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Zhi Yuan, Liana Amaya Moreno, Armin Fügenschuh, Anton Kaier and Swen Schlobach

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Herausgegeben von der
Professur für Angewandte Mathematik
Professor Dr. rer. nat. Armin Fügenschuh

Helmut-Schmidt-Universität / Universität der Bundeswehr Hamburg
Fachbereich Maschinenbau
Holstenhofweg 85
D-22043 Hamburg

Telefon: +49 (0)40 6541 3540
Fax: +49 (0)40 6541 3672

e-mail: appliedmath@hsu-hh.de
URL: http://www.hsu-hh.de/am

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Discrete Speed in Vertical Flight Planning

Zhi Yuan[1], Liana Amaya Moreno[1], Armin Fügenschuh[1], Anton Kaier[2], and Swen Schlobach[2]

[1] Professorship of Applied Mathematics, Department of Mechanical Engineering, Helmut Schmidt University, Hamburg, Germany
{yuanz,fuegenschuh,lamayamo}@hsu-hh.de
[2] Lufthansa Systems AG, Kelsterbach, Germany
{anton.kaier,swen.schlobach}@lhsystems.com

Abstract. Vertical flight planning concerns assigning optimal cruise altitude and speed to each trajectory-composing segment, such that the fuel consumption is minimized, and the arrival time constraints are satisfied. The previous work that assigns continuous speed to each segment leads to prohibitively long computation time. In this work, we propose a mixed integer linear programming model that assigns discrete speed. In particular, an all-but-one speed discretization scheme is found to scale well with problem size with only negligible objective deviation from using continuous speed. Extensive experiments with real-world instances have shown the practical effectiveness and feasibility of the proposed speed discretization approach.

Keywords: Flight planning; mixed integer programming; variable discretization; piecewise linear interpolation

1 Introduction

Air transport is an important component of many international logistics networks, including transportation of goods and people. Planning a fuel-efficient trajectory for each flight is a practically relevant and computationally hard optimization problem. Such a flight trajectory is in general four-dimensional (4D), which consists of horizontally a 2D route on the earth surface, vertically, a number of discrete admissible altitude levels, and a time dimension controlled by aircraft speed such that the flight can arrive within a certain strict time window. Due to the computational difficulty of such a 4D optimization problem, in practice, it is usually approached in two separate phases [2]: a horizontal optimization phase that searches for a trajectory on the earth surface consisting of a set of segments, to which an optimal altitude and speed is assigned to in the subsequent vertical optimization phase.

In this work, we focus on the vertical flight planning problem. The vertical profile of a flight includes five stages: take-off, climb, cruise, descend, and landing. Here we focus on the cruise stage, since it consumes the most fuel and time during a flight, while the other stages are relatively short and usually have fixed procedures due to safety considerations, which leaves little flexibility for fuel optimization. Computing an optimal altitude profile in the absence of wind can also provide estimated altitude for the 2D horizontal trajectory optimization [12]. Such a steady-atmosphere optimal altitude profile increases approximately linearly as fuel burns, however, it becomes irregular if altitude-dependent wind is considered [9]. A recent research by Lovegren and Hansman [10] confirmed a potential fuel saving of up to 3.5% by reassigning only altitude and speed to fixed flight trajectories, based on a study of 257 real flight operations in US. However, no time constraint is taken into account in their computation as in real-world airline operations. In such case, there exists a backward dynamic programming approach to compute fuel-optimal vertical profile [17].

A practical challenge in airline operations is to handle time constraints, especially delays, due to disruptions such as undesirable weather conditions, unexpected maintenance requirements, or waiting for passengers transferring from other already delayed flights. Such delays are typically recovered by increasing cruise speed, such that the next connection for passengers as well as for the aircraft and the crew can be reached [1]. Varying cruise speed may also be useful, e.g., to enter a time-dependent restricted airspace before it is closed (or after it is open), or when an aircraft is reassigned to a flight that used to be served by a faster (or slower) aircraft. The industrial standard suggests using a cost index procedure to vary cruise speed. This requires inputing a value that reflects the importance between time-related cost and fuel-related cost. The use of cost index was criticized due to the difficulty to quantify the time-related cost in the presence of delay, thus a dynamic cost index approach has been proposed to this end [6]. However, such approach still cannot handle explicitly hard time constraints, such as the about-to-close airspace. Aktürk et al. [1] formulate the time constraint explicitly into their MIP model in the context of aircraft rescheduling. Their model uses only constant speed. Yuan et al. [18, 17] explicitly include the time constraint and the use of variable speed in the vertical flight planning.

In [18, 17], the vertical flight planning problem with variable continuous speed is identified as a mixed-integer second-order cone programming (MISOCP) problem. The second-order cone constraints consist in calculating the flight time, and the integer variables consist in the 2D piecewise linear interpolation of the fuel consumption function, as well as the selection of discrete admissible altitude levels. The MISOCP model is reformulated as a mixed-integer linear programming model by applying linear approximation techniques [4, 8] and various piecewise linear approximation techniques. Despite the performance boost by using the linear approximation of the MISOCP model, the long computation time still prevents it from being a practically feasible approach.

In this present work, we study an alternative model for the vertical flight planning problem with discrete speed, i.e., only a set of speed levels can be selected for each segment. The use of speed discretization replaces the quadratic cone constraints by linear constraints, and it also reduces the 2D piecewise linear fuel function to 1D, at the expense of introducing more binary variables. We experimentally investigate the computation scalability of the discrete speed model, and carefully analyze the discretization error that leads to differences in the objective value. In particular, to balance the computational scalability and the discretization error, an all-but-one discretization, which discretizes speed on all but one segments, appears to be the most practically viable approach.

2 Vertical Flight Planning: The Problem Description

In the vertical flight planning problem (VFP), we are given a set of segments that compose the flight trajectory. The wind information for each segment is given in both the track direction (flight direction) and cross-track direction. The task is to assign an altitude and a speed to each segment, such that the flight consumes the least fuel while the arrival time constraints are satisfied. The altitude and the speed on each segment are invariant, and they can only be changed at the beginning of each segment, due to safety requirements. The cruise stage under consideration in this work starts after the initial climb has brought the aircraft above the crossover altitude of around 29 000 feet. Depending on the flight direction (eastwards or westwards), a set of discrete admissible flight altitudes are allowed. We consider IFR RVSM flight levels [13], where two adjacent flight levels usually differ by 1 000 feet, and the eastwards and westwards flights are allowed to fly in alternate flight levels.

The aircraft manufacturers provide the aircraft performance data as unit distance fuel consumption, which depends on three factors: aircraft speed, altitude, and weight. Each aircraft's unit distance fuel consumption data is measured at discrete levels of each of the three factors. For a given value that does not lie on these measured levels, it needs to be linearly interpolated by adjacent grid points. An illustrative example is given in Fig. 1. As can be observed, in general, the heavier the weight, the more fuel is consumed; besides, the higher the altitude, the less fuel is burnt. If no time constraint is considered, a fuel-optimal vertical profile can be determined by a backward dynamic programming approach [17] by enumerating all speed and altitude levels from the last segment to the first segment. However, if the arrival time constraint is enforced, such as to avoid delays and missing connections, assigning speed and altitude is not an easy task. Our previous works [18, 17] modeled the vertical flight planning problem as a mixed-integer nonlinear programming (MINLP) model. We further observed, if the arrival time window enforces speeding up the aircraft from its unconstrained fuel-optimal vertical profile, such as to avoid delays, then the MINLP model can be formulated as a mixed-integer second order cone programming (MISOCP) model. In this work, we compared our proposed discrete speed model to the continuous speed model in MISOCP, and adopted the instances in [18, 17] for speeding up the aircraft. But the discrete speed model can be potentially also applied to cases when the aircraft needs slowing down.

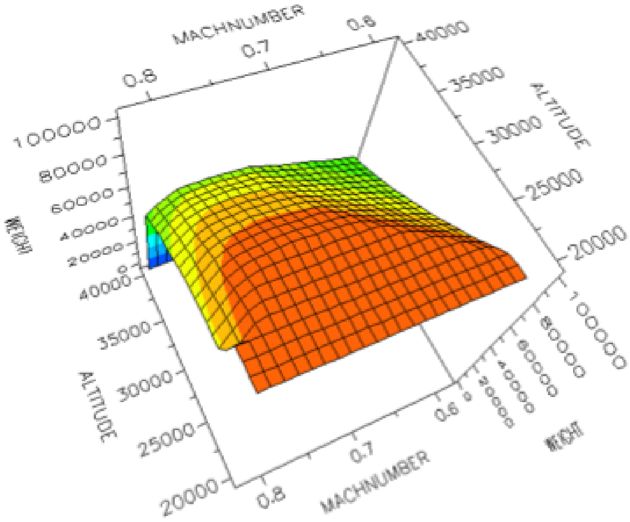

Fig. 1. Unit distance fuel consumption with respect to aircraft weight (in kg), altitude (in feet), and speed (in Mach number).

3 Vertical Flight Planning with Continuous Speed

To the best of our knowledge, the first mathematical programming model for the vertical flight planning problem was proposed in [18], which studies the use of variable speed during a flight in the absence of wind. This model is further extended in [17] to include wind. Both models assign continuous speed to each segment, and can be identified as mixed integer second-order cone programming (MISCOP), if the aircraft needs to be speeded up. These two models are briefly presented in this section.

3.1 Vertical Flight Planning without Wind (VFP-C)

In [18], a mathematical model for vertical flight planning without wind (VFP-C) is presented as follows. The unit distance fuel consumption F of an aircraft is given as measured data at discrete levels of the three dependent factors: speed V, altitude H, and weight W, as illustrated in Figure 1. If no wind is considered, given speed and weight, the optimal altitude can be precomputed by checking all possible altitudes, thus it is not necessary to include its computation in the optimization model. Other input parameters include a set of n segments $S := \{1, \ldots, n\}$ with length L_i for all $i \in S$; the minimum and maximum trip duration $\underline{T}$ and $\overline{T}$; and the dry aircraft weight W^{dry}, i.e. the weight of a loaded aircraft without trip fuel (reserve fuel for safety is included in the dry weight). The variables include the time vector t_i for $i \in S \cup \{0\}$, where t_{i-1} and t_i denote the start and end time of segment i; the travel time Δt_i spent on a segment $i \in S$; the weight vector w_i for $i \in S \cup \{0\}$ and w_i^{mid} for $i \in S$ where w_{i-1}, w_i^{mid}, and w_i denote the start, middle, and end weight at a segment i; the speed v_i on a segment $i \in S$; and the fuel f_i consumed on a segment $i \in S$. A general mathematical model for VFP-C can be stated as follows:

$$
\begin{aligned}
\min \quad & w_0 - w_n && (1)\\
\text{s.t.} \quad & t_0 = 0, \quad \underline{T} \le t_n \le \overline{T} && (2)\\
\forall i \in S: \quad & \Delta t_i = t_i - t_{i-1} && (3)\\
\forall i \in S: \quad & L_i = v_i \cdot \Delta t_i && (4)\\
& w_n = W^{dry} && (5)\\
\forall i \in S: \quad & w_{i-1} = w_i + f_i && (6)\\
\forall i \in S: \quad & w_{i-1} + w_i = 2 \cdot w_i^{mid} && (7)\\
\forall i \in S: \quad & f_i = L_i \cdot \widehat{F}(v_i, w_i^{mid}). && (8)
\end{aligned}
$$

The objective function (1) minimizes the total fuel consumption measured by the difference of aircraft weight before and after the flight; (2) ensures the flight duration within a given interval; time consistency is preserved by (3); the basic equation of motion (4) is enforced on each segment; (5) initializes the weight vector by assuming all trip fuel is burnt during the flight; weight consistency is ensured in (6), and the middle weight of each segment calculated in (7) will be used in the calculation of fuel consumption of each segment in (8), where $\widehat{F}(v, w)$ is a piecewise linear function interpolating F for all the continuous values of v and w within the given grid of $V \times W$. $\widehat{F}$ can be formulated as a MILP submodel using Dantzig's convex combination method [7, 16], a.k.a. lambda method. Our previous work [18] presents a variant of the 2D lambda method tailored for this problem. The quadratic constraint (4) can also be formulated as second-order cone constraint, if the time constraint (2) requires the aircraft to speed up from its unconstrained fuel-optimal travel time. A variable transformation technique to formulate it into a standard second-order cone constraint is presented in [18]. The resulting MISOCP can be solved by applying the linear approximation formulation for the second-order cone constraints that was proposed by Ben-Tal and Nemirovski [4] and refined by Glineur [8] (see [18] for more details).

3.2 Vertical Flight Planning with Wind (VFPW-C)

In practice, wind plays an important roll in planning a fuel-optimal flight trajectory. In vertical flight planning, the wind also depends on the flight altitude. Since the segments S are given, the track wind component $U^t_{i,h}$, i.e., the wind in the flight direction, as well as the cross-track wind component $U^c_{i,h}$, i.e., the wind perpendicular to the flight direction, can be precomputed for each segment i at each altitude h. The mathematical model without wind presented in Section 3.1 is extended in [17] to include wind influence. Firstly, a further binary variable $\mu_{i,h}$ is introduced to indicate whether a segment i is flown on altitude h. Then

$$
\forall i \in S: \quad \sum_{h \in H} \mu_{i,h} = 1 \qquad (9)
$$

guarantees only one altitude is assigned to each segment. With the help of variable μ, the wind can be assigned to each segment by

$$\forall i \in S: \quad u_i^t = \sum_{h \in H} \mu_{i,h} \cdot U_{i,h}^t, \tag{10}$$

$$\forall i \in S: \quad u_i^c = \sum_{h \in H} \mu_{i,h} \cdot U_{i,h}^c. \tag{11}$$

The equation of motion (4) is reformulated based on the wind triangle (Figure 2):

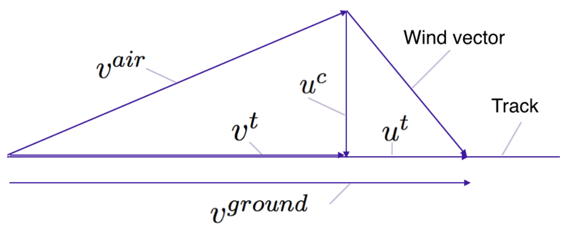

Fig. 2. Wind triangle. v^{ground} denotes the ground speed; v^{air} denotes the aircraft speed; v^t and u^t denote the aircraft speed and wind speed in the track direction, respectively; u^c denotes the cross-track wind speed.

$$\forall i \in S: \quad L_i = v_i^{ground} \cdot \Delta t_i \tag{12}$$

$$\forall i \in S: \quad v_i^{ground} = v_i^t + u_i^t \tag{13}$$

$$\forall i \in S: \quad (v_i^{air})^2 = (v_i^t)^2 + (u_i^c)^2. \tag{14}$$

The two quadratic constraints (12) and (14) can be transformed into second-order cone if speeding up the aircraft is enforced, and thus can be reformulated by linear approximation [17]. Furthermore, the fuel consumption per segment in (8) should be reformulated using the air speed and air distance as

$$\forall i \in S: \quad f_i = \sum_{h \in H} \mu_{i,h} \cdot \widehat{F}_{i,h}^L(v_i^{air}, w_i^{mid}), \tag{15}$$

where $F_{i,h}^L(v,w)$ denotes the fuel consumed by flying a segment i on an altitude h, which can be computed in the preprocessing phase by

$$\forall (v,w) \in V \times W: \quad F_{i,h}^L(v,w) = F(v,w) \cdot L_i \cdot \frac{v}{\sqrt{v^2 - (U_{i,h}^c)^2} + U_{i,h}^t}$$

based on the wind triangle. $\widehat{F}_{i,h}^L(v,w)$ is the 2D piecewise linear interpolation of the data $F_{i,h}^L(v,w)$ for all continuous values of (v,w) in the grid of $V \times W$, and thus can be solved by the 2D piecewise linear function techniques. In particular, the lambda method is found to outperform the delta method for this model [17].

4 Speed Discretization in Vertical Flight Planning

The continuous speed models introduced in Section 3 can be classified as mixed-integer second-order cone programming models. The drawback of such models is their unpractically long computation time. In this section, another modeling alternative by discretizing the aircraft speed is presented.

4.1 Discrete Speed in VFP without Wind (VFP-D)

We first focus on the vertical flight planning model without wind. Given a discrete set of aircraft speed V, we can further introduce binary variables $\mu_{i,v}$, which indicates whether a discrete speed level v is used when flying on segment i. Then only one speed level can be assigned to each segment by

$$\forall i \in S : \quad \sum_{v \in V} \mu_{i,v} = 1. \tag{16}$$

With the discretized speed, the travel time $\Delta T_{i,v}$ for segment i with speed v can be calculated in preprocessing,

$$\forall i \in S, v \in V : \quad \Delta T_{i,v} = \frac{L_i}{v},$$

such that the quadratic constraint (4) can be linearized as:

$$\forall i \in S : \quad \Delta t = \sum_{v \in V} \mu_{i,v} \cdot \Delta T_{i,v}. \tag{17}$$

Besides, the 2D matrix $F(v, w)$ in (8) can be reduced to a 1D vector $F_v(w)$ by precomputing:

$$\forall v \in V : \quad F_v(w) = F(v, w),$$

such that the fuel consumption (8) can be reformulated as

$$\forall i \in S : \quad f_i = L_i \cdot \sum_{v \in V} \mu_{i,v} \cdot \widehat{F}_v(w_i^{mid}). \tag{18}$$

Therefore, the speed discretization is the "stone that kills two birds": it linearizes the quadratic travel time constraint and reduces the 2D piecewise linear fuel function to 1D.

4.2 Discrete Speed in VFP with Wind (VFPW-D)

Similarly as in the VFP-D model in Section 4.1, speed discretization can help to simplify the travel time equation as well as the fuel interpolation in the VFPW-C model. Firstly, the binary variables $\mu_{i,h}$ in the VFPW-C model are extended to $\mu_{i,h,v}$ by one more dimension $v \in V$. Then (9) is replaced by

$$\forall i \in S : \quad \sum_{h \in H, v \in V} \mu_{i,h,v} = 1 \tag{19}$$

to ensure only one altitude and one speed level is assigned to each segment. Then the travel time $\Delta T_{i,h,v}$ for a segment i traveled on altitude h with speed v can be precomputed based on the wind triangle:

$$\forall i \in S, h \in H, v \in V: \quad \Delta T_{i,h,v} = \frac{L_i}{\sqrt{v^2 - (U^c_{i,h})^2} + U^t_{i,h}}.$$

Then the travel time computation given by (12, 13, 14) can be simply replaced by linear constraint

$$\forall i \in S: \quad \Delta t = \sum_{h \in H, v \in V} \mu_{i,h,v} \cdot \Delta T_{i,h,v}. \tag{20}$$

And replacing 2D matrix $F^L_{i,h}(v,w)$ by 1D vector $F^L_{i,h,v}(w)$ as

$$\forall v \in V: \quad F^L_{i,h,v}(w) = F^L_{i,h}(v,w),$$

reduces the 2D piecewise linear function (15) by one dimension:

$$\forall i \in S: \quad f_i = \sum_{h \in H, v \in V} \mu_{i,h,v} \cdot \widehat{F}^L_{i,h,v}(w^{mid}_i), \tag{21}$$

4.3 Univariate Piecewise Linear Interpolation

Here we review three different techniques to model the univariate piecewise linear function such as $\widehat{F}_v$ and $\widehat{F}^L_{i,h,v}$ into mixed integer linear programming. Despite being mathematically equivalent (in the sense that they all describe the same set of feasible solutions), their performances in terms of computation time are problem dependent. Given an index set $K_0 := \{0, 1, \ldots, m\}$, and the values for the parameters $W_0 := \{w_0, w_1, \ldots, w_m\}$ are specified as $F(w_k)$ for $k \in K_0$. We further denote $K := K_0 \setminus \{0\}$ for the index set of intervals. A piecewise linear function $\widehat{F} : [w_0, w_m] \to \mathbb{R}$ interpolating F can be modeled as follows.

The Convex Combination (Lambda) Method. A variant of the *convex combination* or *lambda method* [7] can be formulated as follows. To interpolate F we introduce binary decision variables $\tau_k \in \{0,1\}$ for each $k \in K$, and continuous decision variables $\lambda^l_k, \lambda^r_k \in [0,1]$ for each $k \in K$.

$$\sum_{k \in K} \tau_k = 1 \tag{22a}$$

$$\forall\, k \in K: \quad \lambda^l_k + \lambda^r_k = \tau_k \tag{22b}$$

$$w = \sum_{k \in K} (w_{k-1} \cdot \lambda^l_k + w_k \cdot \lambda^r_k) \tag{22c}$$

$$\widehat{F}(w) = \sum_{k \in K} (F(w_{k-1}) \cdot \lambda^l_k + F(w_k) \cdot \lambda^r_k) \tag{22d}$$

Note that our variant uses twice as many lambda variables compared to the original version of Dantzig [7], but in our numerical experiments it turned out that problem instances can be solved significantly faster.

The Special Ordered Set of Type 2 (SOS2) Method. Instead of introducing decision variables for the selection of a particular interval (the τ_k above), we mark the lambda variables as belonging to a special ordered set of type 2 (SOS2). That is, from an ordered set (or list) of variables $(\lambda_0, \lambda_1, \ldots, \lambda_m)$ it is required, that at most two of them are positive, and these two have to be adjacent with respect to the ordering. This information is implicitly treated by the solver in the solution process when branching on such special ordered set. SOS2 branching was introduced by Beale and Tomlin [3]. We introduce continuous decision variables $0 \leq \lambda_k \leq 1$ for each $k \in K_0$, and the following constraints:

$$\mathrm{SOS2}(\lambda_0, \lambda_1, \ldots, \lambda_m) \tag{23a}$$

$$w = \sum_{k \in K} (w_{k-1} \cdot \lambda_{k-1} + w_k \cdot \lambda_k) \tag{23b}$$

$$\widehat{F}(w) = \sum_{k \in K} (F(w_{k-1}) \cdot \lambda_{k-1} + F(w_k) \cdot \lambda_k) \tag{23c}$$

The Incremental (Delta) Method. The *incremental (delta) method* is the oldest of the three, introduced by Markowitz and Manne [11]. It uses binary decision variable $\tau_k \in \{0, 1\}$ for $k \in K$ and continuous decision variables $\delta_k \in [0, 1]$ for $k \in K$, and the following constraints:

$$\forall\, k \in K: \quad \tau_k \geq \delta_k \tag{24a}$$

$$\forall\, k \in K \setminus \{n\}: \quad \delta_k \geq \tau_{k+1} \tag{24b}$$

$$w = w_0 + \sum_{k \in K} (w_k - w_{k-1}) \cdot \delta_k \tag{24c}$$

$$\widehat{F}(w) = F(w_0) + \sum_{k \in K} (F(w_k) - F(w_{k-1})) \cdot \delta_k \tag{24d}$$

4.4 All-but-one Speed Discretization

Variable discretization often leads to a discretization error and thus a loss of optimality in the objective value. In particular, when an aircraft needs speeding up, the shorter the flight time, the more fuel is consumed. Thus it is usually fuel-optimal to arrive at the exact arrival time upper bound. But this is usually not possible with discretized speed as it is with continuous speed. Therefore, it may result in an unnecessary speedup on some segments, and the speed on some segments may need to be adjusted from its optimal setting in order to arrive as close to the prescribed time boundary as possible. This problem can be solved by leaving one segment with continuous speed while discretizing the speed for all other segments. More specifically, we pick the last segment to use continuous speed by the method described in Section 3, and the discrete speed is used on the rest of the segments and solved as described in this section.

5 Experimental Results

5.1 Experimental Setup

Two of the most common aircraft types, Airbus 320 (A320) and Boeing 737 (B737), are used for the empirical studies in this work. The aircraft performance data and the upper air data are provided by Lufthansa Systems AG. Two vertical flight planning problems are considered, the one without wind (VFP), and the other with wind (VFPW). For VFP, the instances considered in [18] for continuous speed are adopted here, including two speedup factors (flying 2.5% and 5% faster than unconstrained optimal). Five instances sizes are considered for A320, ranging from 15, 20, 25, 30, and 35 segments, each of which is 100 nautical miles (NM) long,[3] which results in flight ranges from 1500 NM to 3500 NM; four B737 instance sizes are considered: 8, 12, 15, 18, i.e., flight ranges from 800 NM to 1800 NM, totalling 18 instances. For VFPW, we adopted the instances considered in [17], with two different wind fields (one for westwards, and one for its eastwards return trip), three speedup factors: 2%, 4%, and 6%, and three different instance sizes: 10, 20, and 40 segments of 75 NM each for A320, and 10, 15, 20 segments of 75 NM each for B737, totaling 36 instances. All experiments ran on a computing node with a 12-core Intel Xeon X5675 CPU at 3.07 GHz and 48 GB RAM. Three MIP solvers are considered: SCIP 3.1, Cplex 12.6, and Gurobi 6.0.0. Each solver run uses 12 threads, and an instance is considered optimally solved, when the MIP gap is within 0.01%, which corresponds to a maximum fuel error of 1 kg for B737, and maximum 2 kg for A320.

5.2 Solver comparison in continuous speed

The MISOCP model for continuous speed is extensively studied in [17]. The use of the linear approximation for the second-order cone constraints plus the lambda method for the 2D piecewise linear fuel interpolation are found to be the best performing MIP model. In this work, we compare three MIP solvers on our best continuous speed model, including SCIP, Cplex and Gurobi. The runtime development plot for this comparison is shown in Figure 3, with the VFP-C on the left and VFPW-C on the right. On the horizontal axis, the solver performance is displayed in terms of computation time in seconds. For VFP-C, Cplex is faster than SCIP by an average factor of 7, while Gurobi is faster than SCIP by an average factor of 30. For VFPW-C, the average speedup factor of Cplex over SCIP is 5, while Gurobi is in average 15 times faster than SCIP. The best performing solver Gurobi also scales best as the instance size grows. Its speedup is more significant for large instances than for small instances. The largest real-world instances can be solved with Gurobi within 100 seconds when no wind is considered, and require around 1 hour when wind is included.

[3] Note that we currently considered segments of maximum 100 NM, based on our previous accuracy studies on using middle-weight segment fuel estimation [18]. In fact, longer segments can also be used to reduce the number of segments, if the segment fuel consumption is precomputed without using middle weight, e.g., using a numerical integral approach of the unit distance fuel consumption function [5].

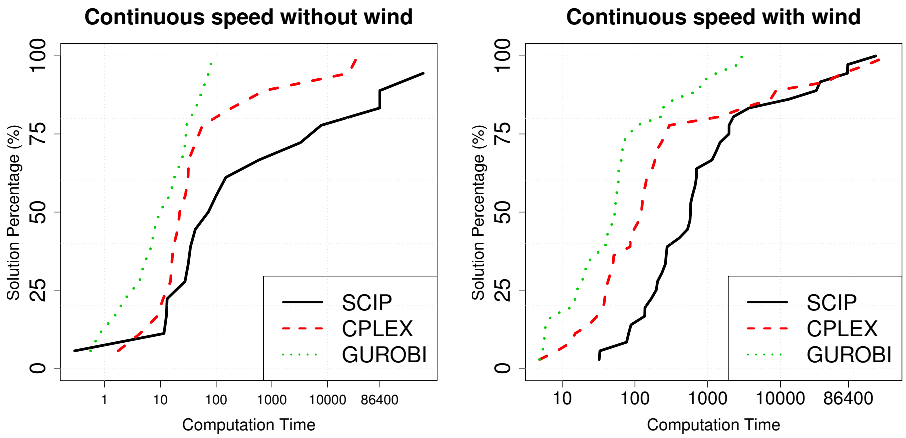

Fig. 3. The comparison of three MIP solvers: SCIP, Cplex, and Gurobi, in the best continuous speed models without wind (VFP-C, left) and with wind (VFPW-C, right).

5.3 Comparison of piecewise linear methods in speed discretization

The use of speed discretization replaces the second-order cone constraints with linear constraints, and it also reduces the 2D piecewise linear function for fuel computation to 1D. The three 1D piecewise linear interpolation techniques, namely, lambda, delta, and SOS methods, are empirically studied in this section with the two commercial MIP solvers Cplex and Gurobi. The comparison can be visualized in the plots in Figure 4, where the model without wind VFP-D is shown on the left, and the right plots with wind VFPW-D. The maximum cutoff time is set to 4 hours (14400 seconds), and their gap between the upper and lower bound after cutoff is also compared. For both models, Gurobi outperforms Cplex in almost all cases. For VFP-D, the delta method solved by Gurobi appears to be the best performing one, and solves all instances within 2 seconds. While the SOS method scales the worst for VFP-D, it appears to be the fastest solver for 90% of the VFPW-D instances as shown on the right of Figure 4. However, it scales poorly for the largest instances, and leaves the largest gap (close to 0.1%) after 4 hours. The best approach for VFP-D, the delta method by Gurobi (Del-G), appears to be the most robust and scalable approach also for VFPW-D, and solves the largest instances to optimality in around one minute.

5.4 Comparison of discrete speed and continuous speed

The best approach for discrete speed studied in Section 5.3, namely Del-G, is compared with the best approach for continuous speed (see Section 5.2), in terms of computational performance as well as discretization error. As shown in Figure 5, the use of discrete speed substantially speeds up the continuous speed across all instances. The average speedup factor is 44 for VFP without wind, and

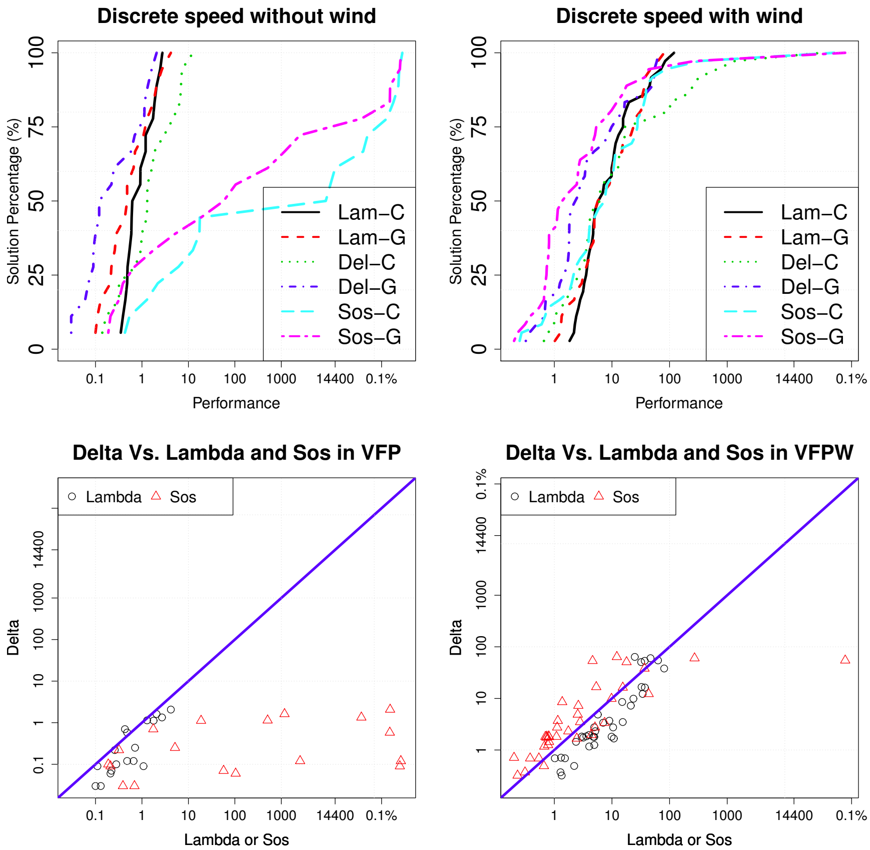

Fig. 4. The comparison of three piecewise linear function techniques (lambda, delta, SOS) solved by two commercial MIP solvers (Cplex and Gurobi) on the vertical flight planning models without wind (left) and with wind (right). The scatter plots show comparison of the delta with lambda and SOS methods with Gurobi.

15 for VFP with wind. Note that the scalability of the discrete speed model is especially noticeable for the largest instances. For instances that take more than 30 seconds by VFP-C, the average speedup of using discrete speed is of factor 80; while for instances that take over 30 minutes by VFPW-C, using discrete speed is in average over 50 times faster. The computation time of the largest instance is shortened from one hour to one minute by applying discrete speed.

However, the drawback of using fully discretized speed is its discretization error. The objective deviation from using continuous speed is shown in the columns of VFP-D and VFPW-D in Fig. 6. The use of discrete speed results in an increase in the objective value of over 0.1% in VFP without wind, and even over 0.5% for VFPW, which translates to a possible fuel increase of 50 kg for aircraft B737 or 100 kg for A320.

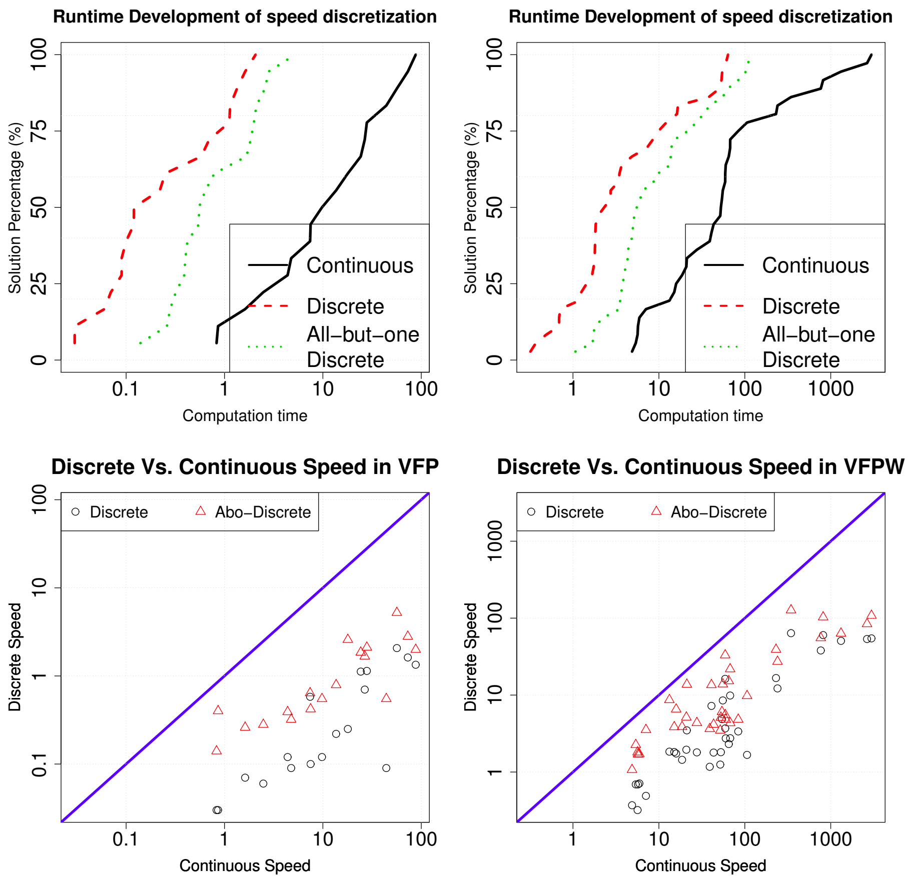

Fig. 5. The comparison of three piecewise linear techniques (lambda, delta, SOS) solved by two commercial MIP solvers (Cplex and Gurobi) on the vertical flight planning models without wind (left) and with wind (right).

The all-but-one (abo) speed discretization proposed in Section 4.4 can be used to reduce the discretization error. The speed discretization leaving one segment with continuous speed significantly lowers the discretization error as shown in Figure 6 in columns VFP-A and VFPW-A. As visualized in the box plot, around 75% of the instances in both models without or with wind have an objective deviation of less than 0.01%, which is the solver termination MIP gap, and it translates to maximum 1 kg fuel consumption for B737 and 2 kg for A320. Besides, the maximum objective deviation is reduced to 0.02% from 0.5%. With such practically negligible discretization error, the abo-discretization is still in average 13 times faster than using continuous speed in VFP without wind, and 6 times faster when wind is considered. Furthermore, the abo approach scales especially well for large instances, as shown in Figure 5, since only one segment is

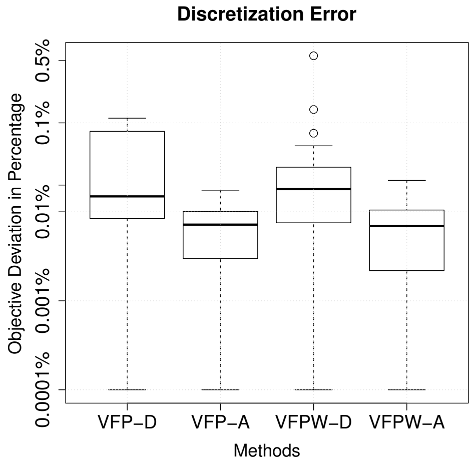

Fig. 6. The percentage discretization error in terms of objective increase over using continuous speed. VFP-D and VFP-A denote the fully discrete and all-but-one-discrete (abo-discrete) approach for VFP without wind, while VFPW-D and VFPW-A denote the fully discrete and abo-discrete approach for VFPW.

assigned with continuous speed. For the largest instances that require more than 30 seconds by VFP-C as well as largest instances that need over 30 minutes by VFPW-C, abo is in average around 30 times faster than continuous speed. The largest instance without wind is solved within 2 seconds, and the largest instance with wind can be solved within 2 minutes. Considering both the computational scalability and discretization error, the Abo-discretization appears to be the most practically viable approach for vertical flight planning.

6 Conclusions and Future Works

In this work, we address the vertical flight planning problem, which concerns assigning optimal altitude and speed to each composing segment of a flight trajectory. The previous work has employed a mixed integer second-order cone programming (MISOCP) model to assign continuous speed to segments. However, such model usually takes hours to solve instances of realistic sizes. In this work, we studied an alternative MIP model by assigning discretized speed. The speed discretization leads to significant speedup, since it not only transforms the quadratic constraints for travel time determination into linear constraints, but also reduce the 2D piecewise linear fuel interpolation into 1D. Computational experiments with various real-world instances have confirmed the effectiveness of the proposed discrete speed model, which can deliver optimal solution within minutes. To cope with the discretization error, an all-but-one (abo) discretization scheme that discretizes speed for all but one segments is proposed. The abo approach is confirmed to scale well to especially large instances, and deliver solution that are under 0.02% discretization error within 2 minutes, thus proves to be a practically viable approach.

Our experiments so far have focused on the instances with time constraint that speeds up the aircraft from its unconstrained fuel-optimal vertical profile, in order to compare with the MISOCP formulation for continuous speed. In the future, it will also be interesting to compute the optimal vertical profile with time constraints that require slowing down the aircraft. Advanced techniques for modeling piecewise linear function such as spatial branching [14] and a logarithmic model [15] may be applied to further speed up our discrete speed model.

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