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Variable Speed in Vertical Flight Planning

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Abstract Vertical flight planning concerns assigning cruise speed and altitude to segments that compose a trajectory, such that the fuel consumption is minimized and the time constraints are satisfied. The fuel consumption over each segment is usually given as a black-box function depending on aircraft speed, weight, and altitude. Without time consideration, it is known that it is fuel-optimal to fly at a constant speed. If an aircraft is under time pressure to speed up, the industrial standard of cost index cannot handle it explicitly, while research literature suggest using a constant speed. In this work, we formulate the vertical flight planning with variable cruise speed into a mixed integer linear programming (MILP) model, and experimentally investigate the fuel saving potential over a constant speed.

1 Introduction and Motivation

Planning a fuel-efficient flight trajectory connecting a departure and an arrival airport is a hard optimization problem. The solution space of a flight trajectory is four-dimensional: a 2D horizontal space on the earth surface, a vertical dimension consisting of discrete altitude levels, and a time dimension controlled by aircraft speed. In practice, the flight planning problem is solved in two separate phases: a horizontal phase that finds an optimal 2D trajectory consisting of a series of segments; followed by a vertical phase that assigns optimal flight altitude and speed to each segment. The altitude and speed can be changed only at the beginning of each segment. This work focuses on the vertical phase. A vertical flight profile consists of five stages: take-off, climb, cruise, descend, and landing. Here we focus on the cruise stage,

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since it consumes most of the fuel and time during a flight, while the other stages are relatively short and have relatively fixed procedures due to safety considerations. Lovegren and Hansman [5] considered assigning optimal speed and altitude for the cruise stage, and comparing the optimal vertical profile to the real operating vertical profiles in USA. A potential fuel saving of up to 3.6% was reported by the vertical profile optimization. However, no time constraint is taken into account in their computation as in real life. Note that in such case, it is known that the fuel-optimal speed assignment is to use a constant optimal cruise speed throughout the flight.

A practical challenge in airline operations is to handle delays due to disruptions such as undesirable weather conditions and unexpected maintenance requirements. Such delays are typically recovered by increasing the cruise speed, such that the next connection for passengers as well as for the aircraft can be caught. Speeding up an aircraft may also be useful, for example, to enter a time-dependent restricted airspace before it is closed, or when an aircraft is reassigned to a flight which used to be served by a faster aircraft. The industrial standard *cost index* was introduced by aircraft manufacturers to input a value (e.g., between 0 to 999) that reflects the importance between time-related cost and fuel-related cost, such that optimal flight speed is controlled. However, this approach cannot handle explicitly hard time constraints such as the about-to-close airspace. Aktürk et al. [1] considered increasing cruise speed in the context of aircraft rescheduling, and handled time constraint explicitly for scheduling purpose. However, their mathematical model only considered assigning a constant speed for the whole flight. It leaves an open research questions: given a flight to be accelerated from its optimal speed, is it more fuel-efficient to allow variable speed on each segment? We formulate this problem as a mixed integer nonlinear programming (MINLP) model, and present linearization techniques in Section 2, examine its computational scalability in Section 3 and empirically investigate the question above using data for various aircrafts.

2 Mathematical Model

The unit distance fuel consumption of an aircraft depends on its speed, altitude, and weight. Each aircraft's unit distance fuel consumption data is measured at discrete levels of each of the three factors. Given speed and weight, the optimal altitude can be precomputed by enumerating all possible altitudes. Thus the unit distance fuel consumption $F_{v,w}$ defined for a speed level $v \in V$ between optimal and maximal speed, and a weight level $w \in W$ can be illustrated in Figure 1. Other input parameters include a set of n segments $S := \{1,\dots,n\}$ with length L_i for all $i \in S$; the minimum and maximum trip duration $\underline{T}$ and $\overline{T}$; and the dry aircraft weight W^{dry}, i.e. the weight of a loaded aircraft without trip fuel (reserve fuel for safety is included in the dry weight). The variables include the time vector t_i for $i \in S \cup \{0\}$, where t_{i-1} and t_i denote the start and end time of segment i; the travel time Δt_i spent on a segment $i \in S$; the weight vector w_i for $i \in S \cup \{0\}$ and w_i^{mid} for $i \in S$ where w_{i-1}, w_i^{mid}, and w_i denote the start, middle, and end weight at a segment i; the speed v_i on a

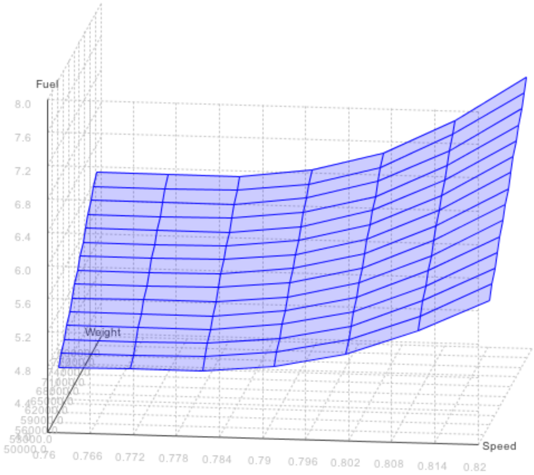

Fig. 1 The unit distance fuel consumption (kg per nautical mile) by aircraft speed (mach number, from optimal speed to maximal speed) and weight (kg) for Airbus 320

segment $i \in S$; and the fuel f_i consumed on a segment $i \in S$. A general mathematical model for the vertical flight planning problem can be stated as follows:

$$\min \quad w_0 - w_n \tag{1}$$
$$\text{s.t.} \quad t_0 = 0, \qquad \underline{T} \leq t_n \leq \overline{T} \tag{2}$$
$$\Delta t_i = t_i - t_{i-1} \qquad \forall i \in S \tag{3}$$
$$L_i = v_i \cdot \Delta t_i \qquad \forall i \in S \tag{4}$$
$$w_n = W^{dry} \tag{5}$$
$$w_{i-1} = w_i + f_i \qquad \forall i \in S \tag{6}$$
$$w_{i-1} + w_i = 2 \cdot w_i^{mid} \qquad \forall i \in S \tag{7}$$
$$f_i = L_i \cdot \tilde{F}(v_i, w_i^{mid}) \qquad \forall i \in S. \tag{8}$$

Equation (1) minimizes the total fuel consumption; (2) enforces the flight duration within a given interval; (3) ensures the time consistency; the basic equation of motion on each segment is given in (4); the weight vector is initialized in (5) by assuming all trip fuel is burnt during the flight; weight consistency is ensured in (6), and the middle weight of each segment calculated in (7) will be used in the calculation of fuel consumption in (8), where $\tilde{F}(v,w)$ is a piecewise linear function interpolating F for all the continuous values of v and w within the given grid of $V \times W$. $\tilde{F}$ can be formulated as a MILP submodel using Danzig's convex combination method [3]. Here we present one of its variants, and drop the index i hereafter for simplification. The grids of $V \times W$ are first partitioned by a set of triangles K. The grid indices of the three vertices of each triangle $k \in K$ is stored in N_k. Each triangle is assigned a binary variable y_k, y_k equals 1 if (v,w) is inside triangle k. We further introduce three continuous variables for each triangle $\lambda_{k,n} \in \mathbb{R}^+$ for $k \in K, n \in N_k$ such that

$$\sum_{k\in K} y_k = 1 \tag{9a}$$

$$\sum_{n\in N_k} \lambda_{k,n} = y_k \qquad \forall k \in K \tag{9b}$$

$$\sum_{k\in K, n\in N_k} \lambda_{k,n} \cdot V_{k,n} = v \tag{9c}$$

$$\sum_{k\in K, n\in N_k} \lambda_{k,n} \cdot W_{k,n} = w \tag{9d}$$

$$\sum_{k\in K, n\in N_k} \lambda_{k,n} \cdot F(V_n, W_n) = \tilde{F}(v,w) \tag{9e}$$

where (9a) ensures only one triangle is selected, (9b) sums λ of each triangle to 1 only if the triangle is selected, together with (9c) and (9d), the value of non-zero lambda is determined, such that the fuel estimation at a (v,w) is given by (9e) as a convex combination of λ and the grid value.

Another difficulty in the model is to handle the quadratic constraint in Equation (4). It can be linearized by quadratic cone approximations. First we can rewrite the equality (4) into an equivalent inequality $L \leq v \cdot \Delta t$, since neither increasing v nor Δt leads to fuel saving. Applying the variable transformation $\alpha = \frac{1}{2}(v - \Delta t)$, $\tau = \frac{1}{2}(v + \Delta t)$, $\beta = \sqrt{L}$ yields $\sqrt{\alpha^2 + \beta^2} \leq \tau$, which defines a second-order cone, and thus can be approximated by linear inequality system as introduced by Ben-Tal and Nemirovski [2] and refined by Glineur [4]. We introduce continuous variables $\alpha_j, \beta_j \in \mathbb{R}$ for $j = 0, 1, \ldots, J$, and initialize by setting $\alpha_0 = \frac{1}{2}(v - \Delta t)$ and $\beta_0 = \sqrt{L}$. The *approximation level* parameter J controls the approximation accuracy. Then the following constraints can be added:

$$\alpha_{j+1} = \cos\left(\frac{\pi}{2^j}\right) \cdot \alpha_j + \sin\left(\frac{\pi}{2^j}\right) \cdot \beta_j, j = 0, 1, \ldots, J-1, \tag{10a}$$

$$\beta_{j+1} \geq -\sin\left(\frac{\pi}{2^j}\right) \cdot \alpha_j + \cos\left(\frac{\pi}{2^j}\right) \cdot \beta_j, j = 0, 1, \ldots, J-1, \tag{10b}$$

$$\beta_{j+1} \geq \sin\left(\frac{\pi}{2^j}\right) \cdot \alpha_j - \cos\left(\frac{\pi}{2^j}\right) \cdot \beta_j, j = 0, 1, \ldots, J-1, \tag{10c}$$

$$\frac{1}{2}(v + \Delta t) = \cos\left(\frac{\pi}{2^J}\right) \cdot \alpha_J + \sin\left(\frac{\pi}{2^J}\right) \cdot \beta_J. \tag{10d}$$

3 Experiments and Results

Four different aircrafts are used for our study: Airbus 320, 380 and Boeing 737 and 777. The characteristics of these aircrafts are listed in Table 1. Our preliminary experiments for fuel estimation accuracy test confirmed that when dividing a longest possible 7500 nautical miles (NM) trip into equidistance segments of 100 NM, the total fuel estimation error is under 1 kg in 200 tons consumption (i.e. a relative error of under $5 \cdot 10^{-6}$). With the same 1 kg error threshold, we experimentally determine $J = 10$ for A320 and B737, $J = 11$ for B777 and $J = 12$ for A380.

Table 1 Four aircraft types, Airbus 320, 380, Boeing 737, 777, and their characteristics, such as optimal and maximal speed (in Mach number), dry weight and maximal weight (in kg), and maximal distance (in NM). The number of speed grids $|V|$ (between optimal and maximal speed) and weight grids $|W|$, and the empirically determined conic approximation level J are also listed.

Type	Opt. Speed	Max. Speed	Dry Weight	Max. Weight	Max. Distance	$\|V\|$	$\|W\|$	J
A320	0.76	0.82	56614	76990	3500	7	15	10
A380	0.83	0.89	349750	569000	7500	7	24	12
B737	0.70	0.76	43190	54000	1800	7	12	10
B777	0.82	0.89	183240	294835	7500	8	16	11

We set up instances for each of the four aircraft types by considering different levels of speed-up and different travel distances. Two levels of speed-up are used: 2.5% and 5%, since the maximum possible speed-up is around 7.2% to 8.5% as shown in Table 1, higher speed-up settings also do not leave much room for speed variation. For each aircraft, different typical travel distances are tested, ranging from 800 NM for B737, which is around the distance from Frankfurt to Madrid, to 7500 NM for A380 and B777, which is around the distance from Frankfurt to west coast of Australia. Each trip is divided into equidistance segments of 100 NM each.

These instances were first tried to be solved without conic reformulation, and SCIP 3.1 was used as a MINLP solver. Each run was performed on a computing node with 12-core Intel Xeon X5675 at 3.07 GHz and 48 GB RAM. Only single thread was used for SCIP. These realistic instances cannot be solved by SCIP within 24 hours. We reduced the number of segments and coarsened the weight grid, and found the largest instance solved is with 10 segments and 4 weight levels ($|W| = 4$).

With the conic reformulation, the MINLP model becomes MILP model, so commercial MILP solver such as CPLEX can be applied. We applied CPLEX 12.6, and each run was performed on the same computing node, with 12 threads per run. The computational results including the computation time and the gap (if cut off at 24 hours) was shown in Table 2. All the real-world instances are solved to provable optimality or near-optimality (less than 0.05%). Instances with no more than 25 segments can typically be solved within one minute. Increasing the number of segments seems to increase the computational difficulty noticeably.

We compared also the optimal value of using variable speed as computed above with an optimal constant speed. Since the fuel consumption is a monotone function of speed, the optimal constant speed can be computed as 2.5% or 5% over the optimal speed, respectively. As shown in Table 2, the potential fuel savings of using variable speed compared to a constant speed are rather small for the relatively mature aircrafts A320, B737, and B777. The rather new A380 shows the highest potential fuel savings of up to 0.18%. Although this number seems small, it means a lot in the highly competitive market of the airline industry. It also shows that there is room for Airbus to improve the performance of their new flagship airplane.

Our current experiments do not consider the influence of the weather, in particular, the wind. As also suggested in [5], a strong head wind may favor higher speed, while flying slower may be advantageous in a strong tail wind. The current

Table 2 Instances and their computational results on four aircraft types, with two speed-up factors 2.5% or 5%, and various numbers of segments $|S|$. Each segment is 100 NM, so the total distance is $100 \times |S|$. Computation time (seconds), gap from optimality, and potential fuel saving are listed.

Aircraft	$\|S\|$	2.5% Speed Up			5% Speed Up		
		Comp. Time	Gap	Fuel Saving	Comp. Time	Gap	Fuel Saving
A320	15	16	0	0.009%	21	0	0.007%
	20	28	0	0.009%	32	0	0.009%
	25	57	0	0.009%	43	0	0.009%
	30	191	0	0.009%	23901	0	0.009%
	35	735	0	0.005%	34631	0	0.010%
A380	30	525	0	0.008%	276	0	0.053%
	40	1691	0	0.005%	360	0	0.093%
	50	524	0	0.012%	14043	0	0.117%
	60	86400	0.02%	0.013%	86400	0.02%	0.121%
	70	86400	0.02%	0.013%	86400	0.03%	0.180%
	75	86400	0.03%	0.017%	86400	0.03%	0.177%
B737	8	2	0	0.013%	4	0	0.020%
	12	9	0	0.015%	11	0	0.014%
	15	15	0	0.011%	22	0	0.016%
	18	17	0	0.013%	31	0	0.023%
B777	25	275	0	0.011%	69	0	0.007%
	35	4759	0	0.001%	425	0	0.005%
	45	86400	0.02%	0.004%	86400	0.03%	0.015%
	55	13552	0	0.001%	86400	0.02%	0.013%
	65	86400	0.04%	0.005%	86400	0.05%	0.020%
	75	86400	0.05%	0.023%	86400	0.03%	0.020%

MILP model can be easily extended to include wind influence. Practically, the current MILP approach may require undesirable long computation time, but its optimal solution may be used to assess the quality of further heuristic approaches.

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References

1. M. S. Aktürk, A. Atamtürk, and S. Gürel. Aircraft rescheduling with cruise speed control. *Operations Research*, 2014. To appear.
2. A. Ben-Tal and A. Nemirovski. On Polyhedral Approximations of the Second-Order Cone. *Mathematics of Operations Research*, 26(2):193 – 205, 2001.
3. G. B. Dantzig. On the significance of solving linear programming problems with some integer variables. *Econometrica*, 28(1):30 – 44, 1960.
4. F. Glineur. Computational Experiments with a Linear Approximation of Second-Order Cone Optimization. Image Technical Report 0001, Faculté Polytechnique de Mons, Belgium, 2000.
5. J. A. Lovegren and R. J. Hansman. Estimation of potential aircraft fuel burn reduction in cruise via speed and altitude optimization strategies. Technical report, ICAT-2011-03, MIT International Center for Air Transportation, 2011.