et al. (2004); the close relation of the Pardalos et al.'s to Glover's formulation was also noted by Hansen and Meyer (2009).

3.5. *Oral-Kettani's Linearization*

Oral and Kettani (1992) proposed two formulations that come with n^2 additional continuous variables, but fewer constraints compared to Glover's formulation; namely, only n^2 additional constraints, compared to (3.5).

The first of the two formulations is:

$$\min\ (3.1),\ \text{s.t.}\ \sum_{j\in G}(L_{j,k}r_j - z_{j,k}) \geq x_k, \quad \forall k \in G, \tag{3.9a}$$

$$z_{j,k} \geq L_{j,k}r_j - \sum_{i\in G} p_{i,j,k}s_i, \quad \forall j,k \in G, \tag{3.9b}$$

$$(3.2), (3.3), s, r, x \in \{0,1\}^G, z \in \mathbb{R}_+^{G\times G}. \tag{3.9c}$$

Consider a fixed k and j, and suppose $r_j = 0$. Then by $(3.9b)$ the value of $z_{j,k}$ is bounded by $-\sum_{i\in G} p_{i,j,k}s_i$ from below. Since the latter is non-positive for any combination of $s_i \in \{0,1\}$, the actual lower bound is $z_{j,k} \geq 0$ from $(3.9c)$. Inequality $(3.9a)$ together with the objective (3.1) then keep $z_{j,k}$ at this lower bound. Now consider the case that $r_j = 1$. Then by $(3.9b)$ the value of $z_{j,k}$ is bounded from below by $L_{j,k} - \sum_{i\in G} p_{i,j,k}s_i$. The latter is now non-negative for any combination of $s_i \in \{0,1\}$. As in the previous case, inequality $(3.9a)$ together with the objective (3.1) keep $z_{j,k}$ at the lower bound. With $z_{j,k}$ being at the lower bound, the $L_{j,k}r_j$ terms in $(3.9a)$ and $(3.9b)$ cancel, and the left-hand side of the inequality in $(3.9a)$ is again the row sum of the matrix $(p_{i,j,k}s_ir_j)_{i,j}$ for the j-th row.

The second Oral-Kettani linearization is:

$$\min\ (3.1),\ \text{s.t.}\ \sum_{j\in G}\left(\sum_{i\in G} p_{i,j,k}s_i - z_{j,k}\right) \geq x_k, \quad \forall k \in G, \tag{3.10a}$$

$$z_{j,k} \geq \sum_{i\in G} p_{i,j,k}s_i - L_{j,k}r_j, \quad \forall j,k \in G, \tag{3.10b}$$

$$(3.2), (3.3), s, r, x \in \{0,1\}^G, z \in \mathbb{R}_+^{G\times G}. \tag{3.10c}$$

In order to understand this formulation, consider again a fixed j and k. If $r_j = 0$, then $z_{j,k} \geq \sum_{i\in G} p_{i,j,k}s_i$ by $(3.10b)$. The objective function (3.1) together with $(3.10a)$ keep $z_{j,k}$ at this lower bound. Then there is a 0 contribution for the outer sum in constraint $(3.10a)$ for this j,k. If $r_j = 1$, then inequality $(3.10b)$ is weak and we have that $z_{j,k} \geq 0$. Then for this j,k the contribution to $(3.10a)$ is $\sum_{i\in G} p_{i,j,k}s_i$.

4. Computational Results

We now compare the above six formulations on a test set of 24 instances. These instances are based on ocean topography data from various regions around the world and were extracted from a global map assembled by Ryan et al. (2009). The computations were carried out on a 2014 MacBookPro with 16 GB RAM and a 2.8 GHz Intel Core i7 processor. We set a time limit of 1,000 seconds and use default settings of the solver IBM ILOG CPLEX 12.7.1 otherwise.

The resulting total computation times appear at the bottom in Table 1 (row "SUM"). Ranking the formulations based on their total computation time across all 24 instances (row "SUMRANK"), we see that the standard formulation (3.7) outperforms the second Oral-Kettani formulation (3.10) by a factor of two. Glover's formulation (3.8) follows, while the first Oral-Kettani formulation (3.9) and the oldest linearization (3.6) are nearly tied. Far behind each of these linearizations comes the nonlinear formulation (3.5), which solves only three of the 24 instances within the time limit. Looking more closely at the results across all instances, we see that the ranking among formulations varies by problem instance. However, the nonlinear formulation (3.5) consistently performs worst.

The ranking of the formulations deviates if we consider the geometric mean (rows

Instance	n	(3.5)	(3.6)	(3.7)	(3.8)	(3.9)	(3.10)
BabAlMandabStrait	29	139.57	1.11	0.48	**0.21**	1.78	0.22
ChoctawhatcheeBay	31	1000.27	7.93	**3.24**	13.55	8.76	7.08
Dardanelles	19	21.14	0.18	0.16	0.07	0.13	**0.03**
EnglishChannel	48	1000.14	840.64	**24.1**	1000.01	1000.02	130.23
FalklandSound	57	1001.1	1.23	2.78	1.1	1000.03	**0.84**
GulfOfAkaba	22	9.94	0.36	0.29	**0.16**	0.51	0.32
GulfOfFinland	37	1000.04	24.21	2.48	19.07	3.27	**1.42**
GulfOfSirte	45	1000.99	16.93	76.01	**11.2**	11.86	43.15
KarkinytskaGulf	34	1000.25	7.48	1.33	9.85	3.18	**1.23**
KerchStrait	36	1000.02	56.29	**0.63**	13.31	8.35	1.37
LagoDeMaracaibo	48	1000.84	0.5	1.37	**0.4**	3.53	0.48
Lesbos	30	1000.01	1.39	0.34	0.79	0.26	**0.08**
MontereyPeninsula	45	1000.21	196.42	**20.36**	83.62	66.4	48.64
NewYork	38	1000.02	39.74	**6.96**	9.91	22.67	32.86
OpenSea-Biscaya	54	1002.94	**5.12**	46.35	22.31	23.38	6.95
Oresund	71	1004.21	1001.35	**105.1**	1000.02	1000.02	495.54
Ruegen	37	1000.41	11.59	18.22	**2.6**	5.52	3.64
Smalandsfarvandet	58	1000.1	1000.02	**77.15**	1000.02	1000.02	166.03
Storebaelt	40	1000.15	57.84	20.58	**9.21**	19.84	18.53
StraitOfGibraltar	52	1000.17	1000.02	**124.29**	168.91	154.69	312.86
StraitOfHormuz	41	1000.08	20.48	6.79	**5.7**	11.37	6.2
TaedongGang	39	1000.13	14.87	15.12	12.32	**11.73**	13.76
Vaeinameri	36	1000.05	172.32	9.1	**1.71**	2.27	8.54
WonsanBay	44	1000.34	26.9	100.67	**12.09**	15.92	27.31
SUM		21183.12	4504.92	663.9	3398.14	4375.51	1327.31
SUMRANK		6	5	1	3	4	2
GEOMEAN		527.22	18.45	6.97	9.27	13.59	6.22
GEOMEANRANK		6	5	2	3	4	1
WINNER		0	1	8	9	1	5
WINNERANK		6	4	2	1	4	3

TABLE 1. Computation time (seconds) of models (3.5)–(3.10) on 24 realistic test instances.

"GEOMEAN" and "GEOMEANRANK") as performance measure. Now the second Oral-Kettani formulation (3.10) is slightly ahead of the standard formulation (3.7), and in the same order as before on ranks 3, and 4, 5 we find Glover's formulation (3.8), the first Oral-Kettani formulation (3.9), and the oldest linearization (3.6), respectively. Still, the nonlinear formulation (3.5) performs worst.

A third way of ranking is by counting the number of wins, that is, how many times is one formulation ahead of all others with respect to the run time (rows "WINNER" and "WINNERRANK"), see also Figure 2. It turns out that from this point of view, Glover's formulation (3.8) with 9 wins is slightly ahead of the standard formulation (3.7) with 8 wins, followed by the second Oral-Kettani formulation (3.10) with 5 wins. The second Oral-Kettani formulation (3.10) as well as the oldest linearization (3.6) only win on 1 instance each. Still, the nonlinear formulation (3.5) performs worst.

An example solution appears in Figure 1.

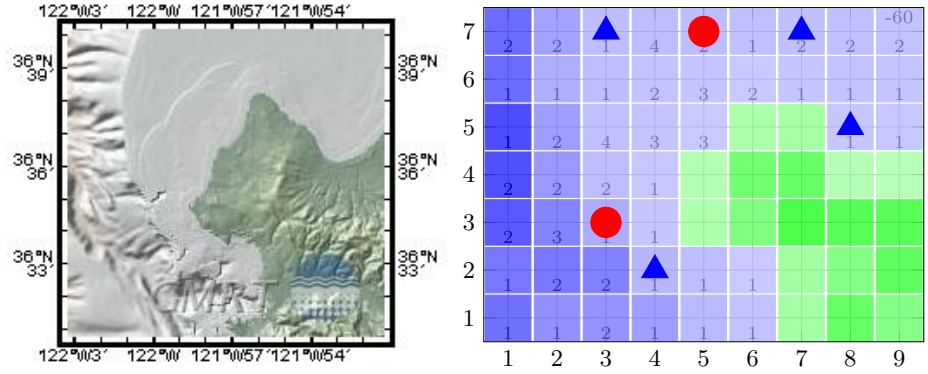

FIGURE 1. Left: The Monterey Peninsula area tile (Ryan et al. 2009) as raw input data (365 cols, 285 rows). Right: Optimal placement of 2 sources (red circles) and 4 receivers (blue triangles) on a 9x7 grid. Numbers ≥ 1 at each coordinate show multiplicity of coverage.

5. Conclusions

When facing a bilinear constraint of the type $x^T Ay \leq b$ with binary variable vectors x, y and an indefinite matrix A, a user may choose among several linearization techniques that have been developed over the last five decades. Today, classical MIP solvers (such as CPLEX) offer features to automatically deal with such nonlinear constraints, lifting the user's burden. However, as our results demonstrate, it is still worthwhile to consider the knowledge of the past, and not to blindly rely on the solver. Because it is difficult to determine *a priori* which method will outperform the others, and this also depends on the kind of measurement for success, it may be necessary to implement and test all of them.

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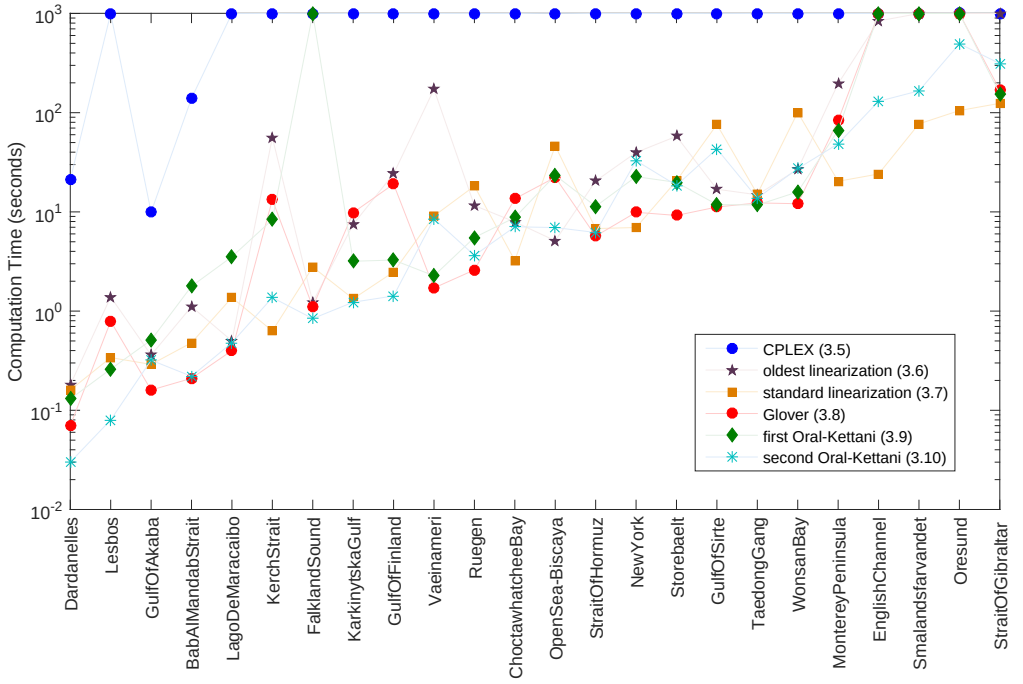

FIGURE 2. Comparison of the solution time for all model formulations and all instances.

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