Applied Mathematics

Helmut Schmidt Universität
Universität der Bundeswehr Hamburg

Angewandte Mathematik und Optimierung Schriftenreihe
Applied Mathematics and Optimization Series
AMOS # 37(2015)

Armin Fügenschuh

Aspects of Time in Mixed-Integer (Non-) Linear Optimization

Herausgegeben von der
Professur für Angewandte Mathematik
Professor Dr. rer. nat. Armin Fügenschuh

Helmut-Schmidt-Universität / Universität der Bundeswehr Hamburg
Fachbereich Maschinenbau
Holstenhofweg 85
D-22043 Hamburg

Telefon: +49 (0)40 6541 3540
Fax: +49 (0)40 6541 3672

e-mail: appliedmath@hsu-hh.de
URL: http://www.hsu-hh.de/am

Angewandte Mathematik und Optimierung Schriftenreihe (AMOS), ISSN-Print 2199-1928
Angewandte Mathematik und Optimierung Schriftenreihe (AMOS), ISSN-Internet 2199-1936

Aspects of Time in Mixed-Integer (Non-) Linear Optimization

Armin Fügenschuh

George Dantzig [6] reported that his linear programming algorithm was criticized right after his inaugural presentation in 1948 [4] for not being able to deal with nonlinear problems that are ubiquitous in real-world applications. However, by using linear inequalities and piecewise linear approximations, most nonlinear functions that are encountered in such applications can be sufficiently approximated. Over the past six decades, several researchers published model formulations to model a piecewise linear function (that approximates a given nonlinear function) in one or several dimensions, for example [22, 5, 2, 1, 28, 24, 21, 23, 29, 30]. These formulations differ in the way they use binary variables and the number of binary variables. All these approaches have in common that the piecewise-linear function has to be defined before the model is formulated, and does not change during the MILP solution process. A more modern approach that avoids this disadvantage was described by Smith and Pantelides [25] and Tawarmalani and Sahinides [27]. Here the approximation of the nonlinear function is created during the branch-and-cut solution process by adding further cutting planes and carrying out spatial branching. A direct comparison on a nonlinear network flow model indicates that this method is actually much faster [13]. For certain types of nonlinear functions, such as second order cone constraints of the form $\sqrt{x_1^2 + \ldots + x_n^2} \leq t$, there exist very accurate linear approximations that embed this cone in a higher dimensional space by introducing auxiliary variables [3, 19]. These reformulations can be used to solve practical nonlinear mixed-integer problems with MILP solvers, such as soft rectangle packing problems [17], or car routing problems in railway freight transportation [16].

Despite the initial critics about the "limited scope" of linear programming (LP), it turned out to become a huge success, and is considered as one of the most important mathematical discoveries (or inventions) of the 20th century. LP is the working horse for solving (mixed-) integer and thus many combinatorial optimization problems, starting from the traveling salesman problem by Dantzig, Fulkerson, and Johnson in 1954 [7]. Soon after, more complex logistic and transportation problems were addressed, such as the truck-use problem by Dantzig and Ramser in 1959 [8]. The transport of goods often requires time synchronization of different work steps. Shipments can be picked up or delivered only in certain time windows, a handover of goods between two players can only take place when both are at the same time in the same place. A useful optimization model for such applications must consider this aspect. While this is evident from a practical side, the integration of the time aspect in MILP models leads to algorithmically difficult problems. Most frequently encountered in the literature is one of the following two ways to include the time aspect in an optimization model.

In the continuous time modeling, time is described by real-valued variables, indicating the time relative to a specified start time in a defined unit (e.g., seconds). For each incident i (for example, an object being at a certain location, or a machine working on a certain job), a continuous time variable t_i describes when this incident is going to happen. For example, in bus or train scheduling application [11, 15], a tour of length d_i is started at time t_i. If tours i and j are connected (i.e., served by the same bus or train directly after another), then $t_j - (t_i + d_i) \geq 0$. The connection is modeled by the binary variable $x_{i,j}$. Hence the precedence relation is formulated by the nonlinear constraint $(t_j - (t_i + d_i))x_{i,j} \geq 0$. In order to apply mixed-integer linear solvers, one has to linearize this constraint. To this end, in the (in-) famous "big-M-method", a parameter M is introduced, which allows to linearize the constraint to $t_j - (t_i + d_i) \geq M(x_{i,j} - 1)$. Although one achieves a mixed-integer linear formulation, it usually has a weak LP-relaxation, which leads to less pruned nodes, thus large branch-and-bound trees, and thus large solution times.

An alternative to the big-M continuous time formulation is provided by a discrete modeling of time. The available finite time horizon is sliced into intervals of fixed unit length (e.g., 5 minutes). Then for each incident i and each time step t a binary variable $x_{i,t}$ is introduced, which indicates, if incident i happens in time step t. In this purely discrete setting, often better LP relaxations are achieved. However, there is a trade-off between accuracy and solution time: If the time is sliced into too many pieces, then the resulting models are of enormous size, which are impractical for a numerical solution process. When carefully used, small to medium size real-world optimization problems can be solved in reasonably short time [26].

A new way of dealing with time called "time-free relaxation" was introduced in [18]. Here the time index of all variables $x_{i,t}$ is projected away, so that variables x_i remain. Then all constraints have to be adapted accordingly. This results in very small, but still mixed-integer models, that usually can be solved very easily. In general, they do not yield feasible solutions to the original problem "with time", and it is in fact a difficult problem to decide if a projected solution has a feasible counterpart with proper times attached to it. When embedded in a branch-and-bound search, where a time-free master problem interacts with time-indexed subproblems to solve this inverse problem, the overall solution process can be faster, compared to a solution over the full time-indexed problem. A refinement of this method is a promising direction of further research.

The individual challenges imposed by mixed-integer optimization over time and mixed-integer optimization with nonlinear constraints are coupled when they are blended with differential equation constraints (ODEs or PDEs). These constraints usually arise in the description of technical systems to model physical properties. Even a simple PDE, such as the transport equation, leads to numerically difficult optimization problems when handed over to a standard MILP solver [20, 12]. One way to include PDEs into a MIP is by a finite difference discretization in time and space of the PDE. Even moderate discretization step sizes lead to instances with a huge number of variables and constraints. Although one can obtain a linear model (for the transport PDE, for instance), the solver tend to have numerical problems on these instances, so certain steps of the classical MILP solution process have to be revisited, for example, the bounds strengthening routines in the presolve phase [9]. Amazingly, putting one of the most simple problems from the area of combinatorial optimization, namely the shortest path problem, together with one of the most simple PDE, namely the heat equation, leads due to their combination to a new and challenging problem, which we call the coolest path problem [10].

ODEs and PDEs describe the physics to a very high degree of accuracy. However, in order to obtain the "right" combinatorial decisions, it can be beneficial not to work with the full complexity of the DE, but with some simpler approximation. To start with, a hierarchy of models must be derived, from fine to coarse, and afterwards the right level to formulate the mixed-integer optimization problem can be chosen among that hierarchy. In [14] this approach was demonstrated using traffic flows in networks.

Clearly, mixed-integer nonlinear dynamic optimization with differential equation constraints is full of challenging problems and has many practical applications, so that it is worth to devote more research on this emerging problem class.

References

[1] E. Beale and J. Forrest. Global Optimization Using Special Ordered Sets. *Mathematical Programming*, 10:52–69, 1976.

[2] E. Beale and J. Tomlin. Special facilities in a general mathematical programming system for nonconvex problems using ordered sets of variables. In J. Lawrence, editor, *Proceedings of the Fifth International Conference on Operations Research*. Tavistock Publications, London, 1970.

[3] A. Ben-Tal and A. Nemirovski. On polyhedral approximations of the second-order cone. *Mathematics of Operations Research*, 26(2):193–205, 2001.

[4] G. Dantzig. Linear Programming. In *Symposium on Numerical Methods, July 1948, Los Angeles, California*. National Bureau of Standards, 1948.

[5] G. Dantzig. *Linear programming and extensions*. Princeton University Press, 1963.

[6] G. Dantzig. Linear Programming. *Operations Research*, 50:42–47, 2002.

[7] G. Dantzig, D. Fulkerson, and S. Johnson. Solution of a large-scale traveling-salesman problem. *Operations Research*, 3:393–410, 1954.

[8] G. Dantzig and J. Ramser. The Truck Dispatching Problem. *Management Science*, 6(1):80–91, 1959.

[9] A. Dittel, A. Fügenschuh, S. Göttlich, and M. Herty. MIP Presolve Techniques for a PDE-based Supply Chain Model. *Optimization Methods & Software*, 24(3):427–445, 2009.

[10] M. Frank, A. Fügenschuh, M. Herty, and L. Schewe. The Coolest Path Problem. *Networks and Heterogeneous Media*, 5(1):143–162, 2010.

[11] A. Fügenschuh. Solving a School Bus Scheduling Problem with Integer Programming. *European Journal of Operational Research*, 193(3):867–884, 2009.

[12] A. Fügenschuh, S. Göttlich, C. Kirchner, M. Herty, and A. Martin. Efficient Reformulation and Solution of a Nonlinear PDE-Controlled Flow Network Model. *Computing*, 85(3):245–265, 2009.

[13] A. Fügenschuh, C. Hayn, and D. Michaels. Mixed-Integer Linear Methods for Layout-Optimization of Screening Systems in Recovered Paper Production. *Optimization and Engineering*, 15(2):533–573, 2014.

[14] A. Fügenschuh, M. Herty, A. Klar, and A. Martin. Combinatorial and Continuous Models and Optimization for Traffic Flow on Networks. *SIAM Journal on Optimization*, 16(4):1155–1176, 2006.

[15] A. Fügenschuh, H. Homfeld, A. Huck, A. Martin, and Z. Yuan. Scheduling Locomotives and Car Transfers in Freight Transport. *Transportation Science*, 42(4):1–14, 2008.

[16] A. Fügenschuh, H. Homfeld, and H. Schülldorf. Single Car Routing in Rail Freight Transport. *Transportation Science*, 49(1):130–148, 2015.

[17] A. Fügenschuh, K. Junosza-Szaniawski, and Z. Lonc. Exact and Approximation Algorithms for a Soft Rectangle Packing Problem. *Optimization*, 63(11):1637–1663, 2014.

[18] A. Fügenschuh, G. Nemhauser, and Y. Zeng. Scheduling and Routing of Fly-in Safari Planes Using a Flow-over-Flow Model. In G. Reinelt M. Jünger, editor, *Facets of Combinatorial Optimization, Series Algorithms and Combinatorics*, pages 419–447. Springer Verlag, Heidelberg, 2013.

[19] F. Glineur. Computational experiments with a linear approximation of second-order cone optimization. Technical report, Image Technical Report 0001, Faculte Polytechnique de Mons, Mons, Belgium, 2000.

[20] S. Göttlich, A. Fügenschuh, M. Herty, A. Klar, and A. Martin. A Discrete Optimization Approach to Large Scale Supply Networks Based on Partial Differential Equations. *SIAM Journal on Scientific Computing*, 30(3):1490–1507, 2008.

[21] J. Lee and D. Wilson. Polyhedral methods for piecewise-linear functions. I: The lambda method. *Discrete Applied Mathematics*, 108(3):269 – 285, 2001.

[22] H. Markowitz and A. Manne. On the solution of discrete programming-problems. *Ecometrica*, 25:84–110, 1957.

[23] A. Martin, M. Möller, and S. Moritz. Mixed-Integer Models for the Stationary Case of Gas Network Optimization. *Mathematical Programming*, 105:563–582, 2006.

[24] M. Padberg. Approximating separable nonlinear functions via mixed zero-one programs. *Operations Research Letters*, 27:1–5, 2000.

[25] E. Smith and C. Pantelides. A Symbolic Reformulation/Spatial Branch-and-Bound Algorithm for the Global Optimization of nonconvex MINLPs. *Computers and Chemical Engineering*, 23:457–478, 1999.

[26] A. Stieber, M. Epp, M. Knapp, A. Fügenschuh, and H. Rothe. The Multiple Traveling Salesmen Problem with Moving Targets. *Optimization Letters*, 2015.

[27] M. Tawarmalani and N. Sahinidis. Global optimization of mixed-integer nonlinear programs: A theoretical and computational study. *Mathematical Programming*, 99:563–591, 2004.

[28] J. Tomlin. A suggested extension of special ordered sets to nonseparable nonconvex programming problems. In *Studies on graphs and discrete programming (Brussels, 1979)*, volume 11 of *Ann. Discrete Math.*, pages 359–370. North-Holland, 1981.

[29] J. Vielma, S. Ahmed, and G. Nemhauser. Mixed-integer models for nonseparable piecewise-linear optimization: Unifying framework and extensions. *Operations Research*, 58(2):303–315, 2010.

[30] J. Vielma and G. Nemhauser. Modeling disjunctive constraints with a logarithmic number of binary variables and constraints. *Mathematical Programming*, 128(1):49–72, 2011.