

## Renormalization in general theories with inter-generation mixing

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### Abstract

We derive general and explicit expressions for the unrenormalized and renormalized dressed propagators of fermions in parity-nonconserving theories with inter-generation mixing. The mass eigenvalues, the corresponding mass counterterms, and the effect of inter-generation mixing on their determination are discussed. Invoking the Aoki-Hioki-Kawabe-Konuma-Muta renormalization conditions and employing a number of very useful relations from Matrix Algebra, we show explicitly that the renormalized dressed propagators satisfy important physical properties.

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## I. INTRODUCTION

The aim of this paper is to derive general and explicit expressions for the unrenormalized and renormalized dressed propagators of fermions in parity-nonconserving theories with inter-generation mixing, and to discuss their important physical properties and implications.

The results presented here immediately apply to the Standard Theory of Elementary Particle Physics, usually referred to as the Standard Model (SM), as well as its extensions. As has been known for a long time, the quark fields are subject to inter-generation mixing, as implemented by the Cabibbo-Kobayashi-Maskawa (CKM) [1] quark mixing matrix. Since neutrino oscillations have been observed experimentally and lower mass bounds have been established, the lepton fields are known to also undergo inter-generation mixing. An early treatment of flavor-changing self-energies, both for leptons and quarks in bound states, which, however, focuses on finite renormalization effects, may be found in Ref. [2]. On the other hand, our treatment is quite general and takes into account the full mixing amplitudes. The renormalization of the CKM matrix has been recently discussed by several authors; see, for example, Ref. [3] and references cited therein. Mixing renormalization has also been worked out for theories involving Majorana neutrinos [4].

This paper is organized as follows. Section II discusses the derivation of the unrenormalized dressed propagators. The mass eigenvalues, the corresponding mass counterterms, and the effect of inter-generation mixing on their determination are also analyzed. Section III discusses the renormalization of the dressed propagators. Invoking the Aoki-Hioki-Kawabe-Konuma-Muta (AHKKM) renormalization conditions and employing very useful relations from Matrix Algebra, it is shown explicitly that the renormalized dressed propagators satisfy important physical properties. Section IV contains our conclusions. The Appendix explains how to derive the two-loop expression for the mass eigenvalues presented in Sec. II, and how to express the mass counterterms in terms of the unrenormalized self-energies.

## II. UNRENORMALIZED DRESSED PROPAGATOR OF MIXED FERMION SYSTEM

As is well known, the unrenormalized mass matrix can be brought to diagonal form with non-negative eigenvalues by means of bi-unitary transformations on the left- and right-

handed fields. On this basis, the unrenormalized inverse propagator is  $-iI_{ij}(\not{p})$ , where

$$I_{ij}(\not{p}) = (\not{p} - m_i^0)\delta_{ij} - \Sigma_{ij}(\not{p}), \quad (1)$$

$i, j$  are flavor indices<sup>1</sup> and the self-energies  $\Sigma_{ij}(\not{p})$  are given by

$$\Sigma_{ij}(\not{p}) = [\not{p}(B_+)_{ij} + (A_+)_{ij}] a_+ + [\not{p}(B_-)_{ij} + (A_-)_{ij}] a_-. \quad (2)$$

In Eq. (2),  $(A_\pm)_{ij}$ ,  $(B_\pm)_{ij}$  are Lorentz-invariant functions of  $p^2$  and  $a_\pm = (1 \pm \gamma_5)/2$  are the chiral projectors.<sup>2</sup>

Equations (1) and (2) can be written in compact form, as

$$I(\not{p}) = (\not{p}S_+ - T_+)a_+ + (\not{p}S_- - T_-)a_-, \quad (3)$$

where  $S_+$  and  $T_+$  are matrices defined by

$$(S_\pm)_{ij} = \delta_{ij} - (B_\pm)_{ij}, \quad (T_\pm)_{ij} = m_i^0\delta_{ij} + (A_\pm)_{ij}. \quad (4)$$

The unrenormalized dressed propagator is  $iP(\not{p}) = i(I(\not{p}))^{-1}$ .<sup>3</sup> Writing  $(I(\not{p}))^{-1} = (\not{p}U_+ - V_+)a_+ + (\not{p}U_- - V_-)a_-$ , we find the relations

$$S_+V_+ + T_-U_+ = 0, \quad (5)$$

$$S_-V_- + T_+U_- = 0, \quad (6)$$

$$p^2S_+U_- + T_-V_- = \mathbb{1}, \quad (7)$$

$$p^2S_-U_+ + T_+V_+ = \mathbb{1}, \quad (8)$$

where  $\mathbb{1}$  stands for the unit matrix.

In order to express  $U_\pm$  and  $V_\pm$  in terms of  $S_\pm$  and  $T_\pm$ , we first solve for  $V_-$  in Eq. (6) and insert the result in Eq. (7). This leads to

$$U_- = [p^2S_+ - T_-(S_-)^{-1}T_+]^{-1}, \quad (9)$$

$$V_- = -(S_-)^{-1}T_+U_-. \quad (10)$$

<sup>1</sup> In this paper, repeated indices are not summed, unless a summation symbol is explicitly included.

<sup>2</sup> Throughout this paper, we adopt the notational conventions of Bjorken and Drell [5].

<sup>3</sup> Here and in the following, the matrix  $iP(\not{p})$  is referred to as the unrenormalized propagator. The particle propagators are the elements of this matrix, namely  $iP_{ij}(\not{p})$ . An analogous denomination is used in Sec. III for the renormalized propagator  $i\hat{P}(\not{p})$  and the renormalized particle propagators  $i\hat{P}_{ij}(\not{p})$ .

Next we solve for  $V_+$  in Eq. (5) and insert the result in Eq. (8), which leads to

$$U_+ = [p^2 S_- - T_+(S_+)^{-1} T_-]^{-1}, \quad (11)$$

$$V_+ = -(S_+)^{-1} T_- U_+. \quad (12)$$

More convenient forms for  $U_{\pm}$  are obtained by writing

$$U_- = [(p^2 - T_-(S_-)^{-1} T_+(S_+)^{-1}) S_+]^{-1} = (S_+)^{-1} (p^2 - DC)^{-1}, \quad (13)$$

$$U_+ = [(p^2 - T_+(S_+)^{-1} T_-(S_-)^{-1}) S_-]^{-1} = (S_-)^{-1} (p^2 - CD)^{-1}, \quad (14)$$

where

$$C = T_+(S_+)^{-1}, \quad D = T_-(S_-)^{-1}. \quad (15)$$

It is also convenient to introduce the matrices

$$E = (S_+)^{-1} T_-, \quad F = (S_-)^{-1} T_+. \quad (16)$$

Using Eqs (9)–(14), (15), and (16), the unrenormalized dressed propagator is given by  $iP$ , where

$$P = (\not{p} + E)(S_-)^{-1} (p^2 - CD)^{-1} a_+ + (\not{p} + F)(S_+)^{-1} (p^2 - DC)^{-1} a_-, \quad (17)$$

which is fully expressed in terms of the self-energy matrices  $S_{\pm}$  and  $T_{\pm}$ . The matrices  $(p^2 - CD)^{-1}$  and  $(p^2 - DC)^{-1}$  are related by similarity transformations, as

$$(p^2 - CD)^{-1} = C(p^2 - DC)^{-1} C^{-1} = D^{-1} (p^2 - DC)^{-1} D. \quad (18)$$

Writing

$$(p^2 - CD)^{-1} = \frac{\alpha_+}{\det(p^2 - CD)}, \quad (p^2 - DC)^{-1} = \frac{\alpha_-}{\det(p^2 - DC)}, \quad (19)$$

where  $\alpha_+$  and  $\alpha_-$  are the corresponding adjoint matrices,<sup>4</sup> we see that the determinants are equal and that  $\alpha_+$  and  $\alpha_-$  are related by the same similarity transformations as in Eq. (18).

Thus, the squared mass eigenvalues  $M_i^2$  are the zeros of  $\det(p^2 - CD)$ , namely they satisfy

$$\det(M_i^2 - Y(M_i^2)) = 0, \quad (20)$$

$$Y(p^2) = (CD)(p^2). \quad (21)$$

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<sup>4</sup> Given a square matrix  $M$ , in this paper the adjoint matrix  $\text{Adj } M$  means the transpose of the matrix whose elements are the cofactors of  $M$  (see, for example, Ref. [6].) We recall that the cofactor  $C_{ij}$  of the element  $m_{ij}$  of  $M$  is  $(-1)^{i+j}$  times the determinant of the matrix obtained by deleting the  $i$ -th row and the  $j$ -th column of  $M$ .

The off-diagonal elements of  $Y(p^2)$  arise from inter-generation mixing and are, therefore, of  $\mathcal{O}(g^2)$  or higher, where  $g$  is a generic weak-interaction gauge coupling. As a consequence, if terms of  $\mathcal{O}(g^4)$  are neglected, only the diagonal elements of  $p^2 - Y(p^2)$  contribute to the determinant, and the eigenvalues are of the form

$$\tilde{M}_i^2 = \tilde{Y}_{ii}(\tilde{M}_i^2) + \mathcal{O}(g^4), \quad (22)$$

where  $\tilde{Y}_{ii}(p^2)$  denotes  $Y_{ii}(p^2)$  in the absence of  $\mathcal{O}(g^4)$  contributions. If, instead, terms of  $\mathcal{O}(g^4)$  are retained, but three-loop contributions and higher are neglected, there are two additional effects: (i) there are now terms of  $\mathcal{O}(g^4)$  in  $Y_{ii}(p^2)$  and (ii) the non-diagonal elements  $Y_{ij}(p^2)$  ( $i \neq j$ ) contribute to the determinant. As a consequence, the mass eigenvalues are now of the form

$$M_i^2 = Y_{ii}(M_i^2) + \sum_{j \neq i} \frac{(Y_{ij} Y_{ji})(M_i^2)}{M_i^2 - M_j^2} + \mathcal{O}(g^6, g^4 \alpha_s). \quad (23)$$

In the Appendix, we outline the derivation of Eq. (23) and show how Eqs. (4), (15), (21), and (23) can be used to express the mass counterterms in terms of the unrenormalized self-energy functions  $A^\pm$  and  $B^\pm$ , in the approximation of neglecting three-loop-contributions.

### III. RENORMALIZED DRESSED PROPAGATOR OF MIXED FERMION SYSTEM

In order to renormalize  $P$  [cf. Eq. (17)], we recall that the unrenormalized propagator is the Fourier transform of  $\langle 0 | T(\Psi^0(x) \bar{\Psi}^0(0)) | 0 \rangle$ , where the zero superscripts denote the unrenormalized fields. In our case, they are column and row fields with components labeled by flavor indices. In the following discussion, we assume for simplicity that the fermions are stable. Decomposing the fields into right- and left-handed components, as

$$\Psi^0 = \Psi_+^0 + \Psi_-^0, \quad \bar{\Psi}^0 = \bar{\Psi}_+^0 + \bar{\Psi}_-^0, \quad (24)$$

where  $\Psi_\pm^0 = a_\pm \Psi$  and  $\bar{\Psi}_\pm^0 = \bar{\Psi} a_\mp$ , and taking into account the effect of the chiral projectors  $a_\pm$ , it is easy to see that the first, second, third, and fourth terms of  $P$  arise from  $\langle 0 | T(\Psi_-^0 \bar{\Psi}_-^0) | 0 \rangle$ ,  $\langle 0 | T(\Psi_+^0 \bar{\Psi}_-^0) | 0 \rangle$ ,  $\langle 0 | T(\Psi_+^0 \bar{\Psi}_+^0) | 0 \rangle$ , and  $\langle 0 | T(\Psi_-^0 \bar{\Psi}_+^0) | 0 \rangle$ , respectively.

Shifting the fields according to

$$\Psi_+^0 = Z_+^{1/2} \Psi_+, \quad \bar{\Psi}_+^0 = \bar{\Psi}_+ (Z_+^{1/2})^\dagger, \quad (25)$$

$$\Psi_-^0 = Z_-^{1/2} \Psi_-, \quad \bar{\Psi}_-^0 = \bar{\Psi}_- (Z_-^{1/2})^\dagger, \quad (26)$$

where  $\Psi_{\pm}$  are the renormalized fields, we see that the four terms in  $P$  are multiplied on the left and right by various combinations of  $Z_{\pm}^{1/2}$  and  $(Z_{\pm}^{1/2})^{\dagger}$  factors. Since the time-ordered products are now expressed in terms of renormalized fields, in order to obtain the renormalized propagator  $i\hat{P}$ , we must divide out such factors. Specifically, the first term in  $P$  must be multiplied on the left by  $Z_-^{-1/2}$  and on the right by  $(Z_-^{-1/2})^{\dagger}$ , the second term by  $Z_+^{-1/2}$  on the left and  $(Z_-^{-1/2})^{\dagger}$  on the right, the third term by  $Z_+^{-1/2}$  on the left and  $(Z_+^{-1/2})^{\dagger}$  on the right, and the fourth term by  $Z_-^{-1/2}$  on the left and  $(Z_+^{-1/2})^{\dagger}$  on the right.

Thus, the renormalized propagator is  $i\hat{P}$ , where

$$\begin{aligned}\hat{P} = & (\not{p}Z_-^{-1/2} + Z_+^{-1/2}E)(S_-)^{-1}(p^2 - CD)^{-1}(Z_-^{-1/2})^{\dagger}a_+ \\ & + (\not{p}Z_+^{-1/2} + Z_-^{-1/2}F)(S_+)^{-1}(p^2 - DC)^{-1}(Z_+^{-1/2})^{\dagger}a_-.\end{aligned}\quad (27)$$

Recalling Eqs. (15) and (16), we see that the third and fourth terms are related to the first and second ones, respectively, by the exchange  $+ \leftrightarrow -$ .

We now note that the  $Z^{-1/2}$  factors in Eq. (27) can be absorbed in a redefinition of the self-energy matrices  $S_{\pm}$  and  $T_{\pm}$ , namely

$$\hat{S}_{\pm} = (Z_{\pm}^{1/2})^{\dagger}S_{\pm}Z_{\pm}^{1/2}, \quad \hat{T}_{\pm} = (Z_{\mp}^{1/2})^{\dagger}T_{\pm}Z_{\pm}^{1/2}.\quad (28)$$

Using Eq. (28),  $\hat{P}$  can be written in the compact form

$$\hat{P} = (\not{p} + \hat{E})(\hat{S}_-)^{-1}(p^2 - \hat{C}\hat{D})^{-1}a_+ + (\not{p} + \hat{F})(\hat{S}_+)^{-1}(p^2 - \hat{D}\hat{C})^{-1}a_-, \quad (29)$$

where

$$\hat{C} = \hat{T}_+\hat{S}_+^{-1}, \quad \hat{D} = \hat{T}_-\hat{S}_-^{-1}, \quad \hat{E} = \hat{S}_+^{-1}\hat{T}_-, \quad \hat{F} = \hat{S}_-^{-1}\hat{T}_+.\quad (30)$$

In particular,  $\hat{C}\hat{D}$  and  $CD$  are related by a similarity transformation, as

$$\hat{C}\hat{D} = (Z_-^{1/2})^{\dagger}CD(Z_-^{1/2})^{\dagger-1}, \quad (31)$$

so that  $\det(p^2 - \hat{C}\hat{D}) = \det(p^2 - CD)$  and the mass eigenvalues are the zeros of either determinant. The matrices  $\hat{S}_{\pm}$ ,  $\hat{T}_{\pm}$ ,  $\hat{C}$ ,  $\hat{D}$ ,  $\hat{E}$ , and  $\hat{F}$  are the renormalized counterparts of  $S_{\pm}$ ,  $T_{\pm}$ ,  $C$ ,  $D$ ,  $E$ , and  $F$ , respectively.

In analogy with Eq. (18), we have the relations

$$(p^2 - \hat{C}\hat{D})^{-1} = \hat{C}(p^2 - \hat{D}\hat{C})^{-1}\hat{C}^{-1} = \hat{D}^{-1}(p^2 - \hat{D}\hat{C})^{-1}\hat{D}.\quad (32)$$

We note that  $\hat{C}\hat{D}$  and  $\hat{F}\hat{E}$  are also related by a similarity transformation and so are  $(p^2 - \hat{C}\hat{D})^{-1}$  and  $(p^2 - \hat{F}\hat{E})^{-1}$ , namely

$$\hat{S}_-^{-1}\hat{C}\hat{D}\hat{S}_- = \hat{F}\hat{E}, \quad \hat{S}_-^{-1}(p^2 - \hat{C}\hat{D})^{-1}\hat{S}_- = (p^2 - \hat{F}\hat{E})^{-1}. \quad (33)$$

Interchanging  $+ \leftrightarrow -$ , we obtain

$$\hat{S}_+^{-1}\hat{D}\hat{C}\hat{S}_+ = \hat{E}\hat{F}, \quad \hat{S}_+^{-1}(p^2 - \hat{D}\hat{C})^{-1}\hat{S}_+ = (p^2 - \hat{E}\hat{F})^{-1}. \quad (34)$$

Using Eqs. (30), (32), (33), and (34), Eq. (29) can be cast in the alternative form

$$\hat{P} = a_-(p^2 - \hat{F}\hat{E})^{-1}\hat{S}_-^{-1}(\not{p} + \hat{C}) + a_+(p^2 - \hat{E}\hat{F})^{-1}\hat{S}_+^{-1}(\not{p} + \hat{D}). \quad (35)$$

It differs from Eq. (29) in that the chiral projectors  $a_{\pm}$  are on the left side of the expression. In both Eqs. (29) and (35), the cofactors of  $a_-$  and  $a_+$  are related by the exchange  $+ \leftrightarrow -$ .

Writing

$$(p^2 - \hat{C}\hat{D})^{-1} = \frac{\hat{\alpha}_+}{\det(p^2 - \hat{C}\hat{D})}, \quad (p^2 - \hat{D}\hat{C})^{-1} = \frac{\hat{\alpha}_-}{\det(p^2 - \hat{D}\hat{C})}, \quad (36)$$

$$(p^2 - \hat{F}\hat{E})^{-1} = \frac{\hat{\beta}_+}{\det(p^2 - \hat{F}\hat{E})}, \quad (p^2 - \hat{E}\hat{F})^{-1} = \frac{\hat{\beta}_-}{\det(p^2 - \hat{E}\hat{F})}, \quad (37)$$

where  $\hat{\alpha}_{\pm}$  and  $\hat{\beta}_{\pm}$  are the corresponding adjoint matrices (cf. Footnote 4), the similarity relations in Eqs. (32)–(34) tell us that

$$\det(p^2 - \hat{C}\hat{D}) = \det(p^2 - \hat{D}\hat{C}) = \det(p^2 - \hat{F}\hat{E}) = \det(p^2 - \hat{E}\hat{F}), \quad (38)$$

and

$$\hat{S}_{\mp}^{-1}\hat{\alpha}_{\pm} = \hat{\beta}_{\pm}\hat{S}_{\mp}^{-1}, \quad (39)$$

a relation that plays an important rôle in our discussion of the propagator's properties. We recall that in the previous equations,  $\hat{S}_{\pm}$ ,  $\hat{T}_{\pm}$ ,  $\hat{C}$ ,  $\hat{D}$ ,  $\hat{E}$ ,  $\hat{F}$ ,  $\hat{\alpha}_{\pm}$ , and  $\hat{\beta}_{\pm}$  are functions of  $p^2$ .

We now turn our attention to the renormalization conditions. As emphasized in the seminal work of AHKKM [7], a fundamental physical property of the renormalized propagator  $i\hat{P}$  is that, as  $\not{p} \rightarrow m_n$ , where  $m_n$  is one of the mass eigenvalues, the pole  $(\not{p} - m_n)^{-1}$  should be present only in the diagonal element  $i\hat{P}_{nn}$  of the renormalized propagator matrix. In order to implement this property, as well as the conventional requirement that the pole residue equals the imaginary unit, AHKKM proposed suitable conditions on the renormalized inverse propagators, which were described both graphically and mathematically.

Recalling Eq. (3), in our general matrix notation the renormalized inverse propagator is  $-i\hat{I}(\not{p})$ , where

$$\hat{I}(\not{p}) = (\not{p}\hat{S}_+ - \hat{T}_+)a_+ + (\not{p}\hat{S}_- - \hat{T}_-)a_-. \quad (40)$$

An alternative expression is

$$\hat{I}(\not{p}) = a_-(\not{p}\hat{S}_+ - \hat{T}_-) + a_+(\not{p}\hat{S}_- - \hat{T}_+), \quad (41)$$

where the chiral projectors  $a_{\pm}$  are placed on the left. The homogeneous AHKKM renormalization conditions read

$$\bar{u}_n(\not{p})\hat{I}_{nl}(\not{p}) = 0, \quad (42)$$

$$\hat{I}_{ln}(\not{p})u_n(\not{p}) = 0, \quad (43)$$

where  $u_n(\not{p})$  is a spinor that satisfies  $\not{p}u_n(\not{p}) = m_n u_n(\not{p})$ ,  $\bar{u}_n(\not{p})$  its hermitian adjoint, and  $n$  and  $l$  are flavor indices.

Inserting Eq. (40) into Eq. (42), we have

$$\left[ m_n \hat{S}_{\pm}(m_n^2) - \hat{T}_{\pm}(m_n^2) \right]_{nl} = 0. \quad (44)$$

Multiplying on the right by  $\hat{S}_{\pm}^{-1}(m_n^2)_{lj}$ , summing over  $l$ , and remembering the definitions in Eq. (30), this becomes

$$\hat{C}_{nj}(m_n^2) = \hat{D}_{nj}(m_n^2) = m_n \delta_{nj}, \quad (45)$$

which implies

$$(\hat{C}\hat{D})_{nm}(m_n^2) = m_n^2, \quad (\hat{C}\hat{D})_{nj}(m_n^2) = 0 \quad (j \neq n), \quad (46)$$

with the analogous result for  $(\hat{D}\hat{C})(m_n^2)$ .

Inserting Eq. (41) into Eq. (43), recalling the definitions in Eq. (30), and carrying out the analogous analysis, we obtain

$$\hat{E}_{in}(m_n^2) = \hat{F}_{in}(m_n^2) = m_n \delta_{in}, \quad (47)$$

which leads to

$$(\hat{E}\hat{F})_{nn}(m_n^2) = m_n^2, \quad (\hat{E}\hat{F})_{in}(m_n^2) = 0 \quad (i \neq n), \quad (48)$$

and the analogous result for  $(\hat{F}\hat{E})(m_n^2)$ .

Equation (46) tells us that, as  $p^2 \rightarrow m_n^2$ , all the elements in the  $n$ -th row of  $p^2 - \hat{C}\hat{D}$  and  $p^2 - \hat{D}\hat{C}$  vanish. Therefore, the only non-vanishing cofactors of  $(p^2 - \hat{C}\hat{D})$  and  $(p^2 - \hat{D}\hat{C})$  are



those corresponding to the elements of that row, namely the cofactors  $C_{nl}$ . Since the adjoint matrices are the transpose of the cofactor matrices (cf. Footnote 4), we conclude that the only non-vanishing elements of  $\hat{\alpha}_{\pm}(m_n^2)$  are those in the  $n$ -th column, namely the elements  $(\hat{\alpha}_{\pm})_{in}(m_n^2)$ . Similarly, from Eq. (48) we see that, as  $p^2 \rightarrow m_n^2$ , all the elements in the  $n$ -th column of  $p^2 - \hat{F}\hat{E}$  and  $p^2 - \hat{E}\hat{F}$  vanish. Consequently, the only non-vanishing elements of  $\hat{\beta}_{\pm}(m_n^2)$  are those in the  $n$ -th row, namely  $(\hat{\beta}_{\pm})_{nj}(m_n^2)$ . In combination with Eq. (39), these results imply that, as  $p^2 \rightarrow m_n^2$ , the only non-vanishing elements of the matrices  $\hat{S}_{\mp}^{-1}\hat{\alpha}_{\pm}$  and  $\hat{\beta}_{\pm}\hat{S}_{\mp}^{-1}$  are the diagonal  $nn$  elements  $(\hat{S}_{\mp}^{-1}\hat{\alpha}_{\pm})_{nn}(m_n^2) = (\hat{\beta}_{\pm}\hat{S}_{\mp}^{-1})_{nn}(m_n^2)$ . Thus,

$$(\hat{S}_{\mp}^{-1}\hat{\alpha}_{\pm})_{ij}(m_n^2) = (\hat{\beta}_{\pm}\hat{S}_{\mp}^{-1})_{ij}(m_n^2) = 0 \quad (i \text{ or } j \neq n). \quad (49)$$

To examine the effect of these results on the renormalized propagators, we insert Eq. (36) and Eq. (37) into Eq. (29) and Eq. (35), respectively. Recalling Eq. (38), we obtain

$$\hat{P} = \frac{(\not{p} + \hat{E})(\hat{S}_-)^{-1}\hat{\alpha}_+a_+ + (\not{p} + \hat{F})(\hat{S}_+)^{-1}\hat{\alpha}_-a_-}{\det(p^2 - \hat{C}\hat{D})} \quad (50)$$

from Eq. (29), and

$$\hat{P} = \frac{a_-\hat{\beta}_+(\hat{S}_-)^{-1}(\not{p} + \hat{C}) + a_+\hat{\beta}_-(\hat{S}_+)^{-1}(\not{p} + \hat{D})}{\det(p^2 - \hat{F}\hat{E})} \quad (51)$$

from Eq. (35).

Using Eqs. (45), (47), and (49), one readily verifies that, as  $p^2 \rightarrow m_n^2$ , the only non-vanishing elements in the numerators of Eqs. (50) and (51) are, in fact, the diagonal  $nn$  elements. Thus, the explicit expressions of the renormalized propagator  $i\hat{P}$ , given in Eqs. (29), (35), (50), and (51), indeed satisfy the fundamental physical property that the  $(\not{p} - m_n)^{-1}$  pole is present only in the diagonal element  $i\hat{P}_{nn}$  of the propagator matrix.

The inhomogeneous AHKKM renormalization conditions are

$$\frac{1}{\not{p} - m_n} \hat{I}_{nn}(\not{p}) u_n(\not{p}) = u_n(\not{p}), \quad (52)$$

$$\bar{u}_n(\not{p}) \hat{I}_{nn}(\not{p}) \frac{1}{\not{p} - m_n} = \bar{u}_n(\not{p}). \quad (53)$$

Inserting Eq. (40) into Eq. (52), expanding the numerator about  $\not{p} = m_n$ , and using Eq. (44), we find the renormalization conditions

$$\left\{ \hat{S}_-(m_n^2) + m_n \left[ m_n(\hat{S}_+ + \hat{S}_-) - \hat{T}_+ - \hat{T}_- \right]' \right\}_{nn} = 1, \quad (54)$$

$$(\hat{S}_-)_{nn}(m_n^2) = (\hat{S}_+)_{nn}(m_n^2),$$

where the prime symbol stands for the derivative with respect to  $p^2$ , evaluated at  $p^2 = m_n^2$ . Inserting Eq. (41) into Eq. (53), we obtain the same result.

In order to analyze the effect of Eq. (54), we evaluate the residue of the  $(\not{p} - m_n)^{-1}$  pole in  $\hat{P}$  using Eq. (50) and focus on the  $a_+$  term. We expand  $\det(p^2 - \hat{C}\hat{D})$  about  $p^2 = m_n^2$  through  $\mathcal{O}(p^2 - m_n^2)$ . Since  $p^2 = m_n^2$  is a zero of the determinant, the first term vanishes, and we have

$$\det(p^2 - \hat{C}\hat{D}) = \left[ \det(p^2 - \hat{C}\hat{D}) \right]' (p^2 - m_n^2) + \dots \quad (55)$$

Using the well known expression

$$(\det M)' = \text{Tr} (M' \text{Adj} M), \quad (56)$$

the r.h.s. of Eq. (55) becomes  $\text{Tr} \left\{ \hat{\alpha}_+(m_n^2) \left[ 1 - (\hat{C}\hat{D})' \right] \right\} (p^2 - m_n^2) + \dots$ . Multiplying by  $\not{p} - m_n$ , taking the limit  $\not{p} \rightarrow m_n$ , and recalling Eqs. (47) and (49), we see that the residue of the  $(\not{p} - m_n)^{-1}$  pole in the  $a_+$  term of Eq. (50) is

$$\text{Res}_+ = \frac{(\hat{S}_-^{-1} \hat{\alpha}_+)_{nn}}{\text{Tr} \left\{ \hat{\alpha}_+ \left[ 1 - (\hat{C}\hat{D})' \right] \right\}}. \quad (57)$$

Here and in the following, it is understood that all the functions are evaluated at  $p^2 = m_n^2$ . To simplify this expression, we insert  $\hat{S}_- \hat{S}_-^{-1} = 1$  in the argument of the trace. Recalling again Eq. (49), we find

$$\text{Tr} \left\{ \hat{S}_- \hat{S}_-^{-1} \hat{\alpha}_+ \left[ 1 - (\hat{C}\hat{D})' \right] \right\} = \left( \hat{S}_-^{-1} \hat{\alpha}_+ \right)_{nn} \left\{ \left[ 1 - (\hat{C}\hat{D})' \right] \hat{S}_- \right\}_{nn}, \quad (58)$$

and the residue becomes

$$\text{Res}_+ = \frac{1}{\left\{ \left[ 1 - (\hat{C}\hat{D})' \right] \hat{S}_- \right\}_{nn}}. \quad (59)$$

Taking into account Eqs. (45)–(47), Eq. (59) becomes

$$\text{Res}_+ = \frac{1}{\left\{ \hat{S}_- + m_n \left[ m_n (\hat{S}_+ + \hat{S}_-) - \hat{T}_+ - \hat{T}_- \right]' \right\}_{nn}}. \quad (60)$$

Thus, the renormalization condition of Eq. (54) indeed implies that

$$\text{Res}_+ = 1. \quad (61)$$

Calling  $\text{Res}_-$  the residue of the  $(\not{p} - m_n)^{-1}$  pole in the  $a_-$  term of Eq. (50), an analogous analysis shows that

$$\text{Res}_- = 1. \quad (62)$$

We conclude that, when the inhomogeneous renormalization condition of Eq. (52) is imposed, the poles in our explicit expressions for the renormalized propagator [cf. Eqs. (29), (35), (50), and (51)] have residues  $i$ .

#### IV. CONCLUSIONS

We derived general and explicit expressions for the unrenormalized and renormalized dressed propagators of fermions in parity-nonconserving theories with inter-generation mixing [cf. Eqs. (17), (29), (35), (50), and (51)]. We analyzed the determination of the mass eigenvalues and the corresponding mass counterterms in the approximation of neglecting three-loop contributions [cf. Eqs. (23) and (A9)]. In particular, we discussed the effect of inter-generation mixing on these determinations. Using the AHKKM renormalization conditions and applying very useful relations from Matrix Algebra, we showed explicitly that our renormalized dressed propagator [cf. Eqs. (29), (35), (50), and (51)], which is valid to all orders in perturbation theory, satisfies important physical properties. In turn, this demonstrates in a clear manner that the AHKKM renormalization conditions are also valid to any order of perturbation theory.

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#### Appendix A: Appendix

In this appendix, we outline the derivation of Eq. (23) in the approximation of neglecting three-loop contributions and show how it can be applied to express the mass counterterms in terms of the basic self-energy functions  $(A_{\pm})_{ij}$  and  $(B_{\pm})_{ij}$  in Eq. (2). For simplicity, we consider the three-generation case.

As explained in the paragraph containing Eqs. (20) and (21), the mass eigenvalues are

the zeros of  $\det(p^2 - Y(p^2))$ , where  $Y(p^2) = (CD)(p^2)$  and the matrices  $C$  and  $D$  are defined in Eq. (15). Using Eqs. (4), (15), and (21), we find

$$Y = (M^0)^2 + Z, \quad (\text{A1})$$

where  $M^0$  is the diagonal bare mass matrix with elements  $m_i^0$  and

$$\begin{aligned} Z = & (M^0)^2 B_- (1 + B_-) + M^0 (A_- + A_- B_- + B_+ A_-) + M^0 B_+ (1 + B_+) M^0 \\ & + A_+ (1 + B_+) M^0 + A_+ M^0 B_- + M^0 B_+ M^0 B_- + A_+ A_- . \end{aligned} \quad (\text{A2})$$

In Eq. (A1), we have separated out the squared bare mass term  $(M^0)^2$  and the one- and two-loop contributions contained in  $Z$ . We recall that  $A_{\pm}$ ,  $B_{\pm}$ ,  $Y$ , and, consequently,  $Z$  are functions of  $p^2$ . It is further convenient to split

$$(M^0)^2 = M^2 + \delta M^2, \quad (\text{A3})$$

where  $M$  is the renormalized mass matrix whose elements are the mass eigenvalues and  $\delta M^2$  is the mass counterterm matrix. Thus,

$$Y = M^2 + X, \quad (\text{A4})$$

where

$$X = \delta M^2 + Z. \quad (\text{A5})$$

We note that, in Eq. (A4),  $M^2$  contains the zeroth-order terms, while  $X$  contains the one- and two-loop contributions.

Neglecting three-loop contributions, in the three-generation case the eigenvalue equation  $\det(p^2 - Y(p^2)) = 0$  becomes

$$(p^2 - Y_{11})(p^2 - Y_{22})(p^2 - Y_{33}) - (p^2 - Y_{11})Y_{23}Y_{32} - (p^2 - Y_{22})Y_{13}Y_{31} - (p^2 - Y_{33})Y_{12}Y_{21} = 0. \quad (\text{A6})$$

Consider the neighborhood of  $p^2 = M_1^2$ , where  $M_1$  is one of the mass eigenvalues: dividing by  $(p^2 - Y_{22})(p^2 - Y_{33})$ , we have

$$(p^2 - Y_{11}) \left[ 1 - \frac{Y_{23}Y_{32}}{(p^2 - Y_{22})(p^2 - Y_{33})} \right] = \frac{Y_{13}Y_{31}}{p^2 - Y_{33}} + \frac{Y_{12}Y_{21}}{p^2 - Y_{22}}. \quad (\text{A7})$$

The factors  $Y_{23}Y_{32}$ ,  $Y_{13}Y_{31}$ , and  $Y_{12}Y_{21}$  are of two-loop order or higher. As  $p^2 \rightarrow M_1^2$ , we see from Eq. (A4) that, to leading order, we have  $p^2 - Y_{22} = M_1^2 - M_2^2$  and  $p^2 - Y_{33} = M_1^2 - M_3^2$ . Thus, neglecting three-loop contributions, as  $p^2 \rightarrow M_1^2$ , Eq. (A7) reduces to

$$M_1^2 = Y_{11}(M_1^2) + \frac{(Y_{12}Y_{21})(M_1^2)}{M_1^2 - M_2^2} + \frac{(Y_{13}Y_{31})(M_1^2)}{M_1^2 - M_3^2}, \quad (\text{A8})$$

which is a particular case of Eq. (23).

Recalling Eqs. (23), (A1), and (A3), the mass counterterms are then

$$\begin{aligned}
\delta M_i^2 &= (m_i^0)^2 - M_i^2 \\
&= (m_i^0)^2 - Y_{ii}(M_i^2) - \sum_{j \neq i} \frac{(Y_{ij} Y_{ji})(M_i^2)}{M_i^2 - M_j^2} \\
&= -Z_{ii}(M_i^2) - \sum_{j \neq i} \frac{(Z_{ij} Z_{ji})(M_i^2)}{M_i^2 - M_j^2},
\end{aligned} \tag{A9}$$

where  $Z$  is defined in Eq. (A2). In the last equality of Eq. (A9), we have replaced  $Y_{ij} \rightarrow Z_{ij}$ , since both are equal when  $i \neq j$  [cf. Eq. (A1)].

We note that, subject to our approximation, the amplitudes involving linear powers of  $A_{\pm}$  and  $B_{\pm}$  in Eq. (A2) contain both one- and two-loop contributions.

Using Eq. (A2), we find for the diagonal terms

$$\begin{aligned}
Z_{ii} &= (m_i^0)^2 (B_+ + B_- + B_+^2 + B_-^2)_{ii} + m_i^0 (A_+ + A_- + A_+ B_+ + A_- B_- + B_+ A_-)_{ii} + (A_+ A_-)_{ii} \\
&\quad + \sum_{j=1}^3 [m_j^0 (A_+)_{ij} (B_-)_{ji} + m_i^0 m_j^0 (B_+)_{ij} (B_-)_{ji}].
\end{aligned} \tag{A10}$$

We note that  $Z_{ii}$  depends not only on the bare fermion masses  $m_i^0$  and  $m_j^0$  displayed in Eq. (A10), but also on additional ones present in the loop diagrams. We refer generically to the latter as  $m_l^0$ . Consistently with our approximation, in the contributions of two-loop order, we replace the bare masses  $m_i^0$ ,  $m_j^0$ , and  $m_l^0$  by the mass eigenvalues  $M_i$ ,  $M_j$ , and  $M_l$ , respectively. In the contributions of one-loop order, we replace

$$m_i^0 = \left[ M_i^2 - Z_{ii}^{(1)}(M_i^2) \right]^{1/2}, \tag{A11}$$

and similarly for  $m_l^0$ . In Eq. (A11), the superscript (1) stands for the one-loop contribution, namely

$$Z_{ii}^{(1)} = M_i^2 (B_+^{(1)} + B_-^{(1)})(M_i^2) + M_i (A_+^{(1)} + A_-^{(1)})(M_i^2), \tag{A12}$$

with an analogous expression for  $Z_{ll}^{(1)}$ .

The contributions involving  $Z_{ij} Z_{ji}$  with  $j \neq i$  in Eq. (A9) are already of two-loop order or higher, so that in the off-diagonal amplitudes  $Z_{ij}$  with  $j \neq i$ , we simply replace  $m_i^0, m_j^0, m_l^0 \rightarrow M_i, M_j, M_l$ . In this way, subject to the approximation of neglecting three-loop contributions,

the mass counterterms  $\delta M_i^2$  given in Eq. (A9) are fully expressed in terms of the basic self-energies  $A_{\pm}(M_i^2)$  and  $B_{\pm}(M_i^2)$  of Eq. (2) and the mass eigenvalues.

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