THE TOPOLOGICAL HOCHSCHILD HOMOLOGY OF ALGEBRAIC K-THEORY OF FINITE FIELDS

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Abstract. Let $\mathrm{K}(\mathbb{F}_q)$ be the algebraic K-theory spectrum of the finite field with q elements and let $p \geq 5$ be a prime number coprime to q. In this paper we study the mod p and v_1 topological Hochschild homology of $\mathrm{K}(\mathbb{F}_q)$, denoted $V(1)_* \mathrm{THH}(\mathrm{K}(\mathbb{F}_q))$, as an $\mathbb{F}_p$-algebra. The computations are organized in four different cases, depending on the mod p behaviour of the function $q^n - 1$. We use different spectral sequences, in particular the Bökstedt spectral sequence and a generalization of a spectral sequence of Brun developed in an earlier paper. We calculate the $\mathbb{F}_p$-algebras $\mathrm{THH}_*(\mathrm{K}(\mathbb{F}_q); H\mathbb{F}_p)$, and we compute $V(1)_* \mathrm{THH}(\mathrm{K}(\mathbb{F}_q))$ in the first two cases.

1. Introduction

Let $q = l^n$ be a prime power and let $\mathrm{K}(\mathbb{F}_q)$ be the algebraic K-theory spectrum of the finite field with q elements. Let p be a prime number with $p \neq l$ and $p \geq 5$. In this paper we study the mod p and v_1 topological Hochschild homology of $\mathrm{K}(\mathbb{F}_q)$, denoted by $V(1)_* \mathrm{THH}(\mathrm{K}(\mathbb{F}_q))$. In [22] we constructed a generalization of a spectral sequence of Brun and we applied it to give a short computation of the mod p and v_1 topological Hochschild homology of p-completed connective complex K-theory $\mathrm{ku}_p \simeq \mathrm{K}(\bar{\mathbb{F}}_l)_p$. In this paper we apply the spectral sequence in a similar fashion to $\mathrm{K}(\mathbb{F}_q)_p$.

Topological Hochschild homology is an ingredient for the computation of topological cyclic homology (TC) via homotopy fixed points and Tate spectral sequences. By [20] and [17] the connective cover of $\mathrm{TC}(\mathrm{K}(\mathbb{F}_q)_p)_p$ is equivalent to $\mathrm{K}(\mathrm{K}(\mathbb{F}_q)_p)_p$. The study of iterated algebraic K-theory is interesting because of the red-shift conjecture predicting that algebraic K-theory increases the chromatic level by one [7].

By Quillen's computations [34] the homotopy of $\mathrm{K}(\mathbb{F}_q)$ is given by

$$\pi_n(\mathrm{K}(\mathbb{F}_q)) = \begin{cases} \mathbb{Z}, & n = 0; \\ \mathbb{Z}/q^i - 1, & n = 2i - 1, i > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Our computations depend on the degree of the first homotopy group with p-torsion and on the order of the p-torsion subgroup. We define r to be the order of q in $(\mathbb{Z}/p)^*$, so that the first p-torsion appears in degree $2r - 1$, and we define $v := v_p(q^r - 1)$ to be the p-adic valuation of $q^r - 1$, so that the p-torsion subgroup of $\pi_{2r-1}(\mathrm{K}(\mathbb{F}_q))$ has order p^v. We distinguish the following four cases:

(1) $r = p - 1$ and $v = 1$,
(2) $r = p - 1$ and $v \geq 2$,
(3) $r < p - 1$ and $v \geq 2$,
(4) $r < p - 1$ and $v = 1$.

We study $V(1)_* \mathrm{THH}(\mathrm{K}(\mathbb{F}_q))$ by means of the Bökstedt spectral sequence, the generalized Brun spectral sequence developed in [22] and a spectral sequence of Veen [39].

The Bökstedt spectral sequence of a commutative S-algebra B and a commutative B-algebra C has the form

$$E^2_{*,*} = \mathbb{H}^{\mathbb{F}_p}_{*,*}\big((H\mathbb{F}_p)_* B; (H\mathbb{F}_p)_* C\big) \Longrightarrow (H\mathbb{F}_p)_* \mathrm{THH}(B; C).$$

Here, $H\mathbb{F}_p$ is the Eilenberg-Mac Lane spectrum of $\mathbb{F}_p$, $(H\mathbb{F}_p)_*(-)$ is mod p homology and $\mathbb{H}^{\mathbb{F}_p}_{*,*}(-;-)$ denotes ordinary Hochschild homology over the ground ring $\mathbb{F}_p$. The Bökstedt spectral sequence is an $(H\mathbb{F}_p)_*H\mathbb{F}_p$-comodule $(H\mathbb{F}_p)_*C$-algebra spectral sequence and under some flatness condition one additionally has a coalgebra structure.

We define $\mathrm{K} \coloneqq \mathrm{K}(\mathbb{F}_q)_p$. We use the following instances of the generalized Brun spectral sequence:

a) $E^2_{*,*} = V(0)_*(H\mathbb{Z}_p \wedge_{\mathrm{K}} H\mathbb{Z}_p) \otimes_{\mathbb{F}_p} \mathrm{THH}_*(H\mathbb{Z}_p; H\mathbb{F}_p) \Longrightarrow V(0)_* \mathrm{THH}(\mathrm{K}; H\mathbb{Z}_p)$,

b) $E^2_{*,*} = V(1)_* \mathrm{K} \otimes_{\mathbb{F}_p} \mathrm{THH}_*(\mathrm{K}; H\mathbb{F}_p) \Longrightarrow V(1)_* \mathrm{THH}(\mathrm{K})$.

Here, $V(0)$ denotes the mod p Moore spectrum. Note that $V(0)_* \mathrm{THH}(\mathrm{K}; H\mathbb{Z}_p)$ is isomorphic to $\mathrm{THH}_*(\mathrm{K}; H\mathbb{F}_p)$, so that the abutment of a) is an input of b).

Veen's spectral sequence has the form

$$E^2_{*,*} = \mathrm{Tor}^{(H\mathbb{F}_p)_* \mathrm{K}}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \mathrm{THH}_*(\mathrm{K}; H\mathbb{F}_p).$$

We examine Veen's spectral sequence in small total degrees and use our result to determine the differentials of a).

In case (1) and (2) the mod p homology of K has an easy form and the Bökstedt spectral sequence converging to $(H\mathbb{F}_p)_* \mathrm{THH}(\mathrm{K})$ has a coalgebra structure. This is useful to compute the differentials. In case (1) the Bökstedt spectral sequence has already been computed by Angeltveit and Rognes [5]. We proceed similarly in case (2) (Subsection 6.2). In case (2) an Ext spectral sequence argument shows that the $(2p-3)$th Postnikov invariant of $V(1) \wedge_S \mathrm{K}$ is in the image of the forgetful functor from the homotopy category of $H\mathbb{F}_p \wedge_S \mathrm{K}$-modules to the homotopy category of K-modules (Lemma 6.12). This implies that $V(1) \wedge_S \mathrm{THH}(\mathrm{K})$ is an $H\mathbb{F}_p$-module and we can identify $V(1)_* \mathrm{THH}(\mathrm{K})$ with the $(H\mathbb{F}_p)_*H\mathbb{F}_p$-comodule primitives in $(H\mathbb{F}_p)_*(V(1) \wedge_S \mathrm{THH}(\mathrm{K}))$. We obtain (see Theorem 6.14):

Theorem. *In case (2) we have an isomorphism of $\mathbb{F}_p$-algebras*

$$V(1)_* \mathrm{THH}(\mathrm{K}) \cong E(x) \otimes_{\mathbb{F}_p} E(\lambda_1, \lambda_2) \otimes_{\mathbb{F}_p} P(\mu_2) \otimes_{\mathbb{F}_p} \Gamma(\gamma'_1).$$

Here, $P(-)$, $E(-)$ and $\Gamma(-)$ denote the polynomial, exterior and divided power algebra over $\mathbb{F}_p$ and the degrees are given by $|x| = 2p-3$, $|\lambda_i| = 2p^i - 1$, $|\mu_2| = 2p^2$ and $|\gamma'_1| = 2p-2$.

We compute $\mathrm{THH}_*(\mathrm{K}; H\mathbb{F}_p)$ via the spectral sequence a) in all the cases except for the subcase of case (4) where $r = 1$ (Subsection 7.2). In case (4) it seems harder to compute $\mathrm{THH}_*(\mathrm{K}; H\mathbb{F}_p)$ via the Bökstedt spectral sequence, because this depends on the mod p homology of K which is complicated in this case. In order to compute the differentials in the spectral sequence a) we only need to examine Veen's spectral sequence in small total degrees. This only depends on low degrees of $(H\mathbb{F}_p)_* \mathrm{K}$.

We determine the spectral sequences b) in case (1) (Subsection 7.3). There is only one possible differential. Its existence follows from the fact that the mod p homology of $V(1) \wedge_S \mathrm{THH}(\mathrm{K})$ has no non-trivial comodule primitives in degree $2p^2 - 1$ (Subsection 6.1). We obtain (see Theorem 7.21):

Theorem. *In case (1) the $V(1)$-homotopy of $\mathrm{THH}(\mathrm{K})$ is the homology of the differential graded algebra*

$$E(x, a, \lambda_2) \otimes_{\mathbb{F}_p} P(\mu_2) \otimes_{\mathbb{F}_p} \Gamma(b), \quad d(\lambda_2) = xa$$

with $|x| = 2p-3$, $|a| = p(2p-2)+1$, $|\lambda_2| = 2p^2 - 1$, $|\mu_2| = 2p^2$ and $|b| = p(2p-2)$.

The last result was also obtained by Angelini-Knoll using a different approach [3]. In [2] Angelini-Knoll also shows that $\mathrm{K}(\mathrm{K}(\mathbb{F}_q)_p)$ detects the β-family in case (1).

There is a fiber sequence of spectra

$$\mathrm{K} \longrightarrow \mathrm{ku}_p \xrightarrow{f} \Sigma^2 \mathrm{ku}_p$$

(see [21]). Denoting by ψ^q the Adams operation, the map f is the unique lift of $\psi^q - \mathrm{id}$ to the 1-connective covering $\Sigma^2 \mathrm{ku}_p$ of ku_p. The relation between f and the multiplication of ku_p can be informally written as

$$f(ab) = f(a)b + af(b) + f(a)(\psi^q - \mathrm{id})(b) \tag{1}$$

(see [40]). We show that this fiber sequence can be constructed in the homotopy category of K-modules and that the equation (1) also holds in this category (Section 3). These observations are very useful for our computations: The K-linearity of the fiber sequence is helpful to determine the multiplicative structure of $V(0)_*\mathrm{K}$ and $(H\mathbb{F}_p)_*\mathrm{K}$ (Section 4 and Section 5). We use the K-linearity of the fiber sequence and equation (1) to compute the ring $V(0)_*(H\mathbb{Z}_p \wedge_{\mathrm{K}} H\mathbb{Z}_p)$ (Subsection 7.1).

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2. Notations and recollections

We work in the setting of Elmendorf, Kriz, Mandell and May [18]. For a commutative S-algebra R let $\mathscr{M}_R$ be the category of R-modules. We denote its symmetric monoidal smash product by $\wedge_R$. The category $\mathscr{M}_R$ is a model category, where the weak equivalences are the π_*-isomorphisms, the cofibrations are the retracts of relative cell R-modules and where all objects are fibrant. We denote its homotopy category by $\mathscr{D}_R$. The category $\mathscr{D}_R$ is a tensor triangulated category (see [11, Definition 1.1]), where the distinguished triangles are given, up to isomorphism, by the images of the cofiber sequences under $\mathscr{M}_R \to \mathscr{D}_R$. The functor $\mathscr{M}_R \to \mathscr{D}_R$ is lax symmetric monoidal and, denoting the tensor product in $\mathscr{D}_R$ by $\wedge^L_R$, the structure map

$$M \wedge^L_R N \longrightarrow M \wedge_R N \tag{2}$$

is an isomorphism in $\mathscr{D}_R$ if M or N is a cofibrant R-module. For a morphism of commutative S-algebras $R \to R'$ the functor $\mathscr{D}_{R'} \to \mathscr{D}_R$ maps distinguished triangles to distinguished triangles and is lax symmetric monoidal. Spheres in $\mathscr{M}_R$ are denoted by S^n_R and homotopy groups are given by $\pi_n(M) = \mathscr{D}_R(S^n_R, M)$ for $M \in \mathscr{M}_R$. We denote the right adjoint of $- \wedge^L_R M$ by $F^L_R(M,-)$. The functor $F^L_R(M,-)$ preserves distinguished triangles. We set $\mathrm{Ext}^*_R(-,-) := \pi_{-*}(F^L_R(-,-))$. Note that we have a natural isomorphism $\mathrm{Ext}^0_R(-,-) \cong \mathscr{D}_R(-,-)$. An R-ring spectrum is an object $A \in \mathscr{D}_R$ with maps $A \wedge^L_R A \to A$ and $R \to A$ in $\mathscr{D}_R$ satisfying the left and right unit laws. We denote the category of commutative R-algebras by $\mathscr{C}\mathscr{A}_R$. It has a model category structure, where the weak equivalences are the π_*-isomorphisms and all objects are fibrant [18, Chapter VII]. The category $\mathscr{C}\mathscr{A}_R$ can be identified with the the category of commutative S-algebras under R and the model category structure on $\mathscr{C}\mathscr{A}_R$ is the one inherited from $\mathscr{C}\mathscr{A}_S$. By [22, Lemma 2.2] the map (2) is an isomorphism of R-ring spectra if $R \to M$ and $R \to N$ are maps between cofibrant commutative S-algebras and if $R \to M$ or $R \to N$ is a cofibration in $\mathscr{C}\mathscr{A}_S$.

For a prime $p \geq 5$ we denote by $V(0)$ and $V(1)$ the mod p Moore spectrum and the mod (p, v_1) Toda-Smith complex. We can assume that $V(0)$ and $V(1)$ are cell S-modules. We have distinguished triangles in $\mathscr{D}_S$

$$S \xrightarrow{p\cdot\mathrm{id}} S \longrightarrow V(0) \longrightarrow \Sigma S \tag{3}$$

$$\Sigma^{2p-2}V(0) \longrightarrow V(0) \longrightarrow V(1) \longrightarrow \Sigma^{2p-1}V(0),$$

where $S \to V(0)$ and $V(0) \to V(1)$ are maps of associative and commutative S-ring spectra, [31], [32], [33].

By [24, Example VI.5.2],[26], [28] and [18, Corollary II.3.6] algebraic K-theory can be realized as a functor $\mathrm{K}(-)$ from the category of commutative rings to $\mathscr{CA}_S$. For a commutative ring R the S-algebra $\mathrm{K}(R)$ is connective. One has $\pi_0(\mathrm{K}(R)) = \mathbb{Z}$ and $\pi_i(\mathrm{K}(R))$ is Quillen's ith algebraic K-theory group for $i > 0$ [27, Example 6.2].

Recall that p-completion is Bousfield localization with respect to $V(0)$. By the proofs of [18, Lemma VII.5.8], [18, Lemma VII.5.2] and [18, Theorem VIII.2.2] the p-completion of a commutative S-algebra can be constructed as a commutative S-algebra in such a way that we get a functor $(-)_p : \mathscr{CA}_S \to \mathscr{CA}_S$ with values in cofibrant commutative S-algebras.

For an abelian group A we denote by HA its Eilenberg-Mac Lane spectrum. Recall that by [28], [25], [24], [18, Corollary II.3.6] and by functoriality of cofibrant replacements [18, LemmaVII.5.8] the Eilenberg-Mac Lane spectrum HR of a commutative ring R can be realized as a cofibrant commutative S-algebra in such a way that we get a functor from the category of commutative rings to $\mathscr{CA}_S$.

We denote by $P(-)$, $E(-)$, $\Gamma(-)$ and $P_k(-)$ the polynomial algebra, the exterior algebra, the divided power algebra and the truncated polynomial algebra over $\mathbb{F}_p$, and we write $\otimes$ for $\otimes_{\mathbb{F}_p}$. Furthermore, we write $b \doteq c$ if b and c are equal up to multiplication by a unit in $\mathbb{F}_p$.

An infinite cycle in a spectral sequence is a class b such that we have $d^s(b) = 0$ for all s. A permanent cycle is an infinite cycle that is not in the image of d^s for any s.

3. The fiber sequence relating $\mathrm{K}(\mathbb{F}_q)_p$ and $\mathrm{K}(\bar{\mathbb{F}}_l)_p$

Throughout this paper we fix a prime power $q = l^n$ for $n \geq 1$ and denote by $\mathbb{F}_q$ the finite field with q elements. Let $p \neq l$ be a prime number with $p \geq 5$ and let $V(0)$ and $V(1)$ be built with respect to this prime.

By [34] we have a fiber sequence of spaces

$$\mathrm{K}(\mathbb{F}_q) \longrightarrow BU \times \mathbb{Z} \xrightarrow{\psi^q - \mathrm{id}} BU,$$

where ψ^q is the Adams operation. After p-completion one gets an analogous fiber sequence of spectra [21]. In this section we construct a fiber sequence of this form in $\mathscr{D}_{\mathrm{K}(\mathbb{F}_q)_p}$.

We define K to be the commutative S-algebra $\mathrm{K}(\mathbb{F}_q)_p$. Quillen's computations [34, Theorem 8] imply that we have

$$V(0)_n(\mathrm{K}) = \begin{cases} \mathbb{F}_p, & n = 2ri, i \geq 0; \\ \mathbb{F}_p, & n = 2ri - 1, i > 0; \\ 0, & \text{otherwise;} \end{cases}$$

where r is the order of q in $(\mathbb{Z}/p)^*$. Let $\bar{\mathbb{F}}_l$ be the algebraic closure of $\mathbb{F}_l$. Then, $\mathrm{K}(\bar{\mathbb{F}}_l)_p$ is equivalent as a ring spectrum to p-completed connective complex K-theory [27, Section7]. We thus have an isomorphism of rings

$$\pi_*(\mathrm{K}(\bar{\mathbb{F}}_l)_p) \cong \mathbb{Z}_p[u]$$

with $|u| = 2$. By [27, Section 7] the Adams operation ψ^q corresponds to the map ϕ^q induced by the Frobenius automorphism $\bar{\mathbb{F}}_l \to \bar{\mathbb{F}}_l, x \mapsto x^q$. In homotopy it is the map given by $u \mapsto qu$ [35, Subsection 5.5.1]. We have isomorphisms of $\mathbb{F}_p$-algebras $V(0)_* \mathrm{K}(\bar{\mathbb{F}}_l)_p = P(u)$ and $V(1)_* \mathrm{K}(\bar{\mathbb{F}}_l)_p = P_{p-1}(u)$.

By functoriality the map $\phi^q : \mathrm{K}(\bar{\mathbb{F}}_l)_p \to \mathrm{K}(\bar{\mathbb{F}}_l)_p$ induced by the Frobenius automorphism is a map of K-algebras and therefore of K-modules. We consider the element

$$\phi^q - \mathrm{id} \in \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p, \mathrm{K}(\bar{\mathbb{F}}_l)_p\big).$$

Note that the 0-connected covering of $\mathrm{K}(\bar{\mathbb{F}}_l)_p$ in K-modules is given by the morphism $\bar{u} : S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \to \mathrm{K}(\bar{\mathbb{F}}_l)_p$ in $\mathscr{D}_{\mathrm{K}}$ defined by

$$S^2_K \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{u \wedge \mathrm{id}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{m} \mathrm{K}(\bar{\mathbb{F}}_l)_p.$$

Here m is the product of the K-ring spectrum $\mathrm{K}(\bar{\mathbb{F}}_l)_p$.

Lemma 3.1. *There exist exactly one element $f \in \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p, S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$ that is mapped to $\phi^q - \mathrm{id}$ under*

$$\mathscr{D}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p, S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \xrightarrow{\bar{u}_*} \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p, \mathrm{K}(\bar{\mathbb{F}}_l)_p\big).$$

Proof. We have an exact sequence:

$$\begin{aligned}\mathrm{Ext}^{-1}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p, H\mathbb{Z}_p\big) \to \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p, S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) &\xrightarrow{\bar{u}_*} \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p, \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)\\ &\to \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p, H\mathbb{Z}_p\big).\end{aligned}$$

By [23] and [12, Theorem 7.1] we have a cohomological strongly convergent spectral sequence of the form

$$E_2^{n,m} = \mathrm{Ext}^{n,m}_{\mathrm{K}_*}\Big(\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big)_*, \mathbb{Z}_p\Big) \Longrightarrow \mathrm{Ext}^{n+m}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p, H\mathbb{Z}_p\big).$$

Since $E_2^{n,m} = 0$ for $m < 0$, we get $\mathrm{Ext}^{-1}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p, H\mathbb{Z}_p\big) = 0$. We have a commutative diagram

$$\begin{array}{ccc} \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p, \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) & \longrightarrow & \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p, H\mathbb{Z}_p\big) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{K}_*}\Big(\pi_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big), \pi_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big)\Big) & \longrightarrow & \mathrm{Hom}_{\mathrm{K}_*}\Big(\pi_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big), \mathbb{Z}_p\Big). \end{array}$$

The right vertical map is an isomorphism because it identifies with the edge homomorphism in the above spectral sequence [23]. Since $(\phi^q)_0$ is the identity, we have that

$$(\phi^q - \mathrm{id})_0 : \pi_0\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \to \pi_0\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$$

is zero. □

The map f is part of a distinguished triangle in $\mathscr{D}_{\mathrm{K}}$:

$$F \longrightarrow \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{f} S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \longrightarrow F \wedge S^1.$$

From the long exact sequence in homotopy we get:

Lemma 3.2. *We have*

$$\pi_n(F) \cong \begin{cases} \mathbb{Z}_p, & n = 0; \\ \mathbb{Z}_p/(q^j - 1)\mathbb{Z}_p, & n = 2j - 1,\ j > 0; \\ 0, & \textit{otherwise}. \end{cases}$$

Lemma 3.3. *We have*

$$V(0)_n F = \begin{cases} \mathbb{F}_p, & n = 2jr,\ j \geq 0; \\ \mathbb{F}_p, & n = 2jr - 1,\ j > 0; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. For $n \in \mathbb{Z}$ let $v_p(n)$ be the p-adic valuation of n. For $j > 0$ we have

$$\mathbb{Z}_p/(q^j - 1)\mathbb{Z}_p = \mathbb{Z}_p/p^{v_p(q^j-1)}\mathbb{Z}_p = \mathbb{Z}/p^{v_p(q^j-1)}\mathbb{Z}.$$

The claim now follows by using the distinguished triangle (3). □

Lemma 3.4. *The map*

$$V(0)_{2rj}(F) \to V(0)_{2rj}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$$

is an isomorphism for $j \geq 0$.

Proof. For $j \geq 0$ there is an exact sequence

$$V(0)_{2jr-1}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \longrightarrow V(0)_{2rj}(F) \longrightarrow V(0)_{2rj}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big).$$

Since the first term is zero, and since the second and third term are both $\mathbb{F}_p$, this proves the lemma. □

We want to show that $F \cong \mathrm{K}$ in $\mathscr{D}_{\mathrm{K}}$. We denote the map $\mathrm{K} \to \mathrm{K}(\bar{\mathbb{F}}_l)_p$ induced by the inclusion of fields $\mathbb{F}_q \to \bar{\mathbb{F}}_l$ by i.

Lemma 3.5. *The map*

$$i_{2rj} : V(0)_{2rj}(\mathrm{K}) \to V(0)_{2rj}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$$

is an isomorphism for $j \geq 0$.

Proof. For $j > 0$ we have $\pi_{2rj}\big(\mathrm{K}(\bar{\mathbb{F}}_l)\big) = 0$ by [34, p.585], so we get a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & V(0)_{2rj}\big(\mathrm{K}(\mathbb{F}_q)\big) \arrow[r] \arrow[d] & \pi_{2rj-1}\big(\mathrm{K}(\mathbb{F}_q)\big) \arrow[r, "\cdot p"] \arrow[d] & \pi_{2rj-1}\big(\mathrm{K}(\mathbb{F}_q)\big) \arrow[d] \\
0 \arrow[r] & V(0)_{2rj}\big(\mathrm{K}(\bar{\mathbb{F}}_l)\big) \arrow[r] & \pi_{2rj-1}\big(\mathrm{K}(\bar{\mathbb{F}}_l)\big) \arrow[r, "\cdot p"] & \pi_{2rj-1}\big(\mathrm{K}(\bar{\mathbb{F}}_l)\big).
\end{tikzcd}$$

Because $V(0)_{2rj}\big(\mathrm{K}(\mathbb{F}_q)\big)$ and $V(0)_{2rj}\big(\mathrm{K}(\bar{\mathbb{F}}_l)\big)$ are both isomorphic to $\mathbb{F}_p$, it suffices to show that $\pi_{2rj-1}\big(\mathrm{K}(\mathbb{F}_q)\big) \to \pi_{2rj-1}\big(\mathrm{K}(\bar{\mathbb{F}}_l)\big)$ is injective. This follows because by [34, p.585] the abelian group $\pi_*(\mathrm{K}(\bar{\mathbb{F}}_l))$ is the filtered colimit of the abelian groups $\pi_*\big(\mathrm{K}(k)\big)$, where k runs over the finite subfields of $\bar{\mathbb{F}}_l$, and because by [34, Theorem 8] the maps $\pi_*\big(\mathrm{K}(k)\big) \to \pi_*\big(\mathrm{K}(k')\big)$ induced by inclusions of finite fields $k \hookrightarrow k'$ are injective. $\square$

Lemma 3.6. *There exist an $h \in \mathscr{D}_{\mathrm{K}}(\mathrm{K}, F)$ that is mapped to i under*

$$\mathscr{D}_{\mathrm{K}}(\mathrm{K}, F) \to \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}, \mathrm{K}(\bar{\mathbb{F}}_l)_p\big).$$

Proof. We have an exact sequence

$$\mathscr{D}_{\mathrm{K}}(\mathrm{K}, F) \longrightarrow \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}, \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \xrightarrow{f_*} \mathscr{D}_{\mathrm{K}}\big(\,\mathrm{K}, S^2_{\mathrm{K}} \wedge_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big).$$

It thus suffices to show that $f_*(i) = 0$. There is an exact sequence

$$\mathrm{Ext}^{-1}_{\mathrm{K}}(\mathrm{K}, H\mathbb{Z}_p) \longrightarrow \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}, S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \xrightarrow{\bar{u}_*} \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}, \mathrm{K}(\bar{\mathbb{F}}_l)_p\big).$$

Using the Ext spectral sequence one gets that $\mathrm{Ext}^{-1}_{\mathrm{K}}(\mathrm{K}, H\mathbb{Z}_p) = 0$. We therefore only have to show that $\bar{u} \circ f \circ i = (\phi^q - \mathrm{id}) \circ i$ is zero. This is clear. $\square$

We will show that $h : \mathrm{K} \to F$ is an isomorphism. We show this by proving that its image under $\mathscr{D}_{\mathrm{K}} \to \mathscr{D}_S$ is a $V(0)_*$-equivalence between $V(0)$-local S-modules.

Lemma 3.7. *The map*

$$h_* : V(0)_* \,\mathrm{K} \xrightarrow{h_*} V(0)_* F$$

is an isomorphism.

Proof. It is clear that h_* is an isomorphism in the degrees $2rj$ for $j \geq 0$. It thus suffices to show that h_* is an isomorphism in the degrees $2rj - 1$ for $j > 0$.

The map h induces a map between the exact couples

$$\begin{tikzcd}
\pi_*(\mathrm{K}) \arrow[rr, "\cdot p"] & & \pi_*(\mathrm{K}) \arrow[dl] \\
& V(0)_*(\mathrm{K}) \arrow[ul, "d"] &
\end{tikzcd} \tag{4}$$

and

$$\begin{tikzcd}
\pi_*(F) \arrow[rr, "\cdot p"] & & \pi_*(F) \arrow[dl] \\
& V(0)_*(F) \arrow[ul, "d"] &
\end{tikzcd} \tag{5}$$

and therefore a map of the associated singly-graded Bockstein spectral sequences. Fix $j > 0$. Let a be a generator of $V(0)_{2jr}(F) = \mathbb{F}_p$. Since $\pi_{2jr}(F) = 0$ we get that

$$d(a) \in \pi_{2jr-1}(F) = \mathbb{Z}/p^{v_p(q^{jr}-1)}\mathbb{Z}$$

is an element of order p. It follows that $d(a)$ has a preimage under the map

$$\pi_{2jr-1}(F) \xrightarrow{\cdot p^{v_p(q^{jr}-1)-1}} \pi_{2jr-1}(F),$$

but not under the map

$$\pi_{2jr-1}(F) \xrightarrow{\cdot p^{v_p(q^{jr}-1)}} \pi_{2jr-1}(F).$$

Hence, a survives to the $E^{v_p(q^{jr}-1)}$-page in the spectral sequence associated to the exact couple (5) and

$$d^{v_p(q^{jr}-1)}(a) \neq 0.$$

Since K has the same homotopy and mod p homotopy groups as F, the same argument as above shows that the preimage b of a under the isomorphism

$$V(0)_{2jr}(\mathrm{K}) \xrightarrow{h_{2jr}} V(0)_{2jr}(F)$$

has to survive to the $E^{v_p(q^{jr}-1)}$-page of the spectral sequence associated to the exact couple (4) and that $d^{v_p(q^{jr}-1)}(b) \neq 0$. We conclude that h_* maps

$$d^{v_p(q^{jr}-1)}(b) \in V(0)_{2jr-1}(\mathrm{K})$$

to

$$d^{v_p(q^{jr}-1)}(a) \in V(0)_{2jr-1}(F).$$

Thus, $h_* : V(0)_* \mathrm{K} \to V(0)_* F$ is an isomorphism. □

By definition, K is $V(0)$-local.

Lemma 3.8. *The S-module F is $V(0)$-local.*

Proof. Let W be a $V(0)$-acyclic S-module. We have an exact sequence

$$\mathrm{Ext}_S^{-1}\big(W, S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \longrightarrow \mathscr{D}_S(W, F) \longrightarrow \mathscr{D}_S\big(W, \mathrm{K}(\bar{\mathbb{F}}_l)_p\big).$$

Because $\mathrm{K}(\bar{\mathbb{F}}_l)_p$ is $V(0)$-local, one has $\mathscr{D}_S\big(W, \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) = 0$. Since desuspensions of $V(0)$-acyclic S-modules are $V(0)$-acyclic, we get

$$\mathrm{Ext}_S^{-1}\big(W, S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) = 0.$$

□

Corollary 3.9. *We have a distinguished triangle of the form*

$$\mathrm{K} \xrightarrow{i} \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{f} S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \longrightarrow \Sigma\,\mathrm{K} \tag{6}$$

in $\mathscr{D}_{\mathrm{K}}$.

The following lemma will be useful later to determine multiplicative structures.

Lemma 3.10. *In $\mathscr{D}_{\mathrm{K}}$ we have the following equality of morphisms:*

$$\begin{aligned}
&\mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{m} \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{f} S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \\
=\,&\mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{f\wedge\mathrm{id}} S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{\Sigma^2 m} S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \\
&+ \mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{\mathrm{id}\wedge f} \mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge^L_{\mathrm{K}} S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{\Sigma^2 m} S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \\
&+ \mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{f\wedge(\phi^q-\mathrm{id})} S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{\Sigma^2 m} S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p.
\end{aligned}$$

Proof. By a Tor spectral sequence argument it follows that $\mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p$ is connective. An Ext spectral sequence computation shows that

$$\mathrm{Ext}^{-1}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p, H\mathbb{Z}_p\big) = 0.$$

It follows that

$$\mathscr{D}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p,\, S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \xrightarrow{\bar{u}_*} \mathscr{D}_{\mathrm{K}}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p,\, \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$$

is injective. It thus suffices to show that the equality holds after composing with $\bar{u}$. Now, we can argue as in [40, Lemma 4.1]. In [40] a similar statement is proven for the category of spectra (instead of $\mathscr{D}_{\mathrm{K}}$) and for q that generates $(\mathbb{Z}/p^2)^*$.

□

4. The algebras $V(0)_* \mathrm{K}(\mathbb{F}_q)$ and $V(1)_* \mathrm{K}(\mathbb{F}_q)$

In this section we determine the multiplicative structure of $V(0)_* \mathrm{K}$ and $V(1)_* \mathrm{K}$. In [14, Theorem 2.6] Browder computes the ring $V(0)_* \mathrm{K}$. Browder works with spaces. In this section we present a computation using the K-linearity of the distinguished triangle (6).

Lemma 4.1. *We have an isomorphism of $\mathbb{F}_p$-algebras*

$$V(0)_* \mathrm{K} \cong E(x) \otimes P(y),$$

where $|x| = 2r-1$ and $|y| = 2r$.

Proof. By Corollary 3.9 we have an $V(0)_* \mathrm{K}$-linear exact sequence

$$\Sigma^2 V(0)_* \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{\ \Delta\ } V(0)_* \mathrm{K} \xrightarrow{\ i_*\ } V(0)_* \mathrm{K}(\bar{\mathbb{F}}_l)_p,$$

where Δ is a map of degree -1 and where i_* is a map of degree 0 that is a map of $\mathbb{F}_p$-algebras.

The map i_{2r} is an isomorphism, so we can define y to be the preimage of u^r under i_*. For $n \geq 0$ we then have $i_*(y^n) = u^{rn}$. In particular, we get $y^n \neq 0$.

For $i \geq 1$ we have

$$V(0)_{2ir-1} \mathrm{K}(\bar{\mathbb{F}}_l)_p = 0.$$

Hence,

$$\big(\Sigma^2 V(0)_* \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)_{2ir} \xrightarrow{\ \Delta\ } V(0)_{2ir-1} \mathrm{K}$$

is an isomorphism. We define $x := \Delta(\Sigma^2 u^{r-1})$. In order to prove the lemma it now suffices to show that $xy^i \neq 0$ for $i \geq 0$. We have

$$xy^i = \Delta(\Sigma^2 u^{r-1}) y^i = \Delta\big(y^i(\Sigma^2 u^{r-1})\big) = \Delta(\Sigma^2 u^{r(i+1)-1}) \neq 0.$$

□

We define $k := \frac{p-1}{r}$.

Lemma 4.2. *We have an isomorphism of $\mathbb{F}_p$-algebras*

$$V(1)_* \mathrm{K} \cong E(x) \otimes P_k(y).$$

Proof. We have a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \Sigma^{2p-2} V(0)_* \mathrm{K} \arrow[r, "v"] \arrow[d, equal] & V(0)_* \mathrm{K} \arrow[r, "\rho"] \arrow[d, equal] & V(1)_* \mathrm{K} \arrow[r] & \cdots, \\
& \Sigma^{2p-2} E(x) \otimes P(y) & E(x) \otimes P(y) & &
\end{tikzcd}$$

where ρ is a map of $\mathbb{F}_p$-algebras. It suffices to show that y^k is in the image of v, or equivalently that $v(\Sigma^{2p-2}1) \neq 0$. For this, we consider the commutative diagram

$$\begin{tikzcd}
\Sigma^{2p-2}V(0)_*\,\mathrm{K} \arrow[r, "v"] \arrow[d, "\Sigma^{2p-2}i_*"'] & V(0)_*\,\mathrm{K} \arrow[d, "i_*"] & \\
\Sigma^{2p-2}V(0)_*\,\mathrm{K}(\bar{\mathbb{F}}_l)_p \arrow[r, "\bar{v}"] \arrow[d, equal] & V(0)_*\,\mathrm{K}(\bar{\mathbb{F}}_l)_p \arrow[r, "\bar{\rho}"] \arrow[d, equal] & V(1)_*\,\mathrm{K}(\bar{\mathbb{F}}_l)_p \arrow[d, equal] \\
\Sigma^{2p-2}P(u) \arrow[r] & P(u) \arrow[r] & P_{p-1}(u).
\end{tikzcd}$$

To show $v(\Sigma^{2p-2}1) \neq 0$, it suffices to prove that $\bar{v}(\Sigma^{2p-2}1) \neq 0$. This follows from $\bar{\rho}(u^{p-1}) = 0$. □

5. The mod p homology of $\mathrm{K}(\mathbb{F}_q)$

In this section we study the mod p homology of K. We define v to be the p-adic valuation of $q^r - 1$. By [19, Lemma VIII.2.4] this is equal to the p-adic valuation of $q^{p-1} - 1$. We distinguish the four different cases:

(1) $r = p - 1$ and $v = 1$,
(2) $r = p - 1$ and $v \geq 2$,
(3) $r < p - 1$ and $v \geq 2$,
(4) $r < p - 1$ and $v = 1$.

The section is in part inspired by Hirata's article [21] which treats the cohomology of algebraic K-theory of finite fields and by Angeltveit's and Rognes' article [5] which treats the mod p homology of K in case (1).

Similarly to [21] we first split the image of the distinguished triangle (6) under $\mathscr{D}_{\mathrm{K}} \to \mathscr{D}_S$ into a wedge of $p-1$ distinguished triangles corresponding to the splitting of $\mathrm{K}(\bar{\mathbb{F}}_l)_p$ into a wedge of suspensions of the p-completed connective Adams summand.

Let q' be a power of a prime l' (possibly different from l) such that q' is a generator of $(\mathbb{Z}/p^2)^*$. We set

$$k' = \cup_{i\geq 0}\mathbb{F}_{q'^{p^i}}.$$

Then, $\ell_p := \mathrm{K}(k')_p$ is a commutative S-algebra model for the p-completed connective Adams summand, [29, Proposition 9.2], [10, Section 2]. We define

$$k = \cup_{i\geq 0}\mathbb{F}_{q'^{p^i(p-1)}}$$

and claim that there is a weak equivalence of commutative S-algebras $\mathrm{K}(k)_p \to \mathrm{K}(\bar{\mathbb{F}}_l)_p$: By Quillen's computations [34, pp.583–585] one gets that $\mathrm{K}(k)_p \to \mathrm{K}(\bar{\mathbb{F}}_{l'})_p$ is an isomorphism in $V(0)$-homotopy and therefore a weak equivalence. By [10] and since $\mathrm{K}(\bar{\mathbb{F}}_{l'})_p$ and $\mathrm{K}(\bar{\mathbb{F}}_l)_p$ are cofibrant commutative S-algebras, we get a weak equivalence $\mathrm{K}(\bar{\mathbb{F}}_{l'})_p \to \mathrm{K}(\bar{\mathbb{F}}_l)_p$. The morphism $j : \ell_p \to \mathrm{K}(k)_p$, given by the inclusion of fields, induces the identification $\pi_*(\ell_p) = \mathbb{Z}_p[u^{p-1}]$ as a subring of $\pi_*(\mathrm{K}(k)_p)$. We have $V(0)_*\ell_p = P(u^{p-1})$ and $V(1)_*\ell_p = \mathbb{F}_p$ as rings. Using the Tor and Ext spectral sequence one concludes that $V(1) \wedge^L_S \ell_p \cong H\mathbb{F}_p$ as S-ring spectra. The maps

$$S^{2i}_S \wedge^L_S \ell_p \xrightarrow{u^i \wedge j} \mathrm{K}(k)_p \wedge^L_S \mathrm{K}(k)_p \longrightarrow \mathrm{K}(k)_p$$

for $i = 0, \dots p-2$ induce an isomorphism in $\mathscr{D}_S$:

$$\bigvee_{i=0}^{p-2} S^{2i}_S \wedge^L_S \ell_p \cong \mathrm{K}(\bar{\mathbb{F}}_l)_p.$$

We get an isomorphism $\bigvee_{i=1}^{p-1} S_S^{2i} \wedge_S^L \ell_p \to S_S^2 \wedge_S^L \mathrm{K}(\bar{\mathbb{F}}_l)_p$ in $\mathscr{D}_S$ which we denote by κ. We claim that the following diagram commutes in $\mathscr{D}_S$:

$$\begin{tikzcd}
\mathrm{K}(\bar{\mathbb{F}}_l)_p \arrow[rr, "f"] & & S_S^2 \wedge_S^L \mathrm{K}(\bar{\mathbb{F}}_l)_p \\
\bigvee_{i=0}^{p-2} S_S^{2i} \wedge_S^L \ell_p \arrow[u, "\cong"] \arrow[r, "\vee f_i"] & (S_S^{2p-2} \wedge_S^L \ell_p) \vee \bigvee_{i=1}^{p-2} S_S^{2i} \wedge_S^L \ell_p. \arrow[ur, phantom] & \arrow[u, "\cong", "\kappa"']
\end{tikzcd}$$

Here, denoting by k_i the inclusion $S_S^{2i}\wedge_S^L\ell_p \to \mathrm{K}(\bar{\mathbb{F}}_l)_p$ and by p_i the projection $\mathrm{K}(\bar{\mathbb{F}}_l)_p \to S_S^{2i}\wedge_S^L\ell_p$, $f_i := p_i \circ (\phi^q - \mathrm{id}) \circ k_i$ for $i = 1, \dots, p-2$ and f_0 is defined to be the unique map $\ell_p \to S_S^{2p-2} \wedge_S^L \ell_p$ that is mapped to $p_0 \circ (\phi^q - \mathrm{id}) \circ k_0$ after composing with

$$\bar{v} : S_S^{2p-2} \wedge_S^L \ell_p \xrightarrow{u^{p-1}\wedge \mathrm{id}} \ell_p \wedge_S^L \ell_p \longrightarrow \ell_p.$$

It suffices to show commutativity after composing with

$$\bar{u} : S_S^2 \wedge_S^L \mathrm{K}(\bar{\mathbb{F}}_l)_p \xrightarrow{u\wedge \mathrm{id}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge_S^L \mathrm{K}(\bar{\mathbb{F}}_l)_p \to \mathrm{K}(\bar{\mathbb{F}}_l)_p.$$

Commutativity then follows because $\bar{u}$ identifies under κ with the map that is $k_0 \circ \bar{v}$ on the wedge summand $S_S^{2p-2} \wedge_S^L \ell_p$ and k_i on the other wedge summands, and because by [1, Corollary 6.4.8] two self-maps of p-completed connective complex K-theory in the stable homotopy category are equal if and only if they induce the same maps on homotopy groups. Let K_i be the fiber of f_i. We get a morphism of distinguished triangles in $\mathscr{D}_S$

$$\begin{tikzcd}
\mathrm{K} \arrow[r] & \mathrm{K}(\bar{\mathbb{F}}_l)_p \arrow[r, "f"] & S_S^2 \wedge_S^L \mathrm{K}(\bar{\mathbb{F}}_l)_p \arrow[r] & \Sigma \mathrm{K} \\
\bigvee_{i=0}^{p-2} \mathrm{K}_i \arrow[r] \arrow[u, dashed] & \bigvee_{i=0}^{p-2} S^{2i} \wedge_S^L \ell_p \arrow[r, "\vee f_i"] \arrow[u, "\cong"] & \bigvee_{i=1}^{p-1} S_S^{2i} \wedge_S^L \ell_p \arrow[r] \arrow[u, "\cong", "\kappa"'] & \Sigma(\bigvee_{i=0}^{p-2} K_i), \arrow[u, dashed]
\end{tikzcd}$$

which is an isomorphism by the five lemma.

Recall that the dual Steenrod algebra $A_* := (H\mathbb{F}_p)_* H\mathbb{F}_p$ is an $\mathbb{F}_p$-Hopf algebra and that for $X \in \mathscr{D}_S$ the mod p homology $(H\mathbb{F}_p)_*X$ has a natural left A_*-comodule structure [9, Theorem 1.1]. We use the letter ν to denote the A_*-coactions. For $X, Y \in \mathscr{D}_S$ the canonical map

$$(H\mathbb{F}_p)_*X \otimes (H\mathbb{F}_p)_*Y \longrightarrow (H\mathbb{F}_p)_*(X \wedge_S^L Y) \tag{7}$$

is an isomorphism of comodules [38, Theorem 17.8.vii]. We get that $(H\mathbb{F}_p)_*X$ is an A_*-comodule algebra if X is an associative S-ring spectrum. If X and Y are associative S-ring spectra then (7) is an isomorphism of comodule algebras. Recall that

$$A_* = P(\xi_1, \xi_2, \dots) \otimes E(\tau_0, \tau_1, \dots) = P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_0, \bar{\tau}_1, \dots)$$

with $|\xi_n| = |\bar{\xi}_n| = 2p^n - 2$ and $|\tau_n| = |\bar{\xi}_n| = 2p^n - 1$ [5, Section 5.1]. Here, ξ_n and τ_n are the generators defined in [30], and $\bar{\xi}_n$ and $\bar{\tau}_n$ are their images under the antipode of A_*. We have

$$\nu(\bar{\xi}_n) = \sum_{i+j=n} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i}, \quad \nu(\bar{\tau}_i) = 1 \otimes \bar{\tau}_n + \sum_{i+j=n} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i},$$

where by convention $\bar{\xi}_0 = 1$. Since the coaction of A_* is the comultiplication, A_* has no non-trivial comodule primitives in positive degrees. Since ℓ_p is a connective, cofibrant commutative S-algebra, we have a map $\ell_p \to H\mathbb{Z}_p$ in $\mathscr{C}\!\mathscr{A}_S$ that is the identity on π_0 [18, Proposition IV.3.1]. The morphisms $\ell_p \to H\mathbb{Z}_p \to H\mathbb{F}_p$ induce the maps

$$P(\bar{\xi}_1, \dots) \otimes E(\bar{\tau}_2, \dots) \subset P(\bar{\xi}_1, \dots) \otimes E(\bar{\tau}_1, \dots) \subset A_*$$

in mod p homology [5, Proposition 5.3]. Let $u \in (H\mathbb{F}_p)_2 \mathrm{K}(\bar{\mathbb{F}}_l)_p$ be the image of $u \in \pi_2(\mathrm{K}(\bar{\mathbb{F}}_l)_p)$ under the Hurewicz map $\pi_*(\mathrm{K}(\bar{\mathbb{F}}_l)_p) \to \pi_*(S \wedge_S^L \mathrm{K}(\bar{\mathbb{F}}_l)_p) \to \pi_*(H\mathbb{F}_p \wedge_S^L \mathrm{K}(\bar{\mathbb{F}}_l)_p)$. Then, j induces an isomorphism

$$P_{p-1}(u) \otimes (H\mathbb{F}_p)_*\ell_p \cong (H\mathbb{F}_p)_* \mathrm{K}(\bar{\mathbb{F}}_l)_p$$

(see [6, Theorem 2.5]).

In the following we compute the mod p homology of K by computing the mod p homology of the K_i separately.

Lemma 5.1. *For $i \in \{0, \dots, p-2\}$ with $r \nmid i$ we have $(H\mathbb{F}_p)_* \mathrm{K}_i = 0$.*

Proof. For $j \geq 0$ the map

$$\pi_{2i+(2p-2)j}(f_i) : \pi_{2i+(2p-2)j}(S_S^{2i} \wedge_S^L \ell_p) \longrightarrow \pi_{2i+(2p-2)j}(S_S^{2i} \wedge_S^L \ell_p)$$

is the multiplication with $q^{i+j(p-1)} - 1$ on $\mathbb{Z}_p$. We have $r \nmid i + j(p-1)$ and thus

$$q^{i+k(p-1)} - 1 \in \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_p/p\mathbb{Z}_p$$

is not zero. Therefore, $q^{i+j(p-1)} - 1$ is a unit in $\mathbb{Z}_p$. We get that $\pi_*(f_i)$ is an isomorphism and that $\mathrm{K}_i \cong *$ in $\mathscr{D}_S$. □

Recall that we defined $k = \frac{p-1}{r}$.

Lemma 5.2. *For $0 < j < k$ we have a short exact sequence*

$$0 \longrightarrow \Sigma^{2jr-1}(H\mathbb{F}_p)_*\ell_p \xrightarrow{\Delta_j} (H\mathbb{F}_p)_* \mathrm{K}_{jr} \xrightarrow{i_j} \Sigma^{2rj}(H\mathbb{F}_p)_*\ell_p \longrightarrow 0.$$

Proof. It suffices to show that $(H\mathbb{F}_p)_*(f_{jr})$ is zero. From the homotopy of $S_S^{2jr} \wedge_S^L \ell_p$ we deduce that

$$\pi_*(\mathrm{K}_{jr}) = \begin{cases} 0, & \text{for } * < 2jr - 1; \\ \mathbb{Z}_p/(q^{rj} - 1) = \mathbb{Z}/p^{v_p(q^{rj}-1)}, & \text{for } * = 2jr - 1. \end{cases}$$

Using that $v_p(q^{rj} - 1) \geq 1$ and the Tor spectral sequence we get that $(H\mathbb{F}_p)_{2rj-1} \mathrm{K}_{jr} = \mathbb{F}_p$. It follows that $(H\mathbb{F}_p)_*(f_{jr})$ is zero in degree $2jr$. We show by induction that $(H\mathbb{F}_p)_n(f_{jr}) = 0$ for all n. Let $n > 2jr$ and suppose that we already know that the claim is true for all $m < n$. Let $b \in (H\mathbb{F}_p)_n(S_S^{2jr} \wedge_S^L \ell_p)$. Using the induction hypothesis one sees that $(H\mathbb{F}_p)_*(f_{jr})(b)$ is an A_*-comodule primitive. Since it has degree $> 2jr$, it has to be zero. □

Lemma 5.3. *Let $m \geq 1$. Then, we have a short exact sequence*

$$0 \longrightarrow (H\mathbb{F}_p)_*H\mathbb{Z} \longrightarrow (H\mathbb{F}_p)_*H(\mathbb{Z}/p^m\mathbb{Z}) \longrightarrow \Sigma(H\mathbb{F}_p)_*H\mathbb{Z} \longrightarrow 0.$$

For $m \geq 2$ the unique class $b \in (H\mathbb{F}_p)_1 H(\mathbb{Z}/p^m\mathbb{Z})$ that is mapped to $\Sigma 1$ is an A_*-comodule primitive.

Proof. Using that $\mathscr{D}_S(X, Y) \cong \mathrm{Hom}_{\mathbb{Z}}(\pi_0(X), \pi_0(Y))$ for S-modules X and Y whose homotopy is concentrated in degree zero, we get a map of distinguished triangles

$$\begin{array}{ccccccc} H\mathbb{Z} & \xrightarrow{p^m} & H\mathbb{Z} & \longrightarrow & H(\mathbb{Z}/p^m\mathbb{Z}) & \longrightarrow & \Sigma H\mathbb{Z} \\ \downarrow{\scriptstyle p^{m-1}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \Sigma p^{m-1}} \\ H\mathbb{Z} & \xrightarrow{p} & H\mathbb{Z} & \longrightarrow & H(\mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \Sigma H\mathbb{Z}, \end{array}$$

where $p^n := p^n\mathrm{id}$. Thus, after applying $(H\mathbb{F}_p)_*(-)$, we get a map of long exact sequences. Since $H\mathbb{F}_p \wedge_S^L (-)$ is additive, this shows the first part of the statement. Now, let $m \geq 2$. We have that

$$\mathbb{F}_p \cong (H\mathbb{F}_p)_0 H(\mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{g_0} (H\mathbb{F}_p)_0 H(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{F}_p$$

is an isomorphism and that

$$\mathbb{F}_p \cong (H\mathbb{F}_p)_1 H(\mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{g_1} (H\mathbb{F}_p)_1 H(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{F}_p$$

is zero. Let b be the generator of $(H\mathbb{F}_p)_1 H(\mathbb{Z}/p^m\mathbb{F}_p)$ that is mapped to 1 under

$$(H\mathbb{F}_p)_1 H(\mathbb{Z}/p^m\mathbb{F}_p) \longrightarrow (H\mathbb{F}_p)_0 H\mathbb{Z}.$$

We can write the coaction of b as $\nu(b) = 1 \otimes b + \bar{\tau}_0 \otimes a$ for an element $a \in (H\mathbb{F}_p)_0 H(\mathbb{Z}/p^m\mathbb{F}_p)$. Because of $g_1(b) = 0$ we have $\bar{\tau}_0 \otimes g_0(a) = 0$ and therefore $a = 0$. $\square$

Lemma 5.4. *We have*

$$\dim_{\mathbb{F}_p}(H\mathbb{F}_p)_{2p-2}\,\mathrm{K}_0 = \begin{cases} 0, & \text{if } v = 1; \\ 1, & \text{if } v \geq 2. \end{cases}$$

Proof. We have an exact sequence

$$\begin{tikzcd} (H\mathbb{F}_p)_{2p-1}(S_S^{2p-2} \wedge_S^L \ell_p) \arrow[r] \arrow[d, equal] & (H\mathbb{F}_p)_{2p-2}\,\mathrm{K}_0 \arrow[r] & (H\mathbb{F}_p)_{2p-2}\ell_p. \arrow[d, equal] \\ 0 & & \mathbb{F}_p\{\bar{\xi}_1\} \end{tikzcd}$$

Therefore, we have $\dim_{\mathbb{F}_p}(H\mathbb{F}_p)_{2p-2}\,\mathrm{K}_0 \leq 1$. Recall that for an $(n-1)$-connected S-module X one has a map $X \to \Sigma^n H\pi_n(X)$ realizing the identity on π_n [36, Theorem II.4.13]. Using this we can inductively construct a Whitehead tower in $\mathscr{D}_S$:

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{K}_0[3] \arrow[r] \arrow[d] & \mathrm{K}_0[2] \arrow[r] \arrow[d] & \mathrm{K}_0[1] \arrow[r] \arrow[d] & \mathrm{K}_0[0] = \mathrm{K}_0 \arrow[d] \\ & \Sigma^3 H\pi_3(\mathrm{K}_0) & \Sigma^2 H\pi_2(\mathrm{K}_0) & \Sigma H\pi_1(\mathrm{K}_0) & H\pi_0(\mathrm{K}_0). \end{tikzcd}$$

Here, the sequences $\mathrm{K}_0[i+1] \to \mathrm{K}_0[i] \to \Sigma^i H\pi_i(\mathrm{K}_0)$ are part of distinguished triangles

$$\mathrm{K}_0[i+1] \to \mathrm{K}_0[i] \to \Sigma^i H\pi_i(\mathrm{K}_0) \to \Sigma\,\mathrm{K}_0[i+1].$$

Applying $(H\mathbb{F}_p)_*(-)$ we get an unrolled exact couple and therefore a spectral sequence $(E^*_{*,*}, d^*)$. Let m_n be the p-adic valuation of $q^{n(p-1)} - 1$. Then, the E^1-page of the spectral sequence is $(H\mathbb{F}_p)_* H\mathbb{Z}_p$ in column 0, $\Sigma^{2(n(2p-2)-1)}(H\mathbb{F}_p)_* H\mathbb{Z}/p^{m_n}$ in column $-(n(2p-2)-1)$ for $n > 0$ and zero in all other columns. We claim that the spectral sequence converges strongly to $(H\mathbb{F}_p)_*\,\mathrm{K}_0$. Since $\mathrm{K}_0[i]$ is $(i-1)$-connected, we have $(H\mathbb{F}_p)_*\,\mathrm{K}_0[i] = 0$ for $* < i$. This implies that the spectral sequence converges conditionally to $(H\mathbb{F}_p)_*\,\mathrm{K}_0$ (see [12, Definition 5.10]). Because $E^1_{*,*}$ is finite in every bidegree, the spectral sequence converges strongly by [12, Theorem 7.1].

Since $(H\mathbb{F}_p)_*(-) \to (H\mathbb{F}_p)_*(- \wedge S^1)$ is compatible with the comodule action, the spectral sequence is a spectral sequence of A_*-comodules.

It is clear that $d^i = 0$ for $i = 1, \dots, 2p-4$. Let $b \in (H\mathbb{F}_p)_1 H(\mathbb{Z}/p^{m_1}\mathbb{Z}) \cong \mathbb{F}_p$ be a non-trivial class. In total degree $2p-2$ the E^{2p-3}-page is a 2-dimensional $\mathbb{F}_p$-vector space generated by $\bar{\xi}_1$ in column 0 and by $\Sigma^{4p-6}b$ in column $-(2p-3)$. The class $\bar{\xi}_1$ survives to the E^∞-page if and only if $d^{2p-3}(\bar{\xi}_1) = 0$. The class $\Sigma^{4p-6}b$ survives to the E^∞-page if and only if for the class $\bar{\tau}_1$ in column zero the equality $d^{2p-3}(\bar{\tau}_1) = 0$ holds. We can write $d^{2p-3}(\bar{\tau}_1) = \lambda \cdot \Sigma^{4p-6}b$ for an element $\lambda \in \mathbb{F}_p$. Let ν denote the coaction of the E^{2p-3}-page. We have

$$\lambda \cdot \nu(\Sigma^{4p-6}b) = 1 \otimes d^{2p-3}(\bar{\tau}_1) + \bar{\tau}_0 \otimes d^{2p-3}(\bar{\xi}_1) + \bar{\tau}_1 \otimes \underbrace{d^{2p-3}(1)}_{=0}. \tag{8}$$

Suppose that $v \geq 2$. Then, we have $m_1 \geq 2$ and $b \in (H\mathbb{F}_p)_1 H(\mathbb{Z}/p^{m_1}\mathbb{Z})$ is an A_*-comodule primitive by Lemma 5.3. It follows that $d^{2p-3}(\bar{\xi}_1) = 0$ and that $\dim_{\mathbb{F}_p}(H\mathbb{F}_p)_{2p-2}\,\mathrm{K}_0 = 1$.

Now, suppose $v = 1$. Then, b is not primitive. If $d^{2p-3}(\bar{\tau}_1)$ was zero, i.e. $\lambda = 0$, the equation (8) would imply that $d^{2p-3}(\bar{\xi}_1) = 0$. One would get $\dim_{\mathbb{F}_p}(H\mathbb{F}_p)_{2p-2}\,\mathrm{K}_0 = 2$, which is a contradiction. Therefore, we have $d^{2p-3}(\bar{\tau}_1) \neq 0$. With equation (8) we get $d^{2p-3}(\bar{\xi}_1) \neq 0$. We conclude that $\dim_{\mathbb{F}_p}(H\mathbb{F}_p)_{2p-2}\,\mathrm{K}_0 = 0$. $\square$

Lemma 5.5. *For $v \geq 2$ we have an exact sequence*

$$0 \longrightarrow \Sigma^{2p-3}(H\mathbb{F}_p)_*\ell_p \xrightarrow{\Delta_0} (H\mathbb{F}_p)_*\,\mathrm{K}_0 \xrightarrow{i_0} (H\mathbb{F}_p)_*\ell_p \longrightarrow 0.$$

Proof. By Lemma 5.4 we have an exact sequence

$$\begin{tikzcd}
(H\mathbb{F}_p)_1(\ell_p) \arrow[r] \arrow[d, equal] & (H\mathbb{F}_p)_{2p-2}\,\mathrm{K}_0 \arrow[r] \arrow[d, equal] & (H\mathbb{F}_p)_{2p-2}\ell_p \arrow[r, "(f_0)_{2p-2}"] \arrow[d, equal] & (H\mathbb{F}_p)_0(\ell_p) \arrow[d, equal] \\
0 \arrow[r] & \mathbb{F}_p \arrow[r] & \mathbb{F}_p\{\bar{\xi}_1\} \arrow[r] & \mathbb{F}_p\{1\}.
\end{tikzcd}$$

We get that $(f_0)_{2p-2}$ is zero. As in Lemma 5.2 it follows by induction that $(f_0)_n = 0$ for all n. $\square$

For $r = p-1$ the following result is in [5, p.1265].

Lemma 5.6. *If $v = 1$ we have an exact sequence*

$$0 \longrightarrow \Sigma^{2p-3}\bar{\xi}_1^{p-1}C_* \xrightarrow{\Delta_0} (H\mathbb{F}_p)_*\,\mathrm{K}_0 \xrightarrow{i_0} C_* \longrightarrow 0,$$

where C_* is given by $P(\bar{\xi}_1^p, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_2, \dots) \subset (H\mathbb{F}_p)_*\ell_p$.

Proof. Because of $(H\mathbb{F}_p)_{2p-2}\,\mathrm{K}_0 = 0$ the map

$$(H\mathbb{F}_p)_{2p-2}\ell_p \xrightarrow{(f_0)_{2p-2}} (H\mathbb{F}_p)_{2p-2}(S_S^{2p-2} \wedge_S^L \ell_p)$$

is an isomorphism. Let λ be the unit in $\mathbb{F}_p$ such that $(f_0)_{2p-2}(\bar{\xi}_1) = \Sigma^{2p-2}(\lambda \cdot 1)$ holds. We claim that the diagram

(9)
$$\begin{tikzcd}
(H\mathbb{F}_p)_*\ell_p \arrow[r, "(f_0)_*"] \arrow[d, hook] & \Sigma^{2p-2}(H\mathbb{F}_p)_*\ell_p \arrow[d, hook] \\
A_* \arrow[r, "\varphi"] & \Sigma^{2p-2}A_*
\end{tikzcd}$$

is commutative, where φ is defined by

$$A_* \xrightarrow{v} A_* \otimes A_* \twoheadrightarrow A_* \otimes \mathbb{F}_p\{\xi_1\} \xrightarrow{\mathrm{id}\otimes(\lambda\cdot\mathrm{id})} A_* \otimes \mathbb{F}_p\{\bar{\xi}_1\} \xrightarrow{\cong} \Sigma^{2p-2}A_*.$$

It is clear that we have commutativity in degrees $\leq 2p-2$. For $V = A_*$ and $V = \mathbb{F}_p\{\bar{\xi}_1\}$ we equip $A_* \otimes V$ with the A_*-coaction given by the coaction of A_*. Then, all the maps in (9) are maps of A_*-comodules. Since the difference of two maps of comodules is a map of comodules, it follows as in Lemma 5.2 that (9) is commutative. We have

$$\varphi(\bar{\xi}_1^{n_1}\bar{\xi}_2^{n_2}\bar{\xi}_3^{n_3}\dots\bar{\tau}_2^{\epsilon_2}\bar{\tau}_3^{\epsilon_3}\dots) = \Sigma^{2p-2}\lambda n_1\bar{\xi}_1^{n_1-1}\bar{\xi}_2^{n_2}\bar{\xi}_3^{n_3}\dots\bar{\tau}_2^{\epsilon_2}\bar{\tau}_3^{\epsilon_3}\dots,$$

where the expression on the right means zero if $n_1 = 0$. It follows that

$$\ker(f_0)_* = P(\bar{\xi}_1^p, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_2, \dots)$$

and

$$\operatorname{coker}(f_0)_* = \Sigma^{2p-2}\bar{\xi}_1^{p-1}P(\bar{\xi}_1^p, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_2, \dots).$$ $\square$

We now consider case (1). In this case q is a generator of $(\mathbb{Z}/p^2)^*$ and we can take $q' = q$ in the definition of ℓ_p. The map $\mathrm{K} \to \mathrm{K}(\bar{\mathbb{F}}_l)_p$ factors through ℓ_p and the diagram

$$\begin{tikzcd}
\mathrm{K}_0 \arrow[r] \arrow[d] & \ell_p \arrow[d, equal] \\
\mathrm{K} \arrow[r] & \ell_p
\end{tikzcd}$$

is commutative in $\mathscr{D}_S$. Furthermore, the left vertical map in this diagram is an isomorphism by the proof of Lemma 5.1. We define $b \in (H\mathbb{F}_p)_{p(2p-2)-1}\,\mathrm{K}$ to be the image of $\Delta_0(\Sigma^{2p-3}\bar{\xi}_1^{p-1})$ under $(H\mathbb{F}_p)_*\,\mathrm{K}_0 \to (H\mathbb{F}_p)_*\,\mathrm{K}$. By Lemma 5.6 there are unique classes $\tilde{\xi}_1^p \in (H\mathbb{F}_p)_{p(2p-2)}\,\mathrm{K}$ and $\tilde{\tau}_2 \in (H\mathbb{F}_p)_{2p^2-1}\,\mathrm{K}$ that map to $\bar{\xi}_1^p$ and $\bar{\tau}_2$ under

$$(H\mathbb{F}_p)_*\,\mathrm{K} \to (H\mathbb{F}_p)_*\ell_p.$$

Recall from [16, Theorem III.1.1] that the mod p homology of a commutative S-algebra R admits natural Dyer-Lashof operations

$$Q^k : (H\mathbb{F}_p)_*R \to (H\mathbb{F}_p)_{*+k(2p-2)}R.$$

For $i \geq 2$ we recursively define

$$\tilde{\tau}_{i+1} := Q^{p^i}(\tilde{\tau}_i) \in (H\mathbb{F}_p)_{2p^{i+1}-1}\,\mathrm{K}$$

Furthermore, we set

$$\tilde{\xi}_i := \beta(\tilde{\tau}_i) \in (H\mathbb{F}_p)_{2p^i-2}\,\mathrm{K}$$

for $i \geq 2$, where β is the mod p homology Bockstein homomorphism. We get by [5, Proposition 7.12]:

Proposition 5.7 (Angeltveit, Rognes). *In case (1) the $\mathbb{F}_p$-algebra map*

$$E(b) \otimes P(\tilde{\xi}_1^p, \tilde{\xi}_2, \dots) \otimes E(\tilde{\tau}_2, \dots) \to (H\mathbb{F}_p)_*\,\mathrm{K}$$

is an isomorphism. The class b is an A_*-comodule primitive and we have

$$\nu(\tilde{\xi}_2) = 1 \otimes \tilde{\xi}_2 + \bar{\xi}_1 \otimes \tilde{\xi}_1^p + \tau_1 \otimes b + \bar{\xi}_2 \otimes 1.$$

The map $(H\mathbb{F}_p)_*\,\mathrm{K} \to (H\mathbb{F}_p)_*\ell_p$ maps $\tilde{\xi}_1^p$, $\tilde{\xi}_i$, $\tilde{\tau}_i$ and b to $\bar{\xi}_1^p$, $\bar{\xi}_i$, $\bar{\tau}_i$ and zero, respectively.

In the following lemma we use the K-linearity of the distinguished triangle (6) to determine the multiplicative structure of $(H\mathbb{F}_p)_*\,\mathrm{K}$ in case (2) and (3). One could also use an argument similar to the one that Angeltveit and Rognes use in case (1).

Proposition 5.8. *In cases (2) and (3) there is an isomorphism of $\mathbb{F}_p$-algebras*

$$(H\mathbb{F}_p)_*\,\mathrm{K} \cong E(x) \otimes P_k(y) \otimes P(\tilde{\xi}_1, \tilde{\xi}_2, \dots) \otimes E(\tilde{\tau}_2, \tilde{\tau}_3, \dots)$$

for certain classes x, y, $\tilde{\xi}_i$ and $\tilde{\tau}_i$ with the following properties:

- *The degrees are $|x| = 2r-1$, $|y| = 2r$, $|\tilde{\xi}_i| = 2p^i - 2$ and $|\tilde{\tau}_i| = 2p^i - 1$.*
- *The map $i_* : (H\mathbb{F}_p)_*\,\mathrm{K} \to (H\mathbb{F}_p)_*\,\mathrm{K}(\bar{\mathbb{F}}_l)_p$ maps x to zero, y to u^r, $\tilde{\xi}_i$ to $\bar{\xi}_i$ and $\tilde{\tau}_i$ to $\bar{\tau}_i$.*
- *For $i \geq 2$ we have $Q^{p^i}(\tilde{\tau}_i) = \tilde{\tau}_{i+1}$ and $\beta(\tilde{\tau}_i) = \tilde{\xi}_i$.*
- *For the A_*-coaction ν on $(H\mathbb{F}_p)_*\,\mathrm{K}$ we have:*

$$\begin{aligned}
\nu(\tilde{\xi}_1) &= 1 \otimes \tilde{\xi}_1 + \bar{\xi}_1 \otimes 1 + a\bar{\tau}_0 \otimes xy^{k-1} \text{ for an } a \in \mathbb{F}_p,\\
\nu(\tilde{\xi}_n) &= \sum_{i+j=n} \bar{\xi}_i \otimes \tilde{\xi}_j^{p^i},\\
\nu(\tilde{\tau}_n) &= 1 \otimes \tilde{\tau}_n + \sum_{i+j=n} \bar{\tau}_i \otimes \tilde{\xi}_j^{p^i}.
\end{aligned}$$

The classes x and y are comodule primitives.

Note that we have $k = 1$ in case (2), so that $P_k(y) = \mathbb{F}_p$.

Proof. We have a commutative diagram with exact rows:

$$\begin{tikzcd}
 & \Sigma(H\mathbb{F}_p)_*\,\mathrm{K}(\bar{\mathbb{F}}_l)_p \arrow[r, "\Delta"] & (H\mathbb{F}_p)_*\,\mathrm{K} \arrow[r, "i_*"] & (H\mathbb{F}_p)_*\,\mathrm{K}(\bar{\mathbb{F}}_l)_p & \\
0 \arrow[r] & \bigoplus_{j=1}^{k} \Sigma^{2jr-1}(H\mathbb{F}_p)_*\ell_p \arrow[r, "\oplus \Delta_j"] \arrow[u, hook] & \bigoplus_{j=0}^{k-1}(H\mathbb{F}_p)_*\,\mathrm{K}_{jr} \arrow[r, "\oplus i_j"] \arrow[u, hook, "\cong"] & \bigoplus_{j=0}^{k-1} \Sigma^{2jr}(H\mathbb{F}_p)_*\ell_p \arrow[r] \arrow[u, hook] & 0.
\end{tikzcd}$$

The vertical maps are injections. We treat them as inclusions.

We set $x := \Delta(\Sigma^{2r-1}1)$. We have that $\Sigma(H\mathbb{F}_p)_*\,\mathrm{K}(\bar{\mathbb{F}}_l)_p$ is zero in degree $2r$. Therefore, for $k > 1$ there is a unique class $y \in (H\mathbb{F}_p)_{2r}\,\mathrm{K}$ such that $i_*(y) = u^r = \Sigma^{2r}1$. For $k = 1$ we set $y = 0$. Since $\Sigma(H\mathbb{F}_p)_*\,\mathrm{K}(\bar{\mathbb{F}}_l)_p$ is zero in degree $2p-2$, there is a unique class $\tilde{\xi}_1 \in (H\mathbb{F}_p)_{2p-2}\,\mathrm{K}$

with $i_*(\tilde{\xi}_1) = \bar{\xi}_1 = \Sigma^0\bar{\xi}_1$. The vector space $\Sigma^{2p-3}(H\mathbb{F}_p)_*\ell_p$ is zero in degree $2p^2-1$. Thus, there is a unique class $\tilde{\tau}_2 \in (H\mathbb{F}_p)_{2p^2-1}\,\mathrm{K}_0$ with $i_0(\tilde{\tau}_2) = \Sigma^0\bar{\tau}_2$. For $i \geq 2$ we define recursively

$$\tilde{\tau}_{i+1} := Q^{p^i}(\tilde{\tau}_i) \in (H\mathbb{F}_p)_*\,\mathrm{K}\,.$$

Furthermore, for $i \geq 2$ we define $\tilde{\xi}_i := \beta(\tilde{\tau}_i)$. Since in the dual Steenrod algebra the analogous equations hold [5, pp.1244–1245], we get that $i_*(\tilde{\tau}_i) = \bar{\tau}_i$ and $i_*(\tilde{\xi}_i) = \bar{\xi}_i$.

We have $i_*(y^k) = u^{p-1} = 0$. Since $\Sigma(H\mathbb{F}_p)_*\,\mathrm{K}(\bar{\mathbb{F}}_l)_p$ is zero in degree $2p-2$ it follows that $y^k = 0$. Thus, we get a map of graded commutative $\mathbb{F}_p$-algebras

$$h : E(x) \otimes P_k(y) \otimes P(\tilde{\xi}_1, \dots) \otimes E(\tilde{\tau}_2, \dots) \longrightarrow (H\mathbb{F}_p)_*\,\mathrm{K}\,.$$

We claim that h is an isomorphism. Given numbers $j \in \{0, \dots, k-1\}$, $n_i \in \mathbb{N}$ for $i \geq 1$ and $\epsilon_i \in \{0,1\}$ for $i \geq 2$ that are almost all equal to zero, we have

$$\begin{aligned} i_*(h(y^j\tilde{\xi}_1^{n_1}\tilde{\xi}_1^{n_2}\dots\tilde{\tau}_2^{\epsilon_2}\tilde{\tau}_3^{\epsilon_3}\dots)) &= u^{rj}\bar{\xi}_1^{n_1}\bar{\xi}_1^{n_2}\dots\bar{\tau}_2^{\epsilon_2}\bar{\tau}_3^{\epsilon_3}\dots \\ &= \Sigma^{2rj}\bar{\xi}_1^{n_1}\bar{\xi}_1^{n_2}\dots\bar{\tau}_2^{\epsilon_2}\bar{\tau}_3^{\epsilon_3}\dots \end{aligned}$$

and

$$\begin{aligned} h(xy^j\tilde{\xi}_1^{n_1}\tilde{\xi}_1^{n_2}\dots\tilde{\tau}_2^{\epsilon_2}\tilde{\tau}_3^{\epsilon_3}\dots) &= \Delta(\Sigma u^{r-1})y^j\tilde{\xi}_1^{n_1}\tilde{\xi}_1^{n_2}\dots\tilde{\tau}_2^{\epsilon_2}\tilde{\tau}_3^{\epsilon_3}\dots \\ &= \pm\Delta(\Sigma u^{r(j+1)-1}\bar{\xi}_1^{n_1}\bar{\xi}_2^{n_2}\dots\bar{\tau}_2^{\epsilon_2}\bar{\tau}_3^{\epsilon_3}\dots) \\ &= \pm\Delta(\Sigma^{2r(j+1)-1}\bar{\xi}_1^{n_1}\bar{\xi}_2^{n_2}\dots\bar{\tau}_2^{\epsilon_2}\bar{\tau}_3^{\epsilon_3}\dots). \end{aligned}$$

Here, the second equality uses that Δ is $(H\mathbb{F}_p)_*\,\mathrm{K}$-linear. Thus, h maps the canonical basis to a basis and is therefore an isomorphism.

It remains to study the comodule structure. We first recall some facts about the Steenrod algebra A^* (see [30]): A basis of A^* is given by

$$\beta^{\epsilon_0}\mathcal{P}^{s_1}\beta^{\epsilon_1}\dots\mathcal{P}^{s_k}\beta^{\epsilon_k},$$

where $\epsilon_i \in \{0,1\}$ and $s_1 \geq ps_2 + \epsilon_1, s_2 \geq ps_3 + \epsilon_2, \dots, s_{k-1} \geq ps_k + \epsilon_{k-1}, s_k \geq 1$. Here, $\mathcal{P}^i \in A^{2i(p-1)}$ denotes the Steenrod reduced pth power and $\beta \in A^1$ denotes the Bockstein. One has $\mathcal{P}^0 = 1$ and A^* is generated as an algebra by

$$\beta, \mathcal{P}^1, \mathcal{P}^p, \mathcal{P}^{p^2}, \dots.$$

We have a right action of the Steenrod algebra on the mod p homology of an S-module X (see [8, p.244]): For $a \in A^*$ and $z \in (H\mathbb{F}_p)_*X$ with coaction $\nu(z) = \sum \gamma_i \otimes z_i$ the element $z \cdot a = a_*(z)$ is defined by

$$z \cdot a = a_*(z) = (-1)^{|a||x|}\,\langle a|\gamma_i\rangle\, z_i.$$

Here, A^* is considered as the $\mathbb{F}_p$-linear dual of A_* and $\langle -,-\rangle : A^* \otimes A_* \to \mathbb{F}_p$ is the dual pairing. Note that because of $\mathscr{D}_S(H\mathbb{F}_p, \Sigma H\mathbb{F}_p) = A^1 \cong \mathbb{F}_p$, we have that β_* and the mod p homology Bockstein β agree degreewise up to a unit.

To determine the comodule structure of $(H\mathbb{F}_p)_*\,\mathrm{K}$ we will compute $a_*(z)$ for certain classes $z \in (H\mathbb{F}_p)_*\,\mathrm{K}$ and $a \in A^*$. We will use that β, Q^i and $\mathcal{P}^i_*$ are linked via the Nishida relations (see [37, Section 6]). This allows to prove the formulas for $\nu(\tilde{\xi}_i)$ and $\nu(\tilde{\tau}_i)$ by an inductive argument.

It is clear that x is a comodule primitive, because Δ is compatible with the comodule action. For $k = 1$ the class y is obviously primitive. For $k > 1$ we can write

$$\nu(y) = 1 \otimes y + \lambda\bar{\tau}_0 \otimes x$$

for an element $\lambda \in \mathbb{F}_p$. To show $\lambda = 0$, we prove $\beta(y) = 0$: The unit $S \to H\mathbb{Z}$ induces a map $V(0)_*\,\mathrm{K} \to (H\mathbb{F}_p)_*\,\mathrm{K}$ that commutes with the Bockstein. Because of $v \geq 2$ the proof of Lemma 3.7 shows that the Bockstein $V(0)_{2r}\,\mathrm{K} \to V(0)_{2r-1}\,\mathrm{K}$ maps y to zero. Thus, it suffices to prove that $V(0)_*\,\mathrm{K} \to (H\mathbb{F}_p)_*\,\mathrm{K}$ maps y to y. Since $(H\mathbb{F}_p)_*\,\mathrm{K} \to (H\mathbb{F}_p)_*\,\mathrm{K}(\bar{\mathbb{F}}_l)_p$ is injective in degree

$2r$, we only need to show that $V(0)_* \mathrm{K}(\bar{\mathbb{F}}_l)_p \to (H\mathbb{F}_p)_* \mathrm{K}(\bar{\mathbb{F}}_l)_p$ maps u^r to u^r. This follows from the commutativity of the diagram

$$\begin{tikzcd} \pi_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \arrow[r] \arrow[dr] & V(0)_* \mathrm{K}(\bar{\mathbb{F}}_l)_p \arrow[d] \\ & (H\mathbb{F}_p)_* \mathrm{K}(\bar{\mathbb{F}}_l)_p. \end{tikzcd}$$

For degree reasons and because of $i_*(\tilde{\xi}_1) = \bar{\xi}_1$ we have

$$\nu(\tilde{\xi}_1) = 1 \otimes \tilde{\xi}_1 + \bar{\xi}_1 \otimes 1 + a\bar{\tau}_0 \otimes xy^{k-1} \tag{10}$$

for an element $a \in \mathbb{F}_p$. Since $i_*(\tilde{\xi}_2) = \bar{\xi}_2$ and since $\ker i_* = \mathbb{F}_p\{x\} \otimes P_k(y) \otimes P(\tilde{\xi}_1, \dots) \otimes E(\tilde{\tau}_2, \dots)$, we get for degree reasons

$$\nu(\tilde{\xi}_2) = \bar{\xi}_2 \otimes 1 + 1 \otimes \tilde{\xi}_2 + \bar{\xi}_1 \otimes \tilde{\xi}_1^p + \sum_{i+j=p} a_{ij}\bar{\tau}_0\bar{\xi}_1^i \otimes \tilde{\xi}_1^j xy^{k-1} + \sum_{i+j=p-1} b_{ij}\bar{\tau}_1\bar{\xi}_1^i \otimes \tilde{\xi}_1^j xy^{k-1} \tag{11}$$

for certain $a_{ij}, b_{ij} \in \mathbb{F}_p$. The classes $\bar{\tau}_0, \bar{\tau}_0\bar{\xi}_1, \dots, \bar{\tau}_0\bar{\xi}_1^p$ lie in the degrees $(2p-2)i+1$ for $i = 0, \dots, p$. The classes $\bar{\tau}_1, \bar{\tau}_1\bar{\xi}_1, \dots, \bar{\tau}_1\bar{\xi}_1^{p-1}$ lie in the degrees $(2p-2)i+1$ for $i = 1, \dots, p$. A basis of the Steenrod algebra in these degrees is given by

$$\{\beta\mathcal{P}^i | 0 \le i \le p\} \cup \{\mathcal{P}^i\beta | 1 \le i \le p\}.$$

To prove $a_{ij} = b_{ij} = 0$ we show $\beta\mathcal{P}^i_*(\tilde{\xi}_2) = 0$ for $i = 1, \dots, p$ and $\mathcal{P}^i_*\beta(\tilde{\xi}_2) = 0$ for $i = 0, \dots, p$. Because of $\beta^2 = 0$ we have $\mathcal{P}^i_*(\beta(\tilde{\xi}_2)) = 0$. Since A_{2p-2} is one-dimensional, (11) implies that $\mathcal{P}^1_*(\tilde{\xi}_2) \doteq \tilde{\xi}_1^p$. By (10) we have

$$\nu(\tilde{\xi}_1^p) = 1 \otimes \tilde{\xi}_1^p + \bar{\xi}_1^p \otimes 1.$$

Thus, $\beta(\tilde{\xi}_1^p) = 0$ and therefore $\beta(\mathcal{P}^1_*(\tilde{\xi}_2)) = 0$. For $2 \le i \le p$ the element $\nu(\tilde{\xi}_2)$ lies in

$$\bigoplus_{n+j=2p^2-2, n \ne i(2p-2)} A_n \otimes (H\mathbb{F}_p)_j \mathrm{K}.$$

This implies that $\mathcal{P}^i_*(\tilde{\xi}_2) = 0$ and therefore that $\beta\mathcal{P}^i_*(\tilde{\xi}_2) = 0$. Because of $i_*(\tilde{\tau}_2) = \bar{\tau}_2$ we have

$$\nu(\tilde{\tau}_2) - \big(1 \otimes \tilde{\tau}_2 + \bar{\tau}_2 \otimes 1 + \bar{\tau}_1 \otimes \tilde{\xi}_1^p + \bar{\tau}_0 \otimes \tilde{\xi}_2\big) \in A_* \otimes \ker i_*.$$

Because of $\tilde{\tau}_2 \in (H\mathbb{F}_p)_* \mathrm{K}_0$ and $\beta(\tilde{\tau}_2) = \tilde{\xi}_2$ and since $(H\mathbb{F}_p)_* \mathrm{K}$ and $(H\mathbb{F}_p)_* \mathrm{K}_0$ are one-dimensional in the degrees 0 and $p(2p-2)$, this class also lies in $A_* \otimes (H\mathbb{F}_p)_* \mathrm{K}_0$. Using that

$$\ker i_* \cap (H\mathbb{F}_p)_* \mathrm{K}_0 = \mathbb{F}_p\{xy^{k-1}\} \otimes P(\tilde{\xi}_1, \dots) \otimes E(\tilde{\tau}_2, \dots)$$

one gets

$$\nu(\tilde{\tau}_2) = 1 \otimes \tilde{\tau}_2 + \bar{\tau}_2 \otimes 1 + \bar{\tau}_1 \otimes \tilde{\xi}_1^p + \bar{\tau}_0 \otimes \tilde{\xi}_2 + \sum_{i+j=p-1} a_{ij}\bar{\tau}_0\bar{\tau}_1\bar{\xi}_1^i \otimes xy^{k-1}\tilde{\xi}_1^j$$

for certain $a_{ij} \in \mathbb{F}_p$. The elements

$$\bar{\tau}_0\bar{\tau}_1, \bar{\tau}_0\bar{\tau}_1\bar{\xi}_1, \dots, \bar{\tau}_0\bar{\tau}_1\bar{\xi}_1^{p-1}$$

lie in the degrees $i(2p-2)+2$ for $i = 1, \dots, p$. A basis of the Steenrod algebra in these degrees is given by

$$\{\beta\mathcal{P}^i\beta | 1 \le i \le p\}.$$

Since $\beta(\mathcal{P}^i_*(\beta(\tilde{\tau}_2))) = \beta\mathcal{P}^i_*(\tilde{\xi}_2) = 0$ for $i = 1, \dots, p$ we get that $a_{ij} = 0$. Therefore, we have proven the formulas

$$\begin{aligned} \nu(\tilde{\xi}_n) &= \sum_{i+j=n} \bar{\xi}_i \otimes \tilde{\xi}_j^{p^i}, \\ \nu(\tilde{\tau}_n) &= 1 \otimes \tilde{\tau}_n + \sum_{i+j=n} \bar{\tau}_i \otimes \tilde{\xi}_j^{p^i} \end{aligned}$$

for $n = 2$. We suppose that $n \geq 2$ and that we have shown these formulas for n. We can write

$$\nu(\tilde{\xi}_{n+1}) = \sum_{i+j=n+1} \bar{\xi}_i \otimes \tilde{\xi}_j^{p^i} + c$$

for an element $c \in A_* \otimes \ker i_* = A_* \otimes \mathbb{F}_p\{x\} \otimes P_k(y) \otimes P(\tilde{\xi}_1, \dots) \otimes E(\tilde{\tau}_2, \dots)$. In order to show $c = 0$ it is enough to prove that

$$h_*(\tilde{\xi}_{n+1}) \in P(\tilde{\xi}_1, \dots) \otimes E(\tilde{\tau}_2, \dots)$$

for all $h \in A^*$. Since by the induction hypothesis

$$h_*(\tilde{\xi}_n^p) \in P(\tilde{\xi}_1, \dots) \otimes E(\tilde{\tau}_2, \dots)$$

for all $h \in A^*$, it suffices to show that $\beta(\tilde{\xi}_{n+1}) = 0$, $\mathcal{P}_*^1(\tilde{\xi}_{n+1}) \doteq \tilde{\xi}_n^p$ and $\mathcal{P}_*^{p^i}(\tilde{\xi}_{n+1}) = 0$ for $i \geq 1$. We have $\beta(\tilde{\xi}_{n+1}) = \beta(\beta(\tilde{\tau}_{n+1})) = 0$. By the Nishida relations [37, Section 6] we have

$$\begin{aligned} \mathcal{P}_*^{p^i}(\tilde{\xi}_{n+1}) &= \mathcal{P}_*^{p^i}(\beta(Q^{p^n}(\tilde{\tau}_n))) \\ &= \sum_j (-1)^{p^i+j} \binom{(p^n - p^i)(p-1) - 1}{p^i - jp} \beta(Q^{p^n - p^i + j}(\mathcal{P}_*^j(\tilde{\tau}_n))) \\ &+ \sum_j (-1)^{p^i+j} \binom{(p^n - p^i)(p-1) - 1}{p^i - jp - 1} Q^{p^n - p^i + j}(\mathcal{P}_*^j(\beta(\tilde{\tau}_n))). \end{aligned}$$

By the induction hypothesis we have $\mathcal{P}_*^j(\tilde{\tau}_n) = 0$ for $j > 0$. Hence, the first sum is equal to

$$-\binom{(p^n - p^i)(p-1) - 1}{p^i} \beta(Q^{p^n - p^i}(\tilde{\tau}_n)).$$

This is zero, because we have $2(p^n - p^i) < |\tilde{\tau}_n|$ which implies that $Q^{p^n - p^i}(\tilde{\tau}_n) = 0$ by [16, Theorem III.1.1]. The summand

$$(-1)^{p^i+j} \binom{(p^n - p^i)(p-1) - 1}{p^i - jp - 1} Q^{p^n - p^i + j}(\mathcal{P}_*^j(\beta(\tilde{\tau}_n)))$$

in the second sum is zero if $p^i - jp - 1 < 0$, because then the binomial coefficient is zero. If $p^i - jp - 1 > 0$ the summand is zero as well, because in this case $2(p^n - p^i + j) < |P_*^j(\beta(\tilde{\tau}_n))|$. The equality $p^i - jp - 1 = 0$ is only possible if $i = 0$ and $j = 0$. In this case the summand is equal to $-Q^{p^n - 1}(\tilde{\xi}_n)$. Because of $2(p^n - 1) = |\tilde{\xi}_n|$ this is equal to $-\tilde{\xi}_n^p$ by [16, Theorem III.1.1]. We now show $\mathcal{P}_*^{p^i}(\tilde{\tau}_{n+1}) = 0$ for all $i \geq 0$. Because of $\beta(\tilde{\tau}_{n+1}) = \tilde{\xi}_{n+1}$ this implies with the same argument as above the formula for the coaction of $\tilde{\tau}_{n+1}$. By the Nishida relations we have

$$\begin{aligned} \mathcal{P}_*^{p^i}(\tilde{\tau}_{n+1}) &= \mathcal{P}_*^{p^i}(Q^{p^n}(\tilde{\tau}_n)) \\ &= \sum_j (-1)^{p^i+j} \binom{(p^n - p^i)(p-1)}{p^i - jp} Q^{p^n - p^i + j}(\mathcal{P}_*^j(\tilde{\tau}_n)). \end{aligned}$$

Since by the induction hypothesis $\mathcal{P}_*^j(\tilde{\tau}_n)$ is zero for $j > 0$, this is equal to

$$(-1)\binom{(p^n - p^i)(p-1)}{p^i} Q^{p^n - p^i}(\tilde{\tau}_n).$$

This is zero, because $2(p^n - p^i) < |\tilde{\tau}_n|$. $\square$

Lemma 5.9. *We consider case (4). Suppose that $r > 1$. Then there is map of graded $\mathbb{F}_p$-algebras*

$$\big(E(x) \otimes P_k(y)\big)/xy^{k-1} \to (H\mathbb{F}_p)_* \mathrm{K}$$

that is an isomorphism in degrees $\leq 2p$. Here, we have $|x| = 2r - 1$ and $|y| = 2r$.

Proof. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & \Sigma(H\mathbb{F}_p)_* \mathrm{K}(\bar{\mathbb{F}}_l)_p \arrow[r, "\Delta"] & (H\mathbb{F}_p)_* \mathrm{K} \arrow[r, "i_*"] & (H\mathbb{F}_p)_* \mathrm{K}(\bar{\mathbb{F}}_l)_p & \\
0 \arrow[r] & \bigoplus_{j=1}^{k-1} \Sigma^{2rj-1}\mathbb{F}_p\{1\} \arrow[u, hook] \arrow[r, "\oplus\Delta_j"] & \big(\bigoplus_{j=0}^{k-1} (H\mathbb{F}_p)_* \mathrm{K}_{jr}\big)_{\le 2p} \arrow[u, hook] \arrow[r, "\oplus i_j"] & \bigoplus_{j=0}^{k-1} \Sigma^{2rj}\mathbb{F}_p\{1\} \arrow[u, hook] \arrow[r] & 0.
\end{tikzcd}$$

The middle vertical map is an isomorphism in degrees $\le 2p$ and we treat it as an inclusion. We define x by $\Delta_1(\Sigma^{2r-1}1)$ and y by the preimage of $\Sigma^{2r}1$ under i_1. Since $(H\mathbb{F}_p)_* \mathrm{K}$ is zero in the degrees $2p-3$ and $2p-2$, we have $xy^{k-1} = 0$ and $y^k = 0$. Hence, we have a map of $\mathbb{F}_p$-algebras

$$\big(E(x) \otimes P_k(y)\big)/xy^{k-1} \to (H\mathbb{F}_p)_* \mathrm{K}\,.$$

In order to show that it is an isomorphism in degrees $\le 2p$, it suffices to show that $xy^j \ne 0$ for $j = 0, \dots, k-2$ and that $y^j \ne 0$ for $j = 0, \dots, k-1$. For $j = 0, \dots, k-1$ we have $i_*(y^j) = u^{rj} \ne 0$ and hence $y^j \ne 0$. For $j = 0, \dots, k-2$ we have

$$xy^j = \Delta(\Sigma u^{r-1})y^j = \Delta(\Sigma u^{r(j+1)-1}) = \Delta_{j+1}(\Sigma^{2r(j+1)-1}1) \ne 0$$

by $(H\mathbb{F}_p)_* \mathrm{K}$-linearity of Δ . □

6. Computations with the Bökstedt spectral sequence

We first recall some facts about (topological) Hochschild homology and the Bökstedt spectral sequence. See [5, Section 2, 3 and 4] and [6, Section 3 and 4] for more details.

Let k be a field, let R be a graded-commutative k-algebra and let Q be a graded-commutative R-algebra. Then, the Hochschild homology $\mathbb{H}^k_{*,*}(R;Q)$ of R with coefficients in Q is the homology of the chain complex associated to the simplicial graded k-vector space given by $[n] \mapsto Q \otimes_k R^{\otimes_k n}$ and the usual face and degeneracy maps. Here, note that we equip the category of graded k-vector spaces with the symmetry $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$, so that the last face map includes signs. We write $\mathbb{H}^{\mathbb{F}_p}_{*,*}(R)$ for $\mathbb{H}^{\mathbb{F}_p}_{*,*}(R;R)$. We have that $\mathbb{H}^k_{*,*}(R;Q)$ is an augmented Q-algebra, and if $\mathbb{H}^k_{*,*}(R;Q)$ is flat over Q, then $\mathbb{H}^k_{*,*}(R;Q)$ is a Q-bialgebra. The map $r \mapsto 1 \otimes r \in Q \otimes_k R$ defines a morphism $\sigma : R \to \mathbb{H}^k_{1,*}(R;Q)$ that satisfies the derivation rule [6, p.1271].

Now, let B be a commutative S-algebra and let C be a commutative B-algebra. We implicitly assume that the necessary cofibrancy conditions are satisfied. The topological Hochschild homology of B with coefficients in C, denoted by $\mathrm{THH}(B;C)$, is the geometric realization of the simplicial S-module $[n] \mapsto C \wedge_S B^{\wedge_S n}$. We have that $\mathrm{THH}(B;C)$ is an augmented commutative C-algebra. Moreover, in the stable homotopy category $\mathrm{THH}(B;C)$ admits the structure of a C-bialgebra. In the stable homotopy category there is a morphism $\sigma : \Sigma B \to \mathrm{THH}(B)$. We denote the composition $\Sigma B \to \mathrm{THH}(B) \to \mathrm{THH}(B;C)$ also by σ. The by σ induced map in mod p homology satisfies the Leibniz rule [5, Proposition 5.10].

Recall that the Bökstedt spectral sequence is a a strongly convergent spectral sequence of the form

$$E^2_{n,m} = \mathbb{H}^{\mathbb{F}_p}_{n,m}\big((H\mathbb{F}_p)_*B; (H\mathbb{F}_p)_*C\big) \Longrightarrow (H\mathbb{F}_p)_* \mathrm{THH}(B;C).$$

The spectral sequence is an A_*-comodule $(H\mathbb{F}_p)_*C$-algebra spectral sequence. We will use that the A_*-comodule structure on the E^2-page is induced by the following map on the Hochschild complex:

$$\begin{aligned}
(H\mathbb{F}_p)_*C \otimes \big((H\mathbb{F}_p)_*B\big)^{\otimes n} &\to A_* \otimes (H\mathbb{F}_p)_*C \otimes \big(A_* \otimes (H\mathbb{F}_p)_*B\big)^{\otimes n} \\
&\to A_*^{\otimes n} \otimes (H\mathbb{F}_p)_*C \otimes \big((H\mathbb{F}_p)_*B\big)^{\otimes n} \\
&\to A_* \otimes (H\mathbb{F}_p)_*C \otimes \big((H\mathbb{F}_p)_*B\big)^{\otimes n}.
\end{aligned}$$

Here, the second map is given by the symmetry (including signs) and the third map is given by the multiplication of A_*. If $(H\mathbb{F}_p)_* \mathrm{THH}(B;C)$ is flat over $(H\mathbb{F}_p)_*C$, it is an A_*-comodule

$(H\mathbb{F}_p)_*C$-bialgebra. If each term $E^r_{*,*}$ is flat over $(H\mathbb{F}_p)_*C$, the Bökstedt spectral sequence is an A_*-comodule $(H\mathbb{F}_p)_*C$-bialgebra spectral sequence, and if only the terms

$$E^2_{*,*}, \ldots, E^r_{*,*}$$

are flat over $(H\mathbb{F}_p)_*C$, then these are A_*-comodule $(H\mathbb{F}_p)_*C$-bialgebras and the differentials respect this structure. We have that the edge homomorphism

$$(H\mathbb{F}_p)_*C = E^2_{0,*} \to E^\infty_{0,*} \to (H\mathbb{F}_p)_* \operatorname{THH}(B;C)$$

is the unit map. As a consequence one gets that the zeroth step of the filtration splits off from $(H\mathbb{F}_p)_* \operatorname{THH}(B;C)$ naturally. For $x \in E^\infty_{1,*}$ we can therefore choose a natural representative $[x]$ in the first step of the filtration. For $x \in (H\mathbb{F}_p)_*B$ we have $\sigma_*(x) = [\sigma x]$ in $(H\mathbb{F}_p)_* \operatorname{THH}(B;C)$ [6, Proposition 4.4]. If $x \in E^\infty_{*,*}$ is an arbitrary class (not necessarily of filtration degree 1) and if there is no non-trivial class in the same total degree and lower filtration we will use the notation $[x]$ to denote the unique representative of x in $(H\mathbb{F}_p)_* \operatorname{THH}(B;C)$.

6.1. **The first case.** In this subsection we consider case (1). In this case Angeltveit and Rognes obtained the following result using the Bökstedt spectral sequence [5, Theorem 7.15]:

Theorem 6.1 (Angeltveit, Rognes)**.** *In case (1) we have an isomorphism of $(H\mathbb{F}_p)_* \mathrm{K}$-algebras*

$$(H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K}) \cong (H\mathbb{F}_p)_* \mathrm{K} \otimes E([\sigma\tilde{\xi}_1^p], [\sigma\tilde{\xi}_2]) \otimes P([\sigma\tilde{\tau}_2]) \otimes \Gamma([\sigma b]).$$

In Subsection 7.3 we apply the generalized Brun spectral sequence to compute the $V(1)$-homotopy of $\operatorname{THH}(\mathrm{K})$ in case (1). For the calculation we need the following lemma:

Lemma 6.2. *In case (1) there is no non-trivial A_*-comodule primitive in*

$$(H\mathbb{F}_p)_{2p^2-1}\big(V(1) \wedge^L_S \operatorname{THH}(\mathrm{K})\big).$$

Proof. We have an isomorphism of comodule algebras:

$$(H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \operatorname{THH}(\mathrm{K})\big) \cong (H\mathbb{F}_p)_*V(1) \otimes (H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K}).$$

Recall that $(H\mathbb{F}_p)_*V(1) \cong E(\epsilon_0, \epsilon_1)$ with $|\epsilon_0| = 1$ and $|\epsilon_1| = 2p-1$ and that $(H\mathbb{F}_p)_*V(1)$ contains no non-trivial comodule primitive in positive degree. This follows from the observation that we can map $(H\mathbb{F}_p)_*V(1)$ injectively into A_* via the map of comodule algebras

$$(H\mathbb{F}_p)_*V(1) \cong (H\mathbb{F}_p)_*(V(1) \wedge^L_S S) \to (H\mathbb{F}_p)_*(V(1) \wedge^L_S \ell_p).$$

With Theorem 6.1 we get that an $\mathbb{F}_p$-basis of $(H\mathbb{F}_p)_{2p^2-1}\big(V(1) \wedge^L_S \operatorname{THH}(\mathrm{K})\big)$ is given by the classes

$$[\sigma\tilde{\xi}_2],\ \ \epsilon_1[\sigma b],\ \ \tilde{\tau}_2,\ \ \epsilon_1\tilde{\xi}_1^p,\ \ \epsilon_0\tilde{\xi}_2,\ \ \epsilon_0\epsilon_1 b.$$

By Proposition 5.7 the class $[\sigma b]$ is a comodule primitive and we have

$$\nu([\sigma\tilde{\xi}_2]) = 1 \otimes [\sigma\tilde{\xi}_2] + \bar{\xi}_1 \otimes [\sigma\tilde{\xi}_1^p] + \tau_1 \otimes [\sigma b].$$

Let $x \in (H\mathbb{F}_p)_{2p^2-1}\big(V(1) \wedge^L_S \operatorname{THH}(\mathrm{K})\big)$ be a comodule primitive. We write

$$x = \lambda_1[\sigma\tilde{\xi}_2] + \lambda_2\epsilon_1[\sigma b] + \lambda_3\tilde{\tau}_2 + \lambda_4\epsilon_1\tilde{\xi}_1^p + \lambda_5\epsilon_0\tilde{\xi}_2 + \lambda_6\epsilon_0\epsilon_1 b$$

with $\lambda_i \in \mathbb{F}_p$. The A_*-coaction of x and the A_*-coaction of

$$\lambda_2\epsilon_1[\sigma b] + \lambda_3\tilde{\tau}_2 + \lambda_4\epsilon_1\tilde{\xi}_1^p + \lambda_5\epsilon_0\tilde{\xi}_2 + \lambda_6\epsilon_0\epsilon_1 b$$

lie in

$$A_* \otimes (H\mathbb{F}_p)_*V(1) \otimes (H\mathbb{F}_p)_* \mathrm{K} \otimes E([\sigma\tilde{\xi}_2]) \otimes \Gamma([\sigma b]).$$

Since this is not true for $\nu([\sigma\tilde{\xi}_2])$ it follows that $\lambda_1 = 0$. The A_*-coaction of x and

$$\lambda_3\tilde{\tau}_2 + \lambda_4\epsilon_1\tilde{\xi}_1^p + \lambda_5\epsilon_0\tilde{\xi}_2 + \lambda_6\epsilon_0\epsilon_1 b$$

lie in

$$A_* \otimes (H\mathbb{F}_p)_*V(1) \otimes (H\mathbb{F}_p)_* \mathrm{K} \ \oplus\ (H\mathbb{F}_p)_*V(1) \otimes \mathbb{F}_p\{[\sigma b]\}.$$

Since ϵ_1 is not primitive, this is not the case for $\epsilon_1[\sigma b]$. We get $\lambda_2 = 0$. Hence, we have

$$x \in (H\mathbb{F}_p)_* V(1) \otimes (H\mathbb{F}_p)_* \mathrm{K} \cong (H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \mathrm{K}\big).$$

The class x has to be in the kernel of

$$(H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \mathrm{K}\big) \longrightarrow (H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \ell_p\big),$$

because there is no non-trivial comodule primitive in $(H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \ell_p\big) \cong A_*$ in positive degree. By Proposition 5.7 the kernel is given by

$$(H\mathbb{F}_p)_* V(1) \otimes \mathbb{F}_p\{b\} \otimes P(\tilde{\xi}_1^p, \tilde{\xi}_2, \dots) \otimes E(\tilde{\tau}_2, \dots).$$

This implies $\lambda_3 = \lambda_4 = \lambda_5 = 0$. Since b is a comodule primitive and $\epsilon_0\epsilon_1$ is not primitive, $\epsilon_0\epsilon_1 b$ is not primitive. We get $\lambda_6 = 0$ and therefore $x = 0$. □

6.2. **The second case.** In this subsection we compute the $V(1)$-homotopy of THH(K) in case (2). We first consider the Bökstedt spectral sequences converging to $(H\mathbb{F}_p)_*\,\mathrm{THH}(\mathrm{K}; H\mathbb{Z}_p)$ and $(H\mathbb{F}_p)_*\,\mathrm{THH}(\mathrm{K})$. In this part we apply methods from [5] and [6]. Furthermore, we use the naturality of the Bökstedt spectral sequence with respect to the morphism $\mathrm{K} \to \mathrm{K}(\bar{\mathbb{F}}_\ell)_p$ and Ausoni's results [6] about the Bökstedt spectral sequences for connective complex K-theory. After computing $(H\mathbb{F}_p)_*\,\mathrm{THH}(\mathrm{K})$ we show that $V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})$ is a module over the S-ring spectrum $H\mathbb{F}_p$ and we deduce the $V(1)$-homotopy of THH(K).

In this subsection we use the map

$$\mathrm{K} \longrightarrow \mathrm{K}(\bar{\mathbb{F}}_l)_p \longrightarrow H\mathbb{Z}_p$$

to build $\mathrm{THH}(\mathrm{K}; H\mathbb{Z}_p)$. We denote by $(E^*_{*,*}, d^*)$ and $(\tilde{E}^*_{*,*}, \tilde{d}^*)$ the Bökstedt spectral sequences converging to $(H\mathbb{F}_p)_*\,\mathrm{THH}(\mathrm{K}; H\mathbb{Z}_p)$ and $(H\mathbb{F}_p)_*\,\mathrm{THH}(\mathrm{K}(\bar{\mathbb{F}}_l)_p; H\mathbb{Z}_p)$. We have a map of spectral sequences $E^*_{*,*} \to \tilde{E}^*_{*,*}$, which we denote by i^*. We define

$$B_* := (H\mathbb{F}_p)_* H\mathbb{Z}_p = P(\bar{\xi}_1, \dots) \otimes E(\bar{\tau}_1, \dots).$$

Recall from [6, p.1283] that the map $(H\mathbb{F}_p)_*\,\mathrm{K}(\bar{\mathbb{F}}_l)_p \to B_*$ is given by $\bar{\xi}_i \mapsto \bar{\xi}_i$, $\bar{\tau}_i \mapsto \bar{\tau}_i$ and $u \mapsto 0$. Using standard facts about Hochschild homology (see [5, Proposition 2.4], [6, Proposition 3.2]) we get the following: We have

$$E^2_{*,*} = B_* \otimes E(\sigma\tilde{\xi}_1, \sigma\tilde{\xi}_2, \dots) \otimes \Gamma(\sigma x, \sigma\tilde{\tau}_2, \sigma\tilde{\tau}_3, \dots)$$

as a B_*-algebra. Every $a \in \{\sigma\tilde{\xi}_1, \sigma\tilde{\xi}_2, \dots\}$ is a coalgebra primitive. For the classes $a \in \{\sigma x, \sigma\tilde{\tau}_2, \sigma\tilde{\tau}_3, \dots\}$ we have the following formula for the comultiplication:

$$\psi^2(\gamma_n(a)) = \sum_{i+j=n} \gamma_i(a) \otimes_{B_*} \gamma_j(a).$$

The class $\gamma_n(\sigma x)$ is represented in the Hochschild complex by the cycle $1 \otimes x^{\otimes n}$. Since $x \in (H\mathbb{F}_p)_*\,\mathrm{K}$ is primitive, $\gamma_n(\sigma x)$ is a comodule primitive. Because σ is a derivation we get that the classes $\sigma\tilde{\xi}_n$ are comodule primitives for $n \geq 2$ and that the coactions on $\sigma\tilde{\xi}_1$ and $\sigma\tilde{\tau}_n$ are given by:

$$\begin{aligned} \nu^2(\sigma\tilde{\xi}_1) &= 1 \otimes \sigma\tilde{\xi}_1 + a\bar{\tau}_0 \otimes \sigma x \\ \nu^2(\sigma\tilde{\tau}_n) &= 1 \otimes \sigma\tilde{\tau}_n + \bar{\tau}_0 \otimes \sigma\tilde{\xi}_n. \end{aligned}$$

Here, a is the element of $\mathbb{F}_p$ defined in Proposition 5.8. Recall from [6, Section 6] that

$$\tilde{E}^2_{*,*} = B_* \otimes E(\sigma u, \sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots) \otimes \Gamma(w, \sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots),$$

where w has the bidegree $|w| = (2, 2p-2)$. For $b \in B_*$ we have $i^2(b) = b$. Furthermore, we have $i^2(\gamma_n(\sigma x)) = 0$ for $n \geq 1$, $i^2(\sigma\tilde{\xi}_n) = \sigma\bar{\xi}_n$ and $i^2(\gamma_j(\sigma\tilde{\tau}_n)) = \gamma_j(\sigma\bar{\tau}_n)$.

Lemma 6.3. *We have $d = 0$ for $i = 2, \dots, p-2$.*

Proof. The E^2-page is generated as a B_*-algebra by

$$T := \{\sigma\tilde{\xi_i} \mid i \geq 1\} \cup \{\gamma_{p^i}(\sigma x) \mid i \geq 0\} \cup \{\gamma_{p^i}(\sigma\tilde{\tau}_j) \mid i \geq 0, j \geq 2\}.$$

Suppose that the spectral sequence has non-trivial differentials. Let s_0 be the minimal number such that $d^{s_0} \neq 0$ and let b be a class of minimal total degree in T with $d^{s_0}(b) \neq 0$. Because the classes $\sigma\tilde{\xi_i}$, σx and $\sigma\tilde{\tau}_j$ lie in the first column they cannot support differentials. Thus, b has filtration degree at least p. Because the differential is compatible with the coalgebra structure $d^{s_0}(b)$ has to be a coalgebra primitive [5, Proposition 4.8]. The B_*-module of coalgebra primitives of $E^2_{*,*}$ is given by

$$B_* \otimes \Big(\bigoplus_{i\geq 1} \mathbb{F}_p\{\sigma\tilde{\xi_i}\} \oplus \mathbb{F}_p\{\sigma x\} \oplus \bigoplus_{i \geq 2} \mathbb{F}_p\{\sigma\tilde{\tau}_2\}\Big).$$

It follows that $d^{s_0}(b)$ has filtration degree one. □

Lemma 6.4. *We have $d^{p-1}(\gamma_n(\sigma x)) = 0$ for all n.*

Proof. We assume that there is an n with $d^{p-1}(\gamma_n(\sigma x)) \neq 0$. Let n_0 be minimal with this property. We must have $n_0 = p^j$ for a $j > 0$. Since $\gamma_{n_0}(\sigma x)$ is a comodule primitive, $d^{p-1}(\gamma_{n_0}(\sigma x))$ is also a comodule primitive. Because of the minimality of n_0 it follows from the formula for $\psi^2(\gamma_{n_0}(\sigma x))$ that $d^{p-1}(\gamma_{n_0}(\sigma x))$ is a coalgebra primitive. The coalgebra primitives have filtration degree one. Therefore, we have $j = 1$. For degree reasons we get

$$d^{p-1}(\gamma_p(\sigma x)) \in B_* \otimes \big(\mathbb{F}_p\{\sigma x\} \oplus \mathbb{F}_p\{\sigma\tilde{\xi_1}\}\big).$$

If $a = 0$ the comodule primitives in this vector space are $\mathbb{F}_p\{\sigma x\} \oplus \mathbb{F}_p\{\sigma\tilde{\xi_1}\}$. If $a \neq 0$ the comodule primitives are $\mathbb{F}_p\{\sigma x\}$. Since the total degree of $d^{p-1}(\gamma_p(\sigma x))$ is different from the total degree of σx and from the total degree of $\sigma\tilde{\xi_1}$ we get a contradiction. □

Lemma 6.5. *We have $d^{p-1}(\gamma_{p+i}(\sigma\tilde{\tau}_n)) \doteq \sigma\tilde{\xi}_{n+1}\gamma_i(\sigma\tilde{\tau}_n)$ for $i \geq 0$ and $n \geq 2$ and therefore*

$$E^p_{*,*} = B_* \otimes E(\sigma\tilde{\xi_1}, \sigma\tilde{\xi_2}) \otimes P_p(\sigma\tilde{\tau}_2, \sigma\tilde{\tau}_3, \dots) \otimes \Gamma(\sigma x).$$

Proof. We first prove by induction on $n \geq 2$ that $d^{p-1}(\gamma_p(\sigma\tilde{\tau}_n)) \doteq \sigma\tilde{\xi}_{n+1}$. By [6, Lemma 6.6] we have $\tilde{d}^{p-1}(\gamma_p(\sigma\bar{\tau}_2)) = \lambda_2\sigma\bar{\xi}_3$ for a unit $\lambda_2 \in \mathbb{F}_p$. Because the kernel of

$$E^{p-1}_{1,*} \xrightarrow{i^{p-1}} \tilde{E}^{p-1}_{1,*}$$

is given by $B_* \otimes \mathbb{F}_p\{\sigma x\}$ we get

$$d^{p-1}(\gamma_p(\sigma\tilde{\tau}_2)) = \lambda_2\sigma\tilde{\xi}_3 + b\sigma x$$

for a class $b \in B_*$ of positive degree. By Lemma 6.4 every class in a total degree less than the total degree of $\gamma_p(\sigma\tilde{\tau}_2)$ has trivial d^{p-1}-differential. This implies that $d^{p-1}(\gamma_p(\sigma\tilde{\tau}_2))$ is a comodule primitive. Therefore, b has to be zero. Assume that we have proven the assertion for all $2 \leq m < n$. By comparing with the spectral sequence $\tilde{E}^*_{*,*}$ we get

$$d^{p-1}(\gamma_p(\sigma\tilde{\tau}_n)) = \lambda_n\sigma\tilde{\xi}_{n+1} + b\sigma x$$

for a unit $\lambda_n \in \mathbb{F}_p$ and a class $b \in B_*$ of positive degree. We get

$$\nu^{p-1}(d^{p-1}(\gamma_p(\sigma\tilde{\tau}_n)) = 1 \otimes \lambda_n\sigma\tilde{\xi}_{n+1} + \nu(b) \cdot 1 \otimes \sigma x.$$

On the other hand we can write

$$\nu^{p-1}(\gamma_p(\sigma\tilde{\tau}_n)) = 1 \otimes \gamma_p(\sigma\tilde{\tau}_n) + \sum_i a_i \otimes b_i$$

for certain $a_i \in A_*$ and certain $b_i \in E^{p-1}_{p,*}$ whose internal degree is less than the internal degree of $\gamma_p(\sigma\tilde{\tau}_n)$. This implies that

$$\nu^{p-1}(d^{p-1}(\gamma_p(\sigma\tilde{\tau}_n))) = 1 \otimes d^{p-1}(\gamma_p(\tilde{\tau}_n)) + \sum_i a_i \otimes d^{p-1}(b_i).$$

By the induction hypothesis and by Lemma 6.4 we have

$$d^{p-1}(b_i) = \sum_{3\leq m\leq n} b_{i,m}\sigma\tilde{\xi}_m$$

for certain $b_{i,m} \in B_*$. It follows that $b = 0$. This proves the induction step.

We now fix $n \geq 2$. Suppose that $i \geq 1$ and that we have already shown

$$d^{p-1}(\gamma_{p+j}(\sigma\tilde{\tau}_n)) = \lambda_n\sigma\tilde{\xi}_{n+1}\gamma_j(\sigma\tilde{\tau}_n)$$

for all $0 \leq j < i$. Then by the induction hypothesis

$$d^{p-1}(\gamma_{p+i}(\sigma\tilde{\tau}_n)) - \lambda_n\sigma\tilde{\xi}_{n+1}\gamma_i(\sigma\tilde{\tau}_n)$$

is a coalgebra primitive. Because it lies in a filtration degree > 1, it has to be zero. This proves the induction step and therefore the lemma. □

Lemma 6.6. *We have $d^s = 0$ for all $s \geq p$. Therefore, we get*

$$E^\infty_{*,*} = B_* \otimes E(\sigma\tilde{\xi}_1, \sigma\tilde{\xi}_2) \otimes P_p(\sigma\tilde{\tau}_2, \sigma\tilde{\tau}_3, \dots) \otimes \Gamma(\sigma x).$$

Proof. Suppose that the statement is wrong. Let $s_0 \geq p$ be the minimal number with $d^{s_0} \neq 0$ and let $i \geq 1$ be the minimal number with $d^{s_0}(\gamma_{p^i}(\sigma x)) \neq 0$. Then, $d^{s_0}(\gamma_{p^i}(\sigma x))$ is a comodule and coalgebra primitive in total degree $p^i(2p-2) - 1$. The coalgebra primitives are given by

$$B_* \otimes \big(\mathbb{F}_p\{\sigma\tilde{\xi}_1\} \oplus \mathbb{F}_p\{\sigma\tilde{\xi}_2\} \oplus \bigoplus_{i\geq 2}\mathbb{F}_p\{\sigma\tilde{\tau}_i\} \oplus \mathbb{F}_p\{\sigma x\}\big).$$

If $a = 0$ the comodule primitives in this $\mathbb{F}_p$-vector space are

$$\mathbb{F}_p\{\sigma\tilde{\xi}_1\} \oplus \mathbb{F}_p\{\sigma\tilde{\xi}_2\} \oplus \bigoplus_{i\geq 3}\mathbb{F}_p\{\sigma\tilde{\tau}_i\} \oplus \mathbb{F}_p\{\sigma x\}.$$

If $a \neq 0$ the comodule primitive are given by

$$\mathbb{F}_p\{\sigma\tilde{\xi}_2\} \oplus \bigoplus_{i\geq 3}\mathbb{F}_p\{\sigma\tilde{\tau}_i\} \oplus \mathbb{F}_p\{\sigma x\}.$$

These classes all lie in total degrees different from $p^i(2p-2)-1$. Thus, we get a contradiction. □

Recall from [6, Proposition 6.7] that we have

$$(H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K}(\bar{\mathbb{F}}_l)_p; H\mathbb{Z}_p) \cong B_* \otimes E([\sigma u], [\sigma\bar{\xi}_1]) \otimes P([w]).$$

For degree reasons $[w]$ has to be a coalgebra primitive.

Theorem 6.7. *In case (2) we have an isomorphism of B_*-algebras*

$$(H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p) \cong B_* \otimes E([\sigma\tilde{\xi}_1], [\sigma\tilde{\xi}_2]) \otimes P([\sigma\tilde{\tau}_2]) \otimes \Gamma([\sigma x]).$$

Proof. By [5, Proposition 5.9] we have $\sigma_* Q^{p^i} = Q^{p^i}\sigma_*$. Since $Q^{p^i}\tilde{\tau}_i = \tilde{\tau}_{i+1}$ one gets as in [5, Theorem 5.12] or [6, Lemma 5.2] that $[\sigma\tilde{\tau}_i]^p = [\sigma\tilde{\tau}_{i+1}]$ for $i \geq 2$. We show by induction on i that we can find a class

$$\gamma_{p^i} \in (H\mathbb{F}_p)_{p^i(2p-2)} \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p)$$

that represents the class $\gamma_{p^i}(\sigma x)$ in $E^\infty_{*,*}$ and that has the property $\gamma^p_{p^i} = 0$. We define $\gamma_1 \coloneqq [\sigma x]$. Then, we have $\gamma_1^p = \sigma_*(Q^{p-1}(x))$. We claim that $Q^{p-1}(x) = 0$. For degree reasons we have $Q^{p-1}(x) \in \mathbb{F}_p\{x\tilde{\xi}_1^{p-1}\}$. The comodule action of $x\tilde{\xi}_1^{p-1}$ is given by

$$\nu(x\tilde{\xi}_1^{p-1}) = \sum_j \binom{p-1}{j}\bar{\xi}_1^j \otimes \tilde{\xi}_1^{p-1-j}x.$$

Since the Steenrod algebra is one-dimensional in degree $(p-1)(2p-2)$ we have $\mathcal{P}^{p-1}_*(x\tilde{\xi}_1^{p-1}) \doteq x$. On the other hand, we have

$$\mathcal{P}^{p-1}_*(Q^{p-1}(x)) = \sum_j (-1)^{p-1+j}\binom{0}{p-1-jp} Q^j\mathcal{P}^j_*(x) = 0$$

by the Nishida relations. Hence, we can conclude $\gamma_1^p = 0$. Suppose that $i > 0$ and that we have already shown the assertion for all $0 \le j < i$. It suffices to show that we can find a representative γ_{p^i} for $\gamma_{p^i}(\sigma x)$ that has the property that $\gamma_{p^i}^p$ is a comodule and coalgebra primitive: Every non-trivial comodule and coalgebra primitive of $(H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p)$ gives a non-trivial comodule and coalgebra primitive in $E^\infty_{*,*}$. By the proof of Lemma 6.6 the simultaneous coalgebra and comodule primitives of E^∞ lie in the total degrees $2p-2$, $2p-1$, $2p^2-1$ and $2p^j$ for $j \ge 3$, which are all different from $p^{i+1}(2p-2)$. By the induction hypothesis we have a map of B_*-algebras

$$B_* \otimes E([\sigma\tilde{\xi}_1], [\sigma\tilde{\xi}_2]) \otimes P([\sigma\tilde{\tau}_2]) \otimes P_p(\gamma_1, \dots, \gamma_{p^{i-1}}) \to (H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p)$$

which is injective and an isomorphism in degree $< p^i(2p-2)$. For a graded-commutative $\mathbb{F}_p$-algebra we denote by I_p the homogeneous ideal of all elements x with $x^p = 0$. The map

$$P(\bar{\xi}_1, \dots) \otimes P([\sigma\tilde{\tau}_2]) \to (H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p)/I_p$$

is an isomorphism in degrees $< p^i(2p-2)$. Furthermore, the map

$$P(\bar{\xi}_1, \dots) \otimes P([w]) \to (H\mathbb{F}_p)_* \operatorname{THH}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p; H\mathbb{Z}_p\big)/I_p$$

and the map from

$$P(\bar{\xi}_1, \dots) \otimes P([w]) \otimes_{P(\bar{\xi}_1,\dots)} P(\bar{\xi}_1, \dots) \otimes P([w])$$

to

$$\Big((H\mathbb{F}_p)_* \operatorname{THH}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p; H\mathbb{Z}_p\big) \otimes_{B_*} (H\mathbb{F}_p)_* \operatorname{THH}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p; H\mathbb{Z}_p\big)\Big)/I_p$$

are isomorphisms. First, let γ_{p^i} be an arbitrary representative for $\gamma_{p^i}(\sigma x)$. We can assume that it is in the kernel of the augmentation

$$\epsilon : (H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p) \to B_*.$$

We consider its image under

$$i_* : (H\mathbb{F}_p)_* \operatorname{THH}\big(\mathrm{K}; H\mathbb{Z}_p\big) \to (H\mathbb{F}_p)_* \operatorname{THH}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p; H\mathbb{Z}_p\big).$$

We have

$$i_*(\gamma_{p^i}) = \sum_j a_j [w]^j \text{ modulo } I_p$$

for certain $a_j \in P(\bar{\xi}_1, \dots)$ with $|a_j| = p^i(2p-2) - 2pj$. Since $i_*(\gamma_{p^i})$ lies in the kernel of the augmentation, we have $a_0 = 0$. We show that $a_j = 0$ for all j that are not divisible by p: The degree of every non-zero class in $P(\bar{\xi}_1, \dots)$ is divisible by $2p-2$. Thus, $P(\bar{\xi}_1, \dots)$ is zero in degree $p^i(2p-2) - 2p$ and we get $a_1 = 0$. We have

$$\psi(i_*(\gamma_{p^i})) = \sum_j a_j \sum_{n=0}^{j} \binom{j}{n} [w]^n \otimes_{B_*} [w]^{j-n} \text{ modulo } I_p. \tag{12}$$

On the other hand, since γ_{p^i} is in $\ker \epsilon$ we can write

$$\psi(\gamma_{p^i}) = 1 \otimes_{B_*} \gamma_{p^i} + \gamma_{p^i} \otimes_{B_*} 1 + \sum_j b_j \otimes_{B_*} c_j$$

for certain $b_j, c_j \in (H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p)$ with $|b_j|, |c_j| < p^i(2p-2)$. It follows that

$$\psi(\gamma_{p^i}) = 1 \otimes_{B_*} \gamma_{p^i} + \gamma_{p^i} \otimes_{B_*} 1 + \sum_{n,m} c_{n,m} [\sigma\tilde{\tau}_2]^n \otimes_{B_*} [\sigma\tilde{\tau}_2]^m \text{ modulo } I_p$$

for certain $c_{n,m} \in P(\bar{\xi}_1, \dots)$. Applying i_* and using that by [6, Lemma 6.5] the relation $[w]^p = [\sigma\bar{\tau}_2]$ holds in $(H\mathbb{F}_p)_* \operatorname{THH}\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p; H\mathbb{Z}_p\big)$, we get

$$\begin{aligned} \psi(i_*(\gamma_{p^i})) \;&=\; 1 \otimes_{B_*} \sum_j a_j [w]^j + \sum_j a_j [w]^j \otimes_{B_*} 1 \\ &\quad + \sum_{n,m} c_{n,m} [w]^{pm} \otimes_{B_*} [w]^{pn} \qquad \text{modulo } I_p. \end{aligned} \tag{13}$$

Let $j \neq 1$ be a natural number that is not divisible by p. In (12) the coefficient of $[w] \otimes [w]^{j-1}$ is $j \cdot a_j$ and in (13) it is zero. We get that $a_j = 0$. We conclude that we have

$$i_*(\gamma_{p^i}) = \sum_{j \geq 1} a_{jp}[w]^{jp} \quad \text{modulo} \;\; I_p,$$

where a_{jp} is zero for $jp > p^{i-1}(p-1)$. The class $\sum_{j\geq 1} a_{jp}[\sigma\tilde{\tau}_2]^j$ lies in filtration $< p^i$. Thus, the element $\gamma_{p^i} - \sum_{j\geq 1} a_{jp}[\sigma\tilde{\tau}_2]^j$ is also a representative for $\gamma_{p^i}(\sigma x)$, it lies in the kernel of the augmentation and it satisfies

$$i_*(\gamma_{p^i} - \sum_j a_{jp}[\sigma\tilde{\tau}_2]^j)^p = 0.$$

We replace γ_{p^i} by $\gamma_{p^i} - \sum_j a_{jp}[\sigma\tilde{\tau}_2]^j$ and denote this class again by γ_{p^i}. Because the maps $(i_* \otimes_{B_*} i_*)/I_p$ and $(\mathrm{id}_{A_*} \otimes i_*)/I_p$ are injective on the images of

$$\bigoplus_{n<p^i(2p-2), m<p^i(2p-2)} (H\mathbb{F}_p)_n \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p) \otimes (H\mathbb{F}_p)_m \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p)$$

and

$$\bigoplus_{m<p^i(2p-2)} A_n \otimes (H\mathbb{F}_p)_m \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p)$$

in

$$\Big(H\mathbb{F}_{p_*} \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p) \otimes_{B_*} (H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p)\Big)/I_p$$

and

$$\Big(A_* \otimes (H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p)\Big)/I_p,$$

$\gamma_{p^i}^p$ now is a comodule and coalgebra primitive. □

We now study the Bökstedt spectral sequence $(E^*_{*,*}, d^*)$ converging to $(H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K})$. Similarly as above, one sees that

$$E^2_{*,*} = (H\mathbb{F}_p)_* \mathrm{K} \otimes E(\sigma\tilde{\xi}_1, \sigma\tilde{\xi}_2, \dots) \otimes \Gamma(\sigma x, \sigma\tilde{\tau}_2, \sigma\tilde{\tau}_3, \dots).$$

The classes $\sigma\tilde{\xi}_i$ are coalgebra primitives. For the classes $a \in \{\sigma x, \sigma\tilde{\tau}_2, \sigma\tilde{\tau}_3, \dots\}$ we have the following formula for the comultiplication:

$$\psi^2(\gamma_n(a)) = \sum_{i+j=n} \gamma_i(a) \otimes_{(H\mathbb{F}_p)_* \mathrm{K}} \gamma_j(a).$$

For $n \geq 2$ the classes $\sigma\tilde{\xi}_n$ are comodule primitives. All the classes $\gamma_n(\sigma x)$ are comodule primitives. The coactions on $\sigma\tilde{\xi}_1$ and $\sigma\tilde{\tau}_n$ are given by:

$$\begin{aligned} \nu^2(\sigma\tilde{\xi}_1) &= 1 \otimes \sigma\tilde{\xi}_1 + a\bar{\tau}_0 \otimes \sigma x \\ \nu^2(\sigma\tilde{\tau}_n) &= 1 \otimes \sigma\tilde{\tau}_n + \bar{\tau}_0 \otimes \sigma\tilde{\xi}_n. \end{aligned}$$

Lemma 6.8. *For $2 \leq s \leq p-2$ the differential d^s vanishes. We have*

$$d^{p-1}(\gamma_n(\sigma x)) = 0$$

for all n and

$$d^{p-1}(\gamma_{p+n}(\sigma\tilde{\tau}_i)) = \sigma\tilde{\xi}_{i+1}\gamma_n(\sigma\tilde{\tau}_i)$$

for all $n \geq 0$ and $i \geq 2$. Therefore, we get

$$E^p_{*,*} = (H\mathbb{F}_p)_* \mathrm{K} \otimes E(\sigma\tilde{\xi}_1, \sigma\tilde{\xi}_2) \otimes P_p(\sigma\tilde{\tau}_2, \sigma\tilde{\tau}_3, \dots) \otimes \Gamma(\sigma x).$$

Proof. By [5, Proposition 5.6] the differentials d^s vanish for $2 \leq s \leq p-2$ and we have

$$d^{p-1}(\gamma_p(\sigma x)) = \sigma\beta Q^{p-1}x \quad \text{and} \quad d^{p-1}(\gamma_p(\sigma\tilde{\tau}_i)) = \sigma\beta Q^{p^i}\tilde{\tau}_i.$$

The proof of Theorem 6.7 shows that $Q^{p-1}(x) = 0$. Using Proposition 5.8 we get

$$d^{p-1}(\gamma_p(\sigma x)) = 0 \quad \text{and} \quad d^{p-1}(\gamma_p(\sigma\tilde{\tau}_i)) = \sigma\tilde{\xi}_{i+1}.$$

Note that the coalgebra primitives of the $E^2 = E^{p-1}$-page are given by

$$(H\mathbb{F}_p)_* \mathrm{K} \otimes \big(\bigoplus_{i\geq 1} \mathbb{F}_p\{\sigma\tilde{\xi}_i\} \oplus \mathbb{F}_p\{\sigma x\} \oplus \bigoplus_{i\geq 2} \mathbb{F}_p\{\sigma\tilde{\tau}_i\}\big).$$

By induction on n one proves that

$$d^{p-1}(\gamma_{p+n}(\sigma x)) = 0 \quad \text{and} \quad d^{p-1}(\gamma_{p+n}(\sigma\tilde{\tau}_i)) - \sigma\tilde{\xi}_{i+1}\gamma_n(\sigma\tilde{\tau}_i) = 0.$$

The induction step follows because the classes are coalgabra primitives in filtration degree > 1. $\square$

Lemma 6.9. *We have $d^s = 0$ for all $s \geq p$. Therefore, we have*

$$E^\infty_{*,*} = (H\mathbb{F}_p)_* \mathrm{K} \otimes E(\sigma\tilde{\xi}_1, \sigma\tilde{\xi}_2) \otimes P_p(\sigma\tilde{\tau}_2, \sigma\tilde{\tau}_3, \dots) \otimes \Gamma(\sigma x).$$

Proof. This follows as in Lemma 6.6 noticing that the coalgebra primitives of the E^p-page are given by

$$(H\mathbb{F}_p)_* \mathrm{K} \otimes \big(\mathbb{F}_p\{\sigma\tilde{\xi}_1\} \oplus \mathbb{F}_p\{\sigma\tilde{\xi}_2\} \oplus \bigoplus_{i\geq 2} \mathbb{F}_p\{\sigma\tilde{\tau}_i\} \oplus \mathbb{F}_p\{\sigma x\}\big)$$

and that the comodule primitives in this vector space are a subspace of

$$\begin{aligned} &\mathbb{F}_p\{\sigma\tilde{\xi}_1\} \oplus \mathbb{F}_p\{x\sigma\tilde{\xi}_1\} \oplus \mathbb{F}_p\{\sigma\tilde{\xi}_2\} \oplus \mathbb{F}_p\{x\sigma\tilde{\xi}_2\} \\ \oplus &\bigoplus_{j\geq 3} \mathbb{F}_p\{\sigma\tilde{\tau}_j\} \oplus \bigoplus_{j\geq 3} \mathbb{F}_p\{x\sigma\tilde{\tau}_j\} \oplus \mathbb{F}_p\{\sigma x\} \oplus \mathbb{F}_p\{x\sigma x\}. \end{aligned}$$

$\square$

Theorem 6.10. *In case (2) we have an isomorphism of $(H\mathbb{F}_p)_* \mathrm{K}$-algebras*

$$(H\mathbb{F}_p)_* \mathrm{THH}(\mathrm{K}) \cong (H\mathbb{F}_p)_* \mathrm{K} \otimes E([\sigma\tilde{\xi}_1], [\sigma\tilde{\xi}_2]) \otimes P([\sigma\tilde{\tau}_2]) \otimes \Gamma([\sigma x]).$$

Proof. As before one shows that $[\sigma\tilde{\tau}_i]^p = [\sigma\tilde{\tau}_{i+1}]$ for $i \geq 2$. We show by induction on $i \geq 0$ that we can find representatives

$$\gamma_{p^i} \in (H\mathbb{F}_p)_{p^i(2p-2)} \mathrm{THH}(\mathrm{K})$$

of $\gamma_{p^i}(\sigma x) \in E^\infty_{*,*}$ such that $\gamma_{p^i}^p = 0$. As in Theorem 6.7 one proves that the element $\gamma_1 = [\sigma x]$ satisfies $\gamma_1^p = 0$. Assume that $i > 0$ and that the assertion has been shown for all $0 \leq j < i$. It suffices to show that we can find a representative γ_{p^i} for $\gamma_{p^i}(\sigma x)$ such that $\gamma_{p^i}^p$ is a comodule and coalgebra primitive: By Lemma 6.9 the simultaneous comodule and coalgebra primitives of E^∞ lie in total degrees different from $p^{i+1}(2p-2)$. First, let γ_{p^i} be an arbitrary representatives of $\gamma_{p^i}(\sigma x)$ that is in the kernel of the augmentation. Let g be the map $\mathrm{THH}(\mathrm{K}) \to \mathrm{THH}(\mathrm{K}; H\mathbb{Z}_p)$. We can write

$$g_*(\gamma_{p^i}) = \sum_j a_j[\sigma\tilde{\tau}_2]^j \text{ modulo } I_p$$

for certain $a_j \in P(\bar{\xi}_1, \dots)$ with $|a_j| = p^i(2p-2) - 2p^2 j$. Since $g_*(\gamma_{p^i})$ lies in the kernel of the augmentation we have $a_0 = 0$. Let $\tilde{a}_j$ be the element of $P(\tilde{\xi}_1, \dots) \subset (H\mathbb{F}_p)_* \mathrm{K}$ that corresponds to a_j under the canonical isomorphism

$$P(\tilde{\xi}_1, \dots) \cong P(\bar{\xi}_1, \dots).$$

Then $\gamma_{p^i} - \sum_j \tilde{a}_j[\sigma\tilde{\tau}_2]^j$ is also a representative for $\gamma_{p^i}(\sigma x)$ that is in the kernel of the augmentation and it satisfies

$$g_*(\gamma_{p^i} - \sum_j \tilde{a}_j[\sigma\tilde{\tau}_2]^j) = 0 \text{ modulo } I_p.$$

We denote this new representative again by γ_{p^i}. As in the proof of Theorem 6.7 one shows that $\gamma_{p^i}^p$ is a comodule and coalgebra primitive. $\square$

We want to deduce the $V(1)$-homotopy of $\mathrm{THH}(\mathrm{K})$ in case (2). We do this by proving that in this case $V(1) \wedge_S^L \mathrm{THH}(\mathrm{K})$ is a module in $\mathscr{D}_S$ over the S-ring spectrum $H\mathbb{F}_p$. Note that this implies that it is isomorphic in $\mathscr{D}_S$ to a coproduct of S-modules of the form $H\mathbb{F}_p \wedge_S^L S_S^n$ and that the Hurewicz morphism induces an isomorphism between $V(1)_* \mathrm{THH}(\mathrm{K})$ and the comodule primitives in $(H\mathbb{F}_p)_*\big(V(1) \wedge_S^L \mathrm{THH}(\mathrm{K})\big)$.

Remark 6.11. Let $R \to R'$ be a morphism of commutative S-algebras, and let M and N be R'-modules. Note that we have a map $\mathrm{Ext}^*_{R'}(M,N) \to \mathrm{Ext}^*_R(M,N)$. We need that it has a compatible map of spectral sequences. Ext spectral sequences can be constructed by applying $\mathrm{Ext}^*_-(-,N)$ to a projective topological resolution of M or by applying $\mathrm{Ext}^*_-(M,-)$ to an injective topological resolution of N [23, Section 6]. Since in [23] conditional convergence is shown for the unrolled exact couples that are constructed from injective topological resolutions, we use these. Let $\pi_*(N) \to I^0_* \xrightarrow{d^0} I^1_* \xrightarrow{d^1} I^2_* \xrightarrow{d^2} \dots$ be an R'_*-injective resolution. We consider a compatible injective topological resolution, i.e. fiber sequences

$$\Omega^{s+1} I^s \xrightarrow{k^s} N^{s+1} \xrightarrow{i^{s+1}} N^s \xrightarrow{j^s} \Omega^s I^s$$

in $\mathscr{D}_{R'}$ for $s \geq 0$ such that $N^0 = N$ and $\pi_* j^s$ is a monomorphism, and such that we have isomorphisms $I^s_* \cong \pi_* I^s$ under which $\pi_* j^0$ corresponds to the augmentation $\pi_*(N) \to I^0_*$ and $\pi_*(j^{s+1} \circ k^s)$ corresponds to $\Sigma^{-(s+1)} d^s$. Analogously, we consider an R_*-injective resolution $\pi_*(N) \to J^*_*$ and a compatible injective topological resolution in $\mathscr{D}_R$. Let $I^*_* \to J^*_*$ be an R_*-linear map of resolutions lifting the identity map of $\pi_*(N)$. Using that $\mathscr{D}_R(K,L) \cong \mathrm{Hom}_{R_*}(\pi_*(K), \pi_*(L))$ if $\pi_*(L)$ is injective [23, Corollary 5.7], one inductively constructs compatible maps of fiber sequences in $\mathscr{D}_R$. Using the natural transformation $\mathrm{Ext}^*_{R'}(M,-) \to \mathrm{Ext}^*_R(M,-)$ we get a map of unrolled exact couples and therefore a map of spectral sequences. On E^1-pages it is in bidegree (s,t) for $s \geq 0$ given by

$$\mathrm{Hom}_{R'_*}(\Sigma^{-t}\pi_*(M), I^s_*) \to \mathrm{Hom}_{R_*}(\Sigma^{-t}\pi_*(M), J^s_*).$$

Now, let $P_{*,*} \to \pi_*(M)$ be an R'_*-projective resolution, let $Q_{*,*} \to \pi_*(M)$ be an R_*-projective resolution and let $Q_{*,*} \to P_{*,*}$ be an R_*-linear chain map lifting the identity map of $\pi_*(M)$. Then, by comparing with the maps on total complexes given by

$$\mathrm{Hom}_{R'_*}(P_{*,*}, \Sigma^t I^*_*) \longrightarrow \mathrm{Hom}_{R_*}(Q_{*,*}, \Sigma^t J^*_*),$$

one sees that the map on E^2-pages is also induced by the maps

$$\mathrm{Hom}_{R'_*}(P_{s,*}, \Sigma^t \pi_*(N)) \longrightarrow \mathrm{Hom}_{R_*}(Q_{s,*}, \Sigma^t \pi_*(N)).$$

Lemma 6.12. *In case (2) the* K*-module* $V(1) \wedge_S \mathrm{K}$ *is isomorphic in* $\mathscr{D}_{\mathrm{K}}$ *to an object in the image of the map*

$$\mathscr{D}_{H\mathbb{F}_p \wedge_S \mathrm{K}} \to \mathscr{D}_{\mathrm{K}}$$

induced by the inclusion of K *into the second smash factor of* $H\mathbb{F}_p \wedge_S \mathrm{K}$.

Proof. Since we have $V(1)^S_* \mathrm{K} = E(x)$ with $|x| = 2p-3$ and

$$\mathscr{D}_{\mathrm{K}}(V(1) \wedge_S \mathrm{K}, H\mathbb{F}_p) = \mathrm{Hom}_{\mathrm{K}_0}(\pi_0(V(1) \wedge_S \mathrm{K}), \mathbb{F}_p)$$

we get a map $V(1) \wedge_S \mathrm{K} \to H\mathbb{F}_p$ that is the identity on π_0 and this is part of a distinguished triangle

$$V(1) \wedge_S \mathrm{K} \longrightarrow H\mathbb{F}_p \xrightarrow{g} \Sigma^{2p-2} H\mathbb{F}_p \longrightarrow \Sigma V(1) \wedge_S \mathrm{K}$$

in $\mathscr{D}_{\mathrm{K}}$. It now suffices to show that there is a $\tilde{g}$ that is mapped to g under

$$\mathscr{D}_{H\mathbb{F}_p \wedge_S \mathrm{K}}(H\mathbb{F}_p, \Sigma^{2p-2} H\mathbb{F}_p) \longrightarrow \mathscr{D}_{\mathrm{K}}(H\mathbb{F}_p, \Sigma^{2p-2} H\mathbb{F}_p).$$

That is because then the image of the fiber of $\tilde{g}$ under $\mathscr{D}_{H\mathbb{F}_p\wedge_S \mathrm{K}} \to \mathscr{D}_{\mathrm{K}}$ is isomorphic to $V(1)\wedge_S \mathrm{K}$. We show that

$$\mathscr{D}_{H\mathbb{F}_p\wedge_S \mathrm{K}}(H\mathbb{F}_p, \Sigma^{2p-2}H\mathbb{F}_p) \longrightarrow \mathscr{D}_{\mathrm{K}}\big(H\mathbb{F}_p, \Sigma^{2p-2}H\mathbb{F}_p\big)$$

is an isomorphism. We have a free resolution of $\mathbb{F}_p$ as an $(H\mathbb{F}_p)_*\,\mathrm{K}$-module

$$\ldots \longrightarrow P_{2,*} \xrightarrow{d^2} P_{1,*} \xrightarrow{d^1} P_{0,*} = (H\mathbb{F}_p)_*\,\mathrm{K} \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

where

$$P_{1,*} = (H\mathbb{F}_p)_*\,\mathrm{K}\{\sigma x\} \oplus \bigoplus_{i\geq 1}(H\mathbb{F}_p)_*\,\mathrm{K}\{\sigma\tilde{\xi}_i\} \oplus \bigoplus_{i\geq 2}(H\mathbb{F}_p)_*\{\sigma\tilde{\tau}_i\}$$

and

$$\begin{aligned} d^1(\sigma x) &= x \\ d^1(\sigma\tilde{\xi}_i) &= \tilde{\xi}_i \text{ for } i \geq 1 \\ d^1(\sigma\tilde{\tau}_i) &= \tilde{\tau}_i \text{ for } i \geq 2, \end{aligned}$$

and where $P_{i,*} = 0$ if $i \geq 2$ and $* \leq 2p-3$. We get

$$\mathrm{Hom}_{(H\mathbb{F}_p)_*\,\mathrm{K}}(P_{n,*}, \Sigma^m\Sigma^{2p-2}\mathbb{F}_p) = \begin{cases} \mathbb{F}_p, & \text{if } (n,m) = (1,-1); \\ 0, & \text{if } n+m = 0 \text{ and } (n,m) \neq (1,-1); \\ 0, & \text{if } (n,m) = (0,-1); \\ 0, & \text{if } n \geq 2 \text{ and } n+m = 1. \end{cases}$$

We have a free resolution of $\mathbb{F}_p$ as a K_*-module

$$\ldots \longrightarrow Q_{2,*} \longrightarrow Q_{1,*} \xrightarrow{d^1} Q_{0,*} = \mathrm{K}_* \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

where

$$Q_{1,*} = \bigoplus_{i\geq 1}\Sigma^{2i(p-1)-1}\,\mathrm{K}_* \oplus \Sigma^0\,\mathrm{K}_*,$$

$d^1(\Sigma^0 1) = p \in \pi_0(\mathrm{K}) = \mathbb{Z}_p$ and

$$d^1(\Sigma^{2i(p-1)-1}1) = 1 \in \pi_{2i(p-1)-1}(\mathrm{K}) = \mathbb{Z}/p^{v_p(q^{i(p-1)}-1)},$$

and where $Q_{i,*} = 0$ if $i \geq 2$ and $* \leq 2p-4$. We get

$$\mathrm{Hom}_{\mathrm{K}_*}(Q_{n,*}, \Sigma^m\Sigma^{2p-2}\mathbb{F}_p) = \begin{cases} \mathbb{F}_p, & \text{if } (n,m) = (1,-1); \\ 0, & \text{if } n+m = 0 \text{ and } (n,m) \neq (1,-1); \\ 0, & \text{if } (n,m) = (0,-1). \end{cases}$$

Furthermore, we have a K_*-linear map of chain complexes

$$\begin{tikzcd} \ldots \arrow[r] & Q_{2,*} \arrow[r] \arrow[d, "f_{2,*}"] & Q_{1,*} \arrow[r] \arrow[d, "f_{1,*}"] & K_* \arrow[r] \arrow[d, "f_{0,*}"] & \mathbb{F}_p \arrow[r] \arrow[d, equal] & 0 \\ \ldots \arrow[r] & P_{2,*} \arrow[r] & P_{1,*} \arrow[r] & (H\mathbb{F}_p)_*\,\mathrm{K} \arrow[r] & \mathbb{F}_p \arrow[r] & 0 \end{tikzcd}$$

with $f_0(1) = 1$ and $f_1(\Sigma^{2p-3}1) = \sigma x$. To prove this, it suffices to show that the Hurewicz map $h_{\mathrm{K}} : \mathrm{K}_* \to (H\mathbb{F}_p)_*\,\mathrm{K}$ maps the element

$$1 \in \pi_{2p-3}(\mathrm{K}) = \mathbb{Z}/p^{v_p(q^{(p-1)}-1)}$$

to x. This follows from the commutativity of the diagram

$$\begin{tikzcd} \Sigma\pi_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \arrow[r] \arrow[d, "\Sigma h_{\mathrm{K}(\bar{\mathbb{F}}_l)_p}"] & \pi_*(\mathrm{K}) \arrow[d, "h_{\mathrm{K}}"] \\ \Sigma(H\mathbb{F}_p)_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \arrow[r, "\Delta"] & (H\mathbb{F}_p)_*\,\mathrm{K}, \end{tikzcd}$$

and from $(\Sigma h_{\mathrm{K}(\bar{\mathbb{F}}_l)_p})(\Sigma u^{p-2}) = \Sigma u^{p-2}$ and $\Delta(\Sigma u^{p-2}) = x$.

Using Remark 6.11 we see that the map

$$\mathrm{Ext}^*_{H\mathbb{F}_p\wedge_S \mathrm{K}}(H\mathbb{F}_p, \Sigma^{2p-2}H\mathbb{F}_p) \longrightarrow \mathrm{Ext}^*_{\mathrm{K}}(H\mathbb{F}_p, \Sigma^{2p-2}H\mathbb{F}_p)$$

has a compatible map of spectral sequence that is an isomorphism on E^∞-pages in total degree zero. Since by [23, Theorem 6.7] and [12, Theorem 7.1] the spectral sequences converge strongly, the claim follows. □

Lemma 6.13. *In case (2) the S-module $V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})$ is isomorphic in $\mathscr{D}_S$ to an $H\mathbb{F}_p$-module. The two $\mathbb{F}_p$-vector spaces $V(1)_* \mathrm{THH}(\mathrm{K})$ and*

$$E(x) \otimes E([\sigma\tilde{\xi}_1], [\sigma\tilde{\xi}_2]) \otimes P([\sigma\tilde{\tau}_2]) \otimes \Gamma([\sigma x])$$

have the same dimension in every degree.

Proof. We have that $V(1) \wedge_S \mathrm{K}$ is a cell K-module [18, Proposition III.4.1]. We therefore have an isomorphism in $\mathscr{D}_{\mathrm{K}}$:

$$V(1) \wedge_S \mathrm{THH}(\mathrm{K}) \cong (V(1) \wedge_S \mathrm{K}) \wedge^L_{\mathrm{K}} \mathrm{THH}(\mathrm{K}).$$

By Lemma 6.12 the latter is isomorphic in $\mathscr{D}_{\mathrm{K}}$ to $M \wedge_{\mathrm{K}} \Gamma^{\mathrm{K}} \mathrm{THH}(\mathrm{K})$, where M is an $(H\mathbb{F}_p, \mathrm{K})$-bimodule and $\Gamma^{\mathrm{K}} \mathrm{THH}(\mathrm{K})$ is a cell approximation of the K-module $\mathrm{THH}(\mathrm{K})$. We get that $V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})$ is isomorphic in $\mathscr{D}_S$ to an $H\mathbb{F}_p$-module. As a consequence, we have

$$(H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big) \cong \bigoplus_{i\in I} \Sigma^{n_i}(H\mathbb{F}_p)_* H\mathbb{F}_p,$$

where the n_i are natural numbers such that for all n the cardinality of

$$\{i \in I : n_i = n\}$$

is equal to the dimension of $V(1)_n \mathrm{THH}(\mathrm{K})$. On the other hand $(H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big)$ is isomorphic to

$$E(\epsilon_0, \epsilon_1) \otimes P(\tilde{\xi}_1, \dots) \otimes E(\tilde{\tau}_2, \dots) \otimes E(x) \otimes E([\sigma\tilde{\xi}_1], [\sigma\tilde{\xi}_2]) \otimes P([\sigma\tilde{\tau}_2]) \otimes \Gamma([\sigma x]).$$

This proves the lemma. □

Theorem 6.14. *In case (2) we have an isomorphism of $\mathbb{F}_p$-algebras*

$$V(1)_* \mathrm{THH}(\mathrm{K}) \cong E(x) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes \Gamma(\gamma'_1)$$

with $|x| = 2p-3$, $|\lambda_i| = 2p^i - 1$, $|\mu_2| = 2p^2$ and $|\gamma'_1| = 2p-2$.

Proof. We compute the A_*-comodule primitives in $(H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big)$. Since we have that $(H\mathbb{F}_p)_* V(1) = E(\epsilon_0, \epsilon_1)$ injects into the dual Steenrod algebra via a map of comodule algebras, we can assume that the A_*-comodule action of ϵ_0 is given by

$$\nu(\epsilon_0) = 1 \otimes \epsilon_0 + \bar{\tau}_0 \otimes 1.$$

We define classes in $(H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big)$ by

$$\begin{aligned} \hat{\xi}_1 &\coloneqq \tilde{\xi}_1 - a\epsilon_0 x, \\ \lambda_1 &\coloneqq [\sigma\tilde{\xi}_1] - a\epsilon_0[\sigma x], \\ \lambda_2 &\coloneqq [\sigma\tilde{\xi}_2], \\ \mu_2 &\coloneqq [\sigma\tilde{\tau}_2] - \epsilon_0[\sigma\tilde{\xi}_2], \end{aligned}$$

where a is the element in $\mathbb{F}_p$ that we defined in Proposition 5.8. Then, the A_*-coaction of $\hat{\xi}_1$ is given by

$$\nu(\hat{\xi}_1) = \bar{\xi}_1 \otimes 1 + 1 \otimes \hat{\xi}_1$$

and the classes λ_1, λ_2 and μ_2 are comodule primitives . Let $\gamma_{p^i} \in (H\mathbb{F}_p)_{(2p-2)p^i} \mathrm{THH}(\mathrm{K})$ be the classes defined in Theorem 6.10. We set

$$A'_* \coloneqq E(\epsilon_0, \epsilon_1) \otimes P(\hat{\xi}_1, \tilde{\xi}_2, \dots) \otimes E(\tilde{\tau}_2, \dots).$$

The map of $\mathbb{F}_p$-algebras

$$A'_* \otimes E(x) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes P_p(\gamma_{p^i} | i \geq 0) \to (H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big)$$

is an isomorphism, because it is surjective and both sides have the same dimension over $\mathbb{F}_p$ in every degree. We treat it as the identity. Note that A'_* is a subcomodule algebra of $(H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big)$, because

$$(\tilde{\xi}_1 - a\epsilon_0 x)^{p^n} = \tilde{\xi}_1^{p^n}$$

for $n \geq 1$. It is isomorphic to $(H\mathbb{F}_p)_* V(1) \otimes (H\mathbb{F}_p)_* \ell \cong A_*$. We show by induction on $i \geq 0$ that we can find classes

$$\gamma'_{p^i} \in (H\mathbb{F}_p)_{(2p-2)p^i}\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big)$$

with the following properties:

- The class γ'_{p^i} is a comodule primitive.
- We have $(\gamma'_{p^i})^p = 0$.
- For $D_* = P_p(\gamma'_1, \dots, \gamma'_{p^i}, \gamma_{p^{i+1}}, \gamma_{p^{i+2}} \dots)$ the map

  $$A'_* \otimes E(x) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes D_* \to (H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big)$$

  is an isomorphism.

We set $\gamma'_1 = \gamma_1 = [\sigma x]$. Suppose that $i > 0$ and that we have already defined γ'_{p^j} for $0 \leq j \leq i-1$. The $\mathbb{F}_p$-vector space

$$W := \big(E(x) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes P_p(\gamma'_1, \dots, \gamma'_{p^{i-1}})\big)_{(2p-2)p^i}$$

is included in the subspace V of primitives in $(H\mathbb{F}_p)_{(2p-2)p^i}\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big)$. By Lemma 6.13 we have

$$\dim_{\mathbb{F}_p} V = \dim_{\mathbb{F}_p} W + 1.$$

Therefore, there is a class $b \in V$ with $b \notin W$. The class b cannot be an element of

$$U := \big(A'_* \otimes E(x) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes P_p(\gamma'_1, \dots, \gamma'_{p^{i-1}})\big)_{p^i(2p-2)},$$

because the comodule primitives in this vector space are the elements of W. Therefore, we have

$$b \doteq \gamma_{p^i} + b'$$

for an $b' \in U$. We have

$$(H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big)/I_p \cong P(\hat{\xi}_1, \tilde{\xi}_2, \dots) \otimes P(\mu_2)$$

and

$$\Big(A_* \otimes (H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big)\Big)/I_p \cong P(\bar{\xi}_1, \dots) \otimes P(\hat{\xi}_1, \tilde{\xi}_1, \dots) \otimes P(\mu_2).$$

The $\mathbb{F}_p$-algebra map $\bar{\nu}$, induced on these quotients by the coaction ν, is given by $\bar{\nu}(\mu_2) = 1 \otimes \mu_2$ and

$$\bar{\nu}(\tilde{\xi}_n) = \bar{\xi}_{n-1} \otimes \hat{\xi}_1^{p^{n-1}} + \sum_{i+j=n,\, j \neq 1} \bar{\xi}_i \otimes \tilde{\xi}_j^{p^i}.$$

Since b is a comodule primitive, we get

$$b = \lambda \cdot \mu_2^{p^{i-2}(p-1)} \text{ modulo } I_p$$

for a $\lambda \in \mathbb{F}_p$ if $i \geq 2$, and

$$b = 0 \text{ modulo } I_p$$

if $i = 1$. We set $\gamma'_{p^i} := b - \lambda \cdot \mu_2^{p^{i-2}(p-1)}$ if $i \geq 2$ and $\gamma'_{p^i} := b$ if $i = 1$. Then γ'_{p^i} has the desired properties. We get

$$(H\mathbb{F}_p)_*\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big) = A'_* \otimes E(x) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes P_p(\gamma'_{p^i} | i \geq 0).$$

This finishes the proof. $\square$

Remark 6.15. The methods we used to compute $V(1)_* \operatorname{THH}(\mathrm{K})$ in case (2) do not apply in the cases (1), (3) and (4):

In case (1) the object $V(1) \wedge_S \mathrm{K} \in \mathscr{D}_S$ is not a module over the S-ring spectrum $H\mathbb{F}_p$: Suppose the contrary. Then, we would have

$$(H\mathbb{F}_p)_*\big(V(1) \wedge_S \mathrm{K}\big) \cong A_* \oplus \Sigma^{2p-3} A_*.$$

This is a contradiction to Proposition 5.7.

In case (3) one can compute $(H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K}; H\mathbb{Z}_p)$ using the above methods. But since we have a tensor factor $P_k(y)$ in $(H\mathbb{F}_p)_* \mathrm{K}$, the Hochschild homology of $(H\mathbb{F}_p)_* \mathrm{K}$ is not flat over $(H\mathbb{F}_p)_* \mathrm{K}$. The Bökstedt spectral sequence converging to $(H\mathbb{F}_p)_* \operatorname{THH}(\mathrm{K})$ therefore has no coalgebra structure.

In case (4) the mod p homology of K has a more complicated form and one needs different methods to compute its Hochschild homology.

7. Computations with the Brun spectral sequence

In [22] we have constructed a generalization of the spectral sequence of Brun in [15, Theorem 6.2.10]. We will refer to this generalization as the Brun spectral sequence. In this section we consider the Brun spectral sequence for $\mathrm{K} = \mathrm{K}(\mathbb{F}_q)_p$. We pursue the same strategy that we used in [22] to compute $V(1)_* \operatorname{THH}(\mathrm{ku}_p)$, where ku_p is p-completed connective complex K-theory.

By [22, Theorem 4.11] and [22, Lemma 4.13] we have a Brun spectral sequence of the form

$$E^2_{n,m} = V(1)_m \mathrm{K} \otimes \operatorname{THH}_n(\mathrm{K}; H\mathbb{F}_p) \Longrightarrow V(1)_{n+m} \operatorname{THH}(\mathrm{K}) \tag{14}$$

which is multiplicative. Here, recall that since K is a connective cofibrant commutative S-algebra, we have a map of commutative S-algebras $\mathrm{K} \to H\pi_0(K) = H\mathbb{Z}_p$ realizing the identity on π_0. We can compose this with the map induced by the ring homomorphism $\mathbb{Z}_p \to \mathbb{F}_p$ to get a map $\mathrm{K} \to H\mathbb{F}_p$. We factor the map $\mathrm{K} \to H\mathbb{Z}_p$ in $\mathscr{C}\mathscr{A}_S$ as a cofibration followed by an acyclic fibration:

$$\mathrm{K} \rightarrowtail \hat{H}\mathbb{Z}_p \xrightarrow{\sim}\!\!\!\!\rightarrow H\mathbb{Z}_p.$$

Analogously to the case of ku_p in [22] we have an isomorphism of S-ring spectra

$$\operatorname{THH}(\mathrm{K}; H\mathbb{F}_p) \cong V(0) \wedge^L_S \operatorname{THH}(\mathrm{K}; \hat{H}\mathbb{Z}_p).$$

Again by [22, Theorem 4.11, Lemma 4.13] we have the Brun spectral sequence

$$E^2_{*,*} = V(0)_*(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \hat{H}\mathbb{Z}_p) \otimes \operatorname{THH}_*(\hat{H}\mathbb{Z}_p; H\mathbb{F}_p) \Longrightarrow V(0)_* \operatorname{THH}(\mathrm{K}; \hat{H}\mathbb{Z}_p). \tag{15}$$

In Subsection 7.1 we will compute the ring $V(0)_*(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \hat{H}\mathbb{Z}_p)$, in Subsection 7.2 we will consider the spectral sequence (15), and finally in Subsection 7.3 we will consider the spectral sequence (14).

7.1. The mod p homotopy of $H\mathbb{Z}_p \wedge^L_{\mathrm{K}(\mathbb{F}_q)_p} H\mathbb{Z}_p$.

In this subsection we will compute the graded $\mathbb{F}_p$-algebra $V(0)_*(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \hat{H}\mathbb{Z}_p)$.

Remark 7.1. A tempting strategy to compute $V(0)_*(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \hat{H}\mathbb{Z}_p)$ would be to use an Eilenberg-Moore type spectral sequence [18, Section IV.6]

$$E^2_{*,*} = \operatorname{Tor}^{V(0)_* \mathrm{K}}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow V(0)_*(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \hat{H}\mathbb{Z}_p).$$

Such a spectral sequence would have to collapse at the E^2-page and would yield

$$V(0)_*(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \hat{H}\mathbb{Z}_p) = \Gamma(\sigma x) \otimes E(\sigma y) \tag{16}$$

with $|\sigma x| = 2r$ and $|\sigma y| = 2r+1$. But this requires $V(0) \wedge_S \mathrm{K}$ to be an S-algebra. As in [4, Example 3.3] one can use that there is a p-fold Massey product $\langle \bar{\tau}_0, \ldots, \bar{\tau}_0 \rangle = -\bar{\xi}_1$ in $(H\mathbb{F}_p)_* H\mathbb{F}_p$ defined with no indeterminacy to show that $V(0) \wedge_S \mathrm{K}$ is no S-algebra in case (1). We therefore cannot use the above strategy, but we still obtain (16).

Lemma 7.2. *We have a multiplicative spectral sequence of the form*

$$E^2_{*,*} = \mathrm{Tor}^{\pi_*(\mathrm{K}(\bar{\mathbb{F}}_l)_p)}_{*,*}\big(B_*, \mathbb{Z}_p\big) \Longrightarrow V(0)_*(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \hat{H}\mathbb{Z}_p), \tag{17}$$

where

$$B_* \cong \pi_*\big((V(0) \wedge_S H\mathbb{Z}_p) \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \cong \pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big).$$

Here, we use that $V(0) \wedge_S H\mathbb{Z}_p$ is an $H\mathbb{Z}_p$-ring spectrum that is isomorphic to $H\mathbb{F}_p$ and that one therefore gets isomorphic K-ring spectra by applying the lax symmetric monoidal functor $\mathscr{D}_{H\mathbb{Z}_p} \to \mathscr{D}_{\mathrm{K}}$.

Proof. The canonical map in $\mathscr{D}_{\mathrm{K}}$

$$\hat{H}\mathbb{Z}_p \wedge^L_{\mathrm{K}} H\mathbb{Z}_p \cong \hat{H}\mathbb{Z}_p \wedge^L_{\mathrm{K}} \hat{H}\mathbb{Z}_p \longrightarrow \hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \hat{H}\mathbb{Z}_p \tag{18}$$

is an isomorphism of K-ring spectra. The diagram

$$\begin{tikzcd} \mathrm{K} \arrow[r] \arrow[d] & H\mathbb{Z}_p \arrow[d, "\mathrm{id}"] \\ \mathrm{K}(\bar{\mathbb{F}}_l)_p \arrow[r] & H\mathbb{Z}_p \end{tikzcd}$$

is homotopy commutative in $\mathscr{C}\mathscr{A}_S$ and therefore in the category of S-modules [18, Proposition VII.2.11]. We get that the left-most K-ring spectrum in (18) is isomorphic to $\hat{H}\mathbb{Z}_p \wedge^L_{\mathrm{K}} H\mathbb{Z}_p$, but where the K-module structure of $H\mathbb{Z}_p$ is now given by

$$\mathrm{K} \longrightarrow \mathrm{K}(\bar{\mathbb{F}}_l)_p \longrightarrow H\mathbb{Z}_p.$$

This is isomorphic as an K-ring spectrum to $\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} H\mathbb{Z}_p$. We have isomorphisms of commutative S-algebras

$$\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} H\mathbb{Z}_p \cong \hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \big(\mathrm{K}(\bar{\mathbb{F}}_l)_p \wedge_{\mathrm{K}(\bar{\mathbb{F}}_l)_p} H\mathbb{Z}_p\big) \cong \big(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \wedge_{\mathrm{K}(\bar{\mathbb{F}}_l)_p} H\mathbb{Z}_p.$$

Since cofibrations are stable under cobase change, the map

$$\mathrm{K}(\bar{\mathbb{F}}_l)_p \longrightarrow \hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p$$

is a cofibration of commutative S-algebras. Therefore, the canonical map

$$\big(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \wedge^L_{\mathrm{K}(\bar{\mathbb{F}}_l)_p} H\mathbb{Z}_p \longrightarrow \big(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \wedge_{\mathrm{K}(\bar{\mathbb{F}}_l)_p} H\mathbb{Z}_p$$

in $\mathscr{D}_{\mathrm{K}(\bar{\mathbb{F}}_l)_p}$ is an isomorphism of $\mathrm{K}(\bar{\mathbb{F}}_l)_p$-ring spectra. Thus, we have an isomorphism of S-ring spectra

$$V(0) \wedge^L_S (\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \hat{H}\mathbb{Z}_p) \cong V(0) \wedge^L_S \Big(\big(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \wedge^L_{\mathrm{K}(\bar{\mathbb{F}}_l)_p} H\mathbb{Z}_p\Big).$$

By [22, Remark 4.10] the latter S-ring spectrum is isomorphic to

$$\Big(V(0) \wedge_S \big(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)\Big) \wedge^L_{\mathrm{K}(\bar{\mathbb{F}}_l)_p} H\mathbb{Z}_p.$$

Here, note that $V(0) \wedge_S \big(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$ is an $\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p$-ring spectrum and that we therefore get a $\mathrm{K}(\bar{\mathbb{F}}_l)_p$-ring spectrum after applying the lax symmetric monoidal functor $\mathscr{D}_{\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p} \to \mathscr{D}_{\mathrm{K}(\bar{\mathbb{F}}_l)_p}$. By [9, Lemma 1.3] we get a multiplicative spectral sequence

$$E^2_{*,*} = \mathrm{Tor}^{\pi_*(\mathrm{K}(\bar{\mathbb{F}}_l)_p)}_{*,*}(B_*, \mathbb{Z}_p) \Longrightarrow V(0)_*(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \hat{H}\mathbb{Z}_p)$$

with $B_* = \pi_*\Big(V(0) \wedge_S \big(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)\Big)$.

We have an isomorphism of S-ring spectra

$$V(0) \wedge_S \big(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \cong V(0) \wedge^L_S \big(\hat{H}\mathbb{Z}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \cong V(0) \wedge^L_S \big(H\mathbb{Z}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big).$$

Again by [22, Remark 4.10] this is isomorphic to $(V(0)\wedge_S H\mathbb{Z}_p)\wedge^L_{\mathrm{K}}\mathrm{K}(\bar{\mathbb{F}}_l)_p$ as an S-ring spectrum. The last three identifications are induced by maps under $\mathrm{K}(\bar{\mathbb{F}}_l)_p$ in $\mathscr{D}_S$. ☐

In the following lemmas we compute the $\pi_*(\mathrm{K}(\bar{\mathbb{F}}_l)_p)$-algebra $\pi_*(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p)$.

Lemma 7.3. *We have*

$$\pi_n(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p) = \begin{cases} \mathbb{F}_p, & n = 2i,\ i \geq 0; \\ 0, & \textit{otherwise}. \end{cases}$$

Moreover, the map $\pi_*(\mathrm{K}(\bar{\mathbb{F}}_l)_p) \to \pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$ *factors as*

$$\begin{tikzcd} \pi_*(\mathrm{K}(\bar{\mathbb{F}}_\ell)_p) \cong \mathbb{Z}_p[u] \arrow[rr] \arrow[dr] & & \pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_\ell)_p\big), \\ & P_r(u) \arrow[ur] & \end{tikzcd}$$

and $P_r(u) \to \pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$ *is an isomorphism in degrees* $< 2r$.

Proof. Define $X \coloneqq V(0) \wedge_S H\mathbb{Z}_p$ and $Y \coloneqq V(0) \wedge_S \mathrm{K}$. The K-module map $g : Y \to X$ defines a map of K-ring spectra. Applying $Y \wedge^L_{\mathrm{K}} (-) \to X \wedge^L_{\mathrm{K}} (-)$ to the distinguished triangle (6) we get a map between long exact sequences

$$\begin{tikzcd} \arrow[r] & \pi_n(X \wedge^L_{\mathrm{K}} \mathrm{K}) \arrow[r] & \pi_n(X \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p) \arrow[r] & \pi_{n-2}(X \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p) \arrow[r] & \cdots \\ \arrow[r] & \pi_n(Y \wedge^L_{\mathrm{K}} \mathrm{K}) \arrow[r] \arrow[u] & \pi_n(Y \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p) \arrow[r] \arrow[u, "(g\wedge \mathrm{id})_n"] & \pi_{n-2}(Y \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p) \arrow[r] \arrow[u, "(g\wedge \mathrm{id})_{n-2}"] & \cdots . \end{tikzcd} \tag{19}$$

Using the Tor spectral sequence one gets that $X \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p$ is connective. Using that $\pi_*(X \wedge^L_{\mathrm{K}} K)$ is $\mathbb{F}_p$ concentrated in degree 0 the above long exact sequence yields

$$\pi_n(X \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p) = \begin{cases} \mathbb{F}_p, & n = 0; \\ 0, & n = 1; \end{cases}$$

and $\pi_n(X \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p) \cong \pi_{n-2}(X \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p)$ for $n \geq 2$. This shows the first part of the statement. We now show the second part of the statement: The map $\mathrm{K}(\bar{\mathbb{F}}_l)_p \to X \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p$ factors in $\mathscr{D}_{\mathrm{K}}$ as

$$\begin{tikzcd} \mathrm{K}(\bar{\mathbb{F}}_l)_p \arrow[r] \arrow[dr] & X \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \\ & Y \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p \arrow[u] \end{tikzcd}$$

The map $\pi_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \to \pi_*\big(Y \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$ identifies with the canonical map $\mathbb{Z}_p[u] \to P(u)$. Hence, it suffices to show that

$$(g \wedge \mathrm{id})_* : \pi_*\big(Y \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \to \pi_*\big(X \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$$

maps u^r to zero and is an isomorphism in degrees $< 2r$. It is clear that it is an isomorphism in odd degrees, because in these degrees both sides are zero. It is also clear that it is an isomorphism in degree zero: In degree zero both sides are equal to $\mathbb{F}_p$, and the map is not zero because it is a map of rings and therefore maps the unit to the unit. We now suppose that $0 < n < 2r$ is even and that we have already shown that $(g \wedge \mathrm{id})_m$ is an isomorphism for all even $0 \leq m < n$. By Lemma 4.1 $\pi_n(Y \wedge^L_{\mathrm{K}} \mathrm{K})$ is zero. Therefore, diagram (19) is given by

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & \mathbb{F}_p \arrow[r] & \mathbb{F}_p \arrow[r] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] \arrow[u] & \mathbb{F}_p \arrow[r] \arrow[u] & \mathbb{F}_p \arrow[r] \arrow[u] & \cdots . \end{tikzcd}$$

The right vertical map is an isomorphism by the induction hypothesis. Thus, the vertical map in the middle is also an isomorphism, and this map is $(g \wedge \mathrm{id})_n$. We conclude that $(g \wedge \mathrm{id})_n$ is an isomorphism for all $n < 2r$. We now consider diagram (19) for $n = 2r$. Since by the proof of Lemma 4.1 the map $\pi_{2r}(Y \wedge^L_{\mathrm{K}} \mathrm{K}) \to \pi_{2r}\big(Y \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$ is an isomorphism, it is given by

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{F}_p \arrow[r, "\cong"] & \mathbb{F}_p \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathbb{F}_p \arrow[r, "\cong"] \arrow[u] & \mathbb{F}_p \arrow[r, "0"] \arrow[u] & \mathbb{F}_p \arrow[r] \arrow[u, "\cong"'] & \cdots .
\end{tikzcd}$$

It follows that $(g \wedge \mathrm{id})_{2r}$ is zero. $\square$

Let F be the map

$$\pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_\ell)_p\big) \xrightarrow{(\mathrm{id}\wedge f)_*} \Sigma^2\pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_\ell)_p\big),$$

where $f : \mathrm{K}(\bar{\mathbb{F}}_\ell)_p \to S^2_{\mathrm{K}} \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_\ell)_p$ is the morphism in the distinguished triangle (6). Let G be the map

$$\pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_\ell)_p\big) \xrightarrow{\mathrm{id}\wedge(\phi^q-\mathrm{id})_*} \pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_\ell)_p\big).$$

The following diagram commutes:

$$\begin{tikzcd}
\pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \arrow[rr, "G"] \arrow[dr, "F"'] & & \pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big). \\
& \Sigma^2\pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \arrow[ur, "(-)\cdot u"'] &
\end{tikzcd} \tag{20}$$

To determine the multiplicative structure of $\pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$ we need the following lemma:

Lemma 7.4. *Let $a, b \in \pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$. The following equations hold:*

$$F(ab) = F(a)b + aF(b) + F(a)G(b) \tag{21}$$

$$F(a^n) = F(a)\Big(\sum_{i=0}^{n-1} \binom{n}{n-1-i} a^{n-1-i}G(a)^i\Big). \tag{22}$$

Proof. Formula (21) follows from Lemma 3.10. Formula (22) follows from formula (21) by induction. $\square$

Lemma 7.5. *There is an isomorphism of $\pi_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$-algebras*

$$\pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \cong P_r(u) \otimes \Gamma(\sigma x),$$

where $|\sigma x| = 2r$.

Proof. By Lemma 7.3 the unit $\pi_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \to \pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$ induces a map $P_r(u) \to \pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$ that is an isomorphism in degrees $< 2r$. For $i \geq 0$ choose a non-zero class

$$\gamma_{p^i}(\sigma x) \in \pi_{2p^ir}\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \cong \mathbb{F}_p.$$

By formula (22) we have

$$F(\gamma_{p^i}(\sigma x)^p) = F\big(\gamma_{p^i}(\sigma x)\big)G\big(\gamma_{p^i}(\sigma x)\big)^{p-1}.$$

Because of the commutativity of the diagram (20) the class $G\big(\gamma_{p^i}(\sigma x)\big)^{p-1}$ is divisible by u^{p-1} and is therefore zero. Since F is injective in positive degrees by the proof of Lemma 7.3, it follows that $\gamma_{p^i}(\sigma x)^p$ is zero. We therefore have a well-defined map of $\pi_*(\mathrm{K}(\bar{\mathbb{F}}_l)_p)$-algebras

$$P_p(\sigma x, \gamma_p(\sigma x), \dots) \otimes P_r(u) \to \pi_*\big(H\mathbb{F}_p \wedge^L_{\mathrm{K}} \mathrm{K}(\bar{\mathbb{F}}_l)_p\big).$$

We claim that it is an isomorphism. Since both sides are one-dimensional over $\mathbb{F}_p$ in each even non-negative degree and zero in all other degrees, it suffices to show the following: For numbers $j \in \{0, \ldots, r-1\}$ and $i_k \in \{0, \ldots, p-1\}$ that are almost all zero, the classes

$$\sigma x^{i_0} \gamma_p(\sigma x)^{i_1} \cdots u^j \in \pi_*\big(H\mathbb{F}_p \wedge_{\mathrm{K}}^L \mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \tag{23}$$

are non-zero. We show this by induction on the degree of $b = \sigma x^{i_0} \gamma_p(\sigma x)^{i_1} \cdots u^j$. If $|b| \in \{0, \ldots, 2r-2\}$, we have $b = u^j$ for some $j \in \{0, \ldots, r-1\}$ and we already know that the claim is true. We now assume that $|b| \geq 2r$ and that we have already proven that all classes of the form (23) with degree less than $|b|$ are non-zero. Let k be minimal with $i_k \neq 0$. We first consider the case $b = \gamma_{p^k}(\sigma x)^{i_k}$. If $i_k = 1$ the claim is true by definition of $\gamma_{p^k}(\sigma x)$. If $i_k \geq 2$ we write

$$F(b) = F\big(\gamma_{p^k}(\sigma x)\big)\bigg(\sum_{i=0}^{i_k-1} \binom{i_k}{i_k-1-i} \gamma_{p^k}(\sigma x)^{i_k-1-i} G\big(\gamma_{p^k}(\sigma x)\big)^i\bigg).$$

Since F is injective in positive degrees, we know by the induction hypothesis that

$$F\big(\gamma_{p^k}(\sigma x)\big) \doteq \Sigma^2 \sigma x^{p-1} \gamma_p(\sigma x)^{p-1} \cdots \gamma_{p^{k-1}}(\sigma x)^{p-1} u^{r-1}.$$

Since $G\big(\gamma_{p^k}(\sigma x)\big)^i$ is divisible by u for $i > 0$, we get

$$F(b) \doteq \Sigma^2 \sigma x^{p-1} \gamma_p(\sigma x)^{p-1} \cdots \gamma_{p^{k-1}}(\sigma x)^{p-1} \gamma_{p^k}(\sigma x)^{i_k-1} u^{r-1}.$$

By the induction hypothesis we know that this is non-zero. Thus, $b \neq 0$ in this case. In the other cases we write

$$\begin{aligned} F(b) = {} & F\big(\gamma_{p^k}(\sigma x)^{i_k}\big) \gamma_{p^{k+1}}(\sigma x)^{i_{k+1}} \cdots u^j \\ & + \gamma_{p^k}(\sigma x)^{i_k} F\big(\gamma_{p^{k+1}}(\sigma x)^{i_{k+1}} \cdots u^j\big) \\ & + F\big(\gamma_{p^k}(\sigma x)^{i_k}\big) G\big(\gamma_{p^{k+1}}(\sigma x)^{i_{k+1}} \cdots u^j\big). \end{aligned} \tag{24}$$

By the induction hypothesis we know that $\gamma_{p^k}(\sigma x)^{i_k}$ is non-zero. Thus, we have

$$F\big(\gamma_{p^k}(\sigma x)^{i_k}\big) \doteq \Sigma^2 \sigma x^{p-1} \cdots \gamma_{p^{k-1}}(\sigma x)^{p-1} \gamma_{p^k}(\sigma x)^{i_k-1} u^{r-1}.$$

Since $G\big(\gamma_{p^{k+1}}(\sigma x)^{i_{k+1}} \cdots u^j\big)$ is divisible by u, the third summand of the right side of (24) is zero. We consider the case $j > 0$. Then the first summand in (24) is zero, too. By the induction hypothesis $\gamma_{p^{k+1}}(\sigma x)^{i_{k+1}} \cdots u^j$ is non-zero, and so we get

$$F\big(\gamma_{p^{k+1}}(\sigma x)^{i_{k+1}} \cdots u^j\big) \doteq \Sigma^2 \gamma_{p^{k+1}}(\sigma x)^{i_{k+1}} \cdots u^{j-1}$$

and

$$F(b) \doteq \Sigma^2 \gamma_{p^k}(\sigma x)^{i_k} \gamma_{p^{k+1}}(\sigma x)^{i_{k+1}} \cdots u^{j-1}.$$

By the induction hypothesis this is non-zero and thus we get $b \neq 0$. We now consider the case $j = 0$. Let $l > k$ be minimal with $i_l \neq 0$. By the induction hypothesis we have

$$F\big(\gamma_{p^{k+1}}(\sigma x)^{i_{k+1}} \cdots u^j\big) \doteq \Sigma^2 \sigma x^{p-1} \cdots \gamma_{p^{l-1}}(\sigma x)^{p-1} \gamma_{p^l}(\sigma x)^{i_l-1} \cdots u^{r-1}.$$

Because of $\gamma_{p^k}(\sigma x)^{i_k+p-1} = 0$ the second summand in (24) is zero. We get that

$$F(b) \doteq \Sigma^2 \sigma x^{p-1} \cdots \gamma_{p^{k-1}}(\sigma x)^{p-1} \gamma_{p^k}(\sigma x)^{i_k-1} \gamma_{p^{k+1}}(\sigma x)^{i_{k+1}} \cdots u^{r-1}.$$

By the induction hypothesis this is not zero. Thus, b is non-zero. □

Lemma 7.5 implies the following:

Lemma 7.6. *We have an isomorphism of graded rings*

$$V(0)_*(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \hat{H}\mathbb{Z}_p) \cong \Gamma(\sigma x) \otimes E(\sigma y)$$

with $|\sigma x| = 2r$ and $|\sigma y| = 2r+1$.

Proof. The sequence

$$0 \longrightarrow \Sigma^2\pi_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \xrightarrow{\cdot u} \pi_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \longrightarrow \mathbb{Z}_p \longrightarrow 0 \tag{25}$$

is a free resolution of $\mathbb{Z}_p$ as a $\pi_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big)$-module. Thus, the E^2-page of the spectral sequence (17) is the homology of

$$0 \longrightarrow \Sigma^2\Gamma(\sigma x) \otimes P_r(u) \xrightarrow{\cdot u} \Gamma(\sigma x) \otimes P_r(u) \longrightarrow 0,$$

which is

$$\Gamma(\sigma x) \otimes \Sigma^2\mathbb{F}_p\{u^{r-1}\}$$

in homological degree one and $\Gamma(\sigma x)$ in homological degree zero. The spectral sequence has to collapse at the E^2-page because the E^2-page is concentrated in columns 0 and 1.

We denote the free resolution (25) by $F_{*,*}$. Let σu be the element $\Sigma^2 1$ in $F_{1,2}$. The usual multiplication on

$$\pi_*\big(\mathrm{K}(\bar{\mathbb{F}}_l)_p\big) \otimes_{\mathbb{Z}} E_{\mathbb{Z}}(\sigma u)$$

defines a map of complexes

$$F_{*,*} \otimes_{\pi_*(\mathrm{K}(\bar{\mathbb{F}}_l)_p)} F_{*,*} \to F_{*,*}$$

that lifts the multiplication of $\mathbb{Z}_p$. This implies that the E^2-page is multiplicatively given by $\Gamma(\sigma x) \otimes E(\sigma y)$, where σy represents $u^{r-1}\sigma u$. Since $\Gamma(\sigma x)$ lies in column zero, there are no multiplicative extensions. □

7.2. **The algebra** $\mathrm{THH}_*(\mathrm{K}(\mathbb{F}_q)_p; H\mathbb{F}_p)$**.** In this subsection we consider the spectral sequence (15):

$$E^2_{*,*} = V(0)_*(\hat{H}\mathbb{Z}_p \wedge_{\mathrm{K}} \hat{H}\mathbb{Z}_p) \otimes \mathrm{THH}_*(\hat{H}\mathbb{Z}_p; H\mathbb{F}_p) \Longrightarrow V(0)_*\,\mathrm{THH}(\mathrm{K}; \hat{H}\mathbb{Z}_p).$$

By Bökstedt's computations [13] and Lemma 7.6 we have

$$E^2_{*,*} = \Gamma(\sigma x) \otimes E(\sigma y) \otimes E(\lambda_1) \otimes P(\mu_1).$$

with $|\sigma x| = (0, 2r)$, $|\sigma y| = (0, 2r+1)$, $|\lambda_1| = (2p-1, 0)$ and $|\mu_1| = (2p, 0)$. We will prove the following result:

Theorem 7.7. *In case (1) we have an isomorphism of rings*

$$\mathrm{THH}_*(\mathrm{K}; H\mathbb{F}_p) \cong P(\mu_1^p) \otimes E(\mu_1^{p-1}\sigma y,\ \lambda_1\sigma x^{p-1}) \otimes \Gamma\big(\gamma_p(\sigma x)\big).$$

In case (2) we have an isomorphism of rings

$$\mathrm{THH}_*(\mathrm{K}; H\mathbb{F}_p) \cong P(\mu_1^p) \otimes E(\mu_1^{p-1}\sigma y,\ \lambda_1) \otimes \Gamma(\sigma x).$$

In case (3) we have an isomorphism of rings

$$\mathrm{THH}_*(\mathrm{K}; H\mathbb{F}_p) \cong P(\mu_1) \otimes E(\sigma y,\ \lambda_1) \otimes \Gamma(\sigma x).$$

In case (4) we have an isomorphism of rings

$$\mathrm{THH}_*(\mathrm{K}; H\mathbb{F}_p) \cong P(\mu_1) \otimes E(\sigma y,\ \lambda_1) \otimes \Gamma(\sigma x),$$

at least if we assume $r \neq 1$.

In order to compute the differentials in the Brun spectral sequence (15) we need an additional spectral sequence:

Lemma 7.8. *Let $B \to C$ be a morphism between cofibrant commutative S-algebras. Then, there is a multiplicative spectral sequence of the form*

$$E^2_{*,*} = \mathrm{Tor}^{\pi_*(C \wedge_S B)}_{*,*}(C_*, C_*) \Longrightarrow \mathrm{THH}_*(B, C).$$

Proof. This follows by using a method of Veen [39]. Writing $S^1 = D^1 \amalg_{S^0} D^1$ one sees that $\mathrm{THH}(B; C) \cong C \wedge^L_{C \wedge_S B} C$ as S-ring spectra. □

Lemma 7.9. *In case (1) we have*

$$V(0)_n \operatorname{THH}(\mathrm{K}, \hat{H}\mathbb{Z}_p) = 0$$

for $1 \leq n \leq 2p$.

Proof. By Lemma 7.8 we have a spectral sequence of the form

$$E^2_{*,*} = \operatorname{Tor}^{(H\mathbb{F}_p)_* \mathrm{K}}_{*,*}\big(\mathbb{F}_p, \mathbb{F}_p\big) \Longrightarrow \operatorname{THH}_*(\mathrm{K}; H\mathbb{F}_p\big).$$

Using Proposition 5.7 one gets

$$E^2_{*,*} = E(\sigma\tilde{\xi}_1^p, \sigma\tilde{\xi}_2, \dots) \otimes \Gamma(\sigma b, \sigma\tilde{\tau}_2, \dots)$$

with $|\sigma x| = (1, |x|)$. Obviously, this bigraded abelian group is zero in the total degrees $n = 1, \dots, 2p$. □

Lemma 7.10. *In case (1) the differentials of the spectral sequence (15) are given by*

$$d^{2p-1}(\lambda_1) \doteq \sigma x \quad \textit{and} \quad d^{2p}(\mu_1) \doteq \sigma y.$$

We have

$$E^\infty_{*,*} = P(\mu_1^p) \otimes E(\mu_1^{p-1}\sigma y,\, \lambda_1 \sigma x^{p-1}) \otimes \Gamma\big(\gamma_p(\sigma x)\big).$$

There are no multiplicative extensions.

Proof. The E^2-page of (15) is multiplicatively generated by the classes λ_1, μ_1, $\gamma_{p^i}(\sigma x)$ and σy. The classes $\gamma_{p^i}(\sigma x)$ and σy are infinite cycles because they lie in column zero. For bidegree reasons the only possible differential on λ_1 and μ_1 are

$$d^{2p-1}(\lambda_1) \doteq \sigma x \quad \text{and} \quad d^{2p}(\mu_1) \doteq \sigma y.$$

We conclude that $d^s = 0$ for $s = 2, \dots, 2p-2$. If $d^{2p-1}(\lambda_1)$ was zero, the class λ_1 would be a permanent cycle. This would contradict the fact that we have

$$V(0)_{2p-1} \operatorname{THH}(\mathrm{K}, \hat{H}\mathbb{Z}_p) = 0$$

by Lemma 7.9. Thus, we have $d^{2p-1}(\lambda_1) \doteq \sigma x$ and

$$E^{2p}_{*,*} = E(\lambda_1 \sigma x^{p-1}) \otimes \Gamma\big(\gamma_p(\sigma x)\big) \otimes P(\mu_1) \otimes E(\sigma y).$$

This algebra is generated by the classes $\lambda_1 \sigma x^{p-1}$, $\gamma_{p^i}(\sigma x)$ for $i \geq 1$, μ_1 and σy. There cannot be any non-trivial differential on $\lambda_1 \sigma x^{p-1}$, because this class lies in column $2p-1$. Thus, the only possible differential on a generator is

$$d^{2p}(\mu_1) \doteq \sigma y.$$

This differential must exist, because otherwise μ_1 would survive to the E^∞-page, which would contradict Lemma 7.9. It follows that

$$E^{2p+1}_{*,*} = P(\mu_1^p) \otimes E(\mu_1^{p-1}\sigma y) \otimes E(\lambda_1 \sigma x^{p-1}) \otimes \Gamma\big(\gamma_p(\sigma x)\big)$$

as $\mathbb{F}_p$-algebras. Now, the spectral sequence has to collapse: The class $\mu_1^{p-1}\sigma y$ has total degree $2p^2 - 1$, and therefore its differentials have total degree $2p^2 - 2$. All non-trivial classes in an even degree $< 2p^2 = |\mu_1^p|$ lie in $\Gamma\big(\gamma_p(\sigma x)\big)$. Since the total degree of every class in $\Gamma\big(\gamma_p(\sigma x)\big)$ is divisible by p, it follows that $\mu_1^{p-1}\sigma y$ is an infinite cycle. The class μ_1^p is an infinite cycle, too: The differentials of μ_1^p have total degree $2p^2 - 1$. There cannot be any differential

$$d^i(\mu_1^p) \doteq \mu_1^{p-1}\sigma y$$

for $i \geq 2p+1$, because μ_1^p lies in column $2p^2$ and $\mu_1^{p-1}\sigma y$ lies in column $2p^2 - 2p$. The classes in $\mathbb{F}_p\{\sigma x^{p-1}\lambda_1\} \otimes \Gamma\big(\gamma_p(\sigma x)\big)$ have total degrees

$$2p - 1 + (2p-2)(p-1) + p(2p-2)i$$

for $i \geq 0$. Since this is always 1 modulo p, and since $2p^2 - 1$ is -1 modulo p, the spectral sequence collapses at the E^{2p+1}-page. Since $\Gamma\big(\gamma_p(\sigma x)\big)$ lies in column zero, there cannot be any multiplicative extensions. □

Lemma 7.11. *In case (2) we have*

$$\dim_{\mathbb{F}_p}\big(V(0)_n\,\mathrm{THH}(\mathrm{K};\hat{H}\mathbb{Z}_p)\big) = \begin{cases} 1, & n = 2p-2; \\ 0, & n = 2p; \\ 2, & n = 2p^2-1; \\ 1, & n = 2p^2. \end{cases}$$

Proof. As in case (1) we consider the spectral sequence

$$E^2_{*,*} = \mathrm{Tor}^{(H\mathbb{F}_p)_*\,\mathrm{K}}_{*,*}\big(\mathbb{F}_p,\mathbb{F}_p\big) \Longrightarrow \mathrm{THH}_*(\mathrm{K};H\mathbb{F}_p).$$

Using Proposition 5.8, we obtain

$$E^2_{*,*} = E(\sigma\tilde{\xi}_1,\dots)\otimes\Gamma(\sigma x,\sigma\tilde{\tau}_2,\dots)$$

with $|\sigma a| = (1,|a|)$. This bigraded abelian group is zero in total degree $2p$, and therefore we have

$$V(0)_{2p}\,\mathrm{THH}(\mathrm{K};\hat{H}\mathbb{Z}_p) = 0.$$

It is zero in total degree $2p-3$, $\mathbb{F}_p\{\sigma x\}$ in total degree $2p-2$ and $\mathbb{F}_p\{\sigma\tilde{\xi}_1\}$ in total degree $2p-1$. Since $\sigma\tilde{\xi}_1$ lies in column 1, it is an infinite cycle. Therefore, we get

$$\dim_{\mathbb{F}_p}\big(V(0)_{2p-2}\,\mathrm{THH}(\mathrm{K};\hat{H}\mathbb{Z}_p)\big) = 1.$$

The E^2-page is given by $\mathbb{F}_p\{\gamma_{p+1}(\sigma x)\}$ in total degree $2p^2-2$, by

$$\mathbb{F}_p\{\sigma\tilde{\xi}_2\}\oplus\mathbb{F}_p\{\sigma\tilde{\xi}_1\gamma_p(\sigma x)\}$$

in total degree $2p^2-1$, by $\mathbb{F}_p\{\sigma\tilde{\tau}_2\}$ in total degree $2p^2$ and by zero in total degree $2p^2+1$. Since $\sigma\tilde{\tau}_2$ lies in column 1, it is an infinite cycle and we get

$$\dim_{\mathbb{F}_p}\big(V(0)_{2p^2}\,\mathrm{THH}(\mathrm{K},\hat{H}\mathbb{Z}_p)\big) = 1.$$

The classes $\sigma\tilde{\xi}_2$ and $\sigma\tilde{\xi}_1\gamma_p(\sigma x)$ are also infinite cycles: For $\sigma\tilde{\xi}_2$ this is clear, because this class lies in column 1. For $\sigma\tilde{\xi}_1\gamma_p(\sigma x)$ it follows since no differential of $\sigma\tilde{\xi}_1\gamma_p(\sigma x)$ can hit $\gamma_{p+1}(\sigma x)$, because both classes lie in column $p+1$. We get

$$\dim_{\mathbb{F}_p}\big(V(0)_{2p^2-1}\,\mathrm{THH}(\mathrm{K};\hat{H}\mathbb{Z}_p)\big) = 2.$$

□

Lemma 7.12. *In case (2) the spectral sequence (15) has the differential*

$$d^{2p}(\mu_1) \doteq \sigma y.$$

We have

$$E^\infty_{*,*} = E(\lambda_1,\,\mu_1^{p-1}\sigma y)\otimes\Gamma(\sigma x)\otimes P(\mu_1^p).$$

There are no multiplicative extensions.

Proof. As in case (1) the only possible differentials on the canonical algebra generators of the E^2-page are

$$d^{2p-1}(\lambda_1) \doteq \sigma x \quad \text{and} \quad d^{2p}(\mu_1) \doteq \sigma y.$$

Hence, we have $d^i = 0$ for $i = 2,\dots,2p-2$. The E^{2p-1}-page is given by $\mathbb{F}_p\{\sigma x\}$ in total degree $2p-2$. If there was a differential $d^{2p-1}(\lambda_1) \doteq \sigma x$, the E^∞-page would be zero in total degree $2p-2$. This would contradict Lemma 7.11. Thus, we have $d^{2p-1} = 0$. If $d^{2p}(\mu_1)$ was zero, the class μ_1 would survive to the E^∞-page. This would contradict $\dim_{\mathbb{F}_p}\big(V(0)_{2p}\,\mathrm{THH}(\mathrm{K},\hat{H}\mathbb{Z}_p)\big) = 0$. Therefore, we get $d^{2p}(\mu_1) \doteq \sigma y$ and

$$E^{2p+1}_{*,*} = P(\mu_1^p)\otimes E(\mu_1^{p-1}\sigma y)\otimes E(\lambda_1)\otimes\Gamma(\sigma x).$$

The E^{2p+1}-page is given by

$$\mathbb{F}_p\{\mu_1^{p-1}\sigma y\}\oplus\mathbb{F}_p\{\lambda_1\cdot\gamma_p(\sigma x)\}$$

in total degree $2p^2-1$ and by $\mathbb{F}_p\{\mu_1^p\}$ in total degree $2p^2$. So, by Lemma 7.11, there cannot be any differentials on μ_1^p or $\sigma y\mu_1^{p-1}$. We conclude that

$$E^\infty_{*,*} = E^{2p+1}_{*,*}.$$

□

Lemma 7.13. *In case (3) we have*

$$\dim_{\mathbb{F}_p} V(0)_{2p-2}\operatorname{THH}(\mathrm{K};\hat{H}\mathbb{Z}_p) = 1.$$

There is a class

$$\sigma x \in V(0)_{2r}\operatorname{THH}(\mathrm{K},\hat{H}\mathbb{Z}_p)$$

and a class

$$\sigma y \in V(0)_{2r+1}\operatorname{THH}(\mathrm{K},\hat{H}\mathbb{Z}_p)$$

such that $\sigma x^{k-1}\sigma y \neq 0$.

Proof. As in the other cases we use the multiplicative spectral sequence

$$E^2_{*,*} = \operatorname{Tor}^{(H\mathbb{F}_p)_*\mathrm{K}}_{*,*}\big(\mathbb{F}_p,\mathbb{F}_p\big) \Longrightarrow \operatorname{THH}_*(\mathrm{K};H\mathbb{F}_p).$$

Using Proposition 5.8 it follows that

$$E^2_{*,*} = E(\sigma y, \sigma\tilde{\xi}_1,\dots)\otimes\Gamma(\sigma x, z, \sigma\tilde{\tau}_2,\dots)$$

with $|z| = (2, 2p-2)$. The E^2-page is $\mathbb{F}_p\{\sigma x^k\}$ in total degree $2p-2$ and

$$\mathbb{F}_p\{\sigma\tilde{\xi}_1\}\oplus\mathbb{F}_p\{\sigma x^{k-1}\sigma y\}$$

in total degree $2p-1$. The classes σx^k, $\sigma\tilde{\xi}_1$ and $\sigma x^{k-1}\sigma y$ are infinite cycles, because they are products of classes in column 1. This implies that

$$\dim_{\mathbb{F}_p} V(0)_{2p-2}\operatorname{THH}(\mathrm{K};\hat{H}\mathbb{Z}_p) = 1.$$

The E^2-page is in total degree $2p$ given by $\mathbb{F}_p\{z\}$ if $r>1$ and by

$$\mathbb{F}_p\{z\}\oplus\mathbb{F}_p\{\gamma_p(\sigma x)\}$$

if $r=1$. The differentials of z cannot hit $\sigma x^{k-1}\sigma y$ because z lies in column 2 and $\sigma x^{k-1}\sigma y$ lies in a column ≥ 2. If $r=1$ the class $\sigma x^{k-1}\sigma y$ lies in column $p-1$. Since $\gamma_p(\sigma x)$ lies in column p, its differentials cannot hit $\sigma x^{k-1}\sigma y$. Therefore, $\sigma x^{k-1}\sigma y$ is a permanent cycle. □

Lemma 7.14. *In case (3) the spectral sequence (15) collapses at the E^2-page and there are no multiplicative extensions.*

Proof. The only possible differentials on the canonical algebra generators of the E^2-page are

$$d^{2p-1}(\lambda_1) \doteq \sigma x^k \quad\text{and}\quad d^{2p}(\mu_1)\doteq\sigma x^{k-1}\sigma y.$$

This implies that $d^i=0$ for $i=2,\dots,2p-2$. In total degree $2p-2$ the E^{2p-1}-page is given by $\mathbb{F}_p\{\sigma x^k\}$. Therefore, by Lemma 7.13, the differential $d^{2p-1}(\lambda_1)\doteq\sigma x^k$ cannot exist. Hence, we have $d^{2p-1}=0$. In total degree $2r$ the E^{2p}-page is generated as an $\mathbb{F}_p$-vector space by σx, in total degree $2r+1$ the E^{2p}-page is generated by σy. Because the classes have filtration degree zero, Lemma 7.13 implies that they survive to the E^∞-page and that $\sigma x^{k-1}\sigma y\neq 0$ in $E^\infty_{*,*}$. Hence, we cannot have $d^{2p}(\mu_1)\doteq\sigma x^{k-1}\sigma y$ and the spectral sequence collapses at the E^2-page. □

In case (4) the mod p homology of K is more complicated than in the other cases and we only consider the subcase $r>1$. In order to be able to compute the E^2-page of the spectral sequence

$$E^2_{*,*} = \operatorname{Tor}^{(H\mathbb{F}_p)_*\mathrm{K}}(\mathbb{F}_p,\mathbb{F}_p)\Longrightarrow\operatorname{THH}_*(\mathrm{K},H\mathbb{F}_p)$$

in the necessary degrees we need a couple of lemmas. The statements are probably well-known, but since we did not find references, we include proofs.

Lemma 7.15. *Let $S_* \to R_*$ be a homomorphism of non-negatively graded-commutative rings that is an isomorphism in degrees $\leq n$. Let M_* and N_* be non-negatively graded R_*-modules. Then, we have*

$$\mathrm{Tor}^{S_*}_{s,t}(M_*, N_*) \cong \mathrm{Tor}^{R_*}_{s,t}(M_*, N_*)$$

for $t \leq n$.

Proof. We construct by induction a commutative diagram of graded S_*-modules

$$\begin{tikzcd}
F_{s-1,*} \arrow[r, "d^{s-1}"] \arrow[d, "\eta_{s-1}"] & F_{s-2,*} \arrow[r, "d^{s-2}"] \arrow[d, "\eta_{s-2}"] & \dots \arrow[r, "d^1"] & F_{0,*} \arrow[r, "d^0"] \arrow[d, "\eta_0"] & F_{-1,*} \arrow[r] \arrow[d, "\eta_{-1}"] & 0 \\
T_{s-1,*} \arrow[r, "d^{s-1}"] & T_{s-2,*} \arrow[r, "d^{s-2}"] & \dots \arrow[r, "d^1"] & T_{0,*} \arrow[r, "d^0"] & T_{-1,*} \arrow[r] & 0
\end{tikzcd}$$

with the following properties:

- We have $F_{-1,*} = M_*$, $T_{-1,*} = M_*$ and $\eta_{-1} = \mathrm{id}_{M_*}$.
- The lines are exact.
- The $F_{i,*}$ are free non-negatively graded S_*-modules for $i \geq 0$.
- The $T_{i,*}$ are free non-negatively graded R_*-modules for $i \geq 0$ and the lower line is a sequence of R_*-modules.
- The maps $\eta_i : F_{i,*} \to T_{i,*}$ are isomorphisms in degrees $* \leq n$.

We start by defining $F_{-1,*} \coloneqq M_*$, $T_{-1,*} \coloneqq M_*$ and $\eta_{-1} \coloneqq \mathrm{id}_{M_*}$. Let d^{-1} be the unique map $M_* \to 0$. Let $s \geq 0$ and suppose that we have constructed the diagram up to $s-1$. We set

$$F_{s,*} = \bigoplus_{a \in \ker d^{s-1} \setminus \{0\}} \Sigma^{|a|} S_*$$

and define d^s to be the obvious map of S_*-modules. Furthermore, we set

$$T_{s,*} = \bigoplus_{a \in \ker d^{s-1} \setminus \{0\}} \Sigma^{|a|} R_*$$

and define d^s to be the obvious map of R_*-modules. Let η_s be the map of S_*-modules that is defined by

$$\Sigma^{|a|} 1 \mapsto \begin{cases} \Sigma^{|\eta_{s-1}(a)|} 1, & \text{if } \eta_{s-1}(a) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

It is then clear that

$$\begin{tikzcd}
F_{s,*} \arrow[r, "d^s"] \arrow[d, "\eta_s"] & F_{s-1,*} \arrow[d, "\eta_{s-1}"] \\
T_{s,*} \arrow[r, "d^s"] & T_{s-1,*}
\end{tikzcd}$$

is commutative. The map η_s is an isomorphism in degrees $\leq n$. This is because by the induction hypothesis we know that η_{s-1} induces a bijection

$$\{a \in \ker d^{s-1} \setminus \{0\} : |a| \leq n\} \to \{a \in \ker d^{s-1} \setminus \{0\} : |a| \leq n\}$$

and because $S_* \to R_*$ is an isomorphism in degree $\leq n$. This shows the induction step. We have commutative diagrams

$$\begin{tikzcd}
F_{s,*} \otimes_{\mathbb{Z}} N_* \arrow[r] & T_{s,*} \otimes_{\mathbb{Z}} N_* \\
F_{s,*} \otimes_{\mathbb{Z}} S_* \otimes_{\mathbb{Z}} N_* \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & T_{s,*} \otimes_{\mathbb{Z}} R_* \otimes_{\mathbb{Z}} N_* \arrow[u, shift left] \arrow[u, shift right]
\end{tikzcd},$$

where the vertical maps are induced by the S_*- and R_*-actions. The horizontal maps are isomorphism in degrees $\leq n$. We get maps on the coequalizers

$$F_{s,*} \otimes_{S_*} N_* \to T_{s,*} \otimes_{R_*} N_*$$

that are isomorphisms in degrees $\leq n$ and that give a map of complexes. Since homology is taken degreewise this shows the claim. $\square$

Lemma 7.16. *Let R_* be a graded-commutative ring and let N_* be a non-negatively graded R_*-module. Let $Q_{*,*}$ be a complex of graded R_*-modules and let $P_{*,*} \to Q_{*,*}$ be a subcomplex with the following properties:*

(1) If $x \in Q_{*,*}$ has total degree $\leq n+1$, then we have $x \in P_{*,*}$.
(2) $P_{m,*}$ is a direct summand of $Q_{m,*}$.

Then, the map

$$H_*(P_{*,*} \otimes_{R_*} N_*) \to H_*(Q_{*,*} \otimes_{R_*} N_*)$$

is an isomorphism in total degrees $\leq n$.

Proof. Note that the maps

$$P_{m,*} \otimes_{R_*} N_* \to Q_{m,*} \otimes_{R_*} N_*$$

are injective. Thus, $P_{*,*} \otimes_{R_*} N_*$ is a subcomplex of $Q_{*,*} \otimes_{R_*} N_*$. Moreover, every class $x \in Q_{*,*} \otimes_{R_*} N_*$ in total degree $\leq n+1$ lies in $P_{*,*} \otimes_{R_*} N_*$. The lemma now follows from the following fact: Let $Z_{*,*} \subseteq W_{*,*}$ a subcomplex with the property that every class $x \in W_{*,*}$ in total degree $\leq n+1$ lies in $Z_{*,*}$, then the induced map

$$H_*(Z_{*,*}) \to H_*(W_{*,*})$$

is an isomorphism in total degrees $\leq n$. $\square$

Lemma 7.17. *Let R_* be a non-negatively graded-commutative ring and let M_* be a graded R_*-module. Suppose that we have a chain complex*

$$\dots \xrightarrow{d^3} P_{2,*} \xrightarrow{d^2} P_{1,*} \xrightarrow{d^1} P_{0,*} \xrightarrow{d^0} M_* \longrightarrow 0$$

with the following properties:

(1) The $P_{m,*}$ are free graded R_*-modules.
(2) The map d^0 is surjective.
(3) If $x \in \ker d^i$ has total degree $\leq n$, then we have $x \in \operatorname{im} d^{i+1}$.

Then there exists a free resolution $Q_{*,*} \to M_*$ such that $P_{*,*}$ is a subcomplex of $Q_{*,*}$ with the properties (1) and (2) in Lemma 7.16.

Proof. We define the resolution $Q_{*,*}$ inductively. We define

$$Q_{0,*} \xrightarrow{d'^0} M_* \longrightarrow 0$$

to be

$$P_{0,*} \xrightarrow{d^0} M_* \longrightarrow 0.$$

Suppose that $i \geq 1$ and that we have already constructed an exact sequence

$$Q_{i-1,*} \xrightarrow{d'^{i-1}} Q_{i-2,*} \xrightarrow{d'^{i-2}} \dots \longrightarrow Q_{0,*} \xrightarrow{d'^0} M_* \longrightarrow 0$$

such that:

- For $0 \leq m \leq i-1$ we have
$$Q_{m,*} = P_{m,*} \oplus \bigoplus_{s \in I_m} \Sigma^{k_s} R_*$$
for a set I_m and natural numbers k_s for $s \in I_m$ with $k_s + m \geq n+2$.
- The diagram
$$\begin{tikzcd} Q_{i-1,*} \arrow[r, "d'^{i-1}"] & Q_{i-2,*} \arrow[r, "d'^{i-2}"] & \dots \arrow[r] & Q_{0,*} \arrow[r, "d'^0"] & M_* \arrow[r] & 0 \\ P_{i-1,*} \arrow[u] \arrow[r, "d^{i-1}"] & P_{i-2,*} \arrow[u] \arrow[r, "d^{i-2}"] & \dots \arrow[r] & P_{0,*} \arrow[u] \arrow[r, "d^0"] & M_* \arrow[u, "\mathrm{id}"] \arrow[r] & 0 \end{tikzcd}$$

commutes.

We define

$$Q_{i,*} = P_{i,*} \oplus \bigoplus_{\substack{x\in \ker {d'}^{i-1}\setminus\{0\} \\ |x|+i-1\geq n+1}} \Sigma^{|x|} R_*.$$

Here, $|x|$ means the internal degree of the element x. Let ${d'}^i : Q_{i,*} \to Q_{i-1,*}$ be the map that is given by

$$P_{i,*} \to P_{i-1,*} \to Q_{i-1,*}$$

on $P_{i,*}$ and that maps $\Sigma^{|x|}1$ to x. Then, the sequence

$$Q_{i,*} \xrightarrow{{d'}^i} Q_{i-1,*} \xrightarrow{{d'}^{i-1}} \dots \longrightarrow Q_{0,*} \xrightarrow{{d'}^0} M_* \longrightarrow 0$$

is exact: If $x \in \ker {d'}^{i-1} \setminus \{0\}$ and $|x| + i - 1 \leq n$, then $x \in P_{i-1,*}$ and we have $x \in \operatorname{im} {d'}^i$ by item (3). This shows the induction step. □

Lemma 7.18. *We consider case (4). If $r > 1$ we have*

$$\dim_{\mathbb{F}_p} V(0)_n \operatorname{THH}(\mathrm{K}; \hat{H}\mathbb{Z}_p) = \begin{cases} 1, & n = 2p-2; \\ 2, & n = 2p-1. \end{cases}$$

Proof. As in the other cases we consider the spectral sequence

$$E^2_{*,*} = \operatorname{Tor}^{(H\mathbb{F}_p)_* \mathrm{K}}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \operatorname{THH}_*(\mathrm{K}; H\mathbb{F}_p). \tag{26}$$

By Lemma 5.9 we have a map

$$E(x) \otimes P_k(y)/xy^{k-1} \to (H\mathbb{F}_p)_* \mathrm{K}$$

that is an isomorphism in degrees $\leq 2p$. By Lemma 7.15 we have

$$E^2_{s,t} \cong \operatorname{Tor}^{E(x)\otimes P_k(y)/xy^{k-1}}_{s,t}(\mathbb{F}_p, \mathbb{F}_p)$$

for $t \leq 2p$. We set $R_* := E(x) \otimes P_k(y)/xy^{k-1}$. We will construct a chain complex

$$\dots \longrightarrow P_{2,*} \xrightarrow{d^2} P_{1,*} \xrightarrow{d^1} P_{0,*} \xrightarrow{d^0} \mathbb{F}_p \longrightarrow 0$$

of free graded R_*-modules with the following properties:

- The map d^0 is surjective.
- If $x \in \ker d^i$ has total degree $\leq 2p$, then we have $x \in \operatorname{im} d^{i+1}$.

By the Lemmas 7.16 and 7.17 we then have

$$H_*(P_{*,*} \otimes_{R_*} \mathbb{F}_p) \cong \operatorname{Tor}^{R_*}_{*,*}(\mathbb{F}_p, \mathbb{F}_p)$$

in total degrees $\leq 2p$ and therefore $E^2_{*,*} \cong H_*(P_{*,*} \otimes_{R_*} \mathbb{F}_p)$ in total degrees $\leq 2p$.

We set $P_{0,*} = R_*$. Let d^0 be the R_*-module map defined by $1 \mapsto 1$. We define

$$P_{1,*} = R_*\gamma_1 \oplus R_*\upsilon_1,$$

$d^1(\gamma_1) = x$ and $d^1(\upsilon_1) = y$. Note that then γ_1 has to have bidegree $(1, 2r-1)$ and that υ_1 has to have bidegree $(1, 2r)$. Obviously,

$$P_{1,*} \xrightarrow{d^1} P_{0,*} \xrightarrow{d^0} \mathbb{F}_p \longrightarrow 0$$

is exact. The kernel of d^1 is given by

$$\bigoplus_{i=0}^{k-2} \mathbb{F}_p\{xy^i\gamma_1\} \oplus \mathbb{F}_p\{y^{k-1}\gamma_1\} \oplus \mathbb{F}_p\{y^{k-1}\upsilon_1\} \oplus \mathbb{F}_p\{xy^{k-2}\upsilon_1\} \oplus \bigoplus_{i=0}^{k-3} \mathbb{F}_p\{y^{i+1}\gamma_1 - xy^i\upsilon_1\}.$$

We set

$$P_{2,*} = R_*\gamma_2 \oplus R_*w_2 \oplus R_*z_2 \oplus R_*a_2 \oplus R_*\upsilon_2,$$

and define d^2 by $d^2(\gamma_2) = x\gamma_1$, $d^2(w_2) = y^{k-1}\gamma_1$, $d^2(z_2) = y^{k-1}\upsilon_1$, $d^2(a_2) = xy^{k-2}\upsilon_1$ and $d^2(\upsilon_2) = y\gamma_1 - x\upsilon_1$. Then, the bidegrees of the generators of $P_{2,*}$ are given by $|\gamma_2| = (2, 2(2r-1))$, $|w_2| = (2, 2p-3)$, $|z_2| = (2, 2p-2)$, $|a_2| = (2, 2p-3)$, $|\upsilon_2| = (2, 2\cdot 2r-1)$ and the sequence

$$P_{2,*} \xrightarrow{d^2} P_{1,*} \xrightarrow{d^1} P_{0,*} \xrightarrow{d^0} \mathbb{F}_p \longrightarrow 0$$

is exact. The $\mathbb{F}_p$-vector space

$$\begin{aligned} &\bigoplus_{i=0}^{k-2} \mathbb{F}_p\{xy^i\gamma_2\} \oplus \mathbb{F}_p\{y^{k-1}\gamma_2\} \oplus \mathbb{F}_p\{y^{k-1}\upsilon_2\} \oplus \mathbb{F}_p\{xy^{k-2}\upsilon_2\} \\ \oplus \ &\bigoplus_{i=0}^{k-3} \mathbb{F}_p\{y^{i+1}\gamma_2 - xy^i\upsilon_2\} \oplus \mathbb{F}_p\{-w_2 + y^{k-2}\upsilon_2 + a_2\} \end{aligned}$$

is included in $\ker d^2$ and contains every element in $\ker d^2$ with total degree $\leq 2p$. We set

$$P_{3,*} = R_*\gamma_3 \oplus R_*w_3 \oplus R_*z_3 \oplus R_*a_3 \oplus R_*\upsilon_3 \oplus R_*b_3,$$

and define d^3 by $d^3(\gamma_3) = x\gamma_2$, $d^3(w_3) = y^{k-1}\gamma_2$, $d^3(z_3) = y^{k-1}\upsilon_2$, $d^3(a_3) = xy^{k-2}\upsilon_2$, $d^3(\upsilon_3) = y\gamma_2 - x\upsilon_2$ and $d^3(b_3) = -w_2 + y^{k-2}\upsilon_2 + a_2$. We then have $|\gamma_3| = (3, 3(2r-1))$, $|w_3| = (3, 2p-2+2r-2)$, $|z_3| = (3, 2p-2+2r-1)$, $|a_3| = (3, 2p-2+2r-2)$, $|\upsilon_3| = (3, 3\cdot 2r-2)$, $|b_3| = (3, 2p-3)$, the composition

$$P_{3,*} \xrightarrow{d^3} P_{2,*} \xrightarrow{d^2} P_{1,*}$$

is zero and every class in $\ker d^2$ with total degree $\leq 2p$ is in the image of d^3. The $\mathbb{F}_p$-vector space

$$\bigoplus_{i=0}^{k-2} \mathbb{F}_p\{xy^i\gamma_3\} \oplus \mathbb{F}_p\{y^{k-1}\gamma_3\} \oplus \mathbb{F}_p\{y^{k-1}\upsilon_3\} \oplus \mathbb{F}_p\{xy^{k-2}\upsilon_3\} \oplus \bigoplus_{i=0}^{k-3} \mathbb{F}_p\{y^{i+1}\gamma_3 - xy^i\upsilon_3\}$$

is included in the kernel of d^3 and contains every element in the kernel that has a total degree $\leq 2p$. For $i \geq 4$ we set

$$P_{i,*} = R_*\gamma_i \oplus R_*w_i \oplus R_*z_i \oplus R_*a_i \oplus R_*\upsilon_i,$$

where the internal degrees of the generators are defined to be $|\gamma_i| = i(2r-1)$, $|w_i| = 2p-2+(i-2)2r-i+1$, $|z_i| = 2p-2+(i-2)2r-i+2$, $|a_i| = 2p-2+(i-2)2r-i+1$ and $|\upsilon_i| = 2ri-i+1$. We set $d^i(\gamma_i) = x\gamma_{i-1}$, $d^i(w_i) = y^{k-1}\gamma_{i-1}$, $d^i(z_i) = y^{k-1}\upsilon_{i-1}$, $d^i(a_i) = xy^{k-2}\upsilon_{i-1}$ and $d^i(\upsilon_i) = y\gamma_{i-1} - x\upsilon_{i-1}$. For $i \geq 4$ the $\mathbb{F}_p$-vector space

$$\bigoplus_{j=0}^{k-2} \mathbb{F}_p\{xy^j\gamma_i\} \oplus \mathbb{F}_p\{y^{k-1}\gamma_i\} \oplus \mathbb{F}_p\{y^{k-1}\upsilon_i\} \oplus \mathbb{F}_p\{xy^{k-2}\upsilon_i\} \oplus \bigoplus_{j=0}^{k-3} \mathbb{F}_p\{y^{j+1}\gamma_i - xy^j\upsilon_i\}$$

is included in $\ker d^i$ and contains every class in $\ker d^i$ that has total degree $\leq 2p$. This shows that for $i \geq 3$ the following holds: The composition

$$P_{i+1,*} \xrightarrow{d^{i+1}} P_{i,*} \xrightarrow{d^i} P_{i-1,*}$$

is zero and every element in $\ker d^i$ with a total degree $\leq 2p$ is in the image of d^{i+1}.

The complex $P_{*,*} \otimes_{R_*} \mathbb{F}_p$ is given by

$$\begin{array}{c}
\mathbb{F}_p \\
{\scriptstyle 0}\big\uparrow \\
\mathbb{F}_p\gamma_1 \oplus \mathbb{F}_p\upsilon_1 \\
{\scriptstyle 0}\big\uparrow \\
\mathbb{F}_p\gamma_2 \oplus \mathbb{F}_p w_2 \oplus \mathbb{F}_p z_2 \oplus \mathbb{F}_p a_2 \oplus \mathbb{F}_p\upsilon_2 \\
{\scriptstyle d^3}\big\uparrow \\
\mathbb{F}_p\gamma_3 \oplus \mathbb{F}_p w_3 \oplus \mathbb{F}_p z_3 \oplus \mathbb{F}_p a_3 \oplus \mathbb{F}_p\upsilon_3 \oplus \mathbb{F}_p b_3 \\
{\scriptstyle 0}\big\uparrow \\
\mathbb{F}_p\gamma_4 \oplus \mathbb{F}_p w_4 \oplus \mathbb{F}_p z_4 \oplus \mathbb{F}_p a_4 \oplus \mathbb{F}_p\upsilon_4 \\
{\scriptstyle 0}\big\uparrow \\
\cdots
\end{array}$$

Here, d^3 maps all generators to zero, except for b_3. It maps b_3 to $-w_2 + a_2$ if $k > 2$ and it maps b_3 to $-w_2 + a_2 + \upsilon_2$ if $k = 2$. The bigraded abelian group $H_*(P_{*,*} \otimes_{R_*} \mathbb{F}_p)$ is in total degree $2p-3$ zero, in total degree $2p-2$ given by $\mathbb{F}_p\gamma_k$, in total degree $2p-1$ given by $\mathbb{F}_p\upsilon_k \oplus \mathbb{F}_p w_2$ and in total degree $2p$ given by $\mathbb{F}_p z_2$. Thus, the same is true for the E^2-page of the spectral sequence (26). The differentials of w_2 and υ_k cannot hit γ_k, because the homological degree of γ_k is greater as or equal to the homological degree of w_2 and υ_k. For the same reason z_2 has to be an infinite cycle. This proves the lemma. $\square$

Lemma 7.19. *We consider case (4). If $r > 1$ the spectral sequence (15) collapses at the E^2-page. There are no multiplicative extensions.*

Proof. As in case (3) the only possible differentials on the canonical algebra generators of the E^2-page are

$$d^{2p-1}(\lambda_1) \doteq \sigma x^k \quad \text{and} \quad d^{2p}(\mu_1) \doteq \sigma x^{k-1}\sigma y.$$

We conclude that $d^i = 0$ for $i = 2, \ldots, 2p-2$. The E^{2p-1}-page is in total degree $2p-2$ given by $\mathbb{F}_p\{\sigma x^k\}$. Therefore, by Lemma 7.18, the differential $d^{2p-1}(\lambda_1) \doteq \sigma x^k$ cannot exist and we get $d^{2p-1} = 0$. In total degree $2p-1$ the E^{2p}-page is given by

$$\mathbb{F}_p\{\lambda_1\} \oplus \mathbb{F}_p\{\sigma x^{k-1}\sigma y\}.$$

Hence, by Lemma 7.18, the differential $d^{2p}(\mu_1) \doteq \sigma x^{k-1}\sigma y$ cannot exist and we conclude that the spectral sequence collapses at the E^2-page. $\square$

Remark 7.20. It seems likely that Lemma 7.19 is also true for $r = 1$. However, the above proof does not work in this case, because some of the degree arguments require $r > 1$.

7.3. **The $V(1)$-homotopy of $\mathrm{THH}\big(\mathrm{K}(\mathbb{F}_q)_p\big)$ in the first case.** In this subsection we consider the spectral sequence (14)

$$E^2_{*,*} \cong V(1)_*\,\mathrm{K} \otimes \mathrm{THH}_*(\mathrm{K}; H\mathbb{F}_p) \Longrightarrow V(1)_*\,\mathrm{THH}(\mathrm{K})$$

in case (1). By Lemma 4.2 and Theorem 7.7 we have

$$E^2_{*,*} \cong E(x) \otimes E(a, \lambda_2) \otimes P(\mu_2) \otimes \Gamma(b)$$

with $|x| = (0, 2p-3)$, $|a| = (p(2p-2)+1, 0)$, $|\lambda_2| = (2p^2-1, 0)$, $|\mu_2| = (2p^2, 0)$ and $|b| = (p(2p-2), 0)$.

Theorem 7.21. *In case (1) the spectral sequence (14) has the differential*

$$d^{2p-2}(\lambda_2) \doteq xa.$$

We have

$$V(1)_* \,\mathrm{THH}(\mathrm{K}) \cong \Omega^\infty_* \otimes P([\mu_2]) \otimes \Gamma([b]),$$

where Ω^∞_* *is the graded-commutative* $\mathbb{F}_p$*-algebra with generators* x, c, d, e *in degrees* $|x| = 2p-3$, $|c| = 2p^2+2p-4$, $|d| = 4p^2-2p$, $|e| = p(2p-2)+1$ *and relations*

$$\begin{aligned} d^2 &= c^2 = 0, \\ xe &= xc = 0, \\ de &= dc = 0, \\ ec &= -xd. \end{aligned} \tag{27}$$

Proof. Note that the spectral sequence only has two non-trivial lines, namely line zero which is

$$C_* := E(a, \lambda_2) \otimes P(\mu_2) \otimes \Gamma(b)$$

and line $2p-3$ which is

$$\mathbb{F}_p\{x\} \otimes E(a, \lambda_2) \otimes P(\mu_2) \otimes \Gamma(b).$$

We claim that the classes $\gamma_{p^i}(b)$ are infinite cycles for $i \geq 0$. We have

$$d^{2p-2}\big(\gamma_{p^i}(b)\big) = xw$$

for a class w in C_* in degree $p^{i+1}(2p-2)-2p+2$. Since w has even degree it lies in

$$P(\mu_2) \otimes \Gamma(b) \;\oplus\; \mathbb{F}_p\{a\} \otimes \mathbb{F}_p\{\lambda_2\} \otimes P(\mu_2) \otimes \Gamma(b).$$

Every class in this graded abelian group has a degree divisible by $2p$. Since $p^{i+1}(2p-2)-2p+2$ is not divisible by $2p$, we get $d^{2p-2}(\gamma_{p^i}(b)) = 0$. The classes a and μ_2 are infinite cycles, because C_* is trivial in degrees $0 < * < p(2p-2)$ and in degree $p(2p-2)+2$.

We claim that there is a differential $d^{2p-2}(\lambda_2) \doteq xa$. Since in total degree $2p^2-2$ the $(2p-3)$th line of the E^2-page is given by $\mathbb{F}_p\{xa\}$, it suffices to show that λ_2 is not an infinite cycle. To prove this, we note that by [22, Lemma 3.15, proofs of Theorem 4.11 and Lemma 4.13] the edge homomorphism

$$V(1)_* \,\mathrm{THH}(\mathrm{K}) \longrightarrow \mathrm{THH}_*(\mathrm{K}; H\mathbb{F}_p)$$

is induced by a map $V(1) \wedge^L_S \mathrm{THH}(\mathrm{K}) \to \mathrm{THH}(\mathrm{K}, H\mathbb{F}_p)$ in $\mathscr{D}_S$. We therefore have a commutative diagram

$$\begin{tikzcd} V(1)_* \,\mathrm{THH}(\mathrm{K}) \arrow[r, "\varepsilon"] \arrow[d, "h"'] & \mathrm{THH}_*(\mathrm{K}; H\mathbb{F}_p) \arrow[d, "h"] \\ H\mathbb{F}_{p*}\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big) \arrow[r] & (H\mathbb{F}_p)_* \,\mathrm{THH}(\mathrm{K}; H\mathbb{F}_p), \end{tikzcd} \tag{28}$$

where the upper horizontal map is the edge homomorphism and where the vertical maps are the Hurewicz homomorphisms. We suppose that λ_2 is an infinite cycle. Let $[\lambda_2] \in V(1)_{2p^2-1} \,\mathrm{THH}(\mathrm{K})$ be a representative of λ_2. We have $\varepsilon([\lambda_2]) = \lambda_2 \neq 0$. Since $\mathrm{THH}(\mathrm{K}; H\mathbb{F}_p)$ is a module over the S-ring spectrum $H\mathbb{F}_p$, the right Hurewicz homomorphism in (28) is injective. We get $h(\varepsilon([\lambda_2])) \neq 0$ and therefore $h([\lambda_2]) \neq 0$. This is a contradiction, because by [38, Corollary 17.14] the image of the Hurewicz morphism is always contained in the subspace of comodule primitives and because by Lemma 6.2 there is no non-trivial comodule primitive in

$$(H\mathbb{F}_p)_{2p^2-1}\big(V(1) \wedge^L_S \mathrm{THH}(\mathrm{K})\big).$$

We conclude that $d^{2p-2}(\lambda_2) \doteq xa$. We get

$$E^\infty_{*,*} = E^{2p-1}_{*,*} = H_*\big((E(x) \otimes E(a, \lambda_2)\big) \otimes P(\mu_2) \otimes \Gamma(b)$$

and one easily sees that as an $\mathbb{F}_p$-vector space one has

$$H_*\big(E(x) \otimes E(a, \lambda_2)\big) \cong \mathbb{F}_p\{1\} \oplus \mathbb{F}_p\{x\} \oplus \mathbb{F}_p\{a\} \oplus \mathbb{F}_p\{x\lambda_2\} \oplus \mathbb{F}_p\{a\lambda_2\} \oplus \mathbb{F}_p\{xa\lambda_2\}.$$

The $(2p-3)$th line of $E^\infty_{*,*}$ is therefore given by

$$\mathbb{F}_p\{x\} \otimes P(\mu_2) \otimes \Gamma(b) \;\oplus\; \mathbb{F}_p\{x\lambda_2\} \otimes P(\mu_2) \otimes \Gamma(b) \;\oplus\; \mathbb{F}_p\{xa\lambda_2\} \otimes P(\mu_2) \otimes \Gamma(b).$$

Thus, line $2p-3$ is zero in total degrees divisible by $2p$. It follows that the classes $\gamma_{p^i}(b)$, μ_2 and $a\lambda_2$ have unique representatives $[\gamma_{p^i}(b)]$, $[\mu_2]$ and d in $V(1)_*\,\mathrm{THH(K)}$. The class a has a unique representative e because $E^{\infty}_{*,2p-3}$ is zero in total degrees $2p-3 < * < 2p^2-3$. The classes x and $x\lambda_2$ also have unique representatives x and c because they lie in line $2p-3$. Since $\gamma_{p^i}(b)^p = 0$ and $(a\lambda_2)^2 = 0$ in $E^{\infty}_{*,*}$ and since the total degrees of $\gamma_{p^i}(b)^p$ and $(a\lambda_2)^2$ are divisible by $2p$, we get $[\gamma_{p^i}(b)]^p = 0$ and $d^2 = 0$ in $V(1)_*\,\mathrm{THH(K)}$. The equations $c^2 = 0$, $xe = 0$, $xc = 0$, $de = 0$, $dc = 0$ and $ec = -xd$ holds, because the corresponding equations in $E^{\infty}_{*,*}$ are true and because c^2, xe, xc de, dc and $ec + xd$ reduce to classes in lines $\geq 2p-3$. Hence, we have a map of $\mathbb{F}_p$-algebras

$$g : \Omega^{\infty}_* \otimes P([\mu_2]) \otimes P_p([b], [\gamma_p(b)], \dots) \longrightarrow V(1)_*\,\mathrm{THH(K)}.$$

Because of the relations (27) the classes 1, x, e. c, d and xd generate Ω^{∞}_* as an $\mathbb{F}_p$-vector space. Thus, g maps a generating set bijectively onto a basis of $V(1)_*\,\mathrm{THH(K)}$ and therefore is an isomorphism. ☐

We mention some ideas for the differentials of the spectral sequence (14) in the other cases:

Remark 7.22. In case (2) the E^2-page of the spectral sequence (14) is given by

$$E^2_{*,*} = E(x) \otimes E(\lambda_1, \lambda_2) \otimes \Gamma(\sigma x) \otimes P(\mu_2),$$

where the total degrees are $|x| = 2p-3$, $|\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$, $|\sigma x| = 2p-2$ and $|\mu_2| = 2p^2$. By our result obtained with the Bökstedt spectral sequence (Theorem 6.14), the spectral sequence has to collapse.

We now consider the cases (3) and (4). In case (4) we assume that $r > 1$. Then, the E^2-page of the spectral sequence (14) is given by

$$E^2_{*,*} = E(x) \otimes P_k(y) \otimes \Gamma(\sigma x) \otimes E(\sigma y) \otimes E(\lambda_1) \otimes P(\mu_1).$$

In case (3) we have the equation $y^k = 0$ in $(H\mathbb{F}_p)_*\,\mathrm{K}$. In case (4) we have the relations $y^k = 0$ and $xy^{k-1} = 0$ in $(H\mathbb{F}_p)_*\,\mathrm{K}$. It seems plausible that, analogous to the case of ku_p (see [22]), we get a differential

$$d^{2p-2r-1}(\mu_1) = y^{k-1}\sigma y$$

in case (3) and differentials

$$d^{2p-2r-2}(\lambda_1) = xy^{k-2}\sigma y \quad \text{and} \quad d^{2p-2r-1}(\mu_1) = y^{k-1}\sigma y$$

in case (4). In case (4) it seems plausible that there are additional differentials, similar to case (1).

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