Ends as tangles

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Every end of an infinite graph G defines a tangle of infinite order in G. These tangles indicate a highly cohesive substructure in the graph if and only if they are closed in some natural topology.

We characterize, for every finite k, the ends ω whose induced tangles of order k are closed. They are precisely the tangles τ for which there is a set of k vertices that decides τ by majority vote. Such a set exists if and only if the vertex degree plus the number of dominating vertices of ω is at least k.

1 Introduction

Tangles were introduced by Robertson and Seymour in [6] and have since become one of the central objects of study in the field of graph minor theory. So far most of the theory of tangles deals with tangles in finite graphs. In [1], it was shown how the set Θ of tangles of infinite order of an arbitrary infinite graph can be used to compactify it, much in the same way as the set Ω of ends of a connected locally finite graph can be used to compactify it. Indeed, if a graph G is connected and locally finite, these compactifications $|G|_\Theta$ and $|G|_\Omega$ of G coincide. This is because every end ω of an infinite graph G induces a tangle $\tau = \tau_\omega$ of order $\aleph_0$ in G, and for locally finite connected G the map $\omega \mapsto \tau_\omega$ is a bijection between the set Ω of ends of G and the set Θ of its $\aleph_0$-tangles. (Graphs that are not locally finite have $\aleph_0$-tangles that are not induced by an end.)

In [1], a natural topology on the set $\vec{S} = \vec{S}_{\aleph_0}(G)$ of separations of finite order of G was defined. A tangle τ induced by an end of G is a closed set in this topology if and only if τ is defined by an $\aleph_0$-block in G, that is, if there is an $\aleph_0$-block K in G with $K \subseteq B$ for all separations (A, B) in τ.

Our research expands on this latter result. Every end ω of a graph induces not only a tangle of infinite order in G, but for each $k \in \mathbb{N}$ the end ω induces a k-tangle in G. The set $\vec{S}_k$ of all separations (A, B) of G with $|A \cap B| < k$ is a closed set in $\vec{S}$, and thus if the tangle τ induced by ω in G is a closed set in $\vec{S}$, the k-tangle $\tau \cap \vec{S}_k$ induced by ω will be closed as well. However, it is possible that a tangle τ in G of infinite order fails to be closed in $\vec{S}$, while its restrictions $\tau \cap \vec{S}_k$ to $\vec{S}_k$ are closed for some, or even all, $k \in \mathbb{N}$. In this paper we characterize the ends of G by the behaviour of their tangles, as follows:

We show that, for an end ω and its induced tangle τ, the restriction $\tau \cap \vec{S}_k$ to $\vec{S}_k$ is a closed set in $\vec{S}$ if and only if

$$\deg(\omega) + \mathrm{dom}(\omega) \geqslant k,$$

where $\deg(\omega)$ and $\mathrm{dom}(\omega)$ denote the vertex degree and number of vertices dominating ω, respectively.

We further show that τ is closed in $\vec{S}$ if and only if ω is dominated by infinitely many vertices.

A question raised in [3] asks whether for a k-tangle τ in a finite graph G one can always find a set X of vertices which decides τ by majority vote, in the sense that $(A, B) \in \tau$ if and only if $|A \cap X| < |B \cap X|$, for all $(A, B) \in \vec{S}_k$. This problem is still open in general, although some process has been made recently ([5]). We establish an analogue in the infinite setting: we show that for an end ω of G and its induced k-tangle $\tau \cap \vec{S}_k$ in G, the existence of a finite set X which decides $\tau \cap \vec{S}_k$ in the above sense is equivalent to $\tau \cap \vec{S}_k$ being a closed set in $\vec{S}$.

This paper is organized as follows: Section 2 contains the basic definitions and some notation. Following that, in Section 3, we recall the core concepts and results from [1] that are relevant to our studies, including the topology defined on $\vec{S}$. Finally, in Section 4, we prove our main results Theorem 4.4 and Theorem 4.5. The first of these characterises the ends of a graph by the behaviour of their tangles, and the second shows that $\tau \cap \vec{S}_k$ being a closed set in $\vec{S}$ for a k-tangle τ induced by some end ω of G is equivalent to both $\deg(\omega) + \mathrm{dom}(\omega) \geqslant k$ and to $\tau \cap \vec{S}_k$ being decided by some finite set of vertices.

2 Separations, tangles, and their topology

Throughout this paper $G = (V, E)$ will be a fixed infinite graph. For any graph-theoretical notation not explained here we refer the reader to [2].

A *separation* of G is a set $\{A, B\}$ with $A \cup B = V$ such that G contains no edge between $A \setminus B$ and $B \setminus A$. We call such a set $\{A, B\}$ an *unoriented* separation with the two orientations (A, B) and (B, A). Informally we think of the oriented separation (A, B) as *pointing towards* B and *pointing away* from A. The *separator* of a separation $\{A, B\}$ is the set $A \cap B$.

The *order* of a separation (A, B) or $\{A, B\}$ is the size $|A \cap B|$ of its separator. For a cardinal κ we write $S_\kappa = S_\kappa(G)$ for the set of all unoriented separations of G of order $< \kappa$. If S is a set of unoriented separations we write $\vec{S}$ for the corresponding set of oriented separations, that is, the set of all separations (A, B) with $\{A, B\} \in S$. Consequently we write $\vec{S}_\kappa$ for the set of all separations (A, B) of G with $|A \cap B| < \kappa$.

If S is a set of unoriented separations of G, an *orientation* of S is a set $O \subseteq \vec{S}$ such that O contains precisely one of (A, B) or (B, A) for every $\{A, B\} \in S$. A *tangle of S in G* is an orientation τ of S such that there are no (A_1, B_1), (A_2, B_2), $(A_3, B_3) \in \tau$ for which $G[A_1] \cup G[A_2] \cup G[A_3] = G$.

Properties of sets of separations of finite graphs, including their tangles, often generalize to sets of separations of infinite graphs but not always. Those sets of separations to which these properties tend to generalize can be identified, however: they are the sets of separations that are closed in a certain natural topology [4]. Let us define this topology next. It is analogous to the topology of a profinite abstract separation system defined in [4].[1]

From here on we denote by $S = S_{\aleph_0}(G)$ the set of all (unoriented) separations of G of finite order. Thus $\vec{S} = \vec{S}_{\aleph_0}(G)$ is the set of all separations (A, B) of G with $A \cap B$ finite.

We define our topology on $\vec{S}$ by giving it the following basic open sets. Pick a finite set $Z \subseteq V$ and an oriented separation (A_Z, B_Z) of $G[Z]$. Then declare as open the set $O(A_Z, B_Z)$ of all $(A, B) \in \vec{S}$ such that $A \cap Z = A_Z$ and $B \cap Z = B_Z$. We shall say that these (A, B) *induce* (A_Z, B_Z) on Z, writing $(A_Z, B_Z) =: (A, B) \restriction Z$, and that (A, B) and (A', B') *agree on* Z if $(A, B) \restriction Z = (A', B') \restriction Z$.

It is easy to see that the sets $O(A_Z, B_Z)$ do indeed form the basis of a topology on $\vec{S}$. Indeed, $(A, B) \in \vec{S}$ induces (A_1, B_1) on Z_1 and (A_2, B_2) on Z_2 if and only if it induces on $Z = Z_1 \cup Z_2$ some separation (A_Z, B_Z) which in turn induces (A_i, B_i) on Z_i for both i. Hence $O(A_1, B_1) \cap O(A_2, B_2)$ is the union of all these $O(A_Z, B_Z)$.

As we shall see, the intuitive property of tangles in finite graphs that they describe, if indirectly, some highly cohesive region of that graph – however 'fuzzy' this may be in terms of concrete vertices and edges – will extend precisely to those tangles of S that are closed in $\vec{S}$.

3 End tangles of S

We think of an oriented separation $(A, B) \in \vec{S}$ as pointing towards B, or being oriented towards B. In the same spirit, given an end ω of G, we say that (A, B) *points towards* ω, and that ω *lives in* B, if some (equivalently: every) ray of ω has a tail in B. Furthermore, if (A, B) points to an end ω, then (B, A) *points away from* ω.

Clearly, for every end of G and every $\{A, B\} \in S$, precisely one orientation of $\{A, B\}$ points towards that end. In this way, every end ω of G defines an orientation of S by orienting each separation in S towards ω:

$$\tau = \tau_\omega := \{(A, B) \in \vec{S} \mid \text{every ray of } \omega \text{ has a tail in } B\}$$

It is reasy to see ([1]) that this is a tangle in G. We call it the *end tangle induced* on S by the end ω.

Note that every end tangle contains all separations of the form (A, V) for finite $A \subseteq V$, and thus no separation of the form (V, B). Furthermore, any two ends induce different end tangles. Our aim in this section is to recall from [1] some properties of the end-tangles of S that we shall later extend to its subsets S_k. For the convenience of the reader, and also in order to correct an inessential but confusing error in [1], we repeat some of the material from [1] here to make our presentation self-contained.

Let us first see an example of an end tangle that is not closed in $\vec{S}$.

[1] Even though $\vec{S}$ itself is not usually profinite in the sense of [4], the topology we define on $\vec{S}$ is the subspace topology of $\vec{S}$ as a subspace of the (profinite) system of all oriented separations of G, equipped with the inverse limit topology from [4].

Example 3.1. *If G is a single ray $v_0v_1\dots$ with end ω, say, then $\tau = \tau_\omega$ is not closed in $\vec{S}$.*

Indeed, τ contains $(\emptyset, V)$, and hence does not contain $(V, \emptyset)$. But for every finite $Z \subseteq V$ the restriction $(Z, \emptyset)$ of $(V, \emptyset)$ to Z is also induced by the separation $(\{v_0, \dots, v_n\}, \{v_n, v_{n+1}, \dots\}) \in \tau$ for every n large enough that $Z \subseteq \{v_0, \dots, v_{n-1}\}$. So $(V, \emptyset) \in \vec{S} \smallsetminus \tau$ has no open neighbourhood in $\vec{S} \smallsetminus \tau$. □

Here is an example of an end tangle that is closed in $\vec{S}$. Unlike our previous example, it describes a highly cohesive part of G.

Example 3.2. *If $K \subseteq V$ spans an infinite complete graph in G, then*

$$\tau = \{\, (A, B) \in \vec{S} \mid K \subseteq B \,\} \tag{1}$$

is a closed set in $\vec{S}$.

We omit the easy proof. But note that τ is indeed an end tangle: it is induced by the unique end of G which contains all the rays in K.

Perhaps surprisingly, it is not hard to characterize the end tangles that are closed. They are all essentially like Example 3.2: we just have to generalize the infinite complete subgraph used appropriately. Of the two obvious generalizations, infinite complete minors [8] or subdivisions of infinite complete graphs [7], the latter turns out to be the right one.

Let κ be any cardinal. A set of at least κ vertices of G is $(<\kappa)$-*inseparable* if no twoof them can be separated in G by fewer than κ vertices. A maximal $(<\kappa)$-inseparable set of vertices is a κ-*block*. For example, the branch vertices of a TK_κ are $(<\kappa)$-inseparable. Conversely:

Lemma 3.3. *When κ is infinite, every $(<\kappa)$-inseparable set of vertices in G contains the branch vertices of some $TK_\kappa \subseteq G$.*

Proof. Let $K \subseteq V$ be $(<\kappa)$-inseparable. Viewing κ as an ordinal we can find, inductively for all $\alpha < \kappa$, distinct vertices $v_\alpha \in K$ and internally disjoint v_α-v_β paths in G for all $\beta < \alpha$ that also have no inner vertices among those v_β or on any of the paths chosen earlier; this is because $|K| \geqslant \kappa$, and no two vertices of K can be separated in G by the $<\kappa$ vertices used up to that time. □

The original statement of Lemma 3.3 in [1, Lemma 5.4] asserted that for infinite κ a set $K \subseteq V$ is a κ-block in G if and only if it is the set of branch vertices of some $TK_\kappa \subseteq G$. It turns out that both directions of that assertion were false: the set of branch vertices of a $TK_\kappa \subseteq G$ is certainly $(<\kappa)$-inseparable, but might not be maximal with this property and hence not a κ-block. Conversely, if K is a κ-block, there might not be a $TK_\kappa \subseteq G$ whose set of branch vertices is precisely K: if $|K| > \kappa$ this is certainly not possible, but even if $|K| = \kappa$ one might not be able to find a TK_κ in G whose branch vertices are all of K. If for instance the graph G is a clique on κ vertices that is missing exactly one edge, then $K = V(G)$ is a κ-block in G but not the set of branch vertices of a $TK_\kappa \subseteq G$.

For our main result of this section we need one more observation:

Lemma 3.4. *If τ is an end tangle of G and $(A,B),(C,D) \in \tau$ then $(A \cup C\,,\, B \cap D) \in \tau$.*

Proof. Observe first that $(A \cup C\,,\, B \cap D)$ is a separation of G with finite order and thus lies in $\vec{S}$. Moreover, if a ray of G has a tail in B and a tail in D, then that ray also has a tail in $B \cap D$, from which it follows that $(A \cup C\,,\, B \cap D) \in \tau$, as claimed. $\square$

We can now prove our main result of this section. Let us say that a set $K \subseteq V$ *defines* an end tangle τ if τ satisfies (1).

Theorem 3.5 ([1])**.** *Let G be any graph. An end tangle of G is closed in $\vec{S}$ if and only if it is defined by an $\mathcal{A}$-block.*

Proof. Suppose first that τ is an end tangle that is defined by an $\mathcal{A}$-block K. To show that τ is closed, we have to find for every $(A,B) \in \vec{S} \setminus \tau$ a finite set $Z \subseteq V$ such that no $(A',B') \in \vec{S}$ that agrees with (A,B) on Z lies in τ. As $(A,B) \notin \tau$, we have $K \subseteq A$; pick $z \in K \setminus B$. Then every $(A',B') \in \vec{S}$ that agrees with (A,B) on $Z := \{z\}$ also also lies in $\vec{S} \setminus \tau$, since $z \in A' \setminus B'$ and this implies $K \nsubseteq B'$.

Conversely, consider any end tangle τ and let

$$K := \bigcap \{\, B \mid (A,B) \in \tau \,\}.$$

No two vertices in K can be separated by in G by a finite-order separation: one orientation (A,B) of this separation would be in τ, which would contradict the definition of K since $A \setminus B$ also meets K. If K is infinite, it will clearly be maximal with this property, and hence be an $\mathcal{A}$-block. This $\mathcal{A}$-block K will define τ: by definition of K we have $K \subseteq B$ for ever $(A,B) \in \tau$, while also every $(A,B) \in \vec{S}$ with $K \subseteq B$ must be in τ: otherwise $(B,A) \in \tau$ and hence $K \subseteq A$ by definition of K, but $K \nsubseteq A \cap B$ because this is finite. Hence τ will be defined by an $\mathcal{A}$-block, as desired for the forward implication.[2]

It thus suffices to show that if K is finite then τ is not closed in $\vec{S}$, which we shall do next.

Assume that K is finite. We have to find some $(A,B) \in \vec{S} \setminus \tau$ that is a limit point of τ, i.e., which agrees on every finite $Z \subseteq V$ with some $(A',B') \in \tau$. We choose $(A,B) := (V,K)$, which lies in $\vec{S} \setminus \tau$ since $(K,V) \in \tau$.

To complete our proof as outlined, let any finite set $Z \subseteq V$ be given. For every $z \in Z \setminus K$ choose $(A_z,B_z) \in \tau$ with $z \in A_z \setminus B_z$: this exists, because $z \notin K$. As $(K,V) \in \tau$, by Lemma 3.4 we have $(A',B') \in \tau$ for

$$A' := K \cup \bigcup_{z \in Z \setminus K} A_z \quad \text{and} \quad B' := V \cap \bigcap_{z \in Z \setminus K} B_z\,.$$

As desired, $(A',B') \restriction Z = (A,B) \restriction Z$ (which is $(Z, Z \cap K)$, since $(A,B) = (V,K)$): every $z \in Z \setminus K$ lies in some A_z and outside that B_z, so $z \in A' \setminus B'$, while every $z \in Z \cap K$ lies in $K \subseteq A'$ and also, by definition of K, in every B_z (and hence in B'), since $(A_z,B_z) \in \tau$. $\square$

[2] Whether or not τ is closed in $\vec{S}$ is immaterial; we just did not use this assumption.

We conclude the section with the remark that an end tangle τ of an infinite graph G is defined by an $\mathcal{A}$-block if and only if it is defined by some set $X \subseteq V$: for if τ is defined by some $X \subseteq V$, then X must be infinite, as otherwise $\{X, V\} \in S$ would witness that X does not define τ. But if X is infinite then so is $K := \bigcap\{B \mid (A, B) \in \tau\} \supseteq X$, and we can follow the proof of Theorem 3.5 to show that K is an $\mathcal{A}$-block defining τ.

4 End tangles in S_k

For $k \in \mathbb{N}$ we write S_k for the subset of S containing all separations of G of order $< k$. In this section we shall extend Theorem 3.5 from the previous section to these S_k, where we will also be able to prove a stronger and wider statement which will allow us to classify ends by the behaviour of their end tangles – specifically, by the set of those $k \in \mathbb{N}$ for which their end tangle is closed in $\vec{S_k}$.

For every $k \in \mathbb{N}$ the set $\vec{S_k}$ is a closed set in $\vec{S}$. Given an end ω of G with end tangle $\tau = \tau_\omega$, the *tangle induced by ω in S_k* is $\tau \cap \vec{S_k}$. We say that ω *induces a closed tangle in S_k* if $\tau \cap \vec{S_k}$ is a closed set in $\vec{S_k}$, where the latter is equipped with the subspace topology of $\vec{S}$.

We seek to classify and characterize ends by the set of k for which their induced tangles are closed in S_k, similarly to Theorem 3.5. For $l < k$ the set $\vec{S_l}$ is closed in $\vec{S_k}$, and hence any end inducing a closed tangle in S_k also induces a closed tangle in S_l. Therefore, the ends of G all fall into one of the following three categories:

(i) Ends inducing a closed tangle in S;

(ii) ends whose induced tangle in S is not closed in $\vec{S}$, but whose induced tangle in S_k is closed in $\vec{S_k}$ for every $k \in \mathbb{N}$;

(iii) ends whose induced tangle in S_k is not closed in $\vec{S_k}$ for some $k \in \mathbb{N}$.

The ends belonging to the first category are characterized in Theorem 3.5: they are those ends whose tangle in $\vec{S}_{\aleph_0}$ is defined by an $\mathcal{A}$-block. In the remainder of this section we shall characterize the ends which fall into the latter two categories. Furthermore, for an end ω belonging to the third category, there exists a least $k \in \mathbb{N}$ for which the tangle induced by ω in S_k is not closed in $\vec{S_k}$. We shall determine this k and show that it depends just on the vertex degree and number of dominating vertices of ω.

Let us see examples of ends belonging to the third and second category, respectively:

Example 4.1. Let G be as in Example 3.1, that is, a single ray $v_0v_1\ldots$ with end ω. The same argument as in Example 3.1 shows that $\tau \cap \vec{S_k}$ is not closed in $\vec{S_k}$ for $k \geqslant 2$. However, ω does induce a closed tangle in S_1: the set $\tau \cap \vec{S_1} = \{(\emptyset, V)\}$ is closed in $\vec{S_1}$.

Example 4.2. Let G be the infinite grid, ω the unique end of G and τ the tangle induced by ω in $S = S(G)$. As G is locally finite it does not contain an $\mathcal{A}$-block. Therefore τ cannot be defined by an $\mathcal{A}$-block and is thus not closed in $\vec{S}$ by Theorem 3.5. However, for every $k \in \mathbb{N}$, it is easy to see that $\tau \cap \vec{S_k}$ is closed in $\vec{S_k}$: indeed, for fixed $k \in \mathbb{N}$, the size of A is bounded in terms of k for every $(A, B) \in \tau \cap \vec{S}_k$. Thus, for every $(A', B') \in \vec{S}_k \smallsetminus \tau$, any $Z \subseteq V(G)$ with $|Z \cap A'|$ sufficiently large witnesses that (A', B') does not lie in the closure of τ: if Z is large enough that $|Z \cap A'| > |A|$ for every $(A, B) \in \tau \cap \vec{S}_k$, then no $(A, B) \in \tau \cap \vec{S}_k$ will agree with (A', B') on Z.

Example 4.2 shows that the end tangle τ of the infinite grid is not defined by a set $X \subseteq V$ in the sense of (1). Moreover, even for any fixed $k \in \mathbb{N}_{\geqslant 5}$, we cannot find a set $X \subseteq V$ with $X \subseteq B$ for all $(A, B) \in \tau \cap \vec{S}_k$: for every $x \in X$ the separation $(\{x\} \cup N(x)\,,\, V \smallsetminus \{x\})$ lies in $\tau \cap \vec{S}_k$. Thus, even the tangles that τ induces in $\vec{S}_k$ are not defined by any subset of V. However, they come reasonably close to it: for $X = V$ and any $(A, B) \in \tau$, the majority of X lies in B, that is, we have $|A \cap X| < |B \cap X|$. In fact, for any fixed $k \in \mathbb{N}$, any finite set $X \subseteq V$ that is at least twice as large as $\max\{|A| \mid (A, B) \in \tau \cap \vec{S}_k\}$ has the property that $|A \cap X| < |B \cap X|$ for every $(A, B) \in \tau \cap \vec{S}_k$. Therefore, even though no $\tau \cap \vec{S}_k$ is defined by a $X \subseteq V$, we can for each $k \in \mathbb{N}$ find a (finite) $X \subseteq V$ which 'defines' $\tau \cap \vec{S}_k$ by simple majority.

To make the above observation formal, for an end tangle τ of G, let us call a set $X \subseteq V$ a *decider set* for τ (resp., for $\tau \cap \vec{S}_k$) if we have $|A \cap X| < |B \cap X|$ for every $(A, B) \in \tau$ (resp., $(A, B) \in \tau \cap \vec{S}_k$). Thus, if we have a decider set for an end tangle τ, given a separation $(A, B) \in \vec{S}$, the decider set tells us which of (A, B) and (B, A) lies in τ by a simple majority vote. By Theorem 3.5, every end tangle that is closed in $\vec{S}$ has a decider set, as the $\mathcal{A}$-block defining it is such a decider set. In analogy with this, we shall show in the remainder of this section that for an end tangle τ its induced tangle $\tau \cap \vec{S}_k$ is closed in $\vec{S}_k$ if and only if $\tau \cap \vec{S}_k$ has a finite decider set. Such a finite decider set can be thought of as a local encoding of the tangle, or a local witness to the tangle being closed in $\vec{S}_k$.

In contrast, it is easy to see that every end tangle has an infinite decider set:

Proposition 4.3. *For any end ω of G the end tangle τ induced by ω has an infinite decider set.*

Proof. For any ray $R \in \omega$ its vertex set $V(R)$ is a decider set for τ. □

As no end tangle of S can have a finite decider set the existence of decider sets for end tangles is thus only interesting for finite decider sets of the tangle's restrictions to some $\vec{S}_k$.

We shall complement this local witness of a given end tangle being closed with a more global type of witness. For this we need to introduce some notation.

The *(vertex) degree* $\deg(\omega)$ of an end ω of G is the largest size of a family of pairwise disjoint ω-rays[3]. A vertex $v \in V$ *dominates* and end ω if it sends infinitely many disjoint paths to some (equivalently: to all) ray in ω. We write $\mathrm{dom}(\omega)$ for the number of vertices of G which dominate ω. An end ω is *undominated* if $\mathrm{dom}(\omega) = 0$; it is *finitely dominated* if finitely many (including zero) vertices of G dominate ω; and, finally, ω is *infinitely dominated* if $\mathrm{dom}(\omega) = \infty$. We will show that the category that an end ω belongs to depends just on these parameters $\deg(\omega)$ and $\mathrm{dom}(\omega)$. Concretely, we will show the following:

[3]Here our notation deviates from that in [2], where $d(\omega)$ is used for the degree of ω.

Theorem 4.4. *Let τ the end tangle induced by an end ω of G. Then the following statements hold:*

(i) *τ is closed in $\vec{S}$ if and only if* $\mathrm{dom}(\omega) = \infty$.

(ii) *τ is not closed in $\vec{S}$ but $\tau \cap \vec{S}_k$ is closed in $\vec{S}_k$ for every $k \in \mathbb{N}$ if and only if* $\deg(\omega) = \infty$ *and* $\mathrm{dom}(\omega) < \infty$.

(iii) *$\tau \cap \vec{S}_k$ is not closed in $\vec{S}_k$ for some $k \in \mathbb{N}$ if and only if* $\deg(\omega) + \mathrm{dom}(\omega) < \infty$.

Theorem 4.4 will be a consequence of Theorem 3.5 and the following theorem, which characterizes for which $k \in \mathbb{N}$ a given end tangle is closed in $\vec{S}_k$ and makes the connection to finite decider sets:

Theorem 4.5. *Let τ be the end tangle induced by an end ω of G and let $k \in \mathbb{N}$. Then the following are equivalent:*

(i) *$\tau \cap \vec{S}_k$ is closed in $\vec{S}_k$;*

(ii) $\deg(\omega) + \mathrm{dom}(\omega) \geqslant k$*;*

(iii) *$\tau \cap \vec{S}_k$ has a finite decider set;*

(iv) *$\tau \cap \vec{S}_k$ has a decider set of size exactly k.*

Let us first prove Theorem 4.4 from Theorem 3.5 and Theorem 4.5:

Proof of Theorem 4.4. We will show (i) using Theorem 3.5 and (ii) with Theorem 4.5. The third statement is then an immediate consequence of the first two and the fact that an end tangle which is closed in $\vec{S}$ is also closed in $\vec{S}_k$.

To see that (i) holds, let us first suppose that τ is closed in $\vec{S}$. Then, by Theorem 3.5, τ is defined by an $\mathcal{A}$-block $K \subseteq V$. It is easy to see that every vertex in K dominates ω, hence $\mathrm{dom}(\omega) = \infty$ as K is infinite.

For the converse, suppose that ω is infinitely dominated, and let us show that τ is closed in $\vec{S}$. For every separation $(A, B) \in \tau$ every vertex dominating ω must lie in B. Therefore $K := \bigcap\{B \mid (A, B) \in \tau\}$ is infinite and thus, as seen in the proof of Theorem 3.5, an $\mathcal{A}$-block defining τ, which is hence closed in $\vec{S}$.

Let us now show (ii). By (i), τ is not closed in $\vec{S}$ if and only if $\mathrm{dom}(\omega) < \infty$. On the other hand, by Theorem 4.5, $\tau \cap \vec{S}_k$ is closed in $\vec{S}_k$ for all k if and only if $\deg(\omega) + \mathrm{dom}(\omega) \geqslant k$ for all $k \in \mathbb{N}$, that is, if at least one summand is infinite. Thus (ii) holds.

Claim (iii) now follows either from claim (i) and (ii), or directly from Theorem 4.5. □

We will conclude this section by proving Theorem 4.5. For this we shall need two lemmas. The first lemma can be seen as an analogue of Menger's Theorem between a vertex set and an end. Given a set $X \subseteq V$ and an end ω, we say that $F \subseteq V$ *separates X from ω* if every ω-ray which meets X also meets F.

Lemma 4.6. *Let ω be an undominated end of G and $X \subseteq V$ a finite set. The largest size of a family of disjoint ω-rays which start in X is equal to the smallest size of a set $T \subseteq V$ separating X from ω.*

Proof. Let T be a set separating X from ω of minimal size. Clearly, a family of disjoint ω-rays which all start in X cannot be larger than T, as every ray in that family must meet T. So let us show that we can find a family of $|T|$ disjoint ω-rays starting in X.

Observe that, as ω is undominated, for every vertex $v \in V$ we can find a finite set $T_v \subseteq V \setminus \{v\}$ which separates v from ω. Thus, for every finite set $Y \subseteq V$ we can find a finite set in $V \setminus Y$ separating Y from ω: for instance, the set $\bigcup\{T_v \setminus Y \mid v \in Y\}$.

Pick a sequence of finite sets $T_n \subseteq V$ inductively by setting $T_0 := T$ and picking as T_n a set of minimal size with the property that $T_{n-1} \cap T_n = \emptyset$ and that T_n separates T_{n-1} from ω; these sets exist by the above observation. Let C_n be the component of $G - T_n$ that contains ω. Clearly $T_n \subseteq C_{n-1}$.

We claim that $C := \bigcap_{n\in\mathbb{N}} C_n = \emptyset$. To see this, consider any $v \in C$ and a shortest v–X path in G. This path must pass through T_n for every $n \in \mathbb{N}$, which is impossible since the separators T_n are pairwise disjoint. Therefore C must be empty.

By the minimality of each T_n, the sets T_n are of non-decreasing size, and furthermore Menger's Theorem yields a family of $|T_n|$ many disjoint paths between T_n and T_{n+1} for each $n \in \mathbb{N}$, as well as $|T_0|$ many disjoint paths between X and T_0. By concatenating these paths we obtain a family of $|T_0| = |T|$ many rays starting in X. To finish the proof we just need to show that these rays belong to ω. To see this, let ω' be another end of G, and T' a finite set separating ω and ω'. As $C = \emptyset$ we have $C_n \cap T' = \emptyset$ for sufficiently large n, which shows that the rays constructed do not belong to ω' and hence concludes the proof. $\square$

An immediate consequence of Lemma 4.6 is that, for an undominated end ω, every finite set $X \subseteq V$ can be separated from ω by at most $\deg(\omega)$ many vertices. In fact, we can state a slightly more general corollary:

Corollary 4.7. *Let ω be a finitely dominated end of G and $X \subseteq V$ a finite set. Then X can be separated from ω by some $T \subseteq V$ with $|T| \leqslant \deg(\omega) + \mathrm{dom}(\omega)$.*

Proof. Let D be the set of vertices dominating ω and consider the graph $G' := G - D$ and the set $X' := X \setminus D$. By Lemma 4.6 there is a set $T' \subseteq V(G')$ of size at most $\deg(\omega)$ separating X' from ω in G'. Set $T := T' \cup D$. Then T separates X from ω in G and has size $|T| = |T'| + |D| \leqslant \deg(\omega) + \mathrm{dom}(\omega)$. $\square$

The second lemma we shall need for our proof of Theorem 4.5 roughly states that, for an end of high degree, we can find a large family of disjoint ω-rays whose set of starting vertices is highly connected in G, even after removing the tails of these rays:

Lemma 4.8. *Let ω be an end of G and $k \leqslant \deg(\omega) + \mathrm{dom}(\omega)$. Then there are a set $X \subseteq V$ of k vertices and a set $\mathcal{R}$ of disjoint ω-rays with the following properties: every vertex in X is either the start-vertex of a ray in $\mathcal{R}$, or dominates ω and does not lie on any $R \in \mathcal{R}$, and furthermore for any two sets $A, B \subseteq X$ there are $\min(|A|, |B|)$ many disjoint A–B-paths in G whose internal vertices meet no ray in $\mathcal{R}$ and no vertex of X.*

Proof. Pick a set D of vertices dominating ω and a set $\mathcal{R}$ of disjoint ω-rays not meeting D such that $|D| + |\mathcal{R}| = k$; we shall find suitable tails of the rays in $\mathcal{R}$ such that their starting vertices together with D are the desired set X.

Using the fact that the vertices in D dominate ω and that the rays in $\mathcal{R}$ belong to ω, we can pick for each pair x_1, x_2 of elements of $D \cup \mathcal{R}$ an x_1–x_2-path in G in such a way that these paths are pairwise disjoint with the exception of possibly having a common end-vertex in D. Let $\mathcal{P}$ be the set of these paths. Now for each ray in $\mathcal{R}$ pick a tail of that ray which avoids all the paths in $\mathcal{P}$. Let $\mathcal{R}'$ be the set of these tails and X the union of their starting vertices and D. We claim that X and $\mathcal{R}'$ are as desired.

To see this, let us show that for any sets $A, B \subseteq X$ we can find $\min(|A|, |B|)$ many disjoint A–B-paths in G whose internal vertices avoid D as well as $V(R')$ for every $R' \in \mathcal{R}'$. Clearly it suffices to show this for disjoint sets A, B of equal size. So let $A, B \subseteq X$ be two disjoint sets with $n := |A| = |B|$ and let $\mathcal{R}_{A,B}$ be the set of all rays in $\mathcal{R}$ that contain a vertex from A or B. For each pair $(a, b) \in A \times B$ there is a unique path $P \in \mathcal{P}$ such that each of its end-vertices either is a or b (if $a \in D$ or $b \in D$) or lies on a ray in $\mathcal{R}_{A,B}$ which contains a or b; let $P_{a,b}$ be the a–b-path obtained from P by extending it, for each of its end-vertices that is not either a or b, along the corresponding ray in $\mathcal{R}$ up to a or b. Let $\mathcal{P}_{A,B}$ be the set of all these paths $P_{a,b}$. Note that the internal vertices of each path $P_{a,b} \in \mathcal{P}_{A,B}$ meet none of the rays in $\mathcal{R}'$ or vertices in X.

We claim that A and B cannot be separated by fewer than $n = |A|$ vertices in

$$G' := \bigcup_{P_{a,b} \in \mathcal{P}_{A,B}} P_{a,b}\,;$$

the claim will then follow from Menger's Theorem. So suppose that some set $T \subseteq V(G')$ of size less than n is given. Every vertex from T can lie on at most $n+1$ paths from $\mathcal{P}_{A,B}$: for any $v \in T$, if v does not lie in D or on some $R \in \mathcal{R}_{A,B}$, then v can meet at most one path from $\mathcal{P}_{A,B}$ as the paths in $\mathcal{P}_{A,B}$ are pairwise internally disjoint; if v is a vertex in D then v meets at most the n paths in $\mathcal{P}_{A,B}$ which have v as one of their end-vertices; and finally, if v lies on a ray $R \in \mathcal{R}_{A,B}$, then v can meet all of the n paths in $\mathcal{P}_{A,B}$ which have an end-vertex on R, and additionally v can be the internal vertex of up to one other path in $\mathcal{P}_{A,B}$ that does not end on R.

Thus, as $|T| \leqslant n-1$, the set T can meet at most

$$|T|\,(n+1) \leqslant (n-1)(n+1) = n^2 - 1$$

of the paths in $\mathcal{P}_{A,B}$. Since $\mathcal{P}_{A,B}$ contains an a–b-path for every pair (a, b) in $A \times B$, there are $|A \times B| = n^2$ paths in $\mathcal{P}_{A,B}$, which proves that T cannot separate A and B in G'.

We can thus apply Menger's Theorem to obtain n disjoint A–B-paths in G', which are the desired disjoint paths in G whose internal vertices avoid the rays in $\mathcal{R}'$ and vertices in X: the only vertices that are contained both in $V(G')$ as well as in either X or a ray from $\mathcal{R}'$ are vertices from A or B, which cannot be internal vertices of the $n = |A| = |B|$ disjoint A–B-paths. $\square$

We would like to thank Christian Elbracht for pointing out the counting argument used in the final part of the proof of Lemma 4.8.

We are now ready to prove Theorem 4.5:

Theorem 4.5. *Let τ be the end tangle induced by an end ω of G and let $k \in \mathbb{N}$. Then the following are equivalent:*

(i) $\tau \cap \vec{S}_k$ *is closed in* $\vec{S}_k$*;*

(ii) $\deg(\omega) + \operatorname{dom}(\omega) \geqslant k$*;*

(iii) $\tau \cap \vec{S}_k$ *has a finite decider set;*

(iv) $\tau \cap \vec{S}_k$ *has a decider set of size exactly* k*.*

Proof. We will show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i).

To see that (i) $\Rightarrow$ (ii), let us first suppose that $\deg(\omega) + \operatorname{dom}(\omega) < k$ and show that $\tau \cap \vec{S}_k$ is not closed in $\vec{S}_k$. Let D be the set of dominating vertices of ω. As $|D| = \operatorname{dom}(\omega) < k$ the separation (V, D) lies in $\vec{S}_k$. By definition of τ we have $(D, V) \in \tau$ and $(V, D) \notin \tau$. Thus it suffices to show that (V, D) lies in the closure of $\tau \cap \vec{S}_k$ in $\vec{S}_k$. This will be the case if for every finite set $X \subseteq V$ there is a separation $(A, B) \in \tau \cap \vec{S}_k$ with $(A, B) \restriction X = (V, D) \restriction X$.

So let X be a finite subset of V. By Corollary 4.7 some set T of size at most $\deg(\omega) + \operatorname{dom}(\omega)$ separates X from ω. Let C be the component of G containing ω. We define the separation (A, B) by setting $A := V \smallsetminus C$ and $B := T \cup C$. Then (A, B) is a separation of G with

$$|(A, B)| = |T| \leqslant \deg(\omega) + \operatorname{dom}(\omega) < k,$$

so $(A, B) \in \vec{S}_k$. In fact (A, B) lies in $\tau \cap \vec{S}_k$ as ω lives in B. Furthermore we have $X \subseteq A$ and $D \subseteq B$ since no vertex dominating ω can be separated from ω by T. Therefore $(A, B) \restriction X = (X, X \cap D) = (V, D) \restriction X$, showing that (V, D) lies in the closure of $\tau \cap \vec{S}_k$ in $\vec{S}_k$.

Let us now show that (ii) $\Rightarrow$ (iv). So let us assume that $\deg(\omega) + \operatorname{dom}(\omega) \geqslant k$. Then by Lemma 4.8 we find a set $X \subseteq V$ of size k and a family $\mathcal{R}$ of ω-rays such that every vertex of X either dominates ω or is the start-vertex of a ray in $\mathcal{R}$, and such that for any $A, B \subseteq X$ we can find $\min(|A|, |B|)$ many disjoint A–B-paths in G whose internal vertices meet neither X nor any ray in $\mathcal{R}$.

We claim that X is the desired decider set for $\tau \cap \vec{S}_k$. To see this, let (A, B) be any separation in $\tau \cap \vec{S}_k$; we need to show that $|A \cap X| < |B \cap X|$. Let us write $X_{A \smallsetminus B} := (A \smallsetminus B) \cap X$ and $X_{B \smallsetminus A} := (B \smallsetminus A) \cap X$ as well as $X_{A \cap B} := (A \cap B) \cap X$. It then suffices to prove $|X_{A \smallsetminus B}| < |X_{B \smallsetminus A}|$.

So suppose to the contrary that $|X_{A \smallsetminus B}| \geqslant |X_{B \smallsetminus A}|$. Note first that no vertex in $X_{A \smallsetminus B}$ dominates ω as witnessed by the finite-order separation $(A, B) \in \tau \subseteq S_k$. Therefore, for every vertex in $X_{A \smallsetminus B}$, we have a ray in $\mathcal{R}$ starting at that vertex. Each of those disjoint rays must pass through the separator $A \cap B$, and none of them hits $X_{A \cap B}$. Furthermore by Lemma 4.8 there are $|X_{B\smallsetminus A}|$ many disjoint $X_{A\smallsetminus B}$–$X_{B\smallsetminus A}$-paths whose internal vertices avoid $\mathcal{R}$ and X. These paths, too, must pass the separator $A\cap B$ without meeting $X_{A\cap B}$ or any of the rays above. Thus we have

$$|A\cap B| \geqslant |X_{A\cap B}| + |X_{A\smallsetminus B}| + |X_{B\smallsetminus A}| = |X| = k,$$

a contradiction since $(A,B)\in S_k$ and hence $|A\cap B|<k$. Therefore we must have $|X_{A\smallsetminus B}|<|X_{B\smallsetminus A}|$, which immediately implies $|A\cap X|<|B\cap X|$.

Finally, let us show that (iii) ⇒ (i). So let $X\subseteq V(G)$ be a finite decider set for $\tau\cap\vec{S_k}$. We need to show that no $(A,B)\in S_k\smallsetminus\tau$ lies in the closure of $\tau\cap\vec{S_k}$. For this let $(A,B)\in S_k\smallsetminus\tau$ be given; then X witnesses that (A,B) does not lie in the closure of τ. To see this, let any $(C,D)\in\tau$ be given. As X is a decider set for τ we have $|C\cap X|<|D\cap X|$, and since $(A,B)\notin\tau$ we have $|A\cap X|\geqslant|B\cap X|$. Therefore (A,B) and (C,D) do not agree on X, which thus witnesses that (A,B) does not lie in the closure of $\tau\cap\vec{S_k}$ in S_k. □

Note that in our proof above that (iii) implies (i) we did not make use of the assumption that the tangle τ was induced by an end of G: indeed, every orientation of S_k that has a finite decider set is closed in $\vec{S_k}$.

For an end tangle τ that is closed in $\vec{S}$ we can say slightly more about its decider sets in $\vec{S_k}$: for every $k\in\mathbb{N}$ the restriction $\tau\cap\vec{S_k}$ has a decider set of size exactly k which is a $(<k)$-inseparable set. Finding these $(<k)$-inseparable decider sets is straightforward: such an end tangle τ is defined by an $\mathcal{A}$-block K by Theorem 3.5, and every subset $X\subseteq K$ of size k is a $(<k)$-inseparable decider set for $\tau\cap\vec{S_k}$. However, having a $(<k)$-inseparable decider set for $\tau\cap\vec{S_k}$ for all $k\in\mathbb{N}$ is not a characterizing property for the closed end tangles of G:

Example 4.9. For $n\in\mathbb{N}$ let K^n be the complete graph on n vertices. Let G be the graph obtained from a ray $R=v_1v_2\ldots$ by replacing each vertex v_n with the complete graph K^n. Then G has a unique end ω; let τ be the end tangle induced by ω. Since ω is undominated τ is not closed in $\vec{S}=\vec{S}(G)$ by Theorem 4.4. However, for every $k\in\mathbb{N}$, the tangle $\tau\cap\vec{S_k}$ has a $(<k)$-inseparable decider set of size k: the clique K^k which replaced the vertex v_k of R is such a $(<k)$-inseparable decider set.

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