The Overshoot Problem in Inflation after Tunneling

Koushik Dutta, Pascal M. Vaudrevange, and Alexander Westphal
DESY, Notkestrasse 85, 22607 Hamburg, Germany
(Dated: November 11, 2018)

We show the absence of the usual parametrically large overshoot problem of small-field inflation if initiated by a Coleman-De Luccia (CDL) tunneling transition from an earlier vacuum in the limit of small inflationary scale compared to the tunneling scale. For low-power monomial exit potentials $V(\phi) \sim \phi^n$, $n < 4$, we derive an expression for the amount of overshoot. This is bounded from above by the width of the steep barrier traversed after emerging from tunneling and before reaching a slow-roll region of the potential. For $n \geq 4$ we show that overshooting is entirely absent. We extend this result through binomials to a general potential written as a series expansion, and to the case of arbitrary finite initial speed of the inflaton. This places the phase space of initial conditions for small-field and large-field inflation on the same footing in a landscape of string theory vacua populated via CDL tunneling.

arXiv:1109.5182v1 [hep-th] 23 Sep 2011

I. INTRODUCTION

The inflationary paradigm has become a part of the concordance model of cosmology due to its huge number of observational post-dictions [1]. Among other things, it can explain the flatness of the universe, the lack of monopoles and the homogeneity, as well as provide seeds of structure formation via perturbations of the inflaton field, in excellent agreement with observations of the temperature anisotropies in the cosmic microwave background (CMB) [2], [3]. An (as of now) unobserved observational consequence of an inflationary epoch are primordial gravitational waves (also known as tensor modes), imprinting themselves in the B-mode polarization of the CMB. Unfortunately, their amplitude is strongly model dependent. Inflationary models can be broadly separated into two classes, large-field ($\Delta\phi > M_P$) and small-field ($\Delta\phi < M_P$) models, giving rise to a small and large amplitude of gravitational waves, respectively [4], where $\Delta\phi$ is the field variation during inflation.

Despite its amazing success, the inflationary paradigm still faces some important challenges. Many inflationary models suffer from a sensitive dependence of the potential on the UV knowledge of our theory. In addition, we still poorly understand the initial conditions for inflation to start with. Successful inflationary models must sustain a sufficient number efolds of inflation, usually taken to be about 60. For example in small-field models, if the scalar field starts off at even a moderate distance uphill from the flat inflationary plateau, or with an even very modest finite initial downhill speed, it stays in the attractor solution for too little time. In this case, the inflaton *overshoots* the inflationary plateau without ever settling into the slow-roll attractor dynamics. This is the so-called overshoot problem first described in [5].

The problem of fine tuning the initial conditions has been studied earlier. It has been found that for small-field models, e.g inflection point inflation, the problem is rather acute [6], [7], [8], [9]. On the other hand, large-field inflationary models are stable under moderate changes of initial conditions [10], [11], [8], [12], [13]. For a review, see [14]. Recently, [15] discusses the fine-tuning of initial conditions in terms of initial kinetic and potential energy for a large class of single-field inflation models. We will restrict our discussions of the overshoot problem to small-field models, where the typical field range is sub-Planckian.

Various solutions to this problem have been proposed in the literature, most often involving the effects of slower red-shifting forms of matter-energy. They start to dominate over the kinetic energy of the scalar field and damp its motion by increasing the Hubble friction. Examples include the use of additional fields behaving like matter or radiation [16], [17] or primordial black holes coupling to the scalar field [18]. The role of non-zero momentum mode in solving the overshoot problem has been pointed out in [19]. It was also advocated that the string degrees of freedom can generate a time dependent potential that can adibatically track the field to the attractor [20].

String theory is emerging as the prime candidate for a UV complete theory of quantum gravity, making it compelling to search for inflationary models in this context. At the same time, the standard low energy approximation of the string theory leads to a 'landscape' of a large number of discrete vacua separated by potential barriers [21], [22]. Recently, there have been a number of successful attempts at obtaining an inflationary phase from constructions in string theory within this vast landscape of true and false vacua. For some recent reviews see e.g. [23], [24], [25].

The question of fine-tuning the initial conditions necessary to achieve a sufficiently long period of slow-roll inflation in an inflection point small-field potential from warped D3-brane inflation in Type IIB string theory has been studied in [26], [27], [28], [29], [30]. In this context, [31] found that almost the whole a priori microscopically allowed phase space of initial conditions leads to successful inflation. Building upon previous work by [30], it has been found in [32] that the overshoot problem for warped D3-brane inflection point inflation cannot be cured by DBI effects.

Finally, [33] has further extended the analysis. They allow for statistical distributions of the higher-dimension corrections to the inflection point potential. Also, they widen the initial condition phase space to include all 5 angular positions and speeds of the D3-brane at the top of throat . This work found a considerably reduced overshoot problem with respect to the microscopically allowed initial conditions in the 12D phase space. This may very well be due to the additional sources of matter-energy stemming from including angular motion in the Hubble friction as well as in the kinetic term for the radial position, i.e. the inflaton.

Instead of focusing on any particular string realization of inflation, we will study the overshoot problem within the context of the landscape of vacua in string theory. We would like to focus our attention on the time immediately preceding the inflationary phase that ended up being our local Hubble patch. At that time, the universe can be assumed to be at a random position in the landscape, probably in a false vacuum. After tunneling away from the false vacuum towards the (hopefully true) vacuum, the inflaton field will generally appear in a position away from the minimum. After rolling away from the exiting part of the potential where the bubble tunnels, our Universe must enter a (slow roll) phase that generates the observed 60 or so efolds of inflation, see e.g. [34]. A priori this is not guaranteed, in particular, if the allowed field space in the slow-roll plateau is sub-Planckian.

Within the field theoretic description, there are only two known mechanism to traverse the landscape: CDL tunneling [35], [36] and Hawking-Moss tunneling [37] for potentials which are tuned to be very flat and wide. Within our setup, the CDL instanton is dominant. Thus we assume that the evolution of the inflaton field inside our Hubble patch was seeded by a CDL tunneling event with its implicated boundary conditions. In particular, the CDL instanton nucleates a bubble containing an infinite open universe with negative spatial curvature and a scale factor $a(t) = t$ for $t \to 0$ where $t = 0$ denotes the time of the tunneling event. At early times, the curvature contribution to the Hubble friction is arbitrarily large. In other words, the interior of the bubble is *entirely* curvature dominated at the very beginning, independent of the initial speed of the scalar field(s).

It was realized first in [34] for the simplified example of a steep linear potential (mimicking the post-tunneling steep downhill potential) matched to a constant piece (mimicking the small-field inflation region) that this total curvature domination ensures a finite overshoot independent of the potential at the exit from tunneling (in the following we call this the 'exit potential'). Consider the sketch of the scalar potential given in Figure 1. Approximate the exit potential by a steep linear downhill piece between $\phi_0 < 0$ and zero, and the inflationary plateau by a constant piece for $\phi > 0$. Due to curvature domination at bubble nucleation, the scalar field can start at arbitrarily high potential $V_0 >> V_-$ (with V_- the potential of the constant inflationary part), and the field will reach slow-roll speeds on the constant piece of the potential without parametrically large overshoot. The field excursion $\Delta\phi_{CDL}$ between the exit from tunneling and

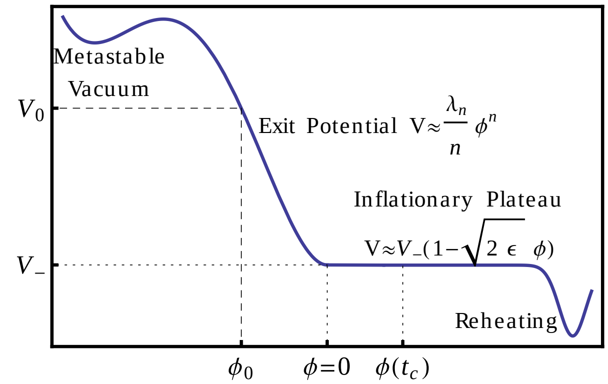

FIG. 1: The overall set-up of our scenario. After tunneling from the metastable vacuum the inflaton field materializes at its exit point ϕ_0 on the exit potential $V \sim \frac{\lambda_n}{n}\phi^n$. Then it rolls towards the minimum of the exit potential at $\phi = 0$. Afterwards it enters the inflationary plateau with potential $V \sim V_-(1 - \sqrt{2\epsilon}\phi)$. For $n = 1, 2, 3$, the field will enter the inflationary phase at position $\phi(t_c) > 0$. For $n \geq 4$, the field will enter the inflationary phase already on the exit potential for $\phi < 0$.

the onset of a potential dominated slow-roll phase is limited and given in terms of of the distance $|\phi_0|$ between the exit from tunneling and the beginning of the plateau region by [34]

$$\Delta\phi_{CDL} = 2|\phi_0| \quad . \tag{1}$$

Here we generalize this result to an arbitrary monomial potential $V(\phi) = \frac{\lambda_n}{n}\phi^n$ describing the exit potential for $\phi < 0$. We chose to describe the inflationary plateau by a linear potential for $\phi > 0$ with slope ϵ, the first slow-roll parameter. We will describe this set-up in more detail in section II. In section III we find that for the three lowest-power monomial potentials ϕ, ϕ^2, ϕ^3 describing the 'exit potential' from tunneling there is a finite amount of overshoot

$$\Delta\phi_{CDL} = \frac{3}{4\sqrt{2}}\sqrt{\epsilon} + \mathcal{O}(1)|\phi_0| \quad , \tag{2}$$

where the $\mathcal{O}(1)$ coefficient turns out to be a decreasing function of n. Surprisingly, for monomial potentials $V \sim \phi^n$, $n \geq 4$ there is no overshoot at all.

We can understand this intuitively in the following way. Increasing n increases the outer steepness of the potential, see Figure 2. This gets the field ϕ down into the flatter part of the potential faster and therefore at larger Hubble friction $\sim 1/t$. Thus the field spends more time with higher friction in the flat part of the potential, leading to a more severe slow-down. For large enough n, the field should come to a complete stop. Thus we expect the overshoot for a given ϕ_0 to decrease with increasing n. Note that for expontential potentials, [38] showed the existence of tracking solutions initiated in the early curvature dominated phase of the bubble. These too track directly into slow-roll inflation on the inflationary plateau.

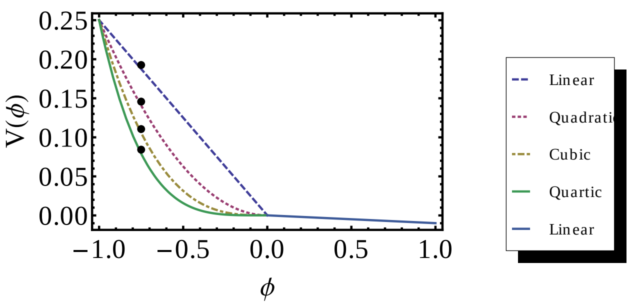

FIG. 2: Plot of $V \sim \phi^n$ potentials. Increasing n makes the potential steeper in the outer part to the left and flatter towards $\phi = 0$.

In section IV we generalize our analysis to a binomial exit potential. It turns out that the overshoot depends on ϕ_0, like the monomials, and is mainly determined by the lowest power monomial in the binomial. This property carries directly over to a general exit potential written as a series expansion around the point $\phi = 0$ where slow-roll flatness has set in.

In section V, we show that this absence of a parametrically large overshoot for small-field inflation is actually *independent* of the initial speed $\dot{\phi}_0$ of the field. The only condition necessary for our results to hold are thus that the CDL tunneling boundary conditions for the negative spatial curvature and the scale factor are valid. However, this implies that no-overshoot results generalize immediately to the case of CDL tunneling in a multi-field landscape. There some of the scalar fields will generically emerge with finite initial speed from the instanton.

In section VI we conclude that the phase space of initial conditions for small-field inflation and the one for large-field inflation are on almost the same footing in a landscape of string theory vacua populated via CDL tunneling.

II. EVOLUTION OF THE UNIVERSE AFTER TUNNELING

In the String Theory "landscape" paradigm, it is conceivable that our Universe originated via bubble nucleation from a nearby metastable vacuum. This picture can be modeled by a single scalar field tunneling from a false vacuum towards the true minimum. Before arriving at the present minimum of our universe, whose potential energy is fixed by the present value of the cosmological constant, the universe must pass through a large vacuum energy dominated phase that can sustain about 60 e-folds of slow-roll inflation. If the vacuum energy is originating from a scalar field potential, the field must have a shallow plateau to sustain this inflation. Now, in the simplest case, the scalar field which is responsible for creating our Universe through bubble nucleation can also be identified as the inflaton. In this case the bubble nucleating part of the scalar potential is followed by a shallow plateau part. The potential energy of the plateau sets the scale of inflation. The amount of inflation crucially depends on the field range available on the plateau and on the initial speed of the scalar field when it approaches the plateau. The initial speed depends on the exiting part of the potential where the bubble nucleation happens, as well as on the energy-matter content of the nucleated bubble, in particular, contributions from the spatial curvature.

The cartoon in Fig. 1 describes the overall set-up of our scenario where ϕ is the scalar field responsible for tunneling and subsequent inflation. The scalar field tunnels from the false vacuum towards ϕ_0 with energy scale V_0 which for our considerations is always much larger than the inflationary scale, denoted by V_-. The tunneling part of the potential is attached to the inflationary shallow potential at $\phi = 0$. The tunneling part of the potential would be modeled by monomial shapes $V(\phi) = \phi^n$ as well as polynomials. The plateau will be modeled by a linear potential $V = V_-(1 - \sqrt{2\epsilon}\phi)$. We will mainly focus on analyzing the dynamics of the field ϕ for different powers of the monomial.

For field distances larger than M_P, *i.e* $\Delta\phi = |\phi_0| > M_P$, between the nucleation point ϕ_0 and the plateau starting at $\phi = 0$, the inflaton will be quickly slowed to its slow-roll speed. Inflation will begin even before reaching to the plateau. In this case, there is no over-shooting issue. The picture here is similar to chaotic inflation or other large-field models for super-Planckian field values.

On the other hand, if the allowed field range is sub-Planckian, *i.e* $|\phi_0| < M_P$, as in small-field models, it is possible that the field reaches the plateau with relatively high speed, potentially overshooting the inflationary part. In this case, the field meets with the inflationary attractor solution too late to sustain enough inflation or does not reach to the attractor solution at all. One of the main goals of this paper is to provide an estimate of this speed for different forms of the exiting part of the potential and of the amount of overshoot. Unless stated otherwise, we will assume that the field range in this exiting part is always sub-Planckian. We will find that the overshoot, while naively expected to be rather severe, turns out be parametrically small or even zero.

Pioneered by Coleman and de Luccia, the process of bubble nucleation is described by an instanton solution of the classical Euclidean equations of motions. Inside the bubble, the universe is open with negatively curved spatial hypersurface with curvature $1/a^2$, where $a(t)$ is the scale factor of the Universe. We describe our Universe by the standard FRW metric, and the equations of motion for the system is given by

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi) \quad , H^2 = \frac{1}{3M_P^2}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) - \frac{k}{a^2} \quad ,$$

where the Hubble constant $H = \dot{a}/a$ and $k = -1$. It is important to note that when the bubble nucleates, H is dominated by the term $1/a^2$. Smoothness of the instanton requires the bubble to start growing with initial conditions $a(t) = t + \mathcal{O}(t^3)$ and $\dot{\phi}(0) = 0$. This in turn says that $H = 1/t$ at the very beginning, rendering the friction term in Eq. (3) divergent. As a consequence, the field moves very slowly in the beginning just after the field tunnels out. This interesting feature due to the negative curvature of the spatial metric inside the bubble has very important implications for the overshoot problem that we are going to discuss soon.

III. ϕ^n MONOMIALS WITH A LINEAR SLOPE ATTACHED

In this section we will develop a detailed analysis of the dynamics of the scalar field after it tunneled out of the metastable vacuum. For simplicity we will assume now that the exiting part of the potential can be approximated by monomials with different power *i.e* $V(\phi) \sim \frac{\lambda_n}{n}\phi^n$. The extremum of the monomial part, chosen for convenience to be at $\phi = 0$, is attached to a shallow linear slope on the right side (see Fig. 1), and given by

$$V(\phi) = V_-(1 - \sqrt{2\epsilon}\ \phi) \quad . \tag{3}$$

This is the plateau of the potential and the scale is set by V_-. We have written the potential in a form such that ϵ can be easily identified with the inflationary slow-roll parameter given by $\epsilon = (V'/\sqrt{2}V)^2$.

We have already seen that the Universe is curvature dominated just after the bubble materializes, thus $H = 1/t$. We will show later that curvature dominates in all the monomial potentials until the field reaches $\phi = 0$ as long as $|\phi_0| < M_P$, and $V_- << V(\phi_0) = V_0$. Therefore we need to solve

$$\ddot{\phi} + \frac{3}{t}\dot{\phi} + (-1)^n \lambda_n \phi^{n-1} \ = \ 0 \quad , \tag{4}$$

subject to the boundary condition $\dot{\phi}(0) = 0, \phi(0) = \phi_0 < 0$. Note that any coefficient in the potential could be absorbed in a rescaling of the time. The factor of $(-1)^n$ ensures that the field always rolls from left to right.

We then use the solution of Eq. (4) as a starting point for solving the motion in the linear part on the right side

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0 \quad , \tag{5}$$

where $V(\phi)$ is given by Eq. (3), and $V_- \ll V(\phi_0) \equiv V_0$. At some time t_c determined from $1/t_c^2 = 1/3V_-$, when the field is on the plateau, the curvature term becomes subdominant compared to the potential. Once the potential starts to dominate inflation sets in.

Before we perform a detailed analysis for the different forms of the monomial potentials, we state the qualitative results that we are going to establish in the remainder of this paper. There are two qualitatively different classes of monomial potentials. For monomials with power $n \geq 4$, the field reaches the junction point $\phi = 0$ with zero velocity. In this case, potential energy even starts to dominate before the field reaches the plateau. Therefore there is no overshooting for $n \geq 4$.

On the other hand, for monomials with $n < 4$, the field reaches the junction point $\phi = 0$ with finite speed and will roll a certain, parametrically small distance along the plateau the right until the potential starts to dominate over the curvature at time t_c and inflation sets in.

It turns out that the distances $\phi(t_c)$ for the different values of $n = 1, 2, 3$ share some common features. Take the evolution of the field on the linear slope on the right

$$\ddot{\phi} + \frac{3}{t}\dot{\phi} - V_-\sqrt{2\epsilon} \ = \ 0 \quad , \tag{6}$$

and assume as initial conditions $\phi(t_f) = 0,\ \dot{\phi}(t_f) = \dot{\phi}_f$, for some generic non-zero time t_f and initial speed $\dot{\phi}_f$. Then it is easy to see that at time t_c *i.e.* when inflation sets in, the field will be at a position

$$\begin{aligned} \phi(t_c) \ &= \ \frac{3}{4\sqrt{2}}\sqrt{\epsilon} + \frac{1}{2}\dot{\phi}_f t_f \\ &\quad -\frac{1}{2\sqrt{2}}\sqrt{\epsilon}V_- t_f^2 - \frac{1}{6}\dot{\phi}_f V_- t_f^3 + \frac{1}{12\sqrt{2}}\sqrt{\epsilon}V_-^2 t_f^4 \quad . \end{aligned} \tag{7}$$

Notice that the first term is independent of the parameters of the potential on the left where tunneling happens which only impacts the values of $\dot{\phi}_f$ and t_f.

In the subsequent analysis we will find that $t_f \sim |\phi_0|/\sqrt{V_0}$, and $\dot{\phi}_f \sim \sqrt{V_0}$. Therefore the second term in Eq. (7) depends parametrically only on the field value ϕ_0 where the bubble nucleated. Now in the approximation of $V_- << V_0$, it is easy to see that all other terms are higher orders and therefore negligible.

In summary, for a potential of form $V(\phi) = \frac{\lambda_n}{n}\phi^n$ with $1 \leq n \leq 3$, the overshoot is given by

$$\phi(t_c) \simeq \frac{3}{4\sqrt{2}}\sqrt{\epsilon} + \mathcal{O}(1)|\phi_0| \quad . \tag{8}$$

We will see later that the numerical prefactor in front of the $\sqrt{\epsilon}$ is universal and independent of the power of the monomials, whereas the coefficient of ϕ_0 depends on the order of the monomial, but is always of $\mathcal{O}(1)$. Now, the value of ϵ is determined by the inflationary requirements, whereas the value ϕ_0 is calculable either exactly when the tunneling potential is known [39], [40], [41], [42], [43] or in the thin-wall approximation [35].

We note that the exiting part of the potential cannot be fully approximated by monomials in a realistic setup of a false vacuum separated from the true one by a potential barrier. The curvature of the potential must change to make the turn around for the barrier. But, as long as the the bubble nucleating point ϕ_0 is in a part where the curvature is positive, our analysis holds fully. Now $V(\phi_0)$ must be smaller than the false vacuum height for Coleman de Luccia instanton to be the consistent solution [35], [36]. Therefore, unless the depth of the false vacuum is too shallow, the field will tunnel out to a field value where the potential is positively curved, making our approximation consistent. If the depth of the false vacuum is small, the field will tunnel to a place where the potential is negatively curved and some hill-top open inflation will ensue with zero initial field velocity.

Also, in this case, the Hawking-Moss instanton might be the dominant process instead of Coleman de Luccia instanton [37]. We are not interested in this regime and neglect it in the subsequent analysis. A realistic potential for the tunneling part would consists of polynomials, instead of the simplified assumption of a monomial that we are making now. In section IV we perform the analysis for polynomials, finding results quantitatively very similar to overshooting dominated by the lower monomial.

A. $n = 1$: The Linear

Assuming curvature domination we can solve Eq. (4) for the linear potential

$$V(\phi) \; = \; V_-(1 - \lambda_1 \phi) \quad . \tag{9}$$

We find the monotonically growing solution in the exiting part of the potential as

$$\phi(t) \; = \; \phi_0 + \frac{\lambda_1 V_-}{8} t^2 \quad . \tag{10}$$

The field reaches at the bottom $\phi = 0$ at $t_f = 2\sqrt{\frac{-2\phi_0}{\lambda_1 V_-}}$ with finite velocity $\dot{\phi}(t_f) = \sqrt{-(\lambda_1 V_- \phi_0)/2}$.

We need to justify the assumption of curvature domination by showing that the contribution of curvature $3/t^2$ to the Friedman equation

$$H^2 = \frac{1}{3}\left[\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) + \frac{3}{t^2}\right] \quad , \tag{11}$$

is larger than the other contributions. Plugging the solution Eq (10) in the expression for the total energy of the scalar field, we obtain

$$\frac{1}{2}\dot{\phi}^2 + V(\phi) \; = \; V_0 - \frac{3}{32} t^2 V_-^2 \lambda_1^2 \quad , \tag{12}$$

which is obviously smaller than V_0. As $t < t_f$ the curvature contribution in this part of the potential $3/t^2$ is always larger than $3/t_f^2$. Therefore to confirm the validity of our assumption we just need to satisfy

$$V_0 < \frac{3}{t_f^2} \; \Rightarrow \; 1 < \frac{3}{8\phi_0^2} \Rightarrow |\phi_0| < \frac{3}{8} \quad , \tag{13}$$

where we assumed that $V_- \ll V_0$. As it has been stated earlier, we are solely discussing small-field models $|\phi_0| < M_P$, so that this condition on curvature domination is always fulfilled.

The finite velocity at the bottom lets the field roll up to

$$\phi(t_c) \; = \; \frac{3}{4\sqrt{2}}\sqrt{\epsilon} - \phi_0 \quad , \tag{14}$$

before the vacuum energy V_- of the plateau starts to dominate. The subsequent field range on the plateau must be large enough to sustain 60 e-folds of inflation. We remind the readers that this finding is in agreement with our general result in Eq. (8). In the limit of $\epsilon = 0$ where the plateau is fully horizontal, we recover the known result of [34]: the field travels a total distance from tunneling exit to beginning of inflation of $2|\phi_0|$ in the field space.

B. $n = 2$: The Quadratic

Now we discuss the situation when the exiting part of the potential where the bubble nucleates is quadratic. Solving Eq. (4) for the quadratic potential

$$V(\phi) = V_- + \frac{1}{2} m^2 \phi^2 \quad , \tag{15}$$

we find the solution for the field

$$\phi(t) \; = \; 2\phi_0 \frac{J_1(mt)}{mt} \quad , \tag{16}$$

where $J_k(z)$ is the Bessel function of the first kind. The field will reach $\phi = 0$ at $t_f \approx 3.83/m$ with a velocity of

$$\dot{\phi}(t_f) \; = \; \phi_0 \frac{J_0(mt_f) - J_1(mt_f)}{t_f} \approx -0.21 m\phi_0 \quad , \tag{17}$$

where we need to keep in mind that $\phi_0 < 0$.

From the solution of Eq. (16), it is easy to show that the potential and the kinetic energy contribution to the Hubble equation is given by

$$\frac{1}{3}\left(V_- + \frac{2\phi_0^2 (J_1(mt)^2 + J_2(mt)^2)}{t^2}\right) \quad , \tag{18}$$

and for $V_- << V_0$, and $\phi_0 < M_P$, it is always subdominant compared to the curvature contribution in the quadratic part of the potential. This can be seen in the following way:

To neglect V_- in Eq. (18), it is sufficient to show the curvature term at the bottom of the quadratic being still larger than V_-,

$$\frac{1}{3} V_- < \frac{3}{t_f^2} \; = \; \frac{3m^2}{3.83^2} \Rightarrow V_- < 0.61 m^2 \quad . \tag{19}$$

Now we have assumed that $V_- \ll V_0$, or in other words

$$\frac{1}{2} m^2 \phi_0^2 \; \gg \; V_- \quad . \tag{20}$$

Now as $\phi_0^2 < 1$ in Planck units, $m^2 \gg V_-$, satisfying the above mentioned condition. Thus neglecting V_-, we note that $|J_\nu(z)| \leq \frac{1}{\sqrt{2}}$ for all $\nu \geq 1, z \in \mathbf{R}$ (see 10.14.1 in [44]), making the total energy of the field smaller than the curvature contribution of $\frac{1}{t^2}$.[1] Thus, assuming curvature domination is consistent.

From the time t_f on when the field reaches $\phi = 0$, it will roll on the right linear slope until curvature becomes subdominant at

$$\phi(t_c) \; = \; \frac{3}{4\sqrt{2}}\sqrt{\epsilon} + \phi_0 J_0(mt_f) \qquad (21)$$

$$= \; 0.53\sqrt{\epsilon} - 0.403\phi_0 \quad . \qquad (22)$$

Let us compare this result with the corresponding result for the linear case, Eq. (14). In the limit of $\epsilon = 0$, the overshooting is less for the quadratic case than the linear slope. Also note that the prefactor of $\sqrt{\epsilon}$ turns out to be the same for the linear and the quadratic potential, in accord with our general result in Eq. (8).

C. $n = 3$: The Cubic

Now we turn to the case when the monomial potential in the exiting part is cubic. As far as we know, there is no known analytical solution to the exact equation of motion

$$\ddot{\phi} + \frac{3}{t}\dot{\phi} - \lambda_3\phi^2 \; = \; 0 \quad . \qquad (23)$$

Therefore we will use the solutions of equations of motion with slightly larger and smaller friction terms to obtain upper and lower bounds on the terminal velocity.

We will prove now that the field reaches the bottom of the potential $\phi = 0$ with non-zero speed. From Eq. (4.24) in [45][2] we know that the equation with slightly larger friction term than Eq. (23).

$$\ddot{\phi} + \frac{10}{3t}\dot{\phi} - \lambda_3\phi^2 \; = \; 0 \quad , \qquad (24)$$

has a first integral given by

$$C_0 = \frac{1}{2}\dot{\phi}^2 t^4 + \frac{4}{3}\phi\dot{\phi}t^3 - \frac{1}{3}\phi^3 t^4 + \frac{2}{9}\phi^2 t^2 - \frac{4}{9}\dot{\phi}t - \frac{28}{27}\phi \quad . \qquad (25)$$

Note that any coefficient in the potential can be absorbed in a rescaling of the time, so without loss of generality we set it to unity.

We can evaluate this first integral for the initial condition at $t = 0, \phi(0) = \phi_0, \dot{\phi}(0) = 0$ to be

$$C_0 \; = \; -\frac{28}{27}\phi_0 \quad . \qquad (26)$$

When the field reaches the minimum at $\phi = 0$, the first integral becomes

$$C_0 \; \equiv \; -\frac{28}{27}\phi_0 = \frac{1}{2}t^2\dot{\phi}^4 - \frac{4}{9}\dot{\phi}t \quad . \qquad (27)$$

leading to a non-zero terminal velocity $\dot{\phi}_f$ at $\phi = 0$.

As the friction term of this equation is larger than the friction term of the equation we are actually trying to solve, the velocity at a given position will be smaller than the velocity of the unknown solution to Eq. (23). Thus even for the solution of Eq. (23), the terminal velocity at $\phi = 0$ is non-zero.

1. *Lower Bound on the Terminal Velocity*

It is also possible to find an exact analytic solution to the equation of motion of the field with a slightly larger friction term, Eq. (24). It is given by [46]

$$\phi(t) \; = \; -\frac{2}{3t^2\lambda_3}\left\{z(t)^2\,\mathfrak{p}\left[\imath z(t), 0, 1\right] + 1\right\} \quad ,$$

where $\mathfrak{p}(z, g_2, g_3)$ denotes the elliptic Weierstrass function[3] and $z(t) = \left(-\frac{3\lambda_3\xi t^2}{2}\right)^{1/6}$. Note we have reintroduced λ_3 again. This solution has $\dot{\phi}(0) = 0$, and $\phi(0) = \phi_0$ determines the integration constant $\xi = 28\phi_0$. The field will reach $\phi = 0$ when

$$\mathfrak{p}(\imath z, 0, 1) \; = \; -\frac{1}{z^2} \quad , \qquad (28)$$

i.e when $z \approx 3.61$. This happens at a time

$$t_f \approx \frac{7.26}{\sqrt{-\lambda_3\phi_0}} \quad , \qquad (29)$$

with a velocity of

$$\dot{\phi}(t_f) \; \approx \; -0.03\;\phi_0\sqrt{-\lambda_3\phi_0} \quad . \qquad (30)$$

Considering that we are working with a system that has larger friction than the exact one, this estimation of terminal velocity is a lower bound. We will comment at the end of the next subsection on the validity of the assumption of curvature domination for the cubic potential.

The finite velocity of Eq. (30) at $\phi = 0$ will lead to the field rolling up to

$$\phi(t_c) = \frac{3}{4\sqrt{2}}\sqrt{\epsilon} - 0.1\phi_0 \quad , \qquad (31)$$

again in accord with our expectations from Eq. (8).

---

[1] Alternatively, one could also plot the total energy of the scalar field as a function of the dimensionless quantity $z = mt$ to see the point.

[2] Notice the different sign in front of ϕ^2 which is compensated by flipping the sign of ϕ in the expression for the first integral Eq. (25).

[3] To prove that this is a solution, we use the defining relation of the Weierstrass function: $\mathfrak{p}'(z, g_2, g_3)^2 = 4\mathfrak{p}(z, g_2, g_3)^3 - g_2\mathfrak{p}(z, g_2, g_3) - g_3$. Specifically, at its zero, we have $\mathfrak{p}'(\zeta) = \pm\sqrt{\frac{1}{81}\phi_0^2}$. To show that it fulfills the boundary condition, use l'Hôpital's rule.

2. Upper Bound on the Terminal Velocity

Now we will estimate the upper bound on the velocity of the field when it reaches at the minimum. It turns out that the case of a slightly smaller friction term in the equation of motion

$$\ddot{\phi} + \frac{5}{3t}\dot{\phi} - \lambda_3\phi^2 \; = \; 0 \quad , \tag{32}$$

also has a closed form solution [46]

$$\phi(t) \; = \; \frac{(8/3\lambda_3)^{1/3}}{t^{2/3}}\mathfrak{p}(t^{2/3}(3\lambda_3/8)^{1/3} + \zeta, 0, \xi) \quad , \tag{33}$$

where again $\mathfrak{p}(z, g_2, g_3)$ denotes the elliptic Weierstrass function, and ζ, ξ are integration constants.

Imposing the initial conditions $\phi(0) = \phi_0 < 0$ and $\dot{\phi}(0) = 0$ implies

$$\zeta \; = \; \int_0^\infty \frac{dx}{\sqrt{4x^3 + \phi_0^2}} = 2^{1/3}\frac{\Gamma(\frac{1}{3})\Gamma(\frac{7}{6})}{\sqrt{\pi}|\phi_0|^{1/3}} \quad , \tag{34}$$

and

$$\xi = -\phi_0^2 \quad , \tag{35}$$

where ζ denotes the first real zero of the Weierstrass function with $\mathfrak{p}(\zeta, 0, -\phi_0^2) = 0$.

The solution $\phi(t)$ reaches the extremum $\phi = 0$ at the second zero of $\mathfrak{p}(z, 0, -\phi_0^2)$ at $z = 2\zeta$, or equivalently at $t_f = \sqrt{8/3\lambda_3}\zeta^{3/2}$. The field velocity at that time is given by

$$\begin{aligned} \dot{\phi}(t_f) \; &= \; -\frac{\pi^{3/4}}{2\sqrt{3}(\Gamma(\frac{1}{3})\Gamma(\frac{7}{6}))^{3/2}}\phi_0\sqrt{-\lambda_3\phi_0} \\ &\approx \; -0.174\phi_0\sqrt{-\lambda_3\phi_0} \quad . \end{aligned} \tag{36}$$

Comparing this terminal velocity with the one for the smaller friction term in Eq. (30), we note that they have the same ϕ_0 and λ_3 dependence. This is an upper bound for the terminal velocity of the cubic potential Eq. (23).

Next we need to check for curvature domination throughout the phase $\phi(t) < 0$. For curvature to dominate, we need to have

$$\frac{1}{2}\dot{\phi}^2 - \frac{\lambda_3}{3}\phi^3 - \frac{3}{t^2} \; < \; 0 \quad . \tag{37}$$

Using the first integral for Eq. (32) (see Eq. (4.21) in [45]) while replacing $t \to \frac{t}{\sqrt{\lambda_3}}$

$$C_1 \; = \; \frac{1}{2}t^2\dot{\phi}^2 + \frac{2}{3}t\phi\dot{\phi} - \frac{\lambda_3}{3}t^2\phi^3 + \frac{2}{9}\phi^2 \equiv \frac{2}{9}\phi_0^2 \quad , \tag{38}$$

we can substitute the first two terms in Eq. (37) to obtain the condition for curvature domination as

$$\frac{2}{9}(\phi_0^2 - \phi^2) - \frac{2}{3}\phi\dot{\phi}t - 3 \; < \; 0 \quad . \tag{39}$$

We are only dealing with $|\phi| < |\phi_0| < 1$ in Planck units, so the first term is always smaller than $\frac{2}{9}$. This gives us the condition for curvature domination to be true if

$$-\frac{2}{3}\phi\dot{\phi}t \; < \; \frac{25}{9} \quad . \tag{40}$$

It is clear that both at the beginning (at $t = 0$) and at the end (at $\phi = 0$) of the trajectory, this condition is fulfilled.

We now discuss the situation in between. We can estimate the maximum velocity the field could have (admittedly a huge over-estimate) as the velocity it would have at the bottom in case of zero Hubble friction. The largest field value is ϕ_0, and the time must be smaller than t_f, giving

$$|\dot{\phi}| \; < \; \sqrt{\frac{2\lambda_3}{3}\phi^3}, \qquad |\phi| < \phi_0 \tag{41}$$

$$t \; < \; t_f = \frac{4}{\sqrt{3}}\frac{(\Gamma(1/3)\Gamma(7/6))^{3/2}}{\pi^{3/4}\sqrt{-\lambda_3\phi_0}} \quad . \tag{42}$$

Plugging all the upper limits into Eq. (40), we obtain

$$\frac{8\sqrt{2}}{9}\frac{\phi_0^2}{\pi^{3/4}}\left(\Gamma(1/3)\Gamma(7/6)\right)^{3/2} \approx 2.09 < \frac{25}{9} \quad . \tag{43}$$

We find the assumption of curvature domination to be correct for the equation of motion with slightly smaller friction term. The velocities for the solution of Eq. (23) are even larger than the velocities for a smaller friction term. Thus we proved that for solutions of Eq. (23), curvature dominates at all times, as long as $\phi_0 < 1$.

The final velocity of Eq. (36) lets the field roll on the right side of the potential until

$$\phi(t_c) = \frac{3}{4\sqrt{2}}\sqrt{\epsilon} - \frac{1}{3}\phi_0 \quad , \tag{44}$$

before the inflationary phase sets in, which is again in accord with our findings in Eq. (8).

Thus we bounded the overshoot for the solution of the equation with correct friction term $\frac{3}{t}$, Eq. (23). The true value of the overshoot lies in between the one for the larger and smaller friction terms, respectively

$$\frac{3}{4\sqrt{2}}\sqrt{\epsilon} - \frac{1}{3}\phi_0 > \phi(t_c) > \frac{3}{4\sqrt{2}}\sqrt{\epsilon} - 0.1\phi_0 \quad . \tag{45}$$

D. $n = 4$: The Quartic

We have seen that for monomial potentials up to order three, the field always reaches the plateau at $\phi = 0$ with nonzero speed. For successful inflation, the plateau needs to be sufficiently long as there is always a chance of overshooting. In the last few subsections we have calculated this amount of overshoot by estimating $\phi(t_c)$ where curvature becomes subdominant over the potential energy.

We will show now that for the quartic potential, under the assumption of curvature domination, the field arrives at the minimum with zero speed, therefore providing ideal initial condition for inflation on the plateau. In fact, curvature domination ceases even before the field reaches the minimum at $\phi = 0$ and inflation takes place even for $\phi < 0$.

For the quartic potential $V(\phi) = \frac{\lambda_4}{4}\phi^4$, Eq. (4) with the boundary conditions $\phi(0) = \phi_0, \dot{\phi}(0) = 0$ is solved exactly by

$$\phi(t) \; = \; \frac{8\phi_0}{8 + t^2\lambda_4\phi_0^2} \quad . \tag{46}$$

This is a non-oscillating function that reaches the bottom of the potential at $\phi = 0$ in infinite time $t_f \to \infty$. Thus zero field velocity at the bottom of the potential will be zero

$$\dot{\phi}(t) \; = \; -\frac{16t\lambda\phi_0^3}{(8 + t^2\lambda\phi_0^2)^2} \stackrel{t\to t_f}{\longrightarrow} 0 \quad . \tag{47}$$

We showed explicitly that there is no overshoot for the quartic potential. The assumption of curvature domination will be justified in section III F.

E. Higher Order Monomials with $n > 4$

To the best of our knowledge there is no closed solution of Eq. (4) for a monomial potential with larger than quartic order subject to the boundary condition $\dot{\phi}(0) = 0$. However, we can make use of the fact that for a differential equation of the form

$$\ddot{\phi} + \frac{\gamma}{t}\dot{\phi} + t^{\frac{\gamma(n-2)-(n+2)}{2}}\phi^{n-1} \; = \; 0 \quad , \tag{48}$$

there exists an expression for a first integral

$$C_3 \; = \; \frac{t^{\gamma-1}}{2}\left(\dot{\phi}^2 t^2 + \dot{\phi}\phi t(\gamma - 1)\right) + \frac{\left(\phi t^{\frac{\gamma-1}{2}}\right)^n}{n} \quad ,\tag{49}$$

with C_3 being a constant, see Eq. (3.25) in [47]. Note again that any coefficient λ_n in the potential $\sim \lambda_n\phi^n$ can be absorbed in a rescaling of the time, so without loss of generality we set $\lambda_n = 1$.

To be consistent with the general form of Eq. (4), the coefficient of ϕ^m in Eq. (48) should be constant, enforcing

$$\gamma \; = \; \frac{n+2}{n-2} \quad . \tag{50}$$

As it turns out, Eq. (50) holds exactly for the quartic potential with $n = 4$ and $\gamma = 3$. Evaluating the first integral of Eq. (49) for the initial conditions $\phi(0) = \phi_0, \dot{\phi}(0) = 0$ at $t = 0$ gives

$$C_3 \; = \; 0 \quad . \tag{51}$$

In order to obtain the velocity of the field at the bottom of the potential, we evaluate Eq. (49) at the time $t > 0$ when $\phi = 0$, immediately giving

$$\dot{\phi}\Big|_{\phi=0} \; = \; 0 \quad , \tag{52}$$

in agreement with the results from the previous subsection.

In order to extract information about higher powers of the potential, we observe that for Eq. (50) to hold for larger $n > 4$, γ needs to go towards unity (for $n = 4 \Rightarrow \gamma = 3$, for $n > 4 \Rightarrow \gamma < 3$). In this case, using the first integral again, the terminal velocity at the bottom is also zero for arbitrary $n \geq 4$. At the same time, we observe that the friction term $\frac{\gamma}{t}\dot{\phi}$ in Eq. (48) for which the first integral holds is always smaller than the true friction term $\frac{3}{t}\dot{\phi}$ for powers $n > 4$. Thus the rolling scalar field in the real model must have a smaller field velocity than the velocity of the field in the above toy model with smaller friction. As the field velocity at the bottom with smaller friction is zero, field velocity with the true friction must also be zero.

In summary, for a quartic or higher monomial potentials, there is no overshoot. Assuming curvature domination, the field reaches the bottom of the potential with zero velocity. In reality, the potential energy of the inflationary plateau V_- will take over before the field reaches the minimum. Thus the field will at most have slow roll speed when entering the linear part of the potential.

F. On Curvature Domination for Higher Order Monomials

So far, we have shown explicitly for $n = 1, 2, 3$ that the assumption of curvature domination in the tunneling part of the potential is fully consistent in our setup as long as $|\phi_0| < M_P$ and $V_- << V_0$. It remains to be shown for $n \geq 4$ that curvature really dominates until the field reaches the junction point at $\phi = 0$. For potentials $\sim \phi^n$ with $n \geq 4$, we can again make use of the first integral C_3 in Eq. (49). We argue that the true field motion must be slower than the one obtained from Eq. (48), and in particular that the true $\phi(t)$ must be smaller than the solution of Eq. (48). Now let us determine some properties of the latter solution. For this part only, we deviate from our usual convention and assume that $\phi > 0$, i.e. the field is rolling from right to left.

Owing to the initial condition at $t = 0, \dot{\phi}(0) = 0$ and $\phi(0) < \infty$, we know that $C_3 = 0$. We can solve Eq. (49) for $\dot{\phi}$

$$\dot{\phi} \; = \; -\frac{2}{n-2}\frac{\phi}{t} \pm \sqrt{\frac{4\phi^2}{(n-2)^2t^2} - \frac{2\phi^n}{n}} \quad , \tag{53}$$

and notice that reality of $\dot{\phi}$ requires the discriminant to be positive. Thus we have the requirement

$$\phi(t) \;<\; 2^{1/(n-2)}\left(\frac{n}{(n-2)^2t^2}\right)^{1/(n-2)} \quad . \tag{54}$$

In particular, this means that the true solution must be even smaller than this, and the potential $V \sim \phi^n$ will depend on time like

$$V(t) \;<\; 2^{n/(n-2)}\left(\frac{n}{(n-2)^2t^2}\right)^{n/(n-2)} \quad , \tag{55}$$

which, for $n \geq 4$, is always smaller than the curvature contribution to the Friedman equation $\propto \frac{1}{t^2}$. In other words, if curvature dominates over potential energy at some point (which is definitely true as $t \to 0$), then it will always dominate over the potential energy.

IV. ANALYSIS FOR POLYNOMIALS

Polynomial potentials will generally not allow for an exact solution of the corresponding equation of motion. We may, however, approximate the full solution by gluing together the solutions of the monomials dominating in a given interval of field values. We shall illustrate this by the most simple yet non-trivial example of a binomial

$$V(\phi) = (-1)^m\frac{\lambda_m}{m}\phi^m + (-1)^n\frac{\lambda_n}{n}\phi^n \quad , \quad n > m \quad . \tag{56}$$

We will see shortly that we need to consider only the situation where $1 \leq m \leq 4$. As tunneling needs to inject the field at a sub-Planckian displacement $|\phi_0| < M_P$ to avoid slow-roll directly emerging from tunneling, we need to distinguish two cases.

In the first case, the higher power monomial has $n > 4$ and thus we have $\lambda_n = \tilde{\lambda}_n/\Lambda^{n-4}$ where $\Lambda \sim M_P$ denotes the UV cutoff of the effective field theory used to derive $V(\phi)$. Effective field theory then tells us – barring further information – to assume $\tilde{\lambda}_n = \mathcal{O}(1)$. This implies a parametric suppression of the n-monomial for sub-Planckian ϕ compared to the lower-order m-monomial, as soon as the n-monomial corresponds an irrelevant operator ($n > 4$). The dynamics thus effectively reduces to the one described in the previous sections for the lower-order m-monomial with $m \leq 4$.

The second case consists of the situation when both $m, n \leq 4$ and thus the effective field theory argument from above cannot be used to automatically suppress the higher-order term by sub-Planckian field values. In this case, we may solve the equation of motion approximately by gluing together the solutions ϕ_{L1} of the pure ϕ^n and ϕ_{L2} of the pure ϕ^m potential, assuming $n > m$. It is clear from

$$\ddot{\phi} + \frac{3}{t}\dot{\phi} = -(-1)^m\lambda_m\phi^{m-1} - (-1)^n\lambda_n\phi^{n-1} \quad , n > m \tag{57}$$

that the matching point ϕ_* may be approximated by the field value where the accelerating 'force' $-V'(\phi)$ has equal contributions from both monomials

$$(-1)^m\lambda_m\phi^{m-1}\big|_{\phi=\phi_*} \cdot (1+\delta) = (-1)^n\lambda_n\phi^{n-1}\big|_{\phi=\phi_*} \quad , \tag{58}$$

where we allowed for perturbing this matching condition by $1+\delta$. This will allow us to determine the quality of the matching condition. Solving the matching equation yields

$$\begin{aligned}\phi_* \;&=\; -\left[\lambda\,(1+\delta)\right]^{\frac{1}{n-m}} = \phi_*^{(0)} \cdot (1+\delta)^{\frac{1}{n-m}} \\ &=\; \phi_*^{(0)} \cdot \left(1 + \frac{1}{n-m}\delta + \mathcal{O}(\delta^2)\right) \quad ,\end{aligned} \tag{59}$$

where $\lambda \equiv \lambda_m/\lambda_n$ and $\phi_*^{(0)} = -\lambda^{\frac{1}{n-m}}$ is the matching field value obtained for $\delta = 0$. We can see that the matching field value is perturbatively stable under moderate changes of the matching condition itself.

As mentioned above, the solution $\phi_{L1,2}$ for $\phi < \phi_*, \phi > \phi_*$ is given by the one for the pure monomial ϕ^n, ϕ^m, matched at

$$\phi_{L1} = \phi_{L2} = \phi_*, \quad \phi_{L2}(t_*) = \phi_{L1}(t_*) \quad , \tag{60}$$

where t_* is defined as $t_* \equiv t(\phi_{L1} = \phi_*)$. ϕ_{L1} will be given by the solution to either a quadratic, cubic, or quartic monomial with ϕ_{L2} the solution of the monomial of lower order, i.e. cubic, quadratic or linear.

The case $n = 2$ is special, as it leaves only $m = 1$. This can be treated in the ensuing equation of motion exactly be shifting $\phi \to \phi - \lambda_2/\lambda_1$.

We will now use the case $n = 4$ and $m = 1$ with its convenient solutions of Eq. (46) and Eq. (10) respectively as an example to demonstrate the parametric sensibility of t_* and $\dot{\phi}_{L1}(t_*)$ on perturbing the matching condition by δ. Demanding from matching that $\phi_{L1}(t_*) = \phi_*$ we get

$$\begin{aligned}t_* \;&=\; \sqrt{\frac{8}{\lambda_4\phi_0^2}\left(\frac{\phi_0}{\phi_*^{(0)}}\,\frac{1}{(1+\delta)^{1/3}} - 1\right)} \\ &=\; t_*^{(0)} \cdot \left(1 + \frac{\phi_0}{6\,(\phi_*^{(0)} - \phi_0)}\,\delta + \mathcal{O}(\delta^2)\right) , \\ \dot{\phi}_{L1}(t_*) \;&=\; \frac{\sqrt{\lambda_4}}{4}(\phi_*^{(0)})^2(1+\delta)^{2/3} \\ &\quad\times\sqrt{8\left(\frac{\phi_0}{\phi_*^{(0)}}\,\frac{1}{(1+\delta)^{1/3}} - 1\right)} \\ &=\; \dot{\phi}_L(t_*^{(0)}) \cdot \left(1 + \frac{\phi_0 - \frac{4}{3}\phi_*^{(0)}}{2(\phi_0 - \phi_*^{(0)})}\,\delta + \mathcal{O}(\delta^2)\right) \quad .\end{aligned} \tag{61}$$

We see that the initial conditions for the subsequent ϕ_{L2}-evolution towards $\phi = 0$, t_* and $\dot{\phi}_L(t_*)$, are stable under moderate perturbations of the matching condition.

We expect qualitatively similar behavior for other values of m, n. The solutions for all values of m, n are polynomial, i.e. non-exponential, in all parameters. This should translate into perturbative stability of the final speed $\dot{\phi}_f(\phi = 0)$ under moderate changes of the matching condition.

A. Binomial $V = V_- - \lambda_1\phi + \frac{1}{4}\lambda_4\phi^4$

We will now proceed to analytically estimate the terminal velocity at $\phi = 0$ for the potential $V = V_- - \lambda_1\phi + \frac{1}{4}\lambda_4\phi^4$. As outlined above, we assume that at early times $t < t_*$, $\lambda_4|\phi|^3 > \lambda_1$, *i.e* the field evolution is dominated by the quartic term, while for $t > t_*$ it is determined by the linear term. The following equations of motions

$$\ddot{\phi}_{L1} + \frac{3}{t}\dot{\phi}_{L1} + \lambda_4\phi_{L1}^3 \ = \ 0 \quad , \tag{62}$$

$$\ddot{\phi}_{L2} + \frac{3}{t}\dot{\phi}_{L2} - \lambda_1 \ = \ 0 \quad , \tag{63}$$

are subject to the boundary conditions

$$\dot{\phi}_{L1}(0) = 0 \quad , \quad \phi_{L1}(0) = \phi_0 \quad , \tag{64}$$

$$\dot{\phi}_{L2}(t_*) = \dot{\phi}_{L1}(t_*) \quad , \quad \phi_{L2}(t_*) = \phi_L(t_*) \quad , \tag{65}$$

are solved by

$$\begin{aligned} \phi_{L1}(t) &= \frac{8\phi_0}{8 + \lambda_4\phi_0^2\, t^2} \quad , \\ \phi_{L2}(t) &= \frac{\lambda\,\lambda_4}{8}t^2 + \frac{8(\lambda^{1/3} + \phi_0)^3}{\lambda_4\phi_0^4}t^{-2} + \frac{2\lambda + 3\lambda^{2/3}\phi_0}{\phi_0^2} \quad , \end{aligned} \tag{66}$$

where $\lambda = \lambda_1/\lambda_4$, making the approximate solution

$$\phi(t) \ = \ \begin{cases} \phi_{L1}(t), & t < t_* \quad , \\ \phi_{L2}(t), & t > t_* \quad . \end{cases} \tag{67}$$

Plugging in some numbers $\lambda_1 = 10^{-10}, \lambda_4 = 1, \phi_0 = -0.01$, we find that the analytical approximation fits quite well to the numerical results, see Figure 3. The terminal speed at time t_f, i.e. at the bottom of the potential, is not clearly visible in this plot, but evaluates to $\dot{\phi}(t_f) = 1.22 \times 10^{-7}$ (analytic) which agrees quite well with the numerical result $\dot{\phi}(t_f) = 1.35 \times 10^{-7}$.

Finally, we can now estimate the amount of overshoot. For this purpose we need to determine the time when the field reaches the bottom, t_f, and its velocity there, $\dot{\phi}_{L2}(t_f)$, by evolving the approximate solution Eq. (66) for until $\phi = 0$.

One then uses $\phi_R(t_f) = 0$, $\dot{\phi}_R(t_f) = \dot{\phi}_{L2}(t_f)$ as initial conditions for the solution to Eq. (5) on the shallow inflationary slope on the right of the matching point $\phi = 0$. The overshoot, i.e. the field position $\phi_R(t_c)$ at time $t_c = \sqrt{3/V_-}$ when curvature becomes subdominant to the vacuum energy driving inflation, evaluates to (only neglecting terms containing V_-)

$$\phi(t_c) = \frac{3}{4\sqrt{2}}\sqrt{\epsilon} - \left(\frac{\phi_*}{\phi_0}\right)^{3/2}\left(4 - 3\frac{\phi_*}{\phi_0}\right)^{1/2}\phi_0 \quad . \tag{68}$$

Writing it in this form, we take the matching point ϕ_* as independent parameter characterizing the two coefficients λ_1, λ_4. If we move the matching point all the way to the left, i.e. to $\phi_* = \phi_0$, we are dealing with a purely linear potential and recover

$$\phi(t_c) \overset{\phi_* \to \phi_0}{\to} \frac{3}{4\sqrt{2}}\sqrt{\epsilon} - \phi_0 \quad , \tag{69}$$

in agreement with Eq. (14). If on the other hand we move ϕ_* all the way to the right, we obtain

$$\phi(t_c) \overset{\phi_* \to 0}{\to} \frac{3}{4\sqrt{2}}\sqrt{\epsilon} \quad , \tag{70}$$

which is the formal limit of the overshoot for a quartic potential: the field reaches the bottom of the potential at zero speed. Neglecting terms with V_- and setting $\dot{\phi}_f = 0$, this agrees with Eq. (8). Note that the overshoot is smaller than the overshoot appearing in a purely linear potential. Indeed, it is rather intuitive that the overshoot of the field in a binomial potential can be at most as large as the overshoot derived from the lowest-power monomial appearing in the scalar potential alone.

Finally, a trinomial can be approximate as the behavior of the highest power monomial matched to the solution of the lower-power binomial, and this procedure may be iterated to higher powers still. Thus, we expect that the above argument concerning the upper limit on the overshoot for the binomial case generalizes to the case of full power-series expansion of the scalar potential in positive powers of ϕ.

B. Binomial $V = V_- + \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda_4\phi^4$

We follow the same strategy as in the previous section, matching solutions where either quadratic part or the quartic part dominate to obtain

$$\begin{aligned} \phi_{L1} &= \frac{8\phi_0}{8 + \lambda_4\phi_0^2 t^2} \quad , \\ \phi_{L2} &= \frac{4\pi\alpha(1-\alpha)}{\sqrt{\lambda_4}t} Y_1(\alpha\sqrt{\lambda_4}\phi_0 t) \\ &\quad \times \left[{}_0F_1(1; 2\alpha(\alpha-1)) - \alpha\ {}_0F_1(2; 2\alpha(\alpha-1))\right] \\ &\quad + \pi\alpha^2\phi_0\ {}_0F_1(2; -\frac{\alpha^2\lambda_4\phi_0^2}{4}t^2) \\ &\quad \times \Big[2(1-\alpha)Y_0(-2\sqrt{2\alpha(1-\alpha)}) \\ &\quad + \sqrt{2\alpha(1-\alpha)}Y_1(-2\sqrt{2\alpha(1-\alpha)})\Big] \quad , \end{aligned} \tag{71}$$

where $\alpha = -\frac{m}{\sqrt{\lambda_4}\phi_0}, 0 < \alpha < 1$. Figure 4 shows the phase space of this solution, with panel (a) using the analytical expression of Eq. (71) and (b) using numerical integration of the differential equation $\ddot{\phi} + \frac{3}{t}\dot{\phi} + m^2\phi + \lambda_4\phi^3 = 0$. We chose $\lambda_4 = 0.1, \phi_0 = -0.1, m = 10^{-5}$. The analytical estimate of terminal velocity is within the right order of magnitude of the numerical result.

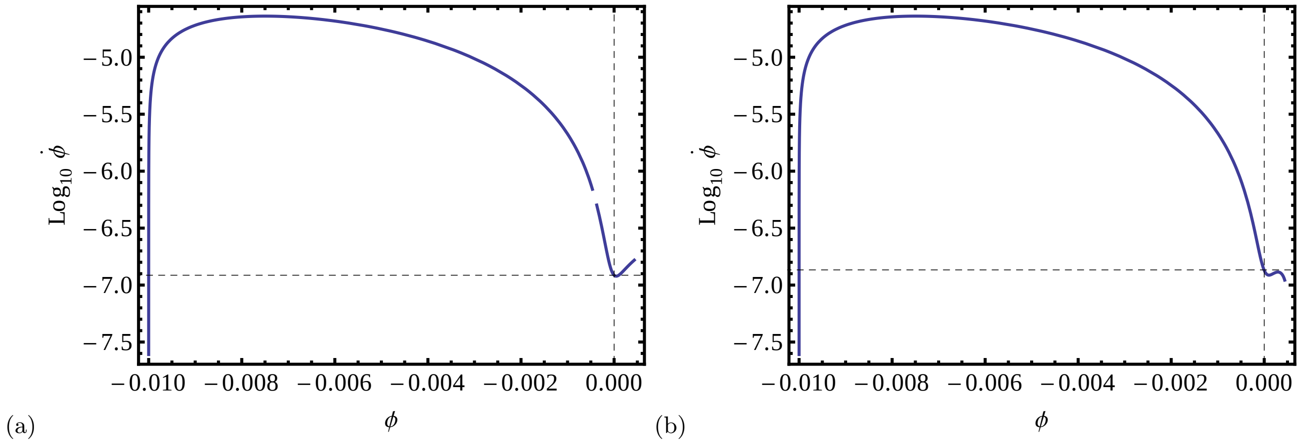

FIG. 3: Phase space for a field rolling in the potential $V = -\lambda_1\phi + \frac{1}{4}\lambda_2\phi^4$ (a) using the analytical expression Eq. (66) and (b) using numerical integration of the differential equation. The analytical approximation agrees quite well with the results from numerics. The terminal speed is not clearly visible in this plot, but evaluates to $\dot\phi(t_f) = 1.22\times10^{-7}$ (analytic) and $\dot\phi(t_f) = 1.35\times10^{-7}$ (numerical).

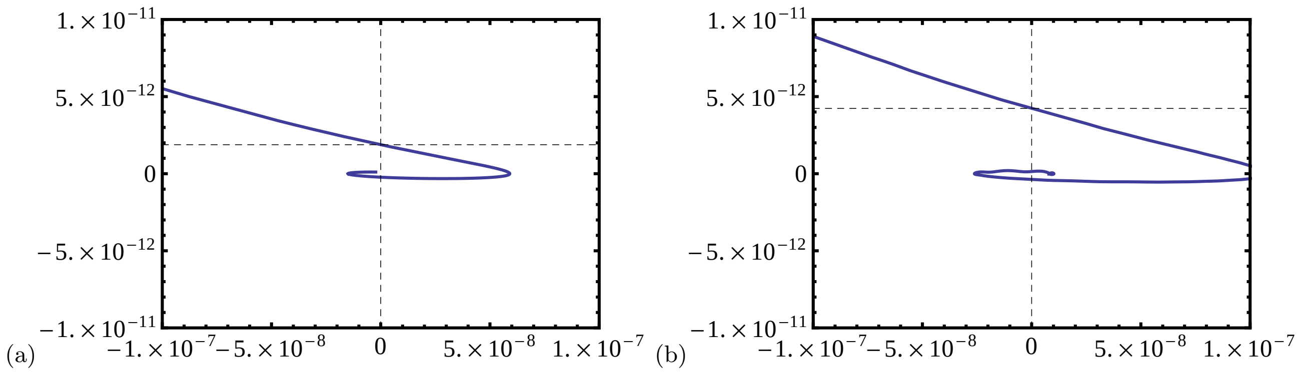

FIG. 4: Phase space for a field rolling in the potential $V = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda_4\phi^4$ (a) using the analytical expression Eq. (71) and (b) using numerical integration of the differential equation. The analytical approximation underestimates the terminal velocity by an order of magnitude (note the different scales on the $\dot\phi$ axes).

In order to estimate the amount of overshoot like in the last subsection, we need to find solve $\phi_{L2}(t) = 0$ with ϕ_{L2} from Eq. (71). This can only be done numerically. Thus we unfortunately cannot give an analytic estimate for the overshoot.

We note that the better agreement of analytics and numerics in the case of previous polynomial of linear and quartic is caused by the fact that the transition region between the quartic and linear part is much shorter than the one between the quartic and quadratic part. For $|\phi| < 1$, the range in ϕ where $\phi^3\lambda_4 \approx \lambda_1$ is much smaller than for $\phi^3\lambda_4 \approx m^2\phi$.

V. NON-ZERO INITIAL SPEED – GENERALIZATION TO THE LANDSCAPE

The discussion so far established parametrically small or even vanishing overshoot in a monotonically falling non-inflationary but otherwise arbitrary potential after tunneling leading into a slow-roll inflationary plateau using the single-field CDL boundary condition of zero initial speed $\dot\phi_0 = 0$. Here we will give a simple argument which generalizes these results to arbitrary finite initial speed $\dot\phi_0 \neq 0$ as long as the post-tunneling evolution starts in a bubble geometry dominated by negative spatial curvature.

Assume the scalar field to emerge from CDL tunneling but with finite $\dot\phi_0 \neq 0$. For sufficiently early times $\epsilon \ll 1$ the friction term in the curvature-dominated equation of motion will completely vanquish the force term $\partial V/\partial\phi$. This leaves at very early times an equation of motion

$$\ddot\phi = -\frac{3}{t}\dot\phi \quad , \tag{72}$$

which is solved by

$$\dot\phi(t) = \dot\phi(\epsilon)\,\left(\frac{\epsilon}{t}\right)^3 \quad . \tag{73}$$

This implies that a given finite initial speed $\dot\phi_0 = \lim_{\epsilon\to0}\dot\phi(\epsilon)$ will fall to a given fraction $\alpha \equiv \dot\phi(t)/\dot\phi_0$ in an arbitrarily short time

$$t_\alpha = \frac{\epsilon}{\alpha^{1/3}} \quad , \tag{74}$$

once we take $\epsilon \to 0$.

Thus a given finite initial speed will approach a value arbitrarily close to zero in arbitrarily short time. The solutions to monomial potentials with finite initial speed asymptote to the our results with zero initial speed at arbitrarily early times. Thus, in a bubble with negative spatial curvature, the overshoot results hold for initial conditions of arbitrary finite initial velocity $\dot{\phi}_0$ as well.

There are immediate consequences of this universal behavior for the multi-field situation in a landscape of local minima. The path of a CDL bounce with minimum action will generically not be one that leads into a basin of classical attraction (determined by the gradient of the scalar potential $\vec{\nabla}\phi$ at the point of exit from tunneling) towards a slow-roll inflationary region.

For instance, the minimum action bounce may very well lead us directly into our present day vacuum, bypassing the inflationary phase completely. Conversely, a CDL bounce may exit into a basin of classical attraction towards a slow-roll inflationary region with a higher-action bounce. However, a bubble universe created this way will undergo rapid inflation. We will not enter the discussion of how to weigh different CDL bounces leading to different amounts of slow-roll inflation post-tunneling. We merely require the existence of ~ 60 e-folds of slow-roll as an anthropic post-selection criterion to select against otherwise smaller bounces leading directly into the final vacuum.

This instanton is generically a curved trajectory in scalar field space which provides some of the scalar fields with a finite initial speed after tunneling. As the initial speed vector may very well point along the inflationary direction in scalar field space, this would generically lead to overshoot of the inflationary plateau region if small or vanishing overshoot were contingent on vanishing initial speed.

Thus the overshoot problem is not as severe even for finite initial velocities. This provides a crucial generalization to a multi-field landscape: If the landscape is populated via CDL tunneling – and this is the only known and controlled mechanism so far – then there never is a large overshoot problem on the *first* slow-roll inflationary plateau reached via steepest-decent after the exit from a CDL tunneling event. Thus, on reaching this first plateau after exit from CDL tunneling, small-field and large-field inflation regions in the landscape are on equal footing with respect to the phase space of initial conditions.

VI. CONCLUSIONS

We have shown that the overshoot problem in inflation after tunneling, i.e. inflation in open universes, is not as severe as it might first seem. We have demonstrated this for arbitrary monomial exit potentials $V(\phi) = \frac{\lambda_n}{n}\phi^n$ – describing the steep downhill part of the potential at $\phi < 0$ after emerging from tunneling – turning into an inflationary plateau with a linear potential for $\phi = 0$ with its slope given by the first slow-roll parameter ϵ. In this setup we have found that for the three lowest-power monomial exit potentials ϕ, ϕ^2, ϕ^3 there is a finite amount of overshoot

$$\Delta\phi_{CDL} = \frac{3}{4\sqrt{2}}\sqrt{\epsilon} + \mathcal{O}(1)|\phi_0| \quad . \tag{75}$$

Surprisingly, for monomials ϕ^n , $n \geq 4$ we get no overshoot at all as the field already enters the inflationary phase on the exit potential. The larger n, i.e. the steeper the exit potential and the faster it asymptotes to slow-roll flatness, the more time the field spends on the flat part of the exit potential. In addition, the Hubble friction $1/t^2$ will be larger as well. These two effects combined result in a more and more efficient slow down until for $n \geq 4$, the field comes to a complete stop.

Next we generalized our analysis to include polynomial exit potentials. For a binomial, the overshoot is controlled by $|\phi_0|$ and determined by the lowest power monomial. This property carries directly over to a general exit potential written as a power series expansion around the point $\phi = 0$ where slow-roll flatness has set in.

Finally, we have shown that this absence of a parametrically large overshoot for small-field inflation is actually *independent* on the initial speed $\dot{\phi}_0$ of the field. This seems to be in contradiction with a comment made in [32] that adding negative curvature in general can be easily overwhelmed by large enough initial field speed. However, our results are actually compatible with this comment: the specific boundary condition of the CDL instanton, namely that $a(t) \to t$ for small t, is crucial for having curvature domination early on. This boundary condition was not used in the comment of [32].

In the limit of small inflationary energy scale compared to the tunneling scale, the CDL tunneling boundary conditions for the negative spatial curvature and the scale factor are the only conditions for our results to be valid. However, this implies that our estimates of the overshoot generalize immediately to the case of CDL tunneling in a multi-field landscape. There, some of the scalar fields will generically have finite initial speed after tunneling.

In the context of the string landscape populated by CDL tunneling, our analysis shows that small-field and large-field inflation have parametrically the same volume of phase space of initial conditions.

Acknowledgements

AW is grateful to L. McAllister and S. Raby for several illuminating comments and discussions. The authors wish to thank N. Kaloper for valuable comments on the draft. This work was supported by the Impuls und Vernetzungsfond of the Helmholtz Association of German Research Centers under grant HZ-NG-603, and German Science Foundation (DFG) within the Collaborative Research Center 676 “Particles, Strings and the Early Universe”.

---

[1] A. D. Linde, “Particle Physics and Inflationary Cosmology”, hep-th/0503203.

[2] **WMAP** Collaboration, E. Komatsu *et al.*, “Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation”, *Astrophys. J. Suppl.* **192** (2011) 18, arXiv:1001.4538.

[3] D. Larson *et al.*, “Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Power Spectra and WMAP-Derived Parameters”, *Astrophys. J. Suppl.* **192** (2011) 16, arXiv:1001.4635.

[4] D. H. Lyth, “What would we learn by detecting a gravitational wave signal in the cosmic microwave background anisotropy?”, *Phys. Rev. Lett.* **78** (1997) 1861–1863, hep-ph/9606387.

[5] R. Brustein and P. J. Steinhardt, “Challenges for superstring cosmology”, *Phys.Lett.* **B302** (1993) 196–201, arXiv:hep-th/9212049.

[6] A. Albrecht, R. H. Brandenberger, and R. Matzner, “Inflation with Generalized Initial Conditions”, *Phys.Rev.* **D35** (1987) 429.

[7] R. H. Brandenberger, H. A. Feldman, and J. H. MacGibbon, “Thermal Fluctuation in New Inflation”, *Phys.Rev.* **D37** (1988) 2071.

[8] D. S. Goldwirth and T. Piran, “Inhomogeneity and the Onset of Inflation”, *Phys. Rev. Lett.* **64** (1990) 2852–2855.

[9] D. S. Goldwirth, “Initial Condition for new Inflation”, *Phys. Lett.* **B243** (1990) 41–44.

[10] V. Belinsky, I. Khalatnikov, L. Grishchuk, and Y. Zeldovich, “Inflationary Stages in Cosmological Models with a Scalar Field”, *Phys.Lett.* **B155** (1985) 232–236.

[11] T. Piran, “On General Conditions for Inflation”, *Phys.Lett.* **B181** (1986) 238–243.

[12] R. H. Brandenberger and J. H. Kung, “Chaotic Inflation as an Attractor in Initial Condition Space”, *Phys. Rev.* **D42** (1990) 1008–1015.

[13] D. S. Goldwirth, “On inhomogeneous initial conditions for inflation”, *Phys.Rev.* **D43** (1991) 3204–3213.

[14] D. S. Goldwirth and T. Piran, “Initial conditions for inflation”, *Phys. Rept.* **214** (1992) 223–291.

[15] S. Bird, H. V. Peiris, and R. Easther, “Fine-tuning criteria for inflation and the search for primordial gravitational waves”, *Phys.Rev.* **D78** (2008) 083518, arXiv:0807.3745, * Brief entry *.

[16] N. Kaloper and K. A. Olive, “Dilatons in string cosmology”, *Astropart.Phys.* **1** (1993) 185–194.

[17] R. Brustein, S. de Alwis, and P. Martens, “Cosmological stabilization of moduli with steep potentials”, *Phys.Rev.* **D70** (2004) 126012, arXiv:hep-th/0408160.

[18] N. Kaloper, J. Rahmfeld, and L. Sorbo, “Moduli entrapment with primordial black holes”, *Phys.Lett.* **B606** (2005) 234–244, arXiv:hep-th/0409226.

[19] M. Dine, “Towards a solution of the moduli problems of string cosmology”, *Phys. Lett.* **B482** (2000) 213–221, hep-th/0002047.

[20] N. Itzhaki and E. D. Kovetz, “Inflection Point Inflation and Time Dependent Potentials in String Theory”, *JHEP* **10** (2007) 054, arXiv:0708.2798.

[21] R. Bousso and J. Polchinski, “Quantization of four form fluxes and dynamical neutralization of the cosmological constant”, *JHEP* **0006** (2000) 006, arXiv:hep-th/0004134.

[22] L. Susskind, “The Anthropic landscape of string theory”, arXiv:hep-th/0302219.

[23] D. Baumann and L. McAllister, “Advances in Inflation in String Theory”, *Ann.Rev.Nucl.Part.Sci.* **59** (2009) 67–94, arXiv:0901.0265.

[24] M. Cicoli and F. Quevedo, “String moduli inflation: An overview”, arXiv:1108.2659.

[25] C. Burgess and L. McAllister, “Challenges for String Cosmology”, arXiv:1108.2660, * Temporary entry *.

[26] S. Kachru, R. Kallosh, A. D. Linde, J. M. Maldacena, L. P. McAllister, *et al.*, “Towards inflation in string theory”, *JCAP* **0310** (2003) 013, arXiv:hep-th/0308055.

[27] D. Baumann, A. Dymarsky, I. R. Klebanov, J. M. Maldacena, L. P. McAllister, *et al.*, “On D3-brane Potentials in Compactifications with Fluxes and Wrapped D-branes”, *JHEP* **0611** (2006) 031, arXiv:hep-th/0607050.

[28] A. Krause and E. Pajer, “Chasing brane inflation in string-theory”, *JCAP* **0807** (2008) 023, arXiv:0705.4682.

[29] D. Baumann, A. Dymarsky, I. R. Klebanov, and L. McAllister, “Towards an Explicit Model of D-brane Inflation”, *JCAP* **0801** (2008) 024, arXiv:0706.0360.

[30] D. Baumann, A. Dymarsky, S. Kachru, I. R. Klebanov, and L. McAllister, “Holographic Systematics of D-brane Inflation”, *JHEP* **0903** (2009) 093, arXiv:0808.2811.

[31] B. Underwood, “Brane Inflation is Attractive”, *Phys.Rev.* **D78** (2008) 023509, arXiv:0802.2117.

[32] S. Bird, H. V. Peiris, and D. Baumann, “Brane Inflation and the Overshoot Problem”, *Phys.Rev.* **D80** (2009) 023534, arXiv:0905.2412.

[33] N. Agarwal, R. Bean, L. McAllister, and G. Xu, “Universality in D-brane Inflation”, *JCAP* **1109** (2011) 002, arXiv:1103.2775, * Temporary entry *.

[34] B. Freivogel, M. Kleban, M. Rodriguez Martinez, and L. Susskind, “Observational consequences of a landscape”, *JHEP* **0603** (2006) 039, arXiv:hep-th/0505232.

[35] S. R. Coleman, “The Fate of the False Vacuum. 1. Semiclassical Theory”, *Phys. Rev.* **D15** (1977) 2929–2936.

[36] S. R. Coleman and F. De Luccia, “Gravitational Effects on and of Vacuum Decay”, *Phys. Rev.* **D21** (1980) 3305.

[37] S. W. Hawking and I. G. Moss, “Supercooled Phase Transitions in the Very Early Universe”, *Phys. Lett.* **B110** (1982) 35.

[38] D. Yamauchi, A. Linde, A. Naruko, M. Sasaki, and T. Tanaka, “Open inflation in the landscape”,

Phys.Rev. **D84** (2011) 043513, `arXiv:1105.2674`, * Temporary entry *.

[39] M. J. Duncan and L. G. Jensen, “Exact tunneling solutions in scalar field theory”, *Phys.Lett.* **B291** (1992) 109–114.

[40] T. Hamazaki, M. Sasaki, T. Tanaka, and K. Yamamoto, “Selfexcitation of the tunneling scalar field in false vacuum decay”, *Phys.Rev.* **D53** (1996) 2045–2061, `arXiv:gr-qc/9507006`.

[41] K. Koyama, K. Maeda, and J. Soda, “An Open universe from valley bounce”, *Nucl.Phys.* **B580** (2000) 409–437, `arXiv:hep-ph/9910556`.

[42] K. Dutta, P. M. Vaudrevange, and A. Westphal, “An Exact Tunneling Solution in a Simple Realistic Landscape”, `arXiv:1102.4742`.

[43] G. Pastras, “Exact Tunneling Solutions in Minkowski Spacetime and a Candidate for Dark Energy”, `arXiv:1102.4567`, * Temporary entry *.

[44] “The Digital Library of Mathematical Functions (DLMF)”, National Institute of Standards and Technology (NIST), Gaithersburg, U.S.A., 2011-07-01.

[45] P. G. L. Leach, “First Integrals For The Modified Emden Equation q+alpha(t)q+qn=O”, *Journal Of Mathematical Physics* **26** (1985), no. 10, 2510–2514.

[46] D. A. Polyanin and F. V. Zaitsev, “Handbook of Exact Solutions for Ordinary Differential Equations”, 2002.

[47] M. R. Feix and H. R. Lewis, “Invariants For Dissipative Nonlinear-Systems By Using Rescaling”, *Journal Of Mathematical Physics* **26** (1985), no. 1, 68–73.