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Finding new relationships between hypergeometric functions by evaluating Feynman integrals

Bernd A. Kniehl[1]*, Oleg V. Tarasov[2]†

[1] Kavli Institute for Theoretical Physics, Kohn Hall, University of Santa Barbara, CA 93106, USA

[2] II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany

Abstract

Several new relationships between hypergeometric functions are found by comparing results for Feynman integrals calculated using different methods. A new expression for the one-loop propagator-type integral with arbitrary masses and arbitrary powers of propagators is derived in terms of only one Appell hypergeometric function F_1. From the comparison of this expression with a previously known one, a new relation between the Appell functions F_1 and F_4 is found. By comparing this new expression for the case of equal masses with another known result, a new formula for reducing the F_1 function with particular arguments to the hypergeometric function ${}_3F_2$ is derived. By comparing results for a particular one-loop vertex integral obtained using different methods, a new relationship between F_1 functions corresponding to a quadratic transformation of the arguments is established. Another reduction formula for the F_1 function is found by analysing the imaginary part of the two-loop self-energy integral on the cut. An explicit formula relating the F_1 function and the Gaussian hypergeometric function ${}_2F_1$ whose argument is the ratio of polynomials of degree six is presented.

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*On leave of absence from II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany.

†On leave of absence from Joint Institute for Nuclear Research, 141980 Dubna (Moscow Region), Russia.

1 Introduction

Radiative corrections to different physical quantities needed for the comparison of theoretical predictions with experimental data to be collected with the CERN Large Hadron Collider (LHC) and, in future, with an International Linear Collider (ILC) and other colliders are expressed in terms of complicated Feynman integrals. In many cases, radiative corrections must be evaluated analytically to achieve reliable accuracies in the calculations. The difficulties in calculating Feynman integrals are usually related to the fact that they depend on several kinematical scales, i.e. they are functions of several variables.

Nowadays, one of the most frequently used methods for calculating Feyman integrals is based on the Mellin-Barnes integral representation [1,2,3]. In many cases, however, this method leads to complicated expressions in terms of hypergeometric functions with many variables. In order to calculate analytically integrals with several kinematical variables and masses, new effective methods are to be developed. Rather promising methods for analytic calculations of Feynman integrals are based on recurrence relations. These can be recurrence relations with respect to the exponent of a propagator of the integral [4] or the parameter of the space-time dimension [5,6,7]. As was already observed in the one-loop case, the solutions of dimensional recurrences are combinations of hypergeometric functions [5,6,8]. This is also true at the two-loop level [7].

As was realized many years ago in Ref. [9], Feynman integrals are generalized hypergeometric functions. This conjecture was confirmed through the evaluations of specific Feynman integrals. Some results for Feynman integrals expressed in terms of hypergeometric functions may be found in Refs. [1,2,3,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24]. These results were obtained using rather different methods, e.g. by directly evaluating the integrals from their Feynman parameter representations, by applying Mellin-Barnes integral representations, by solving recurrence relations, by making use of the negative-dimension approach [25], or by using spectral representations.

As a method for finding relations between hypergeometric functions, the authors of Ref. [26] advocated the evaluation of integrals reducible to hypergeometric functions by several different methods and the comparison of the results thus obtained. In this respect, the evaluation of Feynman integrals may be considered as a rich source for finding relations between hypergeometric functions. New transformation and reduction formulae for hypergeometric functions were derived by calculating Feynman integrals already a long time ago [12]. Several new reduction relations for the Appell hypergeometric functions F_1 and F_4 obtained by comparing different results for the same Feynman integral were presented in Ref. [27].

The analytic evaluation of Feynman integrals offers us a unique possibility to find relations between hypergeometric functions which can be useful in many other applications, far away from high-energy physics. On the other hand, the problems emerging when evaluating Feynman integrals may become interesting for mathematicians, and their participation in the solution of these problems may lead to essential progress in the evaluation of Feynman integrals.

Our paper organized as follows. In section 2, we present a new result for a one-loop propagator-type integral with arbitrary exponents of propagators and arbitrary masses. In section 3, a new formula for the reduction of the Appell function F_4 to the function F_1 is derived. Setting the masses in the result derived in section 2 to be equal and comparing the outcome with a known result, a new formula for the reduction of the Appell function F_1 to the hypergeometric function ${}_3F_2$ is obtained. In section 4, from the results for the one-loop vertex-type integral, a quadratic transformation formula for the Appell function F_1 is derived. In section 5, from the comparison of results for the imaginary part of a two-loop self-energy integral obtained using different methods, a formula for the reduction of the F_1 function to the Gauss hypergeometric function with a complicated argument is obtained. In section 6, we present a short summary of our results.

2 New analytic expression for the one-loop propagator-type integral

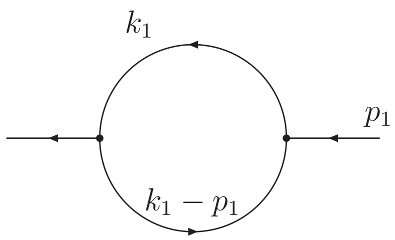

Figure 1: Feynman diagram corresponding to the integral $I^{(d)}_{\nu_1\nu_2}$.

In this section, we consider the evaluation of the one-loop propagator type integral with arbitrary masses and arbitrary powers of propagators,

$$I^{(d)}_{\nu_1\nu_2}(m_1^2, m_2^2;\ s_{12}) = \int \frac{d^dq}{i\pi^{d/2}} \frac{1}{[(q-p_1)^2 - m_1^2]^{\nu_1}[(q-p_2)^2 - m_2^2]^{\nu_2}}. \tag{1}$$

Here and below, it is understood that the usual causal prescription of the propagators is used, i.e. $1/[k^2 - m^2] \leftrightarrow 1/[k^2 - m^2 + i0]$. The Feynman diagram corresponding to this integral is presented in Figure 1. By using the formula

$$\frac{1}{a^{\nu_1}b^{\nu_2}} = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^1 dx \frac{x^{\nu_1-1}(1-x)^{\nu_2-1}}{[ax + b(1-x)]^{\nu_1+\nu_2}}, \tag{2}$$

the product of the propagator factors can be transformed to an integral over Feynman parameters. Changing the integration momentum as $q = t + p_2 + x(p_1 - p_2)$ and applying the formula

$$\int \frac{d^dt}{i\pi^{d/2}} \frac{1}{(t^2 - M^2)^\nu} = (-1)^\nu \frac{\Gamma\left(\nu - \frac{d}{2}\right)}{(M^2)^{\nu-\frac{d}{2}}\Gamma(\nu)}, \tag{3}$$

we obtain the following representation for the integral of Eq. (1):

$$I^{(d)}_{\nu_1\nu_2}(m_1^2, m_2^2;\; s_{12}) = (-1)^{\nu_1+\nu_2} \frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^1 \frac{dx\; x^{\nu_1-1}(1-x)^{\nu_2-1}}{[s_{12}x^2 + x(m_1^2 - m_2^2 - s_{12}) + m_2^2]^{\nu_1+\nu_2-\frac{d}{2}}}. \tag{4}$$

Representing the quadratic polynomial in the denominator as

$$s_{12}x^2 + x(m_1^2 - m_2^2 - s_{12}) + m_2^2 = m_2^2(1-x_1x)(1-x_2x), \tag{5}$$

and then comparing our integral with the integral representation for the Appell function [28],

$$F_1(a, b, b'; c;\; w, z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{du\; u^{a-1}(1-u)^{c-a-1}}{(1-uw)^b(1-uz)^{b'}}, \tag{6}$$

the following result follows:

$$I^{(d)}_{\nu_1\nu_2}(m_1^2, m_2^2;\; s_{12}) = \frac{(-1)^{\nu_1+\nu_2}\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1+\nu_2)(m_2^2)^{\nu_1+\nu_2-d/2}} \times F_1\left(\nu_1, \nu_1+\nu_2-\frac{d}{2}, \nu_1+\nu_2-\frac{d}{2}; \nu_1+\nu_2; x_-, x_+\right), \tag{7}$$

where

$$x_\pm = \frac{1+x-y\pm\sqrt{\Lambda(1,x,y)}}{2}, \qquad x = \frac{s_{12}}{m_2^2}, \qquad y = \frac{m_1^2}{m_2^2}, \tag{8}$$

with

$$\Lambda(x,y,z) = (x-y-z)^2 - 4yz. \tag{9}$$

For the particular case $\nu_1 = \nu_2 = 1$, an expression for the propagator-type integral in terms of the Appell function F_1 was given in Refs. [11,27,29].

The integral $I^{(d)}_{\nu_1\nu_2}(m_1^2, m_2^2;\; s_{12})$ is symmetric with respect to the change $\nu_1, m_1^2 \leftrightarrow \nu_2, m_2^2$. To understand this symmetry, we first observe that, under the change $m_1^2 \leftrightarrow m_2^2$, the arguments of the Appell function F_1 in Eq. (7) transform as

$$x_- \to \frac{x_-}{x_- - 1}, \qquad x_+ \to \frac{x_+}{x_+ - 1}, \tag{10}$$

and then, applying the formula (see, for example, Refs. [28,30,31])

$$F_1\big(\alpha, \beta, \beta', \gamma, w, z\big) = (1-w)^{-\beta}(1-z)^{-\beta'} F_1\big(\gamma-\alpha, \beta, \beta', \gamma, \frac{w}{w-1}, \frac{z}{z-1}\big), \tag{11}$$

we return to the initial expression on the right-hand side of Eq. (7).

3 Relations between the F_1 function and other hypergeometric functions

In Ref. [1], by exploiting the Mellin-Barnes integral representation, the following analytic expression for the considered integral was derived:

$$
\begin{aligned}
I^{(d)}_{\nu_1\nu_2}(m_1^2, m_2^2;\ s_{12}) &= \frac{(-1)^{\nu_1+\nu_2}}{(m_2^2)^{\nu_1+\nu_2-d/2}} \left\{ \frac{\Gamma\left(\frac{d}{2}-\nu_1\right)\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\nu_2)} \right. \\
&\times F_4\left(\nu_1, \nu_1+\nu_2-\frac{d}{2}; \frac{d}{2}, \nu_1-\frac{d}{2}+1; \frac{s_{12}}{m_2^2}, \frac{m_1^2}{m_2^2}\right) \\
&\left. + \left(\frac{m_1^2}{m_2^2}\right)^{\frac{d}{2}-\nu_1} \frac{\Gamma\left(\nu_1-\frac{d}{2}\right)}{\Gamma(\nu_1)} F_4\left(\nu_2, \frac{d}{2}; \frac{d}{2}, \frac{d}{2}-\nu_1+1; \frac{s_{12}}{m_2^2}, \frac{m_1^2}{m_2^2}\right) \right\}. \qquad (12)
\end{aligned}
$$

A hypergeometric representation in terms of Lauricella functions for the one-loop integrals corresponding to diagrams with an arbitrary number of external legs was presented in Refs. [2,3].

Comparing Eqs. (7) and (12), we arrive at the following relation:

$$
\begin{aligned}
& F_1\left(\nu_1, \nu_1+\nu_2-\frac{d}{2}, \nu_1+\nu_2-\frac{d}{2}; \nu_1+\nu_2;\ x_-, x_+\right) = \\
& \frac{\Gamma\left(\frac{d}{2}-\nu_1\right)\Gamma(\nu_1+\nu_2)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\nu_2)} F_4\left(\nu_1, \nu_1+\nu_2-\frac{d}{2}; \frac{d}{2}, \nu_1-\frac{d}{2}+1;\ x, y\right) \\
& + y^{\frac{d}{2}-\nu_1} \frac{\Gamma\left(\nu_1-\frac{d}{2}\right)\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)} F_4\left(\nu_2, \frac{d}{2}; \frac{d}{2}, \frac{d}{2}-\nu_1+1;\ x, y\right). \qquad (13)
\end{aligned}
$$

Here, $x_\pm$ are given by Eq. (8). With the help of the relation given in Ref. [32] (see p. 102),

$$
F_4\left(\alpha, \beta; \gamma, \beta; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right) = [(1-x)(1-y)]^{\alpha} F_1(\alpha, \gamma-\beta, 1+\alpha-\gamma, \gamma; x, xy), \qquad (14)
$$

the second Appell functions F_4 on the right-hand side of Eq. (13) may be expressed in terms of the Appell function F_1. Therefore, the following relation holds:

$$
\begin{aligned}
& F_4(\alpha, \beta, \beta', \alpha-\beta'+1; x, y) \\
&\quad = \frac{\Gamma(\beta')\Gamma(\beta-\alpha+\beta')}{\Gamma(\beta'-\alpha)\Gamma(\beta+\beta')} F_1(\alpha, \beta, \beta, \beta+\beta'; x_-, x_+) \\
&\quad - \frac{\Gamma(\beta')\Gamma(\beta-\alpha+\beta')\Gamma(\alpha-\beta')}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta'-\alpha)}\ y^{\beta'-\alpha}\ (x_+-x)^{\alpha-\beta-\beta'} \\
&\quad \times F_1\left(\beta-\alpha+\beta', 1-\alpha, \beta, \beta'-\alpha+1; \frac{x-x_-}{x}, \frac{x-x_-}{x-x_+}\right). \qquad (15)
\end{aligned}
$$

In Ref. [3], an expression for the integral $I^{(d)}_{\nu_1\nu_2}(m_1^2, m_2^2;\ s_{12})$ in terms of the Kampé de Fériet function was derived:

$$I^{(d)}_{\nu_1\nu_2}(m_1^2, m_2^2;\ s_{12}) = (-1)^{\nu_1+\nu_2}(m_2^2)^{\frac{d}{2}-\nu_1-\nu_2}\frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1+\nu_2)} \times\ F^{2;1;0}_{1;0;0}\left[\begin{matrix}(\nu_1+\nu_2-d/2:1,1),(\nu_1:1,1):(\nu_2:1)\\ (\nu_1+\nu_2:2,1)\end{matrix}\middle|\frac{s_{12}}{m_2^2}, 1-\frac{m_1^2}{m_2^2}\right]. \qquad (16)$$

Comparing this relation with Eq. (7), we obtain the following reduction formula:

$$F^{2;1;0}_{1;0;0}\left[\begin{matrix}(\alpha:1,1),(\nu_1:1,1):(\nu_2:1)\\ (\nu_1+\nu_2:2,1)\end{matrix}\middle|x,y\right] = F_1(\nu_1,\alpha,\alpha,\nu_1+\nu_2;\ z_-,z_+), \qquad (17)$$

where

$$z_\pm = \frac{x+y\pm\sqrt{(x+y)^2-4x}}{2}. \qquad (18)$$

For the case of equal masses $m_2^2 = m_1^2 = m^2$, the following expression was derived in Ref.[1]:

$$I^{(d)}_{\nu_1\nu_2}(m^2, m^2;\ s_{12}) = (-1)^{\nu_1+\nu_2}(m^2)^{d/2-\nu_1-\nu_2} \times \frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1+\nu_2)}\ {}_3F_2\left[\begin{matrix}\nu_1,\nu_2,\nu_1+\nu_2-\frac{d}{2}\,;\\ \frac{\nu_1+\nu_2}{2},\frac{\nu_1+\nu_2+1}{2}\,;\end{matrix}\ \frac{p^2}{4m^2}\right]. \qquad (19)$$

Comparing this formula with Eq. (7) taken at $m_1^2 = m_2^2 = m^2$, we obtain:

$$F_1\left(\alpha,\beta,\beta;\gamma;\ x-\sqrt{x^2-2x},\ x+\sqrt{x^2-2x}\ \right) = {}_3F_2\left[\begin{matrix}\alpha,\gamma-\alpha,\beta\,;\\ \frac{\gamma}{2},\frac{\gamma+1}{2}\,;\end{matrix}\ \frac{x}{2}\right], \qquad (20)$$

which may be rewritten as:

$$F_1\left(\alpha,\beta,\beta;\gamma;\ x,\ \frac{x}{x-1}\ \right) = {}_3F_2\left[\begin{matrix}\alpha,\gamma-\alpha,\beta\,;\\ \frac{\gamma}{2},\frac{\gamma+1}{2}\,;\end{matrix}\ \frac{x^2}{4(x-1)}\right]. \qquad (21)$$

We verified numerically the correctness of this relation setting $\alpha = 1/2$, $\beta = 2$, $\gamma = 3/2$, and $x = -1/2$, and keeping 600 valid digits in the calculations performed using the computer algebra system Maple.

To the best of our knowledge, there is no such a relation in the mathematical literature, i.e. Eq. (21) extends the number of known reduction formulas for the Appell function F_1. In Ref. [33], a relation between F_1 with the same arguments and the Gauss hypergeometric function ${}_2F_1$ is given. That relation corresponds to a particular case of our Eq. (21), taken at $\gamma = 2\alpha$. We would like to recall that the only known transformation of the Appell function F_1 to the hypergeometric function ${}_3F_2$ was known for the case when $x = -y$.

For the Appell hypergeometric function F_1, the following relation holds:

$$F_1(\alpha,\beta,\beta;\gamma;x,-x) = {}_3F_2\left[\begin{matrix}\frac{\alpha}{2},\frac{\alpha+1}{2},\beta\,;\\ \frac{\gamma}{2},\frac{\gamma+1}{2}\,;\end{matrix}\ x^2\right]. \qquad (22)$$

In the case when $s_{12} = m_1^2 - m_2^2$, the relation $x_+ = -x_-$ holds. Therefore, the integral $I^{(d)}_{\nu_1\nu_2}(m_1^2, m_2^2;\ s_{12})$ may be expressed in terms of the hypergeometric function ${}_3F_2$ as

$$I^{(d)}_{\nu_1\nu_2}(m_1^2, m_2^2;\ s_{12})\big|_{s_{12}=m_1^2-m_2^2}$$
$$= \frac{(-1)^{\nu_1+\nu_2}\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1+\nu_2)(m_2^2)^{\nu_1+\nu_2-d/2}}\ {}_3F_2\left[\begin{matrix}\frac{\nu_1}{2}, \frac{\nu_1+1}{2}, \nu_1+\nu_2-\frac{d}{2}\,; \\ \frac{\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2+1}{2}\,;\end{matrix} 1-\frac{m_1^2}{m_2^2}\right]. \tag{23}$$

This formula demonstrates that simplifications of Feynman integrals may also take place for specific values of masses or momenta that are more general than just zero or on-shell.

4 Quadratic transformation for the Appell function F_1

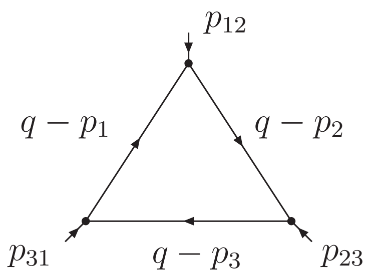

Figure 2: Feynman diagram corresponding to the integral $I_3^{(d)}$.

In this section, we find relations for the Appell function F_1 by comparing the results of different calculations of the one-loop vertex-type integral

$$I_3^{(d)}(m_j^2, m_k^2, m_l^2; p_{kl}, p_{jl}, p_{jk}) = \int \frac{d^d q}{i\pi^{d/2}} \frac{1}{[(q-p_j)^2 - m_j^2][(q-p_k)^2 - m_k^2][(q-p_l)^2 - m_l^2]}, \tag{24}$$

corresponding to the Feynman diagram shown in Figure 2. We consider this integral with a particular set of arguments, namely $I_3^{(d)}(0, m^2, 0;\ 0, s_{13}, s_{12})$. In the case when $s_{12} \le m^2$ and $s_{13} \le m^2$, from its representation as an integral over Feynman parameters, we derive the following analytic expression:

$$I_3^{(d)}(0, m^2, 0;\ 0, s_{13}, s_{12}) = \int_0^1 \int_0^1 \frac{-dx_1 dx_2\ \Gamma\left(3-\frac{d}{2}\right) x_1}{[s_{13}x_1 - s_{13} + x_2(m^2 + s_{13} - s_{12} - x_1 s_{13} + x_1 s_{12})]^{3-\frac{d}{2}}}$$
$$= \frac{1}{m^2}\ I_{11}^{(d)}(0, m^2;\ 0)\ F_1\left(1, 1, 2-\frac{d}{2}, \frac{d}{2}; \frac{s_{12}-s_{13}}{m^2}, \frac{s_{12}}{m^2}\right)$$
$$- \frac{I_{11}^{(d)}(0, 0;\ s_{13})}{m^2}\ {}_2F_1\left[\begin{matrix}1, \frac{d-2}{2}\,; \\ d-2\,;\end{matrix} \frac{s_{12}-s_{13}}{m^2}\right], \tag{25}$$

where F_1 is the Appell hypergeometric function [28] defined by the series

$$F_1(a,b,b';c;w,z) = \sum_{k,l=0}^{\infty} \frac{(a)_{k+l}(b)_k(b')_l}{(c)_{k+l}} \frac{w^k z^l}{k!l!}, \tag{26}$$

and $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the so-called Pochhammer symbol. In our case, the Appell function has a rather simple integral representation, viz.

$$F_1\left(1,1,2-\frac{d}{2},\frac{d}{2};x,y\right) = \frac{(d-2)}{2}\int_0^1 du \frac{[(1-u)(1-yu)]^{\frac{d}{2}-2}}{(1-xu)}. \tag{27}$$

Therefore, by using Eq. (25), one may obtain a result for the integral $I_3^{(d)}$ in terms of the Appell function F_1. The result in terms of function F_1, was previously obtained in Ref. [8] and later on in Ref. [24]. In $d=4$ space-time dimensions, the result for the integral $I_3^{(d)}$ in terms of the function F_3 was given in Ref. [20].

The integral $I_3^{(d)}(0,m^2,0;\ 0,s_{13},s_{12})$ may also be evaluated by another method, based on difference equations with respect to the space-time dimension d. The method of deriving dimensional recurrences is described in detail in Refs. [5,6]. In the case under consideration here, we have

$$\begin{aligned} I_3^{(d+2)}(0,m^2,0;\ 0,s_{13},s_{12}) &= \frac{2m^2 s_{13}(s_{12}-m^2-s_{13})}{(s_{12}-s_{13})^2(d-2)}\ I_3^{(d)}(0,m^2,0;\ 0,s_{13},s_{12}) \\ &+ \frac{m^2}{(d-2)(s_{13}-s_{12})}\ I_{11}^{(d)}(0,m^2;0) + \frac{s_{13}(s_{12}-s_{13}-2m^2)}{(s_{12}-s_{13})^2(d-2)}\ I_{11}^{(d)}(0,0;s_{13}) \\ &+ \frac{(m^2 s_{12}+m^2 s_{13}+s_{12}s_{13}-s_{12}^2)}{(s_{12}-s_{13})^2(d-2)}\ I_{11}^{(d)}(0,m^2;s_{12}). \end{aligned} \tag{28}$$

Denoting its non-homogeneous part as $R^{(d)}$, we rewrite Eq. (28) as:

$$I_3^{(d+2)}(0,m^2,0;\ 0,s_{13},s_{12}) = \frac{2m^2 s_{13}(s_{12}-s_{13}-m^2)}{(s_{12}-s_{13})^2(d-2)}\ I_3^{(d)}(0,m^2,0;0,s_{13},s_{12}) + R^{(d)}. \tag{29}$$

The solution of this equation reads

$$I_3^{(d)}(0,m^2,0;\ 0,s_{13},s_{12}) = \frac{\sigma^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} C_\varepsilon(s_{12},s_{13}) - \frac{(d-2)}{2\sigma}\sum_{k=0}^{\infty}\frac{\left(\frac{d}{2}\right)_k}{\sigma^{2k}} R^{(d+2k)}, \tag{30}$$

where

$$\sigma = \frac{m^2 s_{13}(s_{12}-s_{13}-m^2)}{(s_{12}-s_{13})^2}. \tag{31}$$

An arbitrary periodic function $C_\varepsilon(s_{12}, s_{13})$ emerging in the solution may be found from the following differential equation with respect to the variable s_{12}:

$$\frac{\partial}{\partial s_{12}} I_3^{(d)}(0,m^2,0;0,s_{13},s_{12}) = \frac{(d-2)(s_{13}-s_{12}+2m^2)-2m^2}{2(m^2+s_{13}-s_{12})(s_{13}-s_{12})} I_3^{(d)}(0,m^2,0;0,s_{13},s_{12})$$
$$+ \frac{(d-3)(m^2+s_{13}-2s_{12})}{(m^2-s_{12})(m^2+s_{13}-s_{12})(s_{12}-s_{13})} \; I_{11}^{(d)}(0,m^2;s_{12})$$
$$+ \frac{(d-2)}{2(m^2-s_{12})(m^2+s_{13}-s_{12})} \; I_{11}^{(d)}(0,m^2;0)$$
$$- \frac{(d-3)}{(m^2+s_{13}-s_{12})(s_{12}-s_{13})} \; I_{11}^{(d)}(0,0;s_{13}). \tag{32}$$

Substituting Eq. (30) into Eq. (32), we arrive at the following equation:

$$\frac{\partial}{\partial s_{12}} C_\varepsilon(s_{12},s_{13}) + \frac{(s_{12}-s_{13}-3m^2)}{(s_{12}-s_{13}-m^2)(s_{13}-s_{12})} C_\varepsilon(s_{12},s_{13}) = 0. \tag{33}$$

Taking into account the boundary condition of the integral at $s_{12}=0$, the solution of this equation is

$$C_\varepsilon(s_{12},s_{13}) = 0.$$

Substituting explicit expressions for the propagator integrals $I_2^{(d)}$ into Eq. (30) leads to the following expression:

$$I_3^{(d)}(0,m^2,0;\; 0,s_{13},s_{12}) =$$
$$-\frac{(m^2-s_{12})s_{12}+(m^2+s_{12})s_{13}}{2s_{13}(s_{13}-s_{12}+m^2)(s_{12}-m^2)} \; I_{11}^{(d)}(0,m^2;\; 0) \; F_1\left(\frac{d-2}{2},\frac{1}{2},1,\frac{d}{2};\frac{-4m^2 s_{12}}{(s_{12}-m^2)^2},-\frac{m^2}{\sigma}\right)$$
$$+ \frac{s_{13}-s_{12}}{2(s_{13}-s_{12}+m^2)s_{13}} I_{11}^{(d)}(0,m^2;\; 0) \; {}_2F_1\left[\begin{matrix}1,\frac{d-2}{2};\\ \frac{d}{2};\end{matrix}\frac{-m^2}{\sigma}\right]$$
$$+ \frac{(s_{12}-s_{13}-2m^2)}{2m^2(s_{13}-s_{12}+m^2)} I_{11}^{(d)}(0,0;\; s_{13}) \; {}_2F_1\left[\begin{matrix}1,\frac{d-2}{2};\\ \frac{d-1}{2};\end{matrix}\frac{s_{13}}{4\sigma}\right]. \tag{34}$$

Comparison of the obtained result with Eq. (25) leads to the relation

$$F_1\left(1,1,2-\frac{d}{2},\frac{d}{2};\omega,z\right) = \frac{\omega}{2(\omega-z)(1-\omega)} \; {}_2F_1\left[\begin{matrix}1,\frac{d-2}{2};\\ \frac{d}{2};\end{matrix}\frac{\omega^2}{(\omega-z)(\omega-1)}\right]$$
$$+\frac{(\omega+z\omega-2z)}{2(\omega-z)(1-\omega)(1-z)} \; F_1\left(\frac{d-2}{2},1,\frac{1}{2},\frac{d}{2};\frac{\omega^2}{(\omega-z)(\omega-1)},\frac{-4z}{(z-1)^2}\right). \tag{35}$$

The arguments of the function F_1 on the right-hand side of Eq. (35) are connected with the arguments of the F_1 function on the left-hand side by a quadratic transformation. Therefore, Eq. (35) is the analogue of the quadratic relation for the Gauss hypergeometric function ${}_2F_1$. To the best of our knowledge, there is no such a relation in mathematical literature.

5 New relation between Appel function F_1 and hypergeometric function ${}_2F_1$

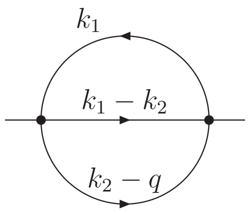

Figure 3: Feynman diagram corresponding to the integral $J_3^{(d)}$.

In this section, we find a relation between the F_1 and ${}_2F_1$ functions by comparing the results evaluated by two different methods for the imaginary part of the integral

$$J_3^{(d)} \equiv \iint \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2}{(i\pi^{d/2})^2} \frac{1}{(k_1^2 - m^2)((k_1 - k_2)^2 - m^2)((k_2 - q)^2 - m^2)}. \tag{36}$$

The Feynman diagram corresponding to this integral is presented in Figure 3. In Ref. [7], the difference equation with respect to d for the master integral $J_3^{(d)}$ was derived. From this equation, the difference equation for the imaginary part of $J_3^{(d)}$ may be obtained. It reads:

$$\begin{aligned} &12x^2(d+1)(d-1)(3d+4)(3d+2)\ \mathrm{Im}J_3^{(d+4)} \\ &-4m^4(x^2-3)(x^4-42x^2+9)(d-1)d\ \mathrm{Im}J_3^{(d+2)} \\ &\qquad -4m^8(x^2-1)^2(x^2-9)^2\ \mathrm{Im}J_3^{(d)} = 0, \end{aligned} \tag{37}$$

where $x = q/m$. The solution of this equation for the imaginary part of $J_3^{(d)}$ was presented in Ref. [7] and reads:

$$\mathrm{Im}J_3^{(d)} = \frac{-4\,\pi^2\sqrt{3}\,m^{2d-6}}{\Gamma(d-1)\,(x^2+3)} \left[\frac{(x^2-9)^2}{27}\right]^{\frac{d-2}{2}} {}_2F_1\left[\begin{matrix}\frac{1}{3},\frac{2}{3};\\ \frac{d}{2};\end{matrix} \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]. \tag{38}$$

An analytic expression for $\mathrm{Im}J_3^{(d)}$ may also be obtained by using another method. In Ref. [34], a one-fold integral representation for the imaginary part of the two-loop sunrise integral with arbitrary masses was derived. For our case, where the masses of all propagators are the same, the imaginary part on the cut is given in Ref. [34] and reads:

$$\mathrm{Im}J_3^{(d)} = \frac{-\pi}{(q^2)^{\frac{d}{2}-1}} \frac{\Gamma^2\left(\frac{d-2}{2}\right)}{\Gamma^2\left(d-2\right)} \int_{4m^2}^{(q-m)^2} \frac{d\theta}{\theta^{\frac{d}{2}-1}} \left(\Lambda(\theta,m^2,m^2)\Lambda(\theta,q^2,m^2)\right)^{\frac{d-3}{2}}, \tag{39}$$

where $\Lambda(x,y,z)$ is defined in Eq. (9). Changing the integration variable in Eq. (39) as $\theta = 4m^2 + (q-3m)(q+m)\beta$ leads to the following expresssion:

$$\begin{aligned}
\mathrm{Im} J_3^{(d)} &= \frac{-\pi}{(q^2)^{\frac{d}{2}-1}} \frac{\Gamma^2\left(\frac{d-2}{2}\right)}{\Gamma^2\left(d-2\right)} \frac{(q-3m)(q+m)}{2m} \left[(q-m)(q+3m)(q+m)^2(q-3m)^2\right]^{\frac{d-3}{2}} \\
&\times \int_0^1 d\beta \Big\{\beta(1-\beta)\big[1 - \frac{(q+m)(q-3m)}{(q-m)(q+3m)}\beta\big]\Big\}^{\frac{d-3}{2}} \frac{1}{\left[1+\frac{(q+m)(q-3m)}{4m^2}\beta\right]^{\frac{1}{2}}}. \quad (40)
\end{aligned}$$

As follows from Eq. (6), the integral on the right-hand side of Eq. (40) is proportional to the Appell function F_1. Using Eq. (6) and comparing Eqs. (40) and (38), we arrive at the following relation:

$$\begin{aligned}
& F_1\left(\frac{d-1}{2}, \frac{3-d}{2}, \frac{1}{2}, d-1; \frac{(x+1)(x-3)}{(x-1)(x+3)}, -\frac{(x+1)(x-3)}{4}\right) \\
&= \frac{2\sqrt{3}}{(x^2+3)} \left[\frac{16}{27}\frac{x^2(x+3)^2}{(x+1)^2}\right]^{\frac{d-2}{2}} \left[(x+3)(x-1)\right]^{\frac{3-d}{2}} {}_2F_1\left[\begin{matrix}\frac{1}{3},\frac{2}{3}; \\ \frac{d}{2};\end{matrix} \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]. \quad (41)
\end{aligned}$$

It is interesting to note that, at $d=2$, the Appell function F_1 in Eq. (41) may be expressed in terms of the ${}_2F_1$ function with the help of the equation (see, for example, Ref. [31])

$$F_1(a,b,b',b+b',w,z) = (1-z)^{-a}\, {}_2F_1\left[\begin{matrix}a,b; \\ b+b';\end{matrix} \frac{w-z}{1-z}\right]. \quad (42)$$

This leads to the relation:

$${}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}; \\ 1;\end{matrix} \frac{(x-3)(x+1)^3}{(x+3)(x-1)^3}\right] = \frac{\sqrt{3(x+3)(x-1)^3}}{(x^2+3)}\, {}_2F_1\left[\begin{matrix}\frac{1}{3},\frac{2}{3}; \\ 1;\end{matrix} \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]. \quad (43)$$

The hypergeometric function ${}_2F_1$ on the left-hand side of this equation is proportional to the complete elliptic integral of the first kind. Relations between hypergeometric functions with parameters $1/2, 1/2, 1$ and $1/3, 2/3, 1$ but with arguments different from that in Eq. (43) were first derived by Ramanujan in Ref. [35].

6 Conclusions

In this section, we briefly summarize the most important results obtained in this paper and point out some topics which may be of interest for future investigations. Specifically,

- a new analytic expression for the one-loop propagator-type Feynnman integral was derived;
- a new formula transforming the Appell function F_1 to the hypergeometric function ${}_3F_2$ was presented;

- a new formula transforming the Appell function F_4 to a combination of two F_1 functions was found;
- a new formula connecting the Kampé de Fériet function and the Appell function F_1 was obtained;
- a formula for the quadratic transformation of the Appell function F_1 was given;
- a new relation between the Appell function F_1 and the elliptic-type Gaussian hypergeometric function ${}_2F_1$ was established.

We expect that the formulae found for the abovementioned hypergeometric functions will be useful, in particular, for finding relations between transcendental numbers. Such relations may be found, for example, via the expansions in $\varepsilon = (4-d)/2$ of the presented results. These relations may also be useful for simplifying the ε expansions of hypergeometric functions involved because some of them may have simpler integral representations than the others.

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