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The Hochschild Complex of a Finite Tensor Category

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Abstract

We express the Hochschild complex of a finite tensor category using a specific projective resolution of the canonical coend of the finite tensor category. This leads to our main result that the Hochschild complex of a not necessarily semisimple modular category carries a canonical homotopy coherent projective action of the mapping class group $\mathrm{SL}(2, \mathbb{Z})$ of the torus. This supports the idea to think of the Hochschild complex of a modular category as a derived conformal block for the torus. In the semisimple case, the (underived) conformal block is known to carry a commutative multiplication. For the derived conformal block (i.e. in the non-semisimple case), we generalize this to an E_2-structure. Our results allow us obtain a notion of Hochschild chains for braided crossed monoidal categories in the sense of Turaev. We prove that these admit an action of an operad built from Hurwitz spaces.

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1 Introduction and outlook

In this article, we investigate the Hochschild complex for a class of linear categories relevant in the representation theory of finite-dimensional Hopf algebras and for the construction of topological field theories, namely *finite tensor categories* as introduced in [EO04]. These are linear Abelian monoidal categories with an exact tensor product which satisfy finiteness conditions and are rigid. They are *not* assumed to be semisimple. By the Hochschild complex of a finite tensor category $\mathcal{C}$ (or more generally any linear category) we understand the differential graded vector space given by the derived coend

$$\int_{\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}} \mathcal{C}(X, X) \tag{1.1}$$

over the endomorphism spaces $\mathcal{C}(X, X)$ of projective objects X (depending on the terminology one prefers, one might also call this the Hochschild complex of the category $\mathsf{Proj}\, \mathcal{C} \subset \mathcal{C}$ of projective objects). Derived coends and Hochschild homology are recalled in Section 2. If $\mathcal{C}$ is written as finite-dimensional modules over a finite-dimensional algebra A, then (1.1) is equivalent to the Hochschild complex of A [MCar94, Kel99].

Let us give a summary of our main results before stating them in detail:

- We construct a canonical homotopy coherent projective action of the mapping class group $\mathrm{SL}(2, \mathbb{Z})$ of the torus on the Hochschild complex of a (not necessarily semisimple) modular category (Theorem 4.7), a special type of finite tensor category intimately related to the theory of modular functors.

- Moreover, we exhibit an E_2-commutative multiplication on this complex (Theorem 5.5).

From the perspective of the theory of modular functors [BK01], these two results suggest to think of the Hochschild complex of a modular category as a *derived conformal block*.

The proof of these results relies on a detailed investigation of the Hochschild complex of a finite tensor category $\mathcal{C}$: As a key tool, we introduce in Section 3.2 a specific projective resolution $\int_{\mathsf{f}\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}} X \otimes X^\vee$ of the canonical coend $\mathbb{F} = \int^{X \in \mathcal{C}} X \otimes X^\vee$ of a finite tensor category from [Lyu95a, Lyu95b, KL01] and prove that we may express the Hochschild chains of $\mathcal{C}$ up to equivalence by

$$\int_{\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}} \mathcal{C}(X, X) \simeq \mathcal{C}\left(I, \int_{\mathsf{f}\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}} X \otimes X^\vee \right) , \tag{1.2}$$

see Corollary 3.7. If $\mathcal{C}$ is pivotal, we have such an equivalence also for *any* projective resolution of $\mathbb{F}$ (Theorem 3.9). The proof of the latter fact uses the modified trace on the tensor ideal of projective objects of a pivotal finite tensor category [GKP18].

The description of the Hochschild complex in terms of a resolution of the canonical coend given in (1.2) enables us to construct a *homotopy coherent* projective action of the mapping class group $\mathrm{SL}(2, \mathbb{Z})$ of the torus on the Hochschild complex of a modular category. This extends previous results in this direction: By [SZ12] there is a projective $\mathrm{SL}(2, \mathbb{Z})$-action on the center of a ribbon factorizable Hopf algebra. Inspired by the fact that the center is just the zeroth Hochschild cohomology, in [LMSS18] a projective $\mathrm{SL}(2, \mathbb{Z})$-action on the Hochschild cohomology of a ribbon factorizable Hopf algebra is constructed. In [Shi18] this is phrased in terms of the representation categories. These actions just exist on the (co)homology. It is a natural question to ask whether this action can be described in a canonical way as *homotopy coherent projective action at the chain level*. We answer this question affirmatively:

Theorem 4.7. *The Hochschild complex $\int_{\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}} \mathcal{C}(X, X)$ of a modular category $\mathcal{C}$ over an algebraically closed field carries a canonical homotopy coherent projective action of the mapping class group $\mathrm{SL}(2, \mathbb{Z})$ of the torus.*

The projectivity of this action arises naturally from the construction. For this reason, we work consistently with projective actions regardless of whether the action can be made linear by appropriate choices, see also Remark 4.5.

The proof of Theorem 4.7 first expresses $\int_{\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}} \mathcal{C}(X, X)$ in a slightly different way which is inspired by a derived version of the factorization axiom for conformal blocks and then uses the projective action of the braid group B_3 on three strands (the mapping class group of the torus with a disk removed) on $\mathbb{F}$ given in [Lyu95a, Lyu95b]. The group B_3 is a central extension of $\mathrm{SL}(2, \mathbb{Z})$, and we finally prove and use a criterion for the projective action of B_3 to descend to a homotopy coherent projective action of $\mathrm{SL}(2, \mathbb{Z})$. This can be seen as a homotopy coherent version of the strategy used in [LMSS19], see Remark 4.13.

The homotopy coherent projective action of the mapping class group suggests the interpretation of the complex $\int_{\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}} \mathcal{C}(X, X)$ as a derived conformal block for the torus. As a further result, we establish a multiplicative structure on $\int_{\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}} \mathcal{C}(X, X)$ which is in line with this topological viewpoint. As a motivation, let us consider a semisimple modular category: In this case, $\int_{\mathbb{L}}^{X\in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ is equivalent to the ordinary coend $\int^{X\in\mathcal{C}} \mathcal{C}(X,X)$, and we can construct from $\mathcal{C}$ an (anomalous) three-dimensional topological field theory whose value on the torus $\mathbb{T}^2$ is precisely $\int^{X\in\mathcal{C}} \mathcal{C}(X,X)$; we refer to the monograph [Tur10-I] for the construction procedure and to [BDSPV15] for the classification of 3-2-1-dimensional topological field theories by semisimple modular categories. If we denote by $P : \mathbb{S}^1 \sqcup \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ the pair of pants bordism, then we can evaluate the topological field theory associated to $\mathcal{C}$ on the bordism $P\times \mathbb{S}^1 : \mathbb{T}^2 \sqcup \mathbb{T}^2 \longrightarrow \mathbb{T}^2$. This yields an associative multiplication on $\int^{X\in\mathcal{C}} \mathcal{C}(X,X)$ which is induced by the tensor product of $\mathcal{C}$. The braiding of $\mathcal{C}$ ensures that the multiplication is commutative. The vector space $\int^{X\in\mathcal{C}} \mathcal{C}(X,X)$ with this multiplication is referred to as the *Verlinde algebra of* $\mathcal{C}$.

The Reshetikhin-Turaev construction of a three-dimensional topological field theory from $\mathcal{C}$ is not available if $\mathcal{C}$ is not semisimple, but the vector space $\int^{X\in\mathcal{C}} \mathcal{C}(X,X)$ can be replaced by the derived coend $\int_{\mathbb{L}}^{X\in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$, and we can ask whether the multiplicative structure present in the semisimple case still exists up to coherent homotopy. We obtain the following result:

THEOREM 5.5. *For every braided finite tensor category $\mathcal{C}$, the Hochschild complex $\int_{\mathbb{L}}^{X\in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ is naturally a non-unital E_2-algebra in differential graded vector spaces. We refer to this E_2-algebra as the derived Verlinde algebra of $\mathcal{C}$ and denote it by $\mathcal{V}^{\mathcal{C}}$.*

In other words, $\int_{\mathbb{L}}^{X\in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ carries a multiplication whose commutativity is controlled by the braid group. One should note that this E_2-structure does *not* just follow from the functoriality of the derived coend applied to the E_2-structure on $\mathcal{C}$ because the differential graded operad obtained from the Hochschild chains of the fundamental groupoid ΠE_2 of E_2 is not an E_2-operad (Remark 5.2).

The strategies used to construct the derived Verlinde algebra and their relation to conformal and topological field theory can be used to obtain, for any finite group G, a candidate for Hochschild chains on braided G-crossed monoidal category $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ in the sense of Turaev [Tur10-II], see [Gal17] for a slightly different definition that we will adopt (Section 5.3). We prove that this equivariant Hochschild complex carries an action of an operad built from Hurwitz spaces, as we will explain now: We propose that the Hochschild chains of $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ come in fact in a family with each member labeled by a central element $z \in Z(G)$. For a fixed such central element z, the Hochschild chains are G-graded with the component (in field theory language: *sector*) for $g\in G$ given by the derived coend

$$\mathcal{V}_g^{\mathcal{C},z} := \int_{\mathbb{L}}^{X\in \mathsf{Proj}\,\mathcal{C}_g} \mathcal{C}_g(z.X,X) \ . \tag{1.3}$$

This definition is relatively intransparent from a purely algebraic point of view, but very natural from the perspective of (extended) G-equivariant topological field theory and its relation to G-equivariant braided monoidal categories [Tur10-II, TV12, TV14, SW19]: The definition ensures that (z,g) defines a G-bundle over the torus making (1.3) the appropriate derived version of the equivariant conformal block associated to the pair (z,g) of commuting holonomies. This field-theoretic viewpoint suggests that the complexes (1.3) should carry a multiplication whose commutativity behaviour is controlled by the homotopy quotient of the braid group action on the space of G-bundles over a punctured plane, i.e. by Hurwitz spaces. We make this statement precise using the *little bundles operad* [MW19], a topological operad built from Hurwitz spaces:

THEOREM 5.10. *Let G be a group and $z\in Z(G)$ an element in its center. Then for any finite braided G-crossed tensor category, the assignment*

$$g \longmapsto \mathcal{V}_g^{\mathcal{C},z} := \int_{\mathbb{L}}^{X\in \mathsf{Proj}\,\mathcal{C}_g} \mathcal{C}_g(z.X,X)$$

defines a non-unital little bundles algebra in differential graded vector spaces. We refer to $(\mathcal{V}_g^{\mathcal{C},z})_{g\in G}$ as the G-equivariant derived Verlinde algebra of $\mathcal{C}$ and the central element $z\in Z(G)$.

The results of this paper are inspired by the idea that the Hochschild complex describes the genus one derived conformal block of a non-semisimple modular category. The construction of a fully-fledged derived modular functor, including a derived version of the sewing axioms and homotopy coherent projective action of mapping class group of higher genus surfaces, will be subject of future work. The mapping class group actions will be the key towards a modular homology for modular categories (Remark 4.14).

Conventions. Throughout this text, we will work over a field k which is *not* assumed to have characteristic zero.

By Ch_k we denote the symmetric monoidal category of differential graded vector spaces over k (aka chain complexes over k) equipped with its projective model structure in which weak equivalences (for short: equivalences) are quasi-isomorphisms and fibrations are degree-wise surjections. Whenever we say that two complexes are canonically equivalent, this will not necessarily mean that there is a canonical map between which is an equivalence, but more generally a zig-zag of such maps. We will denote equivalences and zig-zags thereof by the symbol $\simeq$.

A (small) category enriched over Ch_k will be called a differential graded category. Unless otherwise stated, functors between differential graded categories will automatically be assumed to be enriched. Note that Ch_k is a differential graded category itself.

For a category $\mathcal{C}$, the sets (or in the enriched setting: space, complex) of morphisms from $X \in \mathcal{C}$ to $Y \in \mathcal{C}$ will be denoted by $\mathcal{C}(X, Y)$.

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2 Derived coends and Hochschild homology

In this preliminary section, we discuss a suitable version of a derived coend that we obtain by slightly modifying the derived functor tensor product given in [Shu09] and [Rie14, Chapter 9]. We can see the construction also as the Hochschild-Mitchell chains on a differential graded category [Kel99, CR05] with coefficients in a bimodule. For the convenience of the reader, the presentation will be self-contained.

For differential graded categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\mathcal{C} \otimes \mathcal{D}$ the differential graded category whose objects are pairs $(X, Y) \in \mathcal{C} \times \mathcal{D}$ of objects of $\mathcal{C}$ and $\mathcal{D}$, which we will also denote as $X \times Y$, and whose morphism complexes are given by

$$(\mathcal{C} \otimes \mathcal{D})(X \times Y, X' \times Y') := \mathcal{C}(X, X') \otimes \mathcal{D}(Y, Y') \quad \text{for} \quad X, X' \in \mathcal{C}, \quad Y, Y' \in \mathcal{D} .$$

Definition 2.1. Let $\mathcal{C}$ be a differential graded category. For a functor $F : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathsf{Ch}_k$ (by the above conventions, we will always assume that F is enriched), we define the *(enriched) simplicial bar construction* as the simplicial object in Ch_k with n-simplices

$$B_n F := \bigoplus_{X_0, \dots, X_n \in \mathcal{C}} \mathcal{C}(X_1, X_0) \otimes \cdots \otimes \mathcal{C}(X_n, X_{n-1}) \otimes F(X_0, X_n) .$$

The face and degeneracy maps are defined similarly to [Rie14, Definition 9.1.1]. The j-th face map composes morphisms over the j-th object thereby deleting it from the list of objects indexing the summand and the j-degeneracy maps inserts an identity at the j-th object thereby doubling it in the list of objects indexing the summand. In more detail, we have:

- The face map $\partial_0 : B_n F \longrightarrow B_{n-1} F$ is induced by the map

  $$\mathcal{C}(X_1, X_0) \otimes F(X_0, X_n) \longrightarrow F(X_1, X_n)$$

  which is part of the data of F being an enriched functor.

- For $0 < j < n$ the face map $\partial_j : B_n F \longrightarrow B_{n-1} F$ arises from the composition map

  $$\mathcal{C}(X_j, X_{j-1}) \otimes \mathcal{C}(X_{j+1}, X_j) \longrightarrow \mathcal{C}(X_{j+1}, X_{j-1}) .$$

- The face map $\partial_n : B_n F \longrightarrow B_{n-1} F$ is induced by the map

  $$\mathcal{C}(X_n, X_{n-1}) \otimes F(X_0, X_n) \longrightarrow F(X_0, X_{n-1})$$

  which is part of the data of F being an enriched functor.

- The degeneracy map $s_j : B_n F \longrightarrow B_{n+1} F$ inserts an identity at X_j using the canonical map $k \longrightarrow \mathcal{C}(X, X)$ selecting the identity.

We define the *derived coend* of F as the realization of $B_* F$, i.e. by

$$\int_{\mathbb{L}}^{X \in \mathcal{C}} F(X, X) := |B_* F| = \int^{n \in \Delta^{\mathrm{op}}} N_*(\Delta^n; k) \otimes B_n F \ ,$$

where $N_*(\Delta^n; k)$ are the normalized chains on the standard simplex Δ^n with coefficients in k (equivalently, we may see $B_* F$ as a double complex and totalize).

For a differential graded category $\mathcal{C}$, a functor $F : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathsf{Ch}_k$ will also be referred to as *$\mathcal{C}$-bimodule*. The derived coend $\int_{\mathbb{L}}^{X \in \mathcal{C}} F(X, X)$ will also be called the *derived trace of F*.

The above constructions can be done for more general model categories than chain complexes over a field. But since we want to develop the techniques with an eye towards the intended applications, we deliberately reduce the generality. However, for a slight generalization that we need later on see Remark 2.12.

Example 2.2. For any k-algebra A (by this we always mean an associative and unital k-algebra) the category $\star//A$ with one object whose endomorphisms are given by A is a differential graded category, and a functor $(\star//A)^{\mathrm{op}} \otimes (\star//A) \longrightarrow \mathsf{Ch}_k$ is a differential graded module M over the enveloping algebra $A^{\mathrm{e}} = A^{\mathrm{op}} \otimes A$ of A, i.e. an A-bimodule. Now the derived coend of M over $\star//A$ is just given by the Hochschild chains for the algebra A with coefficients in the A^{e}-module M which we denote by $CH(A; M)$.

Remark 2.3. For a differential graded category $\mathcal{C}$ and a $\mathcal{C}$-bimodule $F : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathsf{Ch}_k$, we obtain a $\mathcal{C}^{\mathrm{op}}$-bimodule $F^{\mathrm{op}} : \mathcal{C} \otimes \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Ch}_k$ by precomposition of F with the flip map. Then by reading backwards the families of objects used for the definition of the bar construction of F, we obtain a reversal isomorphism $\int_{\mathbb{L}}^{X \in \mathcal{C}^{\mathrm{op}}} F^{\mathrm{op}}(X, X) \cong \int_{\mathbb{L}}^{X \in \mathcal{C}} F(X, X)$.

One key feature of the derived coend is its homotopy invariance:

Proposition 2.4. *Let $\mathcal{C}$ be a differential graded category. Then an equivalence $F \xrightarrow{\simeq} G$ between functors $F, G : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathsf{Ch}_k$ induces an equivalence*

$$\int_{\mathbb{L}}^{X \in \mathcal{C}} F(X, X) \xrightarrow{\simeq} \int_{\mathbb{L}}^{X \in \mathcal{C}} G(X, X) \ .$$

The homotopy invariance for the homotopy coend is an extremely crucial built-in property (otherwise the homotopy coend would not deserve its name); it will be used without further mention. The proof of Proposition 2.4 readily follows from the following Lemma, which we obtain from the fact that every simplicial vector space is Reedy cofibrant and [Hir03, Theorem 19.8.4 (1)] applied to the framing obtained by normalized chains on the standard simplices:

Lemma 2.5. *Let $\mathcal{C}$ be a differential graded category and $F : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathsf{Ch}_k$ a functor. Then there is a natural equivalence*

$$\int_{\mathbb{L}}^{X \in \mathcal{C}} F(X, X) \simeq \operatorname*{hocolim}_{n \in \Delta^{\mathrm{op}}} B_n F \ .$$

2.1 Yoneda Lemma and Fubini Theorem

Next we discuss some important tools which will help us to compute with derived coends. They generalize to some extent the calculus for ordinary coends [Mac71]. We will often encounter the requirement that both the differential graded category and the bimodule are concentrated in non-negative degree (or more generally, bounded below). First, we formulate the Yoneda Lemma:

Proposition 2.6. *Let $\mathcal{C}$ be a differential graded category and $H : \mathcal{C} \longrightarrow \mathsf{Ch}_k$ a functor such that both the morphism spaces of $\mathcal{C}$ and the values of H are concentrated in non-negative degree. Then there is a canonical equivalence*

$$\int_{\mathbb{L}}^{X \in \mathcal{C}} \mathcal{C}(X, -) \otimes H(X) \xrightarrow{\simeq} H \ . \tag{2.1}$$

Moreover, this map is surjective and hence a trivial fibration.

Proof. For $Y \in \mathcal{C}$ the maps $\mathcal{C}(X,Y) \otimes H(X) \longrightarrow H(Y)$ provide an augmentation $B_*(\mathcal{C}(-,Y) \otimes H) \longrightarrow H(Y)$ whose realization is a map

$$\int_{\mathbb{L}}^{X \in \mathcal{C}} \mathcal{C}(X,Y) \otimes H(X) \longrightarrow H(Y) \tag{2.2}$$

which is natural in Y and therefore gives us the map (2.1). This map is clearly surjective.

It remains to show that for fixed Y the map (2.2) is an equivalence: Thanks to the assumptions on $\mathcal{C}$ and H, the derived coend $\int_{\mathbb{L}}^{X \in \mathcal{C}} \mathcal{C}(X,-) \otimes H(Y)$ is the realization of the simplicial object $B_*(\mathcal{C}(-,Y) \otimes H)$ in the simplicial model category of non-negatively graded chain complexes over k. By [Rie14, Corollary 4.5.2] the augmentation (2.2) is an equivalence if we can exhibit extra degeneracies for the augmentation, see [GJ09, Section III.5] for a definition of this notion.

We construct extra degeneracies for the augmentation as follows: For $n \geq 0$ the summand of $B_n(\mathcal{C}(-,Y)\otimes H)$ belonging to a family $(X_0, \ldots, X_n)$ of objects in $\mathcal{C}$ is given by

$$\mathcal{C}(X_1, X_0) \otimes \cdots \otimes \mathcal{C}(X_n, X_{n-1}) \otimes \mathcal{C}(X_0, Y) \otimes H(X_n) \ .$$

Using the identity of Y and the symmetric braiding on Ch_k, this summand admits a natural map to

$$\mathcal{C}(X_0, Y) \otimes \mathcal{C}(X_1, X_0) \otimes \cdots \otimes \mathcal{C}(X_n, X_{n-1}) \otimes \mathcal{C}(Y,Y) \otimes H(X_n) \ ,$$

i.e. to the summand of $B_{n+1}(\mathcal{C}(-,Y) \otimes H)$ belonging to $(Y, X_0, \ldots, X_n)$. This yields a map $s_{-1}^n : B_n(\mathcal{C}(-,Y) \otimes H) \longrightarrow B_{n+1}(\mathcal{C}(-,Y) \otimes H)$. It is straightforward to verify that this gives us extra degeneracies for the augmentation. $\square$

For derived coends, there is a Fubini Theorem making a statement about the 'order of integration' for iterated coends:

PROPOSITION 2.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be differential graded categories and $F : (\mathcal{C} \otimes \mathcal{D})^{\mathrm{op}} \otimes \mathcal{C} \otimes \mathcal{D} \longrightarrow \mathsf{Ch}_k$ a functor. Then there is a natural isomorphism*

$$\int_{\mathbb{L}}^{X \in \mathcal{C}} \int_{\mathbb{L}}^{Y \in \mathcal{D}} F(X \times Y, X \times Y) \cong \int_{\mathbb{L}}^{Y \in \mathcal{D}} \int_{\mathbb{L}}^{X \in \mathcal{C}} F(X \times Y, X \times Y) \ .$$

Proof. Note that from the definitions we obtain

$$\begin{aligned}
&\int_{\mathbb{L}}^{X \in \mathcal{C}} \int_{\mathbb{L}}^{Y \in \mathcal{D}} F(X \times Y, X \times Y) \\
&= \int^{m \in \Delta^{\mathrm{op}}} N_*(\Delta^m; k) \otimes \Bigg(\bigoplus_{X_0, \ldots, X_m \in \mathcal{C}} \mathcal{C}(X_1, X_0) \otimes \cdots \otimes \mathcal{C}(X_m, X_{m-1}) \\
&\otimes \int^{n \in \Delta^{\mathrm{op}}} N_*(\Delta^n; k) \otimes \Bigg(\bigoplus_{Y_0, \ldots, Y_n \in \mathcal{D}} \mathcal{D}(Y_1, Y_0) \otimes \cdots \otimes \mathcal{D}(Y_n, Y_{n-1}) \otimes F(X_0 \times Y_0, X_m \times Y_n) \Bigg) \Bigg) \ .
\end{aligned}$$

Using that tensor products of chain complexes, direct sums and coends commute, we see that this is canonically isomorphic to $\int_{\mathbb{L}}^{Y \in \mathcal{D}} \int_{\mathbb{L}}^{X \in \mathcal{C}} F(X \times Y, X \times Y)$. $\square$

The above Proposition tells us that the 'order of integration does not matter'. Therefore, instead of $\int_{\mathbb{L}}^{X \in \mathcal{C}} \int_{\mathbb{L}}^{Y \in \mathcal{D}}$ or $\int_{\mathbb{L}}^{Y \in \mathcal{D}} \int_{\mathbb{L}}^{X \in \mathcal{C}}$ we will just write $\int_{\mathbb{L}}^{\substack{X \in \mathcal{C} \\ Y \in \mathcal{D}}}$.

2.2 Agreement principle

In the sequel, we will often have to evaluate derived coends over (sub)categories of modules over some algebra. It is a pertinent question whether such a derived coend can be reduced to a derived coend over the one-object category associated to that algebra and hence to (ordinary) Hochschild chains (Example 2.2). This leads us to an Agreement Principle that goes back to [MCar94, Kel99], where it appears in a slightly different form. We explain the relation after Corollary 2.10.

We refer to a k-linear category as a *finite-dimensional algebroid over k* if it is equivalent to a k-linear category with finitely many objects and finite-dimensional morphism spaces. The main example is the one-object category whose endomorphisms are given by a finite-dimensional k-algebra.

For a finite-dimensional algebroid $\mathcal{A}$ over k, we denote by $\mathsf{Mod}_k\mathcal{A}$ the k-linear category of all finite-dimensional $\mathcal{A}$-modules. Here, a finite-dimensional $\mathcal{A}$-module is a k-linear functor from $\mathcal{A}$ to finite-dimensional k-vector spaces. By $\mathsf{Proj}_k\, \mathcal{A} \subset \mathsf{Mod}_k\mathcal{A}$ we denote the full k-linear subcategory of finite-dimensional projective

$\mathcal{A}$-modules. We refer to [Wei94, Section 2.2] for the usual equivalent descriptions of projective modules. Following our general conventions for the notation, we denote by $\mathcal{A}(-,-)$, $\mathsf{Mod}_k\mathcal{A}(-,-)$ and $\mathsf{Proj}_k\,\mathcal{A}(-,-)$ the morphism spaces in these k-linear categories.

There is a canonical embedding $\iota_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathsf{Proj}_k\,\mathcal{A}$ sending $a \in \mathcal{A}$ to $\mathcal{A}(a,-)$ along which we can restrict derived coends over $\mathsf{Proj}_k\,\mathcal{A}$. To make a statement about such restricted derived coends, we need the following Lemma:

Lemma 2.8. *For any finite-dimensional algebroid $\mathcal{A}$ over k, any functor $F : (\mathsf{Proj}_k\,\mathcal{A})^{\mathrm{op}} \otimes \mathsf{Proj}_k\,\mathcal{A} \longrightarrow \mathsf{Ch}_k$ whose values are concentrated in non-negative degree and $X, Y \in \mathsf{Proj}_k\,\mathcal{A}$ the natural map*

$$\int_{\mathbb{L}}^{a\in\mathcal{A}} \mathsf{Proj}_k\,\mathcal{A}(X, \iota_{\mathcal{A}}(a)) \otimes F(\iota_{\mathcal{A}}(a), Y) \xrightarrow{\simeq} F(X,Y) \tag{2.3}$$

is an equivalence.

Proof. Without loss of generality, we can assume that $\mathcal{A}$ has finitely many objects and finite-dimensional morphism spaces. We now observe that for a fixed projective $\mathcal{A}$-module Y, the statement that the map (2.3) is an equivalence is true in the following cases:

(1) For $X = \mathcal{A}(b,-)$ for any $b \in \mathcal{A}$ it is true by the Yoneda Lemma (Proposition 2.6),
(2) It is true for finite-dimensional $\mathcal{A}$-modules X and X' if and only if it is true for $X \oplus X'$. Here we use that F by our conventions is always assumed to be enriched. As a consequence, it preserves finite biproducts, i.e. $F(X \oplus X', Y) \cong F(X,Y) \oplus F(X',Y)$.

Now if X is an *arbitrary* finite-dimensional and projective $\mathcal{A}$-module, then the finite-dimensional $\mathcal{A}$-module $Y := \bigoplus_{a\in\mathcal{A}} \mathcal{A}(a,-) \otimes X(a)$ comes with a canonical surjection $\pi : Y \longrightarrow X$, and the short exact sequence

$$0 \longrightarrow \ker \pi \longrightarrow Y \longrightarrow X \longrightarrow 0$$

splits by projectivity of X, and hence $Y \cong X \oplus \ker \pi$. From (1) and (2) it follows that the statement is true for Y and therefore also for X by (2). $\square$

Theorem 2.9 (Agreement principle). *Let $\mathcal{A}$ be a finite-dimensional algebroid over k and $F : (\mathsf{Proj}_k\,\mathcal{A})^{\mathrm{op}} \otimes \mathsf{Proj}_k\,\mathcal{A} \longrightarrow \mathsf{Ch}_k$ a functor whose values are concentrated in non-negative degree. Then the canonical embedding $\iota_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathsf{Proj}_k\,\mathcal{A}$ induces an equivalence*

$$\int_{\mathbb{L}}^{a\in\mathcal{A}} F(\iota_{\mathcal{A}}(a), \iota_{\mathcal{A}}(a)) \xrightarrow{\simeq} \int_{\mathbb{L}}^{X\in\mathsf{Proj}_k\,\mathcal{A}} F(X,X)\ . \tag{2.4}$$

Proof. The map (2.4) is the composition of the reversal isomorphism

$$\int_{\mathbb{L}}^{a\in\mathcal{A}} F(\iota_{\mathcal{A}}(a), \iota_{\mathcal{A}}(a)) \cong \int_{\mathbb{L}}^{a\in\mathcal{A}^{\mathrm{op}}} F^{\mathrm{op}}(\iota_{\mathcal{A}}(a), \iota_{\mathcal{A}}(a))$$

from Remark 2.3 with the map

$$\iota_{\mathcal{A}} : \int_{\mathbb{L}}^{a\in\mathcal{A}^{\mathrm{op}}} F^{\mathrm{op}}(\iota_{\mathcal{A}}(a), \iota_{\mathcal{A}}(a)) \longrightarrow \int_{\mathbb{L}}^{X\in\mathsf{Proj}_k\,\mathcal{A}} F(X,X) \tag{2.5}$$

induced directly by the embedding $\iota_{\mathcal{A}}$. Hence, it suffices to prove that (2.5) is an equivalence. To this end, we note that it fits into the square

$$\begin{array}{ccc}
\displaystyle\int_{\mathbb{L}}^{a\in\mathcal{A}^{\mathrm{op}}} F^{\mathrm{op}}(\iota_{\mathcal{A}}(a), \iota_{\mathcal{A}}(a)) & \xrightarrow{\qquad\iota_{\mathcal{A}}\qquad} & \displaystyle\int_{\mathbb{L}}^{X\in\mathsf{Proj}_k\,\mathcal{A}} F(X,X) \\
{\scriptstyle\alpha}\big\uparrow & & {\scriptstyle\beta}\big\uparrow \\
\displaystyle\int_{\mathbb{L}}^{a\in\mathcal{A}^{\mathrm{op}}} \int_{\mathbb{L}}^{X\in\mathsf{Proj}_k\,\mathcal{A}} \mathsf{Proj}_k\,\mathcal{A}(X, \iota_{\mathcal{A}}(a)) \otimes F^{\mathrm{op}}(\iota_{\mathcal{A}}(a), X) & \xrightarrow{\quad\varphi\quad} & \displaystyle\int_{\mathbb{L}}^{X\in\mathsf{Proj}_k\,\mathcal{A}} \int_{\mathbb{L}}^{a\in\mathcal{A}^{\mathrm{op}}} \mathsf{Proj}_k\,\mathcal{A}(X, \iota_{\mathcal{A}}(a)) \otimes F^{\mathrm{op}}(\iota_{\mathcal{A}}(a), X)\ ,
\end{array}$$

where α is the natural equivalence from the Yoneda Lemma (Proposition 2.6), β is the equivalence from Lemma 2.8 (combined with Remark 2.3), and the isomorphism φ is a consequence of the Fubini Theorem (Proposition 2.7). It remains to prove that the square commutes up to homotopy because then we may conclude that (2.5) is an equivalence.

To prove this, we first note that we can see all the complexes in the above square as realizations of simplicial complexes, namely the simplicial bar constructions that we have used to define derived coends (the two lower complexes are iterated derived coends, hence they are even bisimplicial); moreover, all the maps involved arise as simplicial maps between the simplicial bar constructions. Therefore, we may as well exhibit a simplicial homotopy $\iota_{\mathcal{A}}\alpha \simeq \beta\varphi$, see [Wei94, Definition 8.3.11] for the definition. The complex in the left lower corner of the square can be modeled by the total complex associated to a bisimplicial object, and the latter can be described as the realization of the diagonal simplicial object by the generalized Eilenberg-Zilber Theorem of Dold and Puppe [GJ09, IV.2 Theorem 2.5]. Therefore, the needed simplicial homotopy will run from the simplicial chain complex which in degree n is given by a direct sum of (the reader may ignore the $(*)$-labeled underbraces for the moment)

$$\begin{gathered}\mathcal{A}(a_0,a_1)\otimes\cdots\otimes\mathcal{A}(a_{j-1},a_j)\otimes\underbrace{\mathcal{A}(a_j,a_{j+1})\otimes\cdots\otimes\mathcal{A}(a_{n-1},a_n)}_{(*)}\\ \otimes\underbrace{\mathsf{Proj}_k\,\mathcal{A}(X_1,X_0)\otimes\cdots\otimes\mathsf{Proj}_k\,\mathcal{A}(X_j,X_{j-1})}_{(*)}\otimes\mathsf{Proj}_k\,\mathcal{A}(X_{j+1},X_j)\otimes\cdots\otimes\mathsf{Proj}_k\,\mathcal{A}(X_n,X_{n-1})\\ \otimes\underbrace{\mathsf{Proj}_k\,\mathcal{A}(X_0,\iota_{\mathcal{A}}(a_n))}_{(*)}\otimes F(\iota_{\mathcal{A}}(a_0),X_n)\end{gathered} \tag{2.6}$$

for $a_0,\dots,a_n\in\mathcal{A}$ and $X_0,\dots,X_n\in\mathsf{Proj}_k\,\mathcal{A}$ to B_*F as given in Definition 2.1. For every $0\le j\le n$, the tensor factors marked by $(*)$ admit a map to $\mathsf{Proj}_k\,\mathcal{A}(X_j,\iota_{\mathcal{A}}(a_j))$ which uses composition in $\mathcal{A}$ and $\mathsf{Proj}_k\,\mathcal{A}$. Combining this with the functor $\iota_{\mathcal{A}}$, we obtain a map h_j from the summand (2.6) to the summand

$$\begin{gathered}\mathsf{Proj}_k\,\mathcal{A}(\iota_{\mathcal{A}}(a_1),\iota_{\mathcal{A}}(a_0))\otimes\cdots\otimes\mathsf{Proj}_k\,\mathcal{A}(\iota_{\mathcal{A}}(a_j),\iota_{\mathcal{A}}(a_{j-1}))\otimes\mathsf{Proj}_k\,\mathcal{A}(X_j,\iota_{\mathcal{A}}(a_j))\\ \otimes\mathsf{Proj}_k\,\mathcal{A}(X_{j+1},X_j)\otimes\cdots\otimes\mathsf{Proj}_k\,\mathcal{A}(X_n,X_{n-1})\otimes F(\iota_{\mathcal{A}}(a_0),X_n)\ .\end{gathered}$$

of $B_{n+1}F$ indexed by $\iota_{\mathcal{A}}(a_0),\dots,\iota_{\mathcal{A}}(a_j),X_j,\dots,X_n$. As can be verified by a direct computation, these maps yield a simplicial homotopy from $\partial_0 h_0=\beta\varphi$ to $\partial_{n+1}h_n=\iota_{\mathcal{A}}\alpha$. $\square$

As an important application, Theorem 2.9 yields:

Corollary 2.10. *For any finite-dimensional algebroid $\mathcal{A}$ over k, the canonical map*

$$\int_{\mathbb{L}}^{a\in\mathcal{A}}\mathcal{A}(a,a)\xrightarrow{\simeq}\int_{\mathbb{L}}^{X\in\mathsf{Proj}_k\,\mathcal{A}}\mathsf{Proj}_k\,\mathcal{A}(X,X)$$

is an equivalence.

Example 2.11. If $\mathcal{A}=\star//A$ for a finite-dimensional k-algebra A and if $F:(\mathsf{Proj}_k\,\mathcal{A})^{\mathrm{op}}\otimes\mathsf{Proj}_k\,\mathcal{A}\longrightarrow\mathsf{Ch}_k$ satisfies the hypotheses of Theorem 2.9, we have $CH(A;F(A,A))\simeq\int_{\mathbb{L}}^{X\in\mathsf{Proj}_k\,A}F(X,X)$, where $F(A,A)$ is the A-bimodule that we obtain by evaluation of F on the free A-module A, and $CH(A;F(A,A))$ are the Hochschild chains of A with coefficients in that bimodule (Example 2.2). In particular, we obtain the Agreement Principle from [MCar94] and [Kel99, Theorem 1.5 (a)] that the Hochschild homology of A and the Hochschild-Mitchell homology of the k-linear category of finite-dimensional projective A-modules are isomorphic.

Remark 2.12 (Generalization). Above, we have defined and investigated derived coends of functors going from $\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}$ for some differential graded category $\mathcal{C}$ to chain complexes Ch_k over a field k. In fact, we could have also used chain complexes of modules over an algebra over k instead of Ch_k. Let us sketch this generalization: For a functor $F:\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}\longrightarrow\mathsf{Ch}_R$ to chain complexes of modules over some k-algebra R, we can consider the functor $\mathsf{Q}F:\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}\longrightarrow\mathsf{Ch}_R$ obtained by replacing F cofibrantly pointwise (here we fix again the projective model structure on Ch_R). Using the tensoring of Ch_R over Ch_k, we now define the bar construction $B_*\mathsf{Q}F$ by precisely the same formulae as in Definition 2.1. Its realization

$$\int_{\mathbb{L}}^{X\in\mathcal{C}}F(X,X):=|B_*\mathsf{Q}F|$$

will be referred to as *derived coend of F*. Having replaced F pointwise will ensure that $B_*\mathsf{Q}F$ is cofibrant in each level, which implies that $B_*\mathsf{Q}F$ is Reedy cofibrant. As a consequence, the proof of homotopy invariance goes through. This allows us to prove the generalization of the Yoneda Lemma, the Fubini Theorem and the Agreement Principle.

3 The Hochschild complex of a finite tensor category

In this section, we investigate the Hochschild complex of a finite tensor category. By means of a resolution of the canonical coend of a finite tensor category we express the Hochschild complex in terms of derived class functions (Section 3.2). This will turn out to be the key to a topological interpretation of the Hochschild complex in Section 4, where we construct a homotopy coherent action of the torus mapping class group.

We start by giving definitions and recalling standard terminology: Based on the comparison between derived coends and ordinary Hochschild homology in Section 2.2, it makes sense to make the following Definition:

Definition 3.1. For a k-linear category $\mathcal{C}$ we call the differential graded vector space

$$\int_{\mathbb{L}}^{X\in\operatorname{Proj}\mathcal{C}} \mathcal{C}(X,X)$$

the *Hochschild complex of* $\mathcal{C}$.

This definition reduces to the standard definition of Hochschild homology of a differential graded (here: just linear) category as appearing in [Kel99, CR05]. Sometimes it is also referred to as a *derived trace*. Let us emphasize, however, that in the above definition the derived coend only runs over the *projective objects*.

In [Shi18] a version of Hochschild cohomology of a finite Abelian linear category is proposed using the category of right exact endofunctors. Hochschild homology is then defined indirectly using the Nakayama functor and a dualization. Due to the strong finiteness conditions in [Shi18], all these definitions are equivalent on their common domain of definition.

Definition 3.1 is of course made in such a way that, if $\mathcal{C}$ is a *finite category*, i.e. a linear category given by finite-dimensional modules over some finite-dimensional algebra A, then $\int_{\mathbb{L}}^{X\in\operatorname{Proj}\mathcal{C}} \mathcal{C}(X,X)$ is just equivalent to the ordinary Hochschild chains on A with coefficients in the A-bimodule A.

Since the categories we are interested in will all be of that type, one might ask why it is necessary to consider the complex $\int_{\mathbb{L}}^{X\in\operatorname{Proj}\mathcal{C}} \mathcal{C}(X,X)$ when it is just equivalent to the Hochschild complex of some algebra. The answer is that just knowing that $\int_{\mathbb{L}}^{X\in\operatorname{Proj}\mathcal{C}} \mathcal{C}(X,X)$ is equivalent to a Hochschild complex of *some* algebra is often not very helpful because this presentation in terms of an algebra might be non-canonical; in some sense it corresponds to a choice of coordinates. As a consequence, constructions performed on $\int_{\mathbb{L}}^{X\in\operatorname{Proj}\mathcal{C}} \mathcal{C}(X,X)$ might not have direct counterpart for the Hochschild complex of the randomly chosen algebra (in other cases they might have, but those constructions might be a lot more complicated). This is especially problematic when the category has more structure (tensor product, braiding, ribbon twist). This additional structure might not be reflected on the algebra. Since the main goal of this article is to investigate the additional structure which is present on $\int_{\mathbb{L}}^{X\in\operatorname{Proj}\mathcal{C}} \mathcal{C}(X,X)$ when $\mathcal{C}$ is a monoidal category or braided monoidal category (possibly with more structure or properties), using Definition 3.1 is justified.

Before proceeding let us recall some standard notions from the theory of linear monoidal categories: A *k-linear monoidal category* is a monoidal category with k-linear monoidal product. In a *rigid* k-linear monoidal category every object $X\in\mathcal{C}$ has a left dual $X^\vee$ and a right dual ${}^\vee X$. These give us the natural adjunction isomorphisms

$$\begin{aligned}
\mathcal{C}(X\otimes Y,Z)&\cong\mathcal{C}(X,Z\otimes Y^\vee)\ ,\\
\mathcal{C}(Y^\vee\otimes X,Z)&\cong\mathcal{C}(X,Y\otimes Z)\ ,\\
\mathcal{C}(X\otimes {}^\vee Y,Z)&\cong\mathcal{C}(X,Z\otimes Y)\ ,\\
\mathcal{C}(Y\otimes X,Z)&\cong\mathcal{C}(X,{}^\vee Y\otimes Z)
\end{aligned}$$

for $X,Y,Z\in\mathcal{C}$ (we are following here the conventions of [EGNO17]). A k-linear Abelian rigid monoidal category with simple unit will be referred to as a *tensor category*. A *finite tensor category* [EO04] is a tensor category which is also finite as linear category. Such a category has the important property that $P\otimes X$ is projective for $P\in\operatorname{Proj}\mathcal{C}$ and $X\in\mathcal{C}$ and that the tensor product is exact in both arguments. Furthermore, it is self-injective, i.e. the projective objects are precisely the injective ones.

3.1 Hochschild complex of Drinfeld doubles in finite characteristic

Before investigating the properties of the derived coend $\int_{\mathbb{L}}^{X\in\operatorname{Proj}\mathcal{C}} \mathcal{C}(X,X)$ in general, it is certainly instructive to look at a certain class of finite tensor categories which allow us to perform some concrete computations, namely *Drinfeld doubles*. Recall from e.g. [Kas95, Chapter IX] that for a finite group G the Drinfeld double $D(G)$ is a ribbon factorizable Hopf algebra whose underlying vector space is $k(G)\otimes k[G]$. Here we denote by $k(G)$ the commutative algebra of k-valued functions on G; a basis will be given by the functions $(\delta_g)_{g\in G}$ supported in a single group element. Moreover, we denote by $k[G]$ the group algebra. Now the multiplication of $D(G)$ is given by

$$(\delta_a\otimes b)(\delta_c\otimes d)=\delta_a\delta_{bcb^{-1}}\otimes bd \quad\text{for all}\quad a,b,c,d\in G\ .$$

Modules over $D(G)$ can be equivalently written as Yetter-Drinfeld modules over $k[G]$, see [Kas95, Theorem IX.5.2], and hence as modules over the action groupoid $G//G$ of G acting on itself by conjugation;

$$\mathsf{Mod}_k D(G)\simeq\mathsf{Mod}_k(G//G)\ . \tag{3.1}$$

The groupoid $G//G$ is equivalent to the groupoid $\mathsf{PBun}_G(\mathbb{S}^1)$ of G-bundles over the circle, and in fact this observation will give a topological interpretation to $\int_{\mathbb{L}}^{X\in\mathsf{Proj}_k\, D(G)} \mathrm{Hom}_{D(G)}(X,X)$.

To see this, we introduce some notation: For a groupoid Γ, denote by by $\Lambda\Gamma$ its *loop groupoid*, i.e. the groupoid $\Gamma^{\Pi(\mathbb{S}^1)}$ of functors from the fundamental groupoid $\Pi(\mathbb{S}^1)$ of the circle $\mathbb{S}^1$ to Γ. For a group G and the groupoid BG with one object and automorphism group G, the loop groupoid ΛBG is equivalent to the groupoid of G-bundles over the circle,

$$\Lambda BG = BG^{\Pi(\mathbb{S}^1)} \simeq \mathsf{PBun}_G(\mathbb{S}^1)$$

by the holonomy classification of G-bundles, and hence equivalent to the action groupoid $G//G$. A similar computation shows $\Lambda^n BG \simeq \mathsf{PBun}_G(\mathbb{T}^n)$ and in particular

$$\Lambda(G//G) \simeq \mathsf{PBun}_G(\mathbb{T}^2)\ . \tag{3.2}$$

The chains on the loop groupoid $\Lambda\Gamma$ of any groupoid Γ are equivalent to the Hochschild chains of the free k-linear category $k[\Gamma]$ on Γ. This can be seen as a groupoid version of the classical result [Wei94, Corollary 9.7.5].

Lemma 3.2. *For any groupoid Γ, there is an equivalence*

$$\int_{\mathbb{L}}^{x\in\Gamma} k[\Gamma](x,x) \simeq N_*(\Lambda\Gamma;k)\ .$$

Proof. Up to equivalence, we can describe $\Pi(\mathbb{S}^1)$ as the groupoid $\star//\mathbb{Z}$ with one object and automorphism group $\mathbb{Z}$. As an abbreviation, we will write S_n for the space of n-simplices of the simplicial bar construction of $k[\Gamma](-,-)$. Taking Remark 2.3 into account we can write

$$S_n = \bigoplus_{x_0,\ldots,x_n\in\Gamma} k[\Gamma(x_0,x_1)]\otimes\cdots\otimes k[\Gamma(x_n,x_0)]\ .$$

A string of n morphisms in $\Lambda\Gamma$ is a commutative diagram

$$\begin{array}{ccccccc}
x_0 & \xrightarrow{\varphi_0} & x_1 & \xrightarrow{\varphi_1} & \ldots & \xrightarrow{\varphi_{n-1}} & x_n \\
\big\downarrow{\scriptstyle \alpha_0} & & \big\downarrow{\scriptstyle \alpha_1} & & & & \big\downarrow{\scriptstyle \alpha_n} \\
x_0 & \xrightarrow{\varphi_0} & x_1 & \xrightarrow{\varphi_1} & \ldots & \xrightarrow{\varphi_{n-1}} & x_n\ .
\end{array}$$

Sending this string to the loop

$$x_0 \xrightarrow{\varphi_0} x_1 \longrightarrow \ldots \longrightarrow x_n \xrightarrow{(\varphi_{n-1}\ldots\varphi_0)^{-1}\alpha_n} x_0 \in S_n$$

yields an isomorphism from the free simplicial vector space $k[B\Lambda\Gamma]$ on the nerve $B\Lambda\Gamma$ of the loop groupoid of Γ to S_*. By taking normalized chains, the claim follows. □

Proposition 3.3. *Let G be a finite group. Then there is an equivalence of differential graded vector spaces*

$$\int_{\mathbb{L}}^{X\in\mathsf{Proj}_k\, D(G)} \mathrm{Hom}_{D(G)}(X,X) \simeq N_*(\mathsf{PBun}_G(\mathbb{T}^2);k)\ . \tag{3.3}$$

Proof. The k-linear categories of finite-dimensional representations of $D(G)$ and $G//G$ are equivalent by (3.1), and hence so are their Hochschild chains. If we apply Corollary 2.10 to the free k-linear category $k[G//G]$, we arrive at

$$\int_{\mathbb{L}}^{X\in\mathsf{Proj}_k\, D(G)} \mathrm{Hom}_{D(G)}(X,X) \simeq \int_{\mathbb{L}}^{g\in G//G} k[G//G](g,g)\ .$$

Now we use Lemma 3.2 and (3.2). □

The left hand side of (3.3) is also equivalent to the ordinary Hochschild chains on $D(G)$ with coefficients in $D(G)$ seen as bimodule over itself (Example 2.11). By transporting the geometric mapping class group action from $N_*(\mathsf{PBun}_G(\mathbb{T}^2);k)$ to the left hand side we obtain:

Corollary 3.4. *For any finite group G, the Hochschild chains of the Drinfeld double $D(G)$ carry a homotopy coherent $\mathrm{SL}(2,\mathbb{Z})$-action.*

The category of modules over a Drinfeld double is a very tractable example of a modular category (the definition of a modular category will be recalled at the beginning of Section 4). It is non-semisimple if and only if the characteristic of k divides $|G|$. The above result establishes a homotopy coherent mapping class group action on its Hochschild complex. As one of the main results of this article (Theorem 4.7), we generalize this to arbitrary modular categories. In the general case, a geometric argument as in Proposition 3.3 will not be available.

3.2 Traces, class functions and the Lyubashenko coend

If we are given a finite tensor category $\mathcal{C}$ and consider its Hochschild complex $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$, then the tensor structure is of course not needed to define the complex itself. However, it leads to simplifications when trying to compute the Hochschild homology. The idea is to express a (derived) trace (i.e. a (derived) coend of some sort) via a (derived) space of class functions.

Before making this idea precise below in the case of interest to us, we explain in more detail a related instance where it appears in a different form: Let $\mathcal{C},\mathcal{D}$ and $\mathcal{E}$ be finite categories over k and $F:\mathcal{D}\otimes\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}\longrightarrow\mathcal{E}$ a linear functor. Of course, if $\mathcal{E}$ has sufficiently many colimits, we may consider the coend $\int^{X\in\mathcal{C}} F(-,X,X)$ to obtain a functor $\mathcal{D}\longrightarrow\mathcal{E}$. However, if F is left-exact, we might want to consider a coend of functors such that the result is again left-exact and such that the universality of the coend holds with respect to left-exact functors. Such requirements arise in conformal field theory for the gluing of conformal blocks. Motivated by this problem, a coend $\oint^{X\in\mathcal{C}} F(-,X,X)$ with values in left-exact functors was studied in [Lyu96], see also [FS17] for a review and the relation to conformal field theory. It is a key insight that the coend in left-exact functors can be represented by a canonical object in the following way: Let $\mathcal{C}$ be a finite tensor category. Then one may define the coend

$$\mathbb{F}:=\int^{X\in\mathcal{C}} X\otimes X^\vee$$

which is called the *canonical coend of* $\mathcal{C}$ or also the *Lyubashenko coend* due to its appearance in [Lyu95a, Lyu95b, KL01]. By [Lyu96, Section 8.2] we find

$$\oint^{X\in\mathcal{C}} \mathcal{C}(X,-\otimes X)\cong\mathcal{C}(I,-\otimes\mathbb{F}) \ ,$$

i.e. the coend of the morphism space functor computed in the category of left-exact functors (which is just a type of trace) can be written as the space of morphisms from the monoidal unit to some special object $\mathbb{F}$. If $\mathcal{C}$ arises as finite-dimensional modules over a finite-dimensional Hopf algebra, then $\mathbb{F}$ is the coadjoint representation. For this reason, $\mathcal{C}(I,\mathbb{F})$ should be thought of as a generalized space of class functions. In summary, we see an instance where a trace is expressed as a space of class functions. We should note that the object $\mathbb{F}$ is not only interesting because it provides a description of certain coends in left-exact functors. It also turns out to be the key ingredient for the construction of the mapping class group actions in [Lyu95a, Lyu95b], see also Section 4.

This suggests the question whether we can describe for a finite tensor category $\mathcal{C}$ in an analogous way the derived trace $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$, i.e. the Hochschild complex, as a derived space of class functions using some special differential graded object of $\mathcal{C}$; and if so, whether it is related to the Lyubashenko coend. In order to answer these questions, we consider for any functor $F:\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}\longrightarrow\mathcal{C}$ the derived coend $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)$ by means of the generalizations given in Remark 2.12. Strictly speaking, we cannot see this (as we would like) as a differential graded object in $\mathcal{C}$ because $\mathcal{C}$ does not have infinite coproducts, but the definition of the derived coend involves coproducts over *all* projective objects. Fortunately, we know that up to equivalence we can write $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)$ using some finite collection of projective objects. By the Agreement Principle, even one suitably chosen projective module will suffice. We will denote such a 'finite version' of $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)$ by

$$\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X) \ . \tag{3.4}$$

This is now a differential graded object in $\mathcal{C}$ concentrated in non-negative degree. Up to equivalence, this object is independent of *how* we make the derived coend finite, i.e. two 'finite versions' are related by a canonical zig-zag of equivalences. The computation of (3.4) simplifies when F sends pairs of projective objects to projective objects because then the pointwise cofibrant replacement for F is not necessary (see Remark 2.12). In that case, (3.4) is level-wise projective.

Using such finite derived coends in $\mathcal{C}$ we are able to express $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(G(X),X)$ for any endofunctor G of $\mathcal{C}$ as a morphism space from the monoidal unit to some object:

Theorem 3.5. *Let $\mathcal{C}$ be a finite tensor category. Then for any linear functor $G:\mathcal{C}\longrightarrow\mathcal{C}$ there is a canonical equivalence*

$$\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(G(X),X)\simeq\mathcal{C}\left(I,\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X\otimes G(X)^\vee\right)$$

Proof. By duality and the Agreement Principle we find

$$\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(G(X),X)\cong\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(I,X\otimes G(X)^\vee)\simeq\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(I,X\otimes G(X)^\vee) \ . \tag{3.5}$$

We want to compare this with $\mathcal{C}\left(I, \int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X\otimes G(X)^\vee\right)$, where the object $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X\otimes G(X)^\vee$ is a special case of the construction (3.4). The point-wise cofibrant replacement that would usually be involved in the definition of this derived coend may be omitted because $X\otimes G(X)^\vee$ is projective whenever X is. Since $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(I, X\otimes G(X)^\vee)$ is defined using finite direct sums which are preserved by the hom functor, we now find $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(I, X\otimes G(X)^\vee)\cong \mathcal{C}\left(I, \int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X\otimes G(X)^\vee\right)$ which combined with (3.5) yields the assertion. □

This statement is not very useful unless we can understand the differential graded object $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X\otimes G(X)^\vee$ in $\mathcal{C}$. To this end, we note that for any functor $F:\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}\longrightarrow\mathcal{C}$ there is an augmentation

$$\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)\longrightarrow \int_{\mathrm{f}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)$$

as follows from the definition of the underived coend as a coequalizer. Here the 'f' on the right hand side indicates that the same reduction to a finite coend has been used. This map is surjective, hence a fibration. In fact, the zeroth homology of $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)$ is $\int_{\mathrm{f}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)$ as follows again by definition of the underived coend. But in general, there is no reason why $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)$ should be a projective resolution of $\int_{\mathrm{f}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)$. However, we prove below that this will be true when F is exact and sends pairs of projective objects to projective objects. In this case, we will understand F as a functor $F:\mathcal{C}^{\mathrm{op}}\boxtimes\mathcal{C}\longrightarrow\mathcal{C}$, where $\boxtimes$ denotes the Deligne product. First observe that by [KL01, Proposition 5.1.7] the exactness of F ensures that the canonical map $\int_{\mathrm{f}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)\longrightarrow\int^{X\in\mathcal{C}} F(X,X)$ is an isomorphism (this is a statement about underived coends). Therefore, we will just write $\int^{X\in\mathcal{C}} F(X,X)$ instead of $\int_{\mathrm{f}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)$.

Proposition 3.6. *Let $\mathcal{C}$ be a finite category and $F:\mathcal{C}^{\mathrm{op}}\boxtimes\mathcal{C}\longrightarrow\mathcal{C}$ an exact functor that sends pairs of projective objects to projective objects. Then*

$$\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)\longrightarrow \int^{X\in\mathcal{C}} F(X,X)$$

is a projective resolution. In particular, for any finite tensor category $\mathcal{C}$ and any exact functor $G:\mathcal{C}\longrightarrow\mathcal{C}$

$$\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X\otimes G(X)^\vee\longrightarrow \int^{X\in\mathcal{C}} X\otimes G(X)^\vee$$

is a projective resolution.

Proof. By what has just been explained above, it remains to prove $H_p\left(\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(X,X)\right)=0$ for $p\neq 0$. For the proof of this fact, we write $\mathcal{C}$ as finite-dimensional modules over a finite-dimensional algebra A. By the Agreement Principle (Theorem 2.9) $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} F(A,A)$ is equivalent to the Hochschild chains $A\overset{\mathbb{L}}{\otimes}_{A^\mathrm{e}} F(A,A)$ for the $\mathcal{C}$-valued A-bimodule $F(A,A)$. In order to compute the corresponding Hochschild homology, we consider the object $A\boxtimes A\in\mathcal{C}^{\mathrm{op}}\boxtimes\mathcal{C}$, where A acts by right multiplication on the first copy and by left multiplication on the second copy. But A can additionally act from the left on the first copy and from the right on the second copy. This makes $A\boxtimes A$ an A-bimodule in $\mathcal{C}^{\mathrm{op}}\boxtimes\mathcal{C}$. The Hochschild complex for this bimodule is given by

$$\cdots \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} A^{\otimes 2}\bullet(A\boxtimes A) \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} A\bullet(A\boxtimes A) \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} A\boxtimes A\ ,$$

where $\bullet$ denotes the tensoring of objects in $\mathcal{C}^{\mathrm{op}}\boxtimes\mathcal{C}$ with vector spaces from the left. This is a complex (or simplicial object) in $\mathcal{C}\boxtimes\mathcal{C}$. The underlying complex of vector spaces, however, is just the Hochschild complex for the free A-bimodule. Therefore, the augmentation

$$A\overset{\mathbb{L}}{\otimes}_{A^\mathrm{e}}(A\boxtimes A)\longrightarrow A\otimes_{A^\mathrm{e}}(A\boxtimes A) \tag{3.6}$$

is an equivalence. Since F is linear, the augmentation map

$$A\overset{\mathbb{L}}{\otimes}_{A^\mathrm{e}} F(A,A)\longrightarrow A\otimes_{A^\mathrm{e}} F(A,A) \tag{3.7}$$

of the Hochschild complex $A\overset{\mathbb{L}}{\otimes}_{A^\mathrm{e}} F(A,A)$ is the image of the equivalence (3.6) under F. By exactness of F the map (3.7) is now also an equivalence, which proves the claim. □

Corollary 3.7. *For any finite tensor category $\mathcal{C}$, the object $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X\otimes X^\vee$ is a projective resolution of the canonical coend $\mathbb{F}=\int^{X\in\mathcal{C}} X\otimes X^\vee$ and allows to write the Hochschild complex of $\mathcal{C}$ up to equivalence as*

$$\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \simeq \mathcal{C}\left(I, \int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X\otimes X^\vee\right) .$$

This finally allows us to describe the Hochschild complex as a generalized space of class functions, i.e. as a hom from the monoidal unit to the derived Lyubashenko coend. Note however that Corollary 3.7 does *not* say that $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ is equivalent to $\mathcal{C}(I,\mathsf{Q}\mathbb{F})$ for an *arbitrary* projective resolution $\mathsf{Q}\mathbb{F}$ of $\mathbb{F}$. Of course, $\mathsf{Q}\mathbb{F}\simeq\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X\otimes X^\vee$ by the essential uniqueness of projective resolutions, but $\mathcal{C}(I,-)$ need not preserve this equivalence in general. However, in the following technical Lemma, whose proof is based on the theory of modified traces [GKP18], we find that, under the assumption that $\mathcal{C}$ is pivotal, this will be true.

Lemma 3.8. *Let $\mathcal{C}$ be a pivotal tensor category over an algebraically closed field with finite-dimensional morphism spaces and enough projectives. Then for $X\in\mathcal{C}$ the functor $\mathcal{C}(X,-)$ preserves equivalences between non-negatively differential graded objects which are degree-wise projective.*

Proof. Let D be the projective cover of the monoidal unit and α the socle of D. Consider now the right modified α-trace on the tensor ideal of projective objects [GKP18, Section 5.3]. This trace in particular provides non-degenerate pairings

$$\mathcal{C}(X,P)\otimes\mathcal{C}(\alpha\otimes P,X)\longrightarrow k \tag{3.8}$$

for $X\in\mathcal{C}$ and $P\in\mathsf{Proj}\,\mathcal{C}$ which are moreover natural in X and P. In particular, $\mathcal{C}(X,P)\cong\mathcal{C}(\alpha\otimes P,X)^*$ by natural isomorphisms.

Let now $P\longrightarrow Q$ be an equivalence of non-negatively differential graded objects in $\mathcal{C}$ which are degree-wise projective. We need to show that for $X\in\mathcal{C}$ the induced map $\mathcal{C}(X,P)\longrightarrow\mathcal{C}(X,Q)$ is an equivalence. Using the non-degenerate pairing (3.8) we can equivalently show that the induced map $\mathcal{C}(P,{}^\vee\alpha\otimes X)^*\longrightarrow\mathcal{C}(Q,{}^\vee\alpha\otimes X)^*$ is an equivalence. Since this is a map of finite-dimensional differential graded vector spaces, it suffices to show that the dual map $\mathcal{C}(Q,{}^\vee\alpha\otimes X)\longrightarrow\mathcal{C}(P,{}^\vee\alpha\otimes X)$ is an equivalence. But this is a standard fact from homological algebra, see e.g. [Iv86, Theorem 7.5]. □

Theorem 3.9. *Let $\mathcal{C}$ be a pivotal finite tensor category over an algebraically closed field. Then for any exact functor $G:\mathcal{C}\longrightarrow\mathcal{C}$ there is a canonical equivalence*

$$\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(G(X),X) \simeq \mathcal{C}\left(I, \mathsf{Q}\int^{X\in\mathcal{C}} X\otimes G(X)^\vee\right) , \tag{3.9}$$

where $\mathsf{Q}\int^{X\in\mathcal{C}} X\otimes G(X)^\vee$ is an arbitrary projective resolution of $\int^{X\in\mathcal{C}} X\otimes G(X)^\vee$. In particular,

$$\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \simeq \mathcal{C}\left(I,\mathsf{Q}\mathbb{F}\right) .$$

Proof. By Theorem 3.5 we know

$$\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(G(X),X) \simeq \mathcal{C}\left(I, \int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X\otimes G(X)^\vee\right) ;$$

moreover $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X\otimes G(X)^\vee$ is a projective resolution of $\int^{X\in\mathcal{C}} X\otimes G(X)^\vee$ by Proposition 3.6. However, as explained above, this does not mean that (3.9) holds automatically for any projective resolution of $\int^{X\in\mathcal{C}} X\otimes G(X)^\vee$ because $\mathcal{C}(I,-)$ may not preserve equivalences between two projective resolutions. But this is ensured by Lemma 3.8 if we assume that $\mathcal{C}$ is pivotal and over an algebraically closed field. □

4 Homotopy coherent projective mapping class group action

As one of the main results of this article, we establish a canonical homotopy coherent projective action of the mapping class group $\mathrm{SL}(2,\mathbb{Z})$ of the torus on the Hochschild complex of a modular category. As explained in the introduction, our result provides a homotopy coherent extension of the work of [LM94, Lyu95a, Lyu95b, LMSS18, Shi18].

Let us briefly recall the definition of a modular category: For a braided finite tensor category, one defines the *Müger center* as the subcategory spanned by all objects $X\in\mathcal{C}$ such that the double braiding $c_{Y,X}c_{X,Y}$ with every other object $Y \in \mathcal{C}$ is the identity. A braided finite tensor category is called *non-degenerate* if its Müger center just consists of finitely coproducts of the monoidal unit I, see [Shi19] for different characterizations of non-degeneracy. A *ribbon twist* on a braided finite tensor category $\mathcal{C}$ is a natural automorphism of the identity of $\mathcal{C}$ whose components $\theta_X : X \longrightarrow X$ satisfy

$$\begin{aligned} \theta_{X\otimes Y} &= c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y) \ , \\ \theta_I &= \mathrm{id}_I \ , \\ \theta_{X^\vee} &= \theta_X^\vee \ . \end{aligned}$$

A *finite ribbon category* is a braided finite tensor category equipped with a ribbon twist. Finally, a *modular category* is a finite ribbon category whose underlying braided finite tensor category is non-degenerate. Since a modular category is ribbon and hence pivotal, we may use the techniques developed in the last section.

4.1 Homotopy coherent projective actions

We begin by recalling the notion of a homotopy coherent (projective) group action and by discussing suitable resolutions in order to write such actions down in the case of interest. The reader familiar with homotopy coherent actions can just skim through the lines below and take note of the specific resolutions that will be used.

The idea underlying the notion of a homotopy coherent action ϱ of a group G on a chain complex C is to relax the requirement that for $g, h \in G$ the chain maps $\varrho(gh)$ and $\varrho(g)\varrho(h)$ are equal. Instead, they will just be homotopic by a specific homotopy $H_{g,h} : \varrho(gh) \simeq \varrho(g)\varrho(h)$ that does not only exist, but is part of the data. Moreover, one requires these homotopies to be coherent, i.e. all the different homotopies $H_{g,h}$ for $g, h \in G$ should be related by higher homotopies: For example, for $g, h, \ell \in G$ the diagram

$$\begin{tikzcd} & \varrho(g)\varrho(h\ell) \arrow[dr, "\varrho(g)H_{h,\ell}"] & \\ \varrho(gh\ell) \arrow[ur, "H_{g,h\ell}"] \arrow[dr, "H_{gh,\ell}"'] & & \varrho(g)\varrho(h)\varrho(\ell) \\ & \varrho(gh)\varrho(\ell) \arrow[ur, "H_{g,h}\varrho(\ell)"'] & \end{tikzcd}$$

is required to commute up to homotopy, and again this homotopy is part of the data – and so on and so forth, as will be made precise below; for an introduction to homotopy coherent mathematics we refer to [Rie18].

In the sequel, we will need a slight variation of the above, namely homotopy coherent *projective* representations of a group G. A projective G-representation on a vector space V (or chain complex) is a group morphism $G \longrightarrow \mathrm{P}\,\mathrm{Aut}(V)$, where $\mathrm{P}\,\mathrm{Aut}(V)$ is the quotient of $\mathrm{Aut}(V)$ by the normal subgroup $k^\times \cdot \mathrm{id}_V$. Often it is convenient to assign to each $g \in G$ an actual automorphism $\varrho(g)$ of V by choosing a lift of $\mathrm{Aut}(V) \longrightarrow \mathrm{P}\,\mathrm{Aut}(V)$. Then

$$\varrho(g)\varrho(h) = \xi(g,h)\varrho(gh)$$

for $g, h \in G$ and a cocycle $\xi \in Z^2(G; k^\times)$. We will then say that ϱ is *ξ-projective* because once a lift is chosen, the cocycle ξ controls the projectivity. This point of view is rather helpful because it allows to describe projective actions via the *twisted group algebra* $k_\xi[G]$ of G and $\xi \in Z^2(G; k^\times)$. The underlying vector space of $k_\xi[G]$ is the free vector space on G. The multiplication is given by

$$\langle g\rangle\langle h\rangle = \xi(g,h)\langle gh\rangle \quad \text{for all} \quad g, h \in G \ ,$$

where $\langle g\rangle$ is the basis element corresponding to $g \in G$. The cocycle ξ will be referred to as the *twist*. Now a ξ-projective G-representation on a vector space V is just a $k_\xi[G]$-action on V.

In order to formalize the notion of a homotopy coherent (projective) action, we may as well define the notion of a homotopy coherent action of an algebra (which will then include the case of a twisted group algebra). To this end, we recall the bar construction of an algebra A over k that one constructs via the free-forgetful adjunction

$$F \, : \, \mathsf{Vect}_k \ \rightleftarrows \ \mathsf{Alg}_k \, : \, U \ .$$

Suppressing the forgetful functor in the notation, we will see F as an endofunctor of Alg_k. The algebra A gives rise to a simplicial algebra $\mathrm{Bar}\, A$

$$\cdots \ \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \ F^3 A \ \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \ F^2 A \ \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\!\!\! \ FA \ ,$$

which in level n is given by $F^{n+1}A$ – the *bar resolution of* A. Of course, we can also see this simplicial algebra as a differential graded algebra (via the Dold-Kan correspondence). The bar construction comes with an augmentation $\operatorname{Bar} A \longrightarrow A$ which is also a trivial fibration (on the level of underlying simplicial vector spaces, this augmentation admits extra degeneracies).

The algebra of 0-simplices of $\operatorname{Bar} A$ is the free algebra on the vector space A, i.e. the tensor algebra on the vector space A. For an element $a_1 \otimes \cdots \otimes a_n$ in the free algebra on A we will write $(a_1)\dots(a_n)$. This bracket notation borrowed from [Rie18] is rather convenient because it allows to write the higher simplices of the bar construction by nested brackets. Then the j-the face operator ∂_j deletes the j-th pair of brackets (counted from outside to inside). For example, for $a, b \in A$ we have a 1-simplex $((a)(b))$ with $\partial_0((a)(b)) = (a)(b)$ and $\partial_1((a)(b)) = (ab)$. Hence, the 0-simplices $(a)(b)$ and (ab) are not equal, but there is a path between them.

In order to define homotopy coherent actions, we also need the internal hom of differential graded vector spaces: For differential graded vector spaces and C and D, their internal hom $[C, D]$ is the differential graded vector space with $[C, D]_n := \prod_{m\in\mathbb{Z}} \operatorname{Hom}_k(C_m, D_{m+n})$. As usual, composition endows $[C, C]$ with the structure of a differential graded algebra.

Definition 4.1. For a k-algebra A a *homotopy coherent action of A on a differential graded vector space C* is a map of differential graded algebras $\operatorname{Bar} A \longrightarrow [C, C]$.

Remark 4.2. For the reader familiar with homotopy coherent actions, let us remark that this coincides with the usual definition of the homotopy coherent action of an operad [BM06] because $\operatorname{Bar} A$ cofibrantly resolves the operad whose unary operations are given by A.

As a consequence, given a group G and a cocycle $\xi \in Z^2(G; k^\times)$, a homotopy coherent ξ-projective action of G on a differential graded vector space C is a map $\operatorname{Bar} k_\xi[G] \longrightarrow [C, C]$ of differential graded algebras.

4.2 Homotopy coherent projective actions from central extensions of rank one

Let $0 \longrightarrow J \longrightarrow G \longrightarrow H \longrightarrow 0$ be a short exact sequence of groups. If we are given a representation of G on a vector space, then its is easy to decide whether this representation descends to H: We just have to verify that all elements in the kernel J of $G \longrightarrow H$ are sent to the identity. A similar statement holds for projective actions. If however we are given a (projective) action of G on a chain complex and are able to show that all elements in the kernel J of $G \longrightarrow H$ act by chain maps which are homotopic to the identity, then this is not enough to conclude that we get *in a canonical way* a homotopy coherent (projective) action of the quotient H.

The purpose of this subsection is to highlight at least one case, namely the one of a central extension of rank one, in which we actually get a homotopy coherent (projective) action of the quotient. This result will be key for the construction of the homotopy coherent mapping class group action in the next subsection.

Proposition 4.3. *Let $0 \longrightarrow \mathbb{Z} \longrightarrow G \xrightarrow{\pi} H \longrightarrow 0$ be a central extension of groups and $\xi \in Z^2(H; k^\times)$. We denote the image of $1 \in \mathbb{Z}$ under $\mathbb{Z} \longrightarrow G$ by τ. Suppose we are given a $\pi^*\xi$-projective representation ϱ of G on a chain complex C and a homotopy $\varrho(\tau) \overset{L}{\simeq} \mathrm{id}_C$ such that $L\varrho(g) = \varrho(g)L$ for all $g \in G$. Then this data induces in a canonical way a homotopy coherent ξ-projective representation of H on C.*

The proof of this technical result will require the inductive construction of a simplicial map that will follow a standard procedure that we recall now:

Lemma 4.4. *Let X and Y be simplicial vector spaces. Suppose we are given a family $\phi_n : X_n \longrightarrow Y_n$ of linear maps that is characterized inductively in the following way:*

1. *$\phi_0 : X_0 \longrightarrow Y_0$ is just an arbitrary linear map.*
2. *For some $n \geq 1$ suppose the linear maps $\phi_p : X_p \longrightarrow Y_p$ for $0 \leq p \leq n-1$ are already such that they satisfy $f^*\phi_q\sigma = \phi_p f^*\sigma$ for $f \in \Delta(p, q)$ and $\sigma \in X_q$ for $0 \leq p, q \leq n-1$. Furthermore, the $\phi_n : X_n \longrightarrow Y_n$ are given as follows:*
   - (a) *If $\sigma \in X_n$ is degenerate, then there is a unique iterated degeneracy operator $S = s_{j_1} \dots s_{j_\ell}$ with $j_1 > \cdots > j_\ell$ and a unique non-degenerate $n-\ell$-simplex τ such that write $\sigma = S\tau$. Then set $\phi_n(\sigma) = S\phi_{n-\ell}(\tau)$.*
   - (b) *If σ is non-degenerate, then denote by $\phi_{n-1}(\partial\sigma) : k[\partial\Delta^n] \longrightarrow Y$ the simplicial map that we obtain by evaluation of ϕ_{n-1} on the faces on σ (the fact that this map is well-defined is due to the fact that the ϕ_p respect the simplicial operators for $0 \leq p \leq n-1$). Now $\phi_n(\sigma)$ is given as a lift*

$$\begin{tikzcd} k[\partial\Delta^n] \arrow[r, "\phi_{n-1}(\partial\sigma)"] \arrow[d, hook] & Y \\ k[\Delta^n] \arrow[ur, dashed, "\phi_n(\sigma)"'] & \end{tikzcd},$$

where the existence of such a lift is an assumption.

Then the maps $\phi_n : X_n \longrightarrow Y_n$ are simplicial.

The proof of the Lemma is straightforward if one takes the statements [Hir03, Lemma 15.8.3&4] on degeneracy operators into account.

Before we prove Proposition 4.3, we need to introduce some notation: Fix a set-theoretic section $s : H \longrightarrow G$ of π. The deviation of s from being a group morphism is described by the classifying cocycle $\alpha \in Z^2(H;\mathbb{Z})$ of the central extension $0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow H \longrightarrow 0$; i.e.

$$sh_1 sh_2 = \tau^{\alpha(h_1,h_2)} s(h_1h_2) \quad \text{for} \quad h_1, h_2 \in H \ .$$

For three elements in H, for example, we have

$$sh_1sh_2sh_3 = \tau^{\alpha(h_1,h_2)} s(h_1h_2)sh_3 = \tau^{\alpha(h_1,h_2)+\alpha(h_1h_2,h_3)} s(h_1h_2h_3)$$

by applying the cocycle first to h_1 and h_2 and then to h_1h_2 and h_3. Alternatively, we could have applied it to h_2 and h_3 and then to h_1 and h_2h_3. The cocycle condition

$$\alpha(h_1,h_2) + \alpha(h_1h_2,h_3) = \alpha(h_2,h_3) + \alpha(h_1,h_2h_3) \tag{4.1}$$

tells us that both ways yield the same result. We will therefore denote any of the sides of (4.1) by $\alpha(h_1,h_2,h_3)$. More generally, we can define $\alpha(h_1,\ldots,h_n)$ for n elements in H such that

$$sh_1 \ldots sh_n = \tau^{\alpha(h_1,\ldots,h_n)} s(h_1 \ldots h_n) \ . \tag{4.2}$$

Such a multi-element notation will also be used for the cocycle $\xi \in Z^2(H;k^\times)$ describing the projectivity.

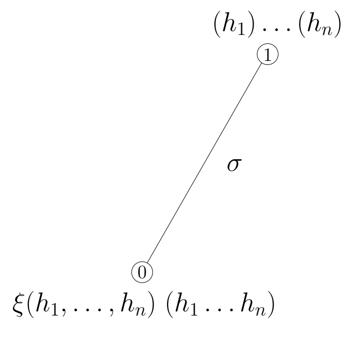
Proof of Proposition 4.3. Following Definition 4.1 we have to build a map $\varphi : \mathrm{Bar}\, k_\xi[H] \longrightarrow [C,C]$ of differential graded algebras. We will equivalently describe it as a map of simplicial algebras and construct the underlying simplicial map following Lemma 4.4. Additionally, we will make sure that in every degree the algebra structure is respected such that we actually obtain a map of simplicial algebras.

First we set $\varphi(h) = \varrho(sh)$ for $h \in H$ and our fixed set-theoretic section $s : H \longrightarrow G$ (up to homotopy, the construction will not depend on the choice of the section). This assignment extends to an algebra map

$$\mathrm{Bar}_0\, k_\xi[H] = Fk[H] \longrightarrow \mathsf{Ch}_k(C,C)$$

because $\mathrm{Bar}_0\, k_\xi[H]$ is freely generated as an algebra by the elements of H. This way, we obtain the definition of φ on 0-simplices.

Next, we consider a 1-simplex $\sigma \in \mathrm{Bar}_1\, k_\xi[H] = F^2k[H]$ of the form $\sigma = ((h_1)\ldots(h_n))$, where we use the bracket notation explained on page 15. These freely generate $F^2k[H]$ as an algebra. We can see σ as a path in the bar construction $\mathrm{Bar}\, k_\xi[H]$ from $\partial_0\sigma = (h_1)\ldots(h_n)$ to $\partial_1\sigma = \xi(h_1,\ldots,h_n)(h_1\ldots h_r)$, where $\xi(h_1,\ldots,h_n)$ is the multi-element notation for the cocycle ξ just introduced. Therefore, we depict the 1-simplex σ by



.

If σ is degenerate, then Lemma 4.4 tells us how to define φ on it. Suppose now that σ is non-degenerate. By definition of φ on 0-simplices, we have

$$\begin{aligned}
\varphi(\partial_0\sigma) &= \varphi(h_1)\ldots\varphi(h_n) = \varrho(sh_1)\ldots\varrho(sh_n) \\
&= \xi(h_1,\ldots,h_n)\varrho(sh_1\ldots sh_n) \\
&= \xi(h_1,\ldots,h_n)\varrho(\tau)^{\alpha(h_1,\ldots,h_n)}\varrho(s(h_1\ldots h_n)) \ , \quad \text{see (4.2)} \ ; \\
\varphi(\partial_1\sigma) &= \xi(h_1,\ldots,h_n)\varrho(s(h_1\ldots h_n)) \ .
\end{aligned}$$

In summary, we arrive at

$$\varphi(\partial_0\sigma) = \varrho(\tau)^{\alpha(h_1,\ldots,h_n)}\varphi(\partial_1\sigma) \ .$$

Now we can assign to σ the homotopy $L^{\alpha(h_1,\dots,h_n)}\varrho(\partial_1\sigma)$, i.e. the $\alpha(h_1,\dots,h_n)$-th power of L combined with the identity homotopy of $\varrho(\partial_1\sigma)$. Since $L^{\alpha(h_1,\dots,h_n)}\varrho(\partial_1\sigma) = \varrho(\partial_1\sigma)L^{\alpha(h_1,\dots,h_n)}$ by assumption, we will suppress the identity homotopy in the notation. This assignment extends to an algebra map

$$\mathrm{Bar}_1\, k_\xi[H] = F^2k[H] \longrightarrow \mathsf{Ch}_k(C\otimes N_*(\Delta^1;k), C)$$

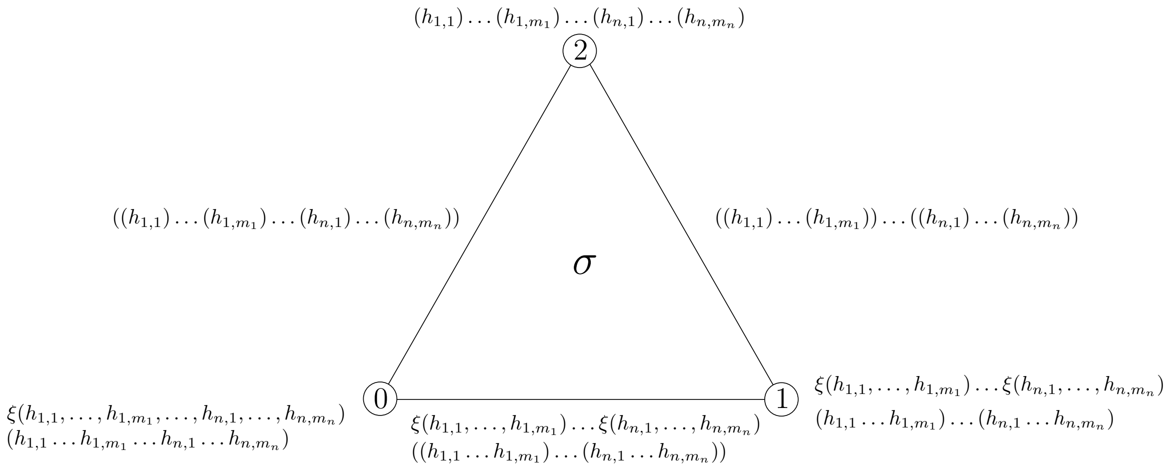
because the 1-simplices that we considered freely generate $\mathrm{Bar}_1\, k_\xi[H]$ as an algebra. This concludes the definition of φ on 1-simplices.

Consider now a 2-simplex $\sigma \in \mathrm{Bar}_2\, k_\xi[H] = F^3k[H]$ of the form

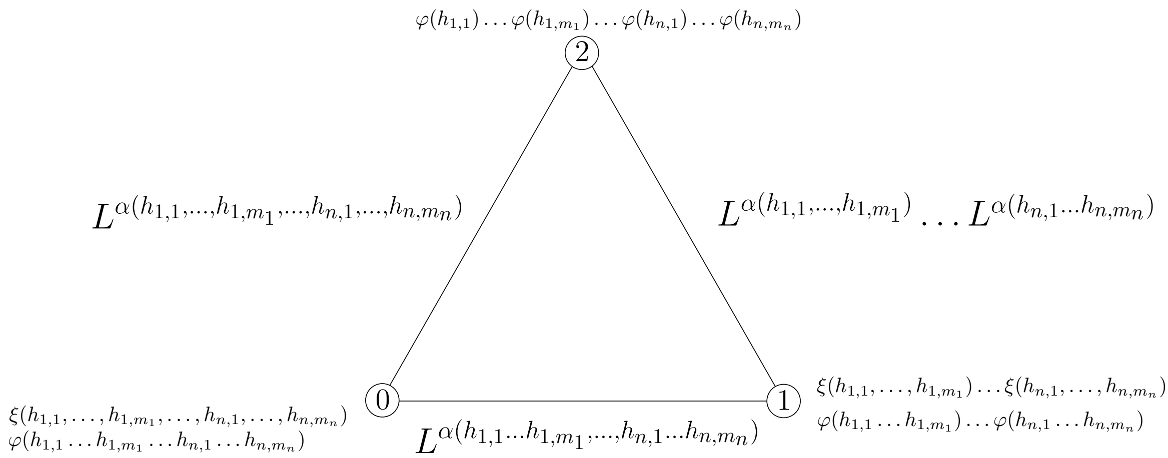
$$\sigma = (((h_{1,1})\dots(h_{1,m_1}))\dots((h_{n,1})\dots(h_{n,m_n})))\ ;$$

again these freely generate $\mathrm{Bar}_2\, k_\xi[H]$ as an algebra. We will now prove that the homotopies that we obtain by evaluation of φ on the boundary of σ provide a strictly commuting triangle. This allows us to define φ on σ as the identity 2-homotopy. Then we extend multiplicatively and hence obtain the definition of φ on 2-simplices. This way we still follow the construction principle from Lemma 4.4 and make sure that φ respects the multiplicative structure. Since to the 2-simplices we have just assigned the identity, the definition on higher simplices can be completed trivially by assigning again identities, thereby completing the definition of φ as a map of simplicial algebras.

It still remains to show that the homotopies assigned by φ to the boundary of σ form a strictly commuting triangle. To this end, we depict the faces and vertices of σ as follows:



By the definition of φ on 0- and 1-simplices the image of the boundary of σ under φ is given by:



(4.3)

Since L commutes with φ by assumption, we are allowed to omit the identity homotopies in the notation. By the cocycle condition on α, we have

$$\begin{aligned}\alpha(h_{1,1},\dots,h_{1,m_1},\dots,h_{n,1},\dots,h_{n,m_n}) =& \alpha(h_{1,1}\dots h_{1,m_1},\dots,h_{n,1}\dots h_{n,m_n})\\ &+\alpha(h_{1,1},\dots,h_{1,m_1})+\dots+\alpha(h_{n,1},\dots,h_{n,m_n})\ ,\end{aligned}$$

which allows us to conclude that (4.3) forms a strictly commutative triangle. This finishes the proof. $\square$

4.3 Homotopy coherent projective $\mathrm{SL}(2,\mathbb{Z})$-action on the Hochschild complex of a modular category

Having discussed the notion of a homotopy coherent projective action and some tools for its construction, we now finally exhibit a homotopy coherent projective action of the mapping class group of the torus on the Hochschild complex of a modular category.

The central non-homotopical ingredient will be the Lyubashenko-Majid action on the canonical coend: Let $\mathcal{C}$ be a modular category and $\mathbb{F}$ the canonical (underived) coend. As one of the main results of [LM94, Lyu95a, Lyu95b], Lyubashenko and Majid construct a projective action of the mapping class group of the punctured torus, i.e. the braid group on three strands $B_3 = \langle s,t,r \mid (st)^3 = s^2, s^4 = r\rangle$, on $\mathbb{F}$. These authors give explicit automorphisms of $\mathbb{F}$ for each generator such that r is sent to the inverse twist $\theta_{\mathbb{F}}^{-1}$ of $\mathbb{F}$. The B_3-action on $\mathbb{F}$ descends to an action of the mapping class group of the torus $\mathrm{SL}(2,\mathbb{Z}) = \langle s,t \mid (st)^3 = s^2, s^4 = 1\rangle$ on $\mathcal{C}(I,\mathbb{F})$, as follows from the naturality of the twist and $\theta_I = \mathrm{id}_I$.

Remark 4.5. The projective B_3-action on $\mathbb{F}$ can be turned into a linear one since $H^2(B_3; k^\times) = 0$. However, then the generator r might not be sent to the inverse twist any longer. Therefore, we refrain from getting rid of the projectivity.

Using the following Lemma the cocycle description of the projectivity of the representation of B_3 on $\mathbb{F}$ can be simplified.

Lemma 4.6. *Let $\pi : G \longrightarrow H$ an epimorphism of groups and X and Y objects in a k-linear category such that $\mathcal{C}(Y,X) \neq 0$. Then any projective representation $\varphi : G \longrightarrow \mathrm{P}\,\mathrm{Aut}\,X$ of G on X for which the composition*

$$G \overrightarrow{\varphi} \mathrm{P}\,\mathrm{Aut}\,X \longrightarrow \mathrm{P}\,\mathrm{Aut}\,\mathcal{C}(Y,X)$$

is trivial on $\ker\pi$ can be described by a family of automorphisms $\varrho(g)$ for $g \in G$ such that the cocycle controlling the projectivity with respect to these maps is the pullback $\pi^*\xi$ of a cocycle $\xi \in Z^2(H;k^\times)$ along π.

Proof. The projective representation of G on X is a group morphism $\varphi : G \longrightarrow \mathrm{P}\,\mathrm{Aut}\,X$. Its concatenation $\psi := \mathcal{C}(Y,-) \circ \varphi : G \longrightarrow \mathrm{P}\,\mathrm{Aut}\,\mathcal{C}(Y,X)$ with $\mathcal{C}(Y,-)$ sends $\ker\pi$ to 1. Now we choose a lift for each element in $\mathrm{P}\,\mathrm{Aut}\,\mathcal{C}(Y,X)$ that sends the unit to the identity of $\mathcal{C}(Y,X)$ and denote the lift of $\psi(g)$ for $g \in G$ by $\widetilde{\psi}(g)$. If $\widehat{\varphi}(g)$ is any lift of $\varphi(g)$, then $\mathcal{C}(Y,-)$ maps $\widehat{\varphi}(g)$ to $c_g\widetilde{\psi}(g)$ for a unique invertible scalar $c_g \in k^\times$. We now set $\widetilde{\varphi}(g) := \widehat{\varphi}(g)/c_g$ and thereby ensure that $\mathcal{C}(Y,-)$ maps $\widetilde{\varphi}(g)$ to $\widetilde{\psi}(g)$. If we let $\nu \in Z^2(G;k^\times)$ be the cocycle describing the projectivity of φ with respect to the representatives $\widetilde{\varphi}(g)$, we find by definition

$$\widetilde{\varphi}(g)\widetilde{\varphi}(g') = \nu(g,g')\widetilde{\varphi}(gg')$$

for $g,g' \in G$. When applying the functor $\mathcal{C}(Y,-)$ in the special case $g' \in \ker\pi$, we obtain

$$\widetilde{\psi}(g) = \widetilde{\psi}(g)\widetilde{\psi}(g') = \nu(g,g')\widetilde{\psi}(gg') = \nu(g,g')\widetilde{\psi}(g)$$

by the choice of our lifts and the assumption that ψ sends $\ker\pi$ to the unit. Since $\mathcal{C}(Y,X) \neq 0$, we conclude $\nu(g,g') = 1$. Hence, ν is trivial on $G \times \ker\pi$ and similarly on $\ker\pi \times G$. Now a direct computation shows that $\xi(h,h') := \nu(s(h),s(h'))$ for $h,h' \in H$ and any set-theoretic section $s : H \longrightarrow G$ of π defines a 2-cocycle ξ on H with such $\pi^*\xi = \nu$. $\square$

If we apply this Lemma to the epimorphism $B_3 \longrightarrow \mathrm{SL}(2,\mathbb{Z})$, $X = \mathbb{F}$ and $Y = I$, then thanks to $\mathcal{C}(I,\mathbb{F}) \neq 0$ and the fact that the B_3-action descends to an $\mathrm{SL}(2,\mathbb{Z})$-action on $\mathcal{C}(I,\mathbb{F})$, we conclude that the projectivity of the B_3-action on $\mathbb{F}$ can be described by the pullback of a cocycle $\xi \in Z^2(\mathrm{SL}(2,\mathbb{Z});k^\times)$. By looking at the proof of Lemma 4.6 we see that we can still arrange that the generator r of the kernel of the projection $B_3 \longrightarrow \mathrm{SL}(2,\mathbb{Z})$ is sent to $\theta_{\mathbb{F}}^{-1}$.

Let us now state the main Theorem:

Theorem 4.7. *The Hochschild complex $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ of a modular category $\mathcal{C}$ over an algebraically closed field carries a homotopy coherent projective action of the mapping class group $\mathrm{SL}(2,\mathbb{Z})$ of the torus which is induced in a canonical way by the action of the braid group on three strands on the canonical coend of $\mathcal{C}$.*

Remark 4.8. We may actually include a statement about the concrete nature of the projectivity of this action: Using Lemma 4.6 we have established that the projectivity B_3-action on $\mathbb{F}$ is controlled by the pullback of a cocycle $\xi \in Z^2(\mathrm{SL}(2,\mathbb{Z});k^\times)$. The homotopy coherent projective $\mathrm{SL}(2,\mathbb{Z})$-action that we construct will be an action of the bar resolution of the ξ-twisted group algebra of $\mathrm{SL}(2,\mathbb{Z})$.

Before going into the proof of Theorem 4.7, we discuss one immediate consequence following from the fact that taking finite-dimensional modules over a ribbon factorizable Hopf algebra provides an example of a modular category:

Corollary 4.9. *There is a canonical homotopy coherent projective* $\mathrm{SL}(2,\mathbb{Z})$*-action on the Hochschild complex of a ribbon factorizable Hopf algebra.*

When stated in this form, our result is a direct homotopy coherent extension of [LMSS18].

For the proof of Theorem 4.7 we will need a few further technical results. One key step will be to replace the Hochschild complex by an equivalent complex.

Proposition 4.10. *For any pivotal finite tensor category* $\mathcal{C}$ *over an algebraically closed field, the Hochschild complex of* $\mathcal{C}$ *is canonically equivalent to the derived coend* $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(I,P)\otimes\mathcal{C}(P,\mathbb{F})$.

The proof of this Proposition will need the following Lemma:

Lemma 4.11. *Let* $\mathcal{C}$ *be a finite category. Then for any* $X,Y\in\mathcal{C}$ *there is a natural equivalence*

$$\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(Y,P)\otimes\mathcal{C}(P,X)\simeq\mathcal{C}(Y,\mathsf{Q}X)\ .$$

Proof. Since $\mathcal{C}(P,X)\simeq\mathcal{C}(P,\mathsf{Q}X)$ for any $P\in\mathsf{Proj}\,\mathcal{C}$, it suffices to prove that the natural map

$$\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(Y,P)\otimes\mathcal{C}(P,\mathsf{Q}X)\longrightarrow\mathcal{C}(Y,\mathsf{Q}X) \tag{4.4}$$

is an equivalence. For this we realize that the left hand side is the total complex of the first quadrant double complex given in degree (m,n) by

$$\bigoplus_{P_0,\dots,P_m\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(P_1,P_0)\otimes\dots\otimes\mathcal{C}(P_m,P_{m-1})\otimes\mathcal{C}(P_0,\mathsf{Q}_nX)\otimes\mathcal{C}(Y,P_m)\ . \tag{4.5}$$

Hence, the n-th row is $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}}\mathcal{C}(P,\mathsf{Q}_nX)\otimes\mathcal{C}(Y,P)$; and since Q_nX is projective, this is equivalent to $\mathcal{C}(Y,\mathsf{Q}_nX)$ by the Yoneda Lemma 2.6 applied to $\mathsf{Proj}\,\mathcal{C}$. Now the spectral sequence associated to the filtration of the double complex (4.5) by rows shows that (4.4) is an equivalence. □

Proof of Proposition 4.10. We have established in Theorem 3.9 that the Hochschild complex of $\mathcal{C}$ is canonically equivalent to $\mathcal{C}(I,\mathsf{Q}\mathbb{F})$, where $\mathsf{Q}\mathbb{F}$ is a projective resolution of $\mathbb{F}$. Now the assertion follows from Lemma 4.11. □

The idea for the proof of Theorem 4.7 is to apply Proposition 4.3 to the B_3-action on the complex $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}}\mathcal{C}(I,P)\otimes\mathcal{C}(P,\mathbb{F})$ induced by the B_3-action on $\mathbb{F}$. The following statement asserts that the assumptions of Proposition 4.3 are met:

Proposition 4.12. *Let* $\mathcal{C}$ *be a finite ribbon category and* $X\in\mathcal{C}$*. Consider the action of the ribbon twist* θ_X *of* X *via postcomposition on the complex* $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}}\mathcal{C}(I,P)\otimes\mathcal{C}(P,X)$*. The corresponding chain map* θ_{X*} *is homotopic to the identity via a canonical homotopy* h*. If* $f:X\longrightarrow X$ *is any endomorphism of* X*, then* $f_*h=hf_*$ *for the chain map* f_* *that* f *gives rise to.*

Proof. We treat $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}}\mathcal{C}(I,P)\otimes\mathcal{C}(P,X)$ as a simplicial vector space and construct h as a simplicial homotopy. An n-simplex of $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}}\mathcal{C}(I,P)\otimes\mathcal{C}(P,X)$ is a string

$$\underline{f}=\left(I\xrightarrow{f_n}P_n\xrightarrow{f_{n-1}}P_{n-1}\xrightarrow{f_{n-2}}\dots\xrightarrow{f_0}P_0\xrightarrow{f_{-1}}X\right)$$

of morphisms in $\mathcal{C}$, where the objects P_j for $0\le j\le n$ are projective (it lives in the summand indexed by $P_0,\dots,P_n$). For $0\le j\le n$ we define $h_j\underline{f}$ as the $n+1$-simplex

$$I\xrightarrow{f_n}P_n\xrightarrow{f_{n-1}}P_{n-1}\xrightarrow{f_{n-2}}\dots\xrightarrow{f_j}P_j\xrightarrow{\theta_{P_j}}P_j\xrightarrow{f_{j-1}}\dots\xrightarrow{f_0}P_0\xrightarrow{f_{-1}}X\ ,$$

i.e. h_j inserts the ribbon twist of P_j. A direct computation using the naturality of the twist and $\theta_I=\mathrm{id}_I$ shows that the maps

$$h_j:\left(\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}}\mathcal{C}(I,P)\otimes\mathcal{C}(P,X)\right)_n\longrightarrow\left(\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}}\mathcal{C}(I,P)\otimes\mathcal{C}(P,X)\right)_{n+1}$$

form a simplicial homotopy [Wei94, Definition 8.3.11] between θ_{X*} and the identity of $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}}\mathcal{C}(I,P)\otimes\mathcal{C}(P,X)$. For any endomorphism $f:X\longrightarrow X$ we see $h_jf_*=f_*h_j$. The same is true for the chain homotopy that the h_j give rise to. □

We can now tie all the technical results together:

Proof of Theorem 4.7. Recall that we can describe the ordinary projective action of B_3 on $\mathbb{F}$ as a $\pi^*\xi$-projective action for a cocycle $\xi \in Z^2(\mathrm{SL}(2,\mathbb{Z}); k^\times)$, where $\pi : B_3 \longrightarrow \mathrm{SL}(2,\mathbb{Z})$ is the canonical projection (this was a consequence of Lemma 4.6). By Proposition 4.10 we know that the Hochschild complex $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ of $\mathcal{C}$ is equivalent to the derived coend $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(I,P)\otimes\mathcal{C}(P,\mathbb{F})$. Hence, we may as well exhibit a homotopy coherent projective $\mathrm{SL}(2,\mathbb{Z})$-action on $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(I,P)\otimes\mathcal{C}(P,\mathbb{F})$.

To this end, we note that by postcomposition we obtain a $\pi^*\xi$-projective action of B_3 on $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(I,P)\otimes\mathcal{C}(P,\mathbb{F})$. The strategy is now to apply Proposition 4.3 to this projective B_3-action and the central extension $0 \longrightarrow \mathbb{Z} \longrightarrow B_3 \xrightarrow{\pi} \mathrm{SL}(2,\mathbb{Z}) \longrightarrow 0$ to conclude that the projective B_3-action descends to a homotopy coherent projective $\mathrm{SL}(2,\mathbb{Z})$-action. Then from the proof of Proposition 4.3 we can read off that this will be constructed as an action of the bar resolution the $\pi^*\xi$-twisted group algebra of $\mathrm{SL}(2,\mathbb{Z})$, which justifies the additional statement on the projectivity of the action that we have included in Remark 4.8.

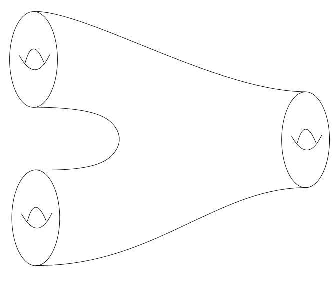
In order to apply Proposition 4.3, it remains to prove the generator r of B_3 acts by a map which is homotopic to the identity by a chain homotopy that commutes with all chain maps that constitute the projective B_3-action on $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(I,P)\otimes\mathcal{C}(P,\mathbb{F})$. But r acts on $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(I,P)\otimes\mathcal{C}(P,\mathbb{F})$ by postcomposition with the inverse ribbon twist of $\mathbb{F}$, as follows from the properties of the Lyubashenko action recalled on page 18. Hence, the desired statement can be deduced from Proposition 4.12. □

Remark 4.13. The fact that in the proof of Theorem 4.7 we use the complex $\int_{\mathbb{L}}^{P\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(I,P)\otimes\mathcal{C}(P,\mathbb{F})$, exhibit a projective B_3-action on it and prove that it descends up to coherent homotopy to $\mathrm{SL}(2,\mathbb{Z})$ is not by accident: In the language of conformal field theory, $\mathcal{C}(P,\mathbb{F})$ is the conformal block for torus with one boundary disk labeled by P. If we think of the Hochschild complex $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ as derived conformal block for the torus, then one can formulate a derived factorization property for the gluing of a disk to the torus with one boundary circle (it is crucial that the gluing is implemented via a derived coend over projective objects). In this process, the B_3-action descends up to coherent homotopy to a $\mathrm{SL}(2,\mathbb{Z})$-action, i.e. from the mapping class group of the torus with one boundary circle to the mapping class group of the closed torus – just as one would expect. We will not make precise the idea of derived conformal blocks here, but still the heuristics just presented makes the strategy in the proof of Theorem 4.7 more transparent. A version of this reasoning without homotopy coherence, i.e. on the level of homology, is used in [LMSS19].

Remark 4.14 (Modular homology). It is natural to ask about the homotopy orbits of the homotopy coherent projective mapping class group action on the Hochschild complex of a modular category that we obtain from Theorem 4.7. This leads to a complex whose homology one might call the *modular homology* of a modular category. Unlike Hochschild homology, this new algebraic invariant should be sensitive to more structure of the modular category than just the underlying linear category, e.g. the braiding and the ribbon twist. For Drinfeld doubles, the modular homology relates to mapping class group orbits of bundles. A detailed investigation is beyond the scope of this paper and will be subject of future work.

5 The derived Verlinde algebra

For a semisimple modular category $\mathcal{C}$, we obtain an algebra structure on the conformal block $\int^{X\in\mathcal{C}} \mathcal{C}(X,X)$ on the torus. This can be seen most conceptually by constructing the (anomalous) 3-2-1-dimensional topological field theory associated to $\mathcal{C}$. Then the multiplication on $\int^{X\in\mathcal{C}} \mathcal{C}(X,X)$ comes from the evaluation of this topological field theory on the bordism $P\times\mathbb{S}^1 : \mathbb{T}^2 \sqcup \mathbb{T}^2 \longrightarrow \mathbb{T}^2$, where $P : \mathbb{S}^1\sqcup\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is the pair of pants. We depict $P\times\mathbb{S}^1$ suggestively as:



This multiplication is easily seen to be commutative thanks to the braiding on $\mathcal{C}$. The resulting algebra is sometimes called the *Verlinde algebra* of $\mathcal{C}$.

When considering the complex $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ in the non-semisimple case, we cannot argue via topological field theory to obtain the multiplication. Still, we will make $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ into a differential graded algebra below and prove that it is a non-unital algebra over the little disk operad E_2, i.e. a homotopy commutative algebra whose commutativity behavior is controlled by the braid group. We will refer to this E_2-algebra as the *derived Verlinde algebra*.

For an introduction to the homotopy theory of operads that we will need in the sequel and in particular to little disk operads we refer to [Fre17].

5.1 The loop operator

The construction of the multiplicative structure on the derived Verlinde algebra will rely on a more general construction that we will explain in this subsection: For a k-linear category $\mathcal{D}$, the derived coend $\int_{\mathbb{L}}^{X\in\mathcal{D}} \mathcal{D}(X,X)$ is the realization of the simplicial vector space $\mathcal{L}\mathcal{D}$ which in degree n is given by

$$\mathcal{L}_n\mathcal{D} = \bigoplus_{X_0,\dots,X_n\in\mathcal{D}} \mathcal{D}(X_1,X_0)\otimes\dots\otimes\mathcal{D}(X_n,X_{n-1})\otimes\mathcal{D}(X_0,X_n) \ , \tag{5.1}$$

i.e. by the space of loops of morphisms in $\mathcal{D}$ through $n+1$ objects (as one finds by specializing Definition 2.1 to the case of a hom functor). In other words, $\int_{\mathbb{L}}^{X\in\mathcal{D}} \mathcal{D}(X,X)$ is given by normalized chains on $\mathcal{L}\mathcal{D}$; in formulae

$$\int_{\mathbb{L}}^{X\in\mathcal{D}} \mathcal{D}(X,X) \cong N_*(\mathcal{L}\mathcal{D}) \ . \tag{5.2}$$

The assignment $\mathcal{D}\longmapsto\mathcal{L}\mathcal{D}$ yields a symmetric monoidal functor $\mathsf{Cat}_k \longrightarrow \mathsf{sVect}_k$ from k-linear categories to simplicial k-vector spaces that we will refer to as the *loop operator*. Before proving the next statement, let us introduce some notation: For a groupoid-valued operad $\mathcal{O}$, we denote by $B\mathcal{O}$ the simplicial operad obtained by taking the nerve of $\mathcal{O}$ and by $k[B\mathcal{O}]$ the corresponding operad in simplicial vector spaces.

Theorem 5.1. *Let $\mathcal{O}$ be an operad in groupoids. Then any $k[\mathcal{O}]$-algebra structure on a k-linear category $\mathcal{D}$ induces an $k[B\mathcal{O}]$-algebra structure on the simplicial vector space $\mathcal{L}\mathcal{D}$.*

Remark 5.2. The above Theorem does *not* just follow from the fact that $\mathcal{L}$ is a symmetric monoidal functor. This would only tell us that for a $k[\mathcal{O}]$-algebra $\mathcal{D}$, the simplicial vector space $\mathcal{L}\mathcal{D}$ is a $\mathcal{L}k[\mathcal{O}]$-algebra, but in general, $\mathcal{L}k[\mathcal{O}]\not\simeq k[B\mathcal{O}]$.

Proof. 1. The structure of a $k[B\mathcal{O}]$-algebra on $\mathcal{L}\mathcal{D}$ consists of maps

$$k[B\mathcal{O}](n)\otimes(\mathcal{L}\mathcal{D})^{\otimes n}\longrightarrow\mathcal{L}\mathcal{D} \tag{5.3}$$

of simplicial vector spaces for $n\geq 0$ which, in turn, consist of linear maps

$$k[B_p\mathcal{O}](n)\otimes(\mathcal{L}_p\mathcal{D})^{\otimes n}\longrightarrow\mathcal{L}_p\mathcal{D} \tag{5.4}$$

in level $p\geq 0$. In the first step, we define these maps. A canonical basis of $k[B_p\mathcal{O}](n)$ is, of course, given by the set $B_p\mathcal{O}(n)$. We denote by

$$g := \left(\varphi_0 \xrightarrow{c_0} \varphi_1 \xrightarrow{c_1} \dots \xrightarrow{c_{p-1}} \varphi_p\right) . \tag{5.5}$$

an element of the set $B_p\mathcal{O}(n)$. Here the φ_j for $0\leq j\leq p$ are objects in the groupoid $\mathcal{O}(n)$. By assumption they act by k-linear functors $\varphi_j:\mathcal{D}^{\otimes n}\longrightarrow\mathcal{D}$ (denoted by the same symbol by a slight abuse of notation). The c_j for $0\leq j\leq p-1$ are morphisms in $\mathcal{O}(n)$, and by assumption they act by natural isomorphisms $\varphi_j \overset{c_j}{\cong} \varphi_{j+1}$ of k-linear functors $\mathcal{D}^{\otimes n}\longrightarrow\mathcal{D}$. We define the additional natural isomorphism $c_p := c_{p-1}\dots c_0$ from φ_0 to φ_p. A pure tensor of $(\mathcal{L}_p\mathcal{D})^{\otimes n}$ is a family of n loops of length $p+1$ in $\mathcal{D}$. We write such a pure tensor by

$$\mathcal{X} := \left\{ \begin{array}{ccccccccc} X_{1,p} & \xrightarrow{f_{1,p}} & X_{1,p-1} & \xrightarrow{f_{1,p-1}} & \dots & \xrightarrow{f_{1,1}} & X_{1,0} & \xrightarrow{f_{1,0}} & X_{1,p} \\ \vdots & & & & \ddots & & & & \vdots \\ X_{n,p} & \xrightarrow{f_{n,p}} & X_{n,p-1} & \xrightarrow{f_{n,p-1}} & \dots & \xrightarrow{f_{n,1}} & X_{n,0} & \xrightarrow{f_{n,0}} & X_{n,p} \end{array} \right\} . \tag{5.6}$$

We use the shorthands $\underline{X}_j := (X_{1,j},\dots,X_{n,j})\in\mathcal{D}^{\otimes n}$ and $\underline{f}_j := f_{1,j}\otimes\dots\otimes f_{n,j}$ for $0\leq j\leq p$.

Using the established notation we obtain the map (5.4) by defining the image $g.\mathcal{X}$ of $g\otimes\mathcal{X}$ under (5.4) (where g is from (5.5) and $\mathcal{X}$ from (5.6)) as the loop

$$g.\mathcal{X} := \left\{ \varphi_p(\underline{X}_p) \xrightarrow{c_{p-1}^{-1}\varphi_p(\underline{f}_p)} \varphi_{p-1}(\underline{X}_{p-1}) \longrightarrow \dots \longrightarrow \varphi_0(\underline{X}_0) \xrightarrow{c_p\varphi_0(\underline{f}_0)} \varphi_p(\underline{X}_p) \right\} .$$

For readability, we suppress the objects that the natural isomorphisms are evaluated on. They are clear from the context.

2. In the next step, we show that for fixed n the maps (5.4) actually define a simplicial map (5.3). First we show the compatibility with the face maps:

   - We observe
   $$\partial_0(g.\mathcal{X}) = \left\{ \ldots \longrightarrow \varphi_1(\underline{X}_1) \xrightarrow{c_p\varphi_0(\underline{f}_0)c_0^{-1}\varphi_1(\underline{f}_1)} \varphi_p(\underline{X}_p) \right\} ,$$
   where we only display those parts of the loop affected by ∂_0. Using functoriality and naturality we find
   $$c_p\varphi_0(\underline{f}_0)c_0^{-1}\varphi_1(\underline{f}_1) = c_{p-1}\ldots c_1\varphi_1(\underline{f}_0\underline{f}_1) ,$$
   which allows us to conclude $\partial_0(g.\mathcal{X}) = (\partial_0 g).(\partial_0\mathcal{X})$.
   - For $0 < j < p$ we obtain
   $$\partial_j(g.\mathcal{X}) = \left\{ \ldots \longrightarrow \varphi_{j+1}(\underline{X}_{j+1}) \xrightarrow{c_{j-1}^{-1}\varphi_j(\underline{f}_j)c_j^{-1}\varphi_{j+1}(\underline{f}_{j+1})} \varphi_{j-1}(\underline{X}_{j-1}) \longrightarrow \ldots \right\} .$$
   Again, by functoriality and naturality we have
   $$c_{j-1}^{-1}\varphi_j(\underline{f}_j)c_j^{-1}\varphi_{j+1}(\underline{f}_{j+1}) = (c_jc_{j-1})^{-1}\varphi_{j+1}(\underline{f}_j\underline{f}_{j+1})$$
   and hence $\partial_j(g.\mathcal{X}) = (\partial_j g).(\partial_j\mathcal{X})$.
   - Similarly,
   $$\partial_p(g.\mathcal{X}) = \left\{ \varphi_{p-1}(\underline{X}_{p-1}) \longrightarrow \ldots \longrightarrow \varphi_0(\underline{X}_0) \xrightarrow{c_{p-1}^{-1}\varphi_p(\underline{f}_p)c_p\varphi_0(\underline{f}_0)} \varphi_{p-1}(\underline{X}_{p-1}) \right\} .$$
   From
   $$c_{p-1}^{-1}\varphi_p(\underline{f}_p)c_p\varphi_0(\underline{f}_0) = c_{p-2}\ldots c_0\varphi_0(\underline{f}_p\underline{f}_0)$$
   we deduce $\partial_p(g.\mathcal{X}) = (\partial_p g).\partial_p\mathcal{X}$.

   The compatibility with the degeneracy maps also holds because $s_j(g.\mathcal{X})$ for $1 \leq j \leq n$ arises from $g.\mathcal{X}$ by adding an identity at $\varphi_j(\underline{X}_j)$, which we can write as the image of the identity of the object $\underline{X}_j$ under φ_j combined with the identity transformation. This shows $(s_jg).(s_j\mathcal{X}) = s_j(g.\mathcal{X})$ and concludes the proof that the maps (5.3) are simplicial.

3. Finally, we have to prove that the simplicial maps (5.3) form an operad action. While unitality and equivariance follow directly from the construction, the preservation of composition requires a small computation: For
   $$g_\ell := \left(\varphi_0^\ell \xrightarrow{c_0^\ell} \varphi_1^\ell \xrightarrow{c_1^\ell} \ldots \xrightarrow{c_{p-1}^\ell} \varphi_p^\ell \right) \in B_p\mathcal{O}(n_\ell) , \quad 1 \leq \ell \leq m ,$$
   $$g := \left(\varphi_0 \xrightarrow{c_0} \varphi_1 \xrightarrow{c_1} \ldots \xrightarrow{c_{p-1}} \varphi_p \right) \in B_p\mathcal{O}(m)$$
   and pure tensors $\mathcal{X}_\ell = (\underline{f}_0^\ell, \ldots, \underline{f}_p^\ell)$ in $(\mathcal{L}_p\mathcal{D})^{\otimes n_\ell}$ for $1 \leq \ell \leq m$ (notation as in (5.6)) we need to prove
   $$g.(g_1.\mathcal{X}_1 \otimes \cdots \otimes g_m.\mathcal{X}_m) = (g(g_1 \otimes \cdots \otimes g_m)).(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_m) , \tag{5.7}$$
   where $(g(g_1 \otimes \cdots \otimes g_m))$ is the composition in $B\mathcal{O}$, which is explicitly given by
   $$\begin{aligned} &(g(g_1 \otimes \cdots \otimes g_m)) \\ &= \left\{ \varphi_0(\varphi_0^1 \otimes \cdots \otimes \varphi_0^m) \xrightarrow{c_0\varphi_0(c_0^1 \otimes \cdots \otimes c_0^m)} \varphi_1(\varphi_1^1 \otimes \cdots \otimes \varphi_1^m) \longrightarrow \ldots \longrightarrow \varphi_p(\varphi_p^1 \otimes \cdots \otimes \varphi_p^m) \right\} \end{aligned}$$

Indeed, using that $\mathcal{D}$ is an $\mathcal{O}$-algebra we arrive at

$$
\begin{aligned}
& g.(g_1.\mathcal{X}_1 \otimes \cdots \otimes g_m.\mathcal{X}_m) \\
= & \Big\{ \varphi_p(\varphi_p^1(\underline{X}_p^1), \ldots, \varphi_p^m(\underline{X}_p^m)) \xrightarrow{c_{p-1}^{-1}\varphi_p((c_{p-1}^1)^{-1}\varphi_p^1(\underline{f}_p^1)\otimes\cdots\otimes(c_{p-1}^m)^{-1}\varphi_p^m(\underline{f}_p^m))} \\
& \varphi_{p-1}(\varphi_{p-1}^1(\underline{X}_{p-1}^1), \ldots, \varphi_{p-1}^m(\underline{X}_{p-1}^m)) \longrightarrow \ldots \longrightarrow \varphi_0(\varphi_0^1(\underline{X}_0^1), \ldots, \varphi_0^m(\underline{X}_0^m)) \\
& \xrightarrow{c_p\varphi_0(c_p^1\varphi_0^1(\underline{f}_0^1)\otimes\cdots\otimes c_p^m\varphi_0^m(\underline{f}_0^m))} \varphi_p(\varphi_p^1(\underline{X}_p^1), \ldots, \varphi_p^m(\underline{X}_p^m)) \Big\} \\
= & \Big\{ \varphi_p(\varphi_p^1 \otimes \cdots \otimes \varphi_p^m)(\underline{X}_p^1, \ldots, \underline{X}_p^m) \xrightarrow{(c_{p-1}\varphi_{p-1}(c_{p-1}^1\otimes\cdots\otimes c_{p-1}^m))^{-1}\varphi_p(\varphi_p^1\otimes\cdots\otimes\varphi_p^m)(\underline{f}_p^1\otimes\cdots\otimes\underline{f}_p^m)} \\
& \varphi_{p-1}(\varphi_{p-1}^1 \otimes \cdots \otimes \varphi_{p-1}^m)(\underline{X}_{p-1}^1, \ldots, \underline{X}_{p-1}^m) \longrightarrow \ldots \longrightarrow \varphi_0(\varphi_0^1 \otimes \cdots \otimes \varphi_0^m)(\underline{X}_0^1, \ldots, \underline{X}_0^m) \\
& \xrightarrow{c_{p-1}\varphi_{p-1}(c_{p-1}^1\otimes\cdots\otimes c_{p-1}^m)\ldots c_0\varphi_0(c_0^1\otimes\cdots\otimes c_0^m)\varphi_0(\varphi_0^1\otimes\cdots\otimes\varphi_0^m)(\underline{f}_0^1\otimes\cdots\otimes\underline{f}_0^m)} \varphi_p(\varphi_p^1 \otimes \cdots \otimes \varphi_p^m)(\underline{X}_p^1, \ldots, \underline{X}_p^m) \Big\} \\
= & (g(g_1 \otimes \cdots \otimes g_m)).(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_m) \ .
\end{aligned}
$$

This yields the proof of (5.7) and hence the proof of the Theorem.

□

5.2 The multiplicative structure on the derived Verlinde algebra

We will obtain the multiplicative structure on the derived Verlinde algebra by applying Theorem 5.1 to the *little disk operad* E_2 whose space $E_2(n)$ of arity n operations is given by the space of affine embeddings from n disks into another disk, see [Fre17, Chapter 4] for details. This operad describes algebraic structures with a homotopy associative multiplication whose commutativity behavior is controlled by the braid group.

Definition 5.3. Denote by E_2 the *little disk operad* in the category of simplicial sets (or spaces). By $\mathit{\Pi} E_2$ we denote the operad in groupoids resulting from application of the fundamental groupoid functor $\mathit{\Pi}$ to E_2.

The space $E_2(n)$ is an Eilenberg-MacLane space $K(P_n, 1)$, where P_n is the pure braid group on n strands [Fre17, Chapter 5]. In particular, the unit

$$E_2 \xrightarrow{\simeq} B\mathit{\Pi} E_2 \tag{5.8}$$

of the adjunction $\mathit{\Pi} \dashv B$ between fundamental groupoid $\mathit{\Pi}$ and the nerve B is a weak equivalence when evaluated arity-wise on E_2. We can also use an alternative description of $E_2(n)$ based on the short exact sequence

$$0 \longrightarrow P_n \longrightarrow B_n \longrightarrow \mathit{\Sigma}_n \longrightarrow 0$$

featuring besides the pure braid group P_n also the braid group B_n on n strands and the permutation group $\mathit{\Sigma}_n$ on n letters. The projection $B_n \longrightarrow \mathit{\Sigma}_n$ defines a transitive action of B_n on $\mathit{\Sigma}_n$, and $\mathit{\Pi} E_2(n)$ is equivalent to the corresponding action groupoid,

$$\mathit{\Pi} E_2(n) \simeq \mathit{\Sigma}_n // B_n \ . \tag{5.9}$$

A permutation $\sigma \in \mathit{\Sigma}_n$ describes the affine embedding which aligns n disks next to each other on the equator of a bigger disk with the order prescribed by σ.

As explained in [Fre17, Chapter 5 and 6], algebras over $\mathit{\Pi} E_2$ are equivalent to braided monoidal categories; and in the description (5.9) of $\mathit{\Pi} E_2(2)$ we have the correspondences

$$\left\{ \begin{array}{rcl} \text{identity permutation} & \longleftrightarrow & \text{tensor product,} \\ \text{transposition of two letters} & \longleftrightarrow & \text{opposite tensor product,} \\ \text{generator of } B_2 \text{ braiding the two strands} & \longleftrightarrow & \text{braiding.} \end{array} \right\} \tag{5.10}$$

Using the free functor $k[-]$ from sets to k-vector spaces we obtain from $\mathit{\Pi} E_2$ an operad $k[\mathit{\Pi} E_2]$ in k-linear categories. Algebras over this operad are equivalent to k-linear braided monoidal categories.

There is a non-unital version of the E_2-operad that we call $\bar{E}_2$. It has the same arity n operations as E_2 for $n \geq 1$, but in arity zero, $\bar{E}_2$ is empty unlike E_2 which is given by a point in arity zero. Categorical $\bar{E}_2$-algebras are *non-unital* braided monoidal categories, i.e. they do not necessarily have a monoidal unit. This leads to the following elementary observation:

Lemma 5.4. *For a finite braided tensor category $\mathcal{C}$ over k, the subcategory $\mathsf{Proj}\,\mathcal{C} \subset \mathcal{C}$ of projective objects is a $k[\mathit{\Pi}\bar{E}_2]$-algebra in k-linear categories.*

Proof. Duality and exactness of the monoidal product ensure that the tensor product of two projective objects is again projective making $\mathsf{Proj}\,\mathcal{C}$ a k-linear non-unital monoidal category. If $\mathcal{C}$ is additionally braided, $\mathsf{Proj}\,\mathcal{C}$ is a k-linear non-unital braided monoidal category, which proves the assertion. $\square$

With these preparations, we may now state the main result on the multiplicative structure of the derived Verlinde algebra:

Theorem 5.5. *For every finite braided category $\mathcal{C}$, the derived coend $\mathcal{V}^{\mathcal{C}} = \oint_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ is naturally a differential graded non-unital E_2-algebra that we refer to as derived Verlinde algebra.*

Proof. Since $\mathcal{C}$ is a finite braided category over k, $\mathsf{Proj}\,\mathcal{C}$ is a $k[\Pi\bar{E}_2]$-algebra in k-linear categories by Lemma 5.4. By Theorem 5.1 applied to the groupoid-valued operad $\Pi\bar{E}_2$ we obtain a $k[B\Pi\bar{E}_2]$-algebra structure on $\mathcal{L}\,\mathsf{Proj}\,\mathcal{C}$. By taking the operadic pullback along the equivalence $k[\bar{E}_2] \xrightarrow{\simeq} k[B\Pi\bar{E}_2]$ induced by (5.8), we obtain a $k[\bar{E}_2]$-algebra structure on $\mathcal{L}\,\mathsf{Proj}\,\mathcal{C}$. Finally, we apply the symmetric lax monoidal functor N_* and use (5.2) to see that $\oint_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ is an algebra over the chains on $\bar{E}_2$. This proves the claim. $\square$

In order to write down the product of the homotopy associative differential graded algebra underlying the derived Verlinde algebra, we need to establish some notation: Recall from (5.1) that elements $\underline{f}, \underline{g} \in \mathcal{L}_n\,\mathsf{Proj}\,\mathcal{C}$ are loops of $n+1$ morphisms between projective objects in $\mathcal{C}$. Using the monoidal product $\bullet$ of $\mathcal{C}$ we can tensor the morphisms of $\underline{f}$ and $\underline{g}$ together to obtain an element in $\mathcal{L}_n\,\mathsf{Proj}\,\mathcal{C}$ that we denote by $\underline{f}\bullet\underline{g}$. Next recall that for $0 \le j \le n$ the degeneracy map $s_j : \mathcal{L}_n\,\mathsf{Proj}\,\mathcal{C} \longrightarrow \mathcal{L}_{n+1}\,\mathsf{Proj}\,\mathcal{C}$ inserts the identity of the j-th object. For a (p,q)-shuffle $(\mu,\nu) = (\mu_1,\dots,\mu_p,\nu_1,\dots,\nu_q)$, i.e. a permutation of $\{1,\dots,p+q\}$ such that $\mu_1 < \mu_2 < \dots < \mu_p$ and $\nu_1 < \nu_2 < \dots < \nu_q$, we define the compositions

$$s_\mu := s_{\mu_p - 1} \circ \dots \circ s_{\mu_1 - 1}\ , \quad s_\nu := s_{\nu_q - 1} \circ \dots \circ s_{\nu_1 - 1}$$

of degeneracy maps.

Corollary 5.6. *The multiplication $*$ of the derived Verlinde algebra $\mathcal{V}^{\mathcal{C}}$ from Theorem 5.5 is explicitly given by the formula*

$$\underline{f} * \underline{g} = \sum_{\substack{(p,q)\text{-shuffles } (\mu,\nu) \\ \text{of } p+q}} \operatorname{sign}(\mu,\nu)\ \ s_\nu(\underline{f}) \bullet s_\mu(\underline{g}) \quad \textit{for} \quad \underline{f} \in \mathcal{V}^{\mathcal{C}}_p, \quad \underline{g} \in \mathcal{V}^{\mathcal{C}}_q\ . \tag{5.11}$$

Proof. From (5.10) and the construction of the operadic action in the proof of Theorem 5.1 it follows that the E_2-multiplication on $\mathcal{L}\,\mathsf{Proj}\,\mathcal{C}$ comes from tensoring loops of morphisms together using the monoidal product of $\mathcal{C}$. In order to obtain a formula for this multiplication on $\oint_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$, we use the structure maps of the symmetric lax monoidal functor N_*, namely the Eilenberg-Zilber maps [Wei94, 8.5.4], and arrive at (5.11). $\square$

Remark 5.7 (Gerstenhaber bracket). By [Co76] the homology of an E_2-algebra is a Gerstenhaber algebra. Invoking the non-unital version of this result, we find on the homology of the derived Verlinde algebra the structure of a non-unital Gerstenhaber algebra, particular a Gerstenhaber bracket. A further investigation of this bracket is beyond the scope of this article.

Example 5.8 (Boundary conditions and the Swiss-Cheese operad). Theorem 5.1 has also applications to topological field theories with boundary conditions: Consider a braided finite tensor category $\mathcal{C}$ and a finite tensor category $\mathcal{W}$ together with a braided monoidal functor $F : \mathcal{C} \longrightarrow Z(\mathcal{W})$. Such a structure appears in the description of boundary condition in three-dimensional topological field theory [FSV12]. By one of the main results of [Idr17], this structure precisely amounts to $(\mathcal{C},\mathcal{W},F)$ being a categorical algebra over the Swiss-Cheese operad introduced by Voronov [Vor99]. Now we can conclude from Theorem 5.1 (strictly speaking, from a colored version thereof) that the Hochschild chains $\oint_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ and $\oint_{\mathbb{L}}^{Y\in\mathsf{Proj}\,\mathcal{W}} \mathcal{W}(Y,Y)$ of $\mathcal{C}$ and $\mathcal{W}$ and the map between those induced by F form a differential graded Swiss-Cheese algebra. By [Hoe09] the corresponding homology yields an algebraic structure closely related to the homotopy algebras used by Kajiura and Stasheff [KS06] for the description of open-closed string field theories.

Remark 5.9 (Application to finite ribbon categories). If we are given a finite ribbon category $\mathcal{C}$, then its projective objects form a non-unital *framed* E_2-algebra in k-linear categories. Using Theorem 5.1 and arguments similar to those in the proof of Theorem 5.5 we can conclude that $\oint_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ is a non-unital framed E_2-algebra making its homology a non-unital Batalin-Vilkovisky algebra. For the framed little disk operad, its categorical algebras and its homology we refer to [SW03].

5.3 The equivariant case: Differential graded little bundles algebras

Based on the discussion of the derived Verlinde algebra and its motivation by topological field theory, we can suggest, for a given finite group G, a reasonable candidate for a Hochschild complex of a braided G-crossed monoidal category in the sense of Turaev and exhibit an interesting multiplicative structure on it. We will first write down the candidate for the Hochschild complex and guess a multiplicative structure based on the ties of braided crossed monoidal categories to equivariant field theories. Then, we will turn this intuition into a precise statement using the little bundles operad defined in [MW19] motivated by its relation to $(\infty,1)$-G-equivariant topological field theories [MW20].

The notion of a braided G-crossed monoidal category is based on [Tur00, Tur10-II]. In the semisimple case, these categories are well-studied objects in equivariant representation theory [Müg04, Kir04, GNN09]. We follow the definition of [Gal17], where in comparison to [Tur10-II] more general coherence conditions are considered: For a finite group G, a *braided G-crossed category* is a k-linear category $\mathcal{C}$ that comes with a decomposition $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ and is equipped with the following data:

- A homotopy coherent action of G on $\mathcal{C}$ making $h \in G$ act as an equivalence $\mathcal{C}_g \longrightarrow \mathcal{C}_{hgh^{-1}}, X \longmapsto h.X$.
- A k-linear monoidal product sending $\mathcal{C}_g \otimes \mathcal{C}_h$ to $\mathcal{C}_{gh}$.
- A G-braiding consisting of natural isomorphisms

$$X \otimes Y \cong g.Y \otimes X$$

  for $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ (this does *not* yield a braiding on $\mathcal{C}$).

For the details on the compatibilities and coherence requirements we refer to [Gal17], see also [MNS12] and, additionally, [MW19] for a description of braided G-crossed categories as algebras over the G-colored operad of parenthesized G-braids. We define a *finite braided G-crossed tensor category* as a k-linear braided G-crossed monoidal category whose underlying k-linear monoidal category is a finite tensor category.

Categories of this type are intimately related to three-dimensional G-equivariant topological field theory [Tur10-II], a flavor of topological field theory in which all manifolds are equipped with principal G-bundles. The decoration with G-bundles leads to interesting phenomena which are not present in the non-equivariant case.

Semisimple G-modular categories (a special type of finite braided G-crossed tensor categories) are used in [TV14] to construct a three-dimensional topological field theory. Conversely, given an extended three-dimensional G-equivariant topological field theory, its evaluation on the circle is a semisimple G-(multi)modular category [SW19]. In the non-semisimple case, this interpretation of braided G-crossed tensor categories in terms of topological field theory breaks down as in the non-equivariant case.

Still, the perspective of topological field theory yields some tools for the study of non-semisimple finite braided G-crossed tensor categories: If $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ is a semisimple G-modular category, then the evaluation of the three-dimensional topological field theory built from $\mathcal{C}$ on the torus decorated with the bundle specified by the two commuting holonomies $g, z \in G$ is given by the coend $\int^{X\in\mathcal{C}_g} \mathcal{C}_g(z.X, X)$ as explained in a different language in [Tur10-II, Section VII.3] and worked out in terms of coends in [SW19, Section 4.6]. This suggests that in the non-semisimple case the collection of derived coends

$$\int_{\mathbb{L}}^{X\in \mathsf{Proj}\, \mathcal{C}_g} \mathcal{C}_g(z.X, X) \tag{5.12}$$

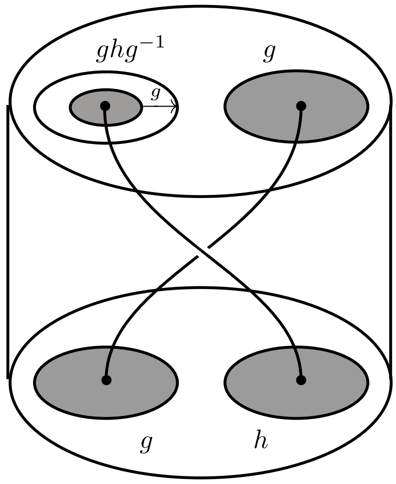
provide a reasonable generalization of Hochschild chains to the equivariant case. More importantly, the topological intuition gives us an idea of the multiplicative structure that we should discover: Following the ideas laid out at the beginning of Section 5, crossing the pair of pants with a circle yields a bordism $\mathbb{T}^2 \sqcup \mathbb{T}^2 \longrightarrow \mathbb{T}^2$. In the equivariant case, this bordism has to be decorated with G-bundles. Upon fixing a central element $z \in Z(H)$, each pair of group elements $g_1, g_2 \in G$ will provide the holonomies for a G-bundle on the bordism $\mathbb{T}^2 \sqcup \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ (for this we need z to commute with g_1 and g_2); i.e. when denoting the bundle specified by the holonomies z and some g by (z, g), we obtain a decorated bordism

$$\left(\mathbb{T}^2, (z, g_1)\right) \sqcup \left(\mathbb{T}^2, (z, g_2)\right) \longrightarrow \left(\mathbb{T}^2, (z, g_1 g_2)\right) . \tag{5.13}$$

Note that we treat here the two $\mathbb{S}^1$-factors of the torus differently. The fact that the holonomies g_1 and g_2 multiply is a consequence of the fundamental group of the pair of pants. The decorated bordism (5.13) should give us a multiplication

$$\int_{\mathbb{L}}^{X\in \mathsf{Proj}\, \mathcal{C}_{g_1}} \mathcal{C}_{g_1}(z.X, X) \otimes \int_{\mathbb{L}}^{X\in \mathsf{Proj}\, \mathcal{C}_{g_1}} \mathcal{C}_{g_2}(z.X, X) \longrightarrow \int_{\mathbb{L}}^{X\in \mathsf{Proj}\, \mathcal{C}_{g_1 g_2}} \mathcal{C}_{g_1 g_2}(z.X, X) \tag{5.14}$$

compatible with the group multiplication. The commutativity behavior should be determined by the braid group action on the groupoid of G-bundles over the complement of little disk embeddings. For instance, consider an embedding of two disks into a bigger disk and a bundle on the complement of this embedding. Such a bundle is determined by the holonomies g and h around the boundaries of the two embedded disks. When moving the disks past each other, these holonomies transform to ghg^{-1} and h:



On the left disk we have indicated the homotopy of classifying maps that acts as a gauge transformation $h \xrightarrow{g} ghg^{-1}$. The multiplication (5.14) should have a symmetry behavior reflecting the topological situation.

As the main result of this section, we prove that indeed this topological intuition describes the multiplicative structure (5.14) accurately. We do this by showing that for a *fixed* central element $z \in Z(G)$ the assignment $g \longmapsto \int_{\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}_g} \mathcal{C}_g(z.X, X)$ provides a (non-unital) algebra over the differential graded little bundles operad E_2^G introduced in [MW19].

The little bundles operad is an aspherical topological operad whose colors are the bundles over the circle (modeled as loops in BG) and whose operations $E_2^G\binom{\psi}{\underline{\varphi}}$ from a family $\underline{\varphi} = (\varphi_1, \ldots, \varphi_n)$ to ψ are given by affine embeddings $f \in E_2(n)$ equipped with a bundle on the complement of the image of f restricting to the bundle $(\varphi_1, \ldots, \varphi_n)$ on the n inner boundary circles and to ψ on the outer boundary circle. The little bundles operad can be seen as an operad built from Hurwitz spaces, i.e. from the homotopy quotient of the braid group action on the moduli space of G-bundles over a punctured plane.

In [MW19, Theorem 4.11 and 4.13] ΠE_2^G is shown to be equivalent to the G-colored operad PBr^G of parenthesized G-braids whose categorical algebras are precisely braided G-crossed monoidal categories. This turns the latter into a 'topological object'. In the following we will suppress the difference between ΠE_2^G and PBr^G in the notation.

Using the little bundles operad we can now make a precise statement about the family of complexes (5.12):

Theorem 5.10. *Let G be a finite group and $z \in Z(G)$ a fixed element in its center. Then for any finite braided G-crossed tensor category $\mathcal{C}$, the assignment*

$$g \longmapsto \mathcal{V}_g^{\mathcal{C},z} := \int_{\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}_g} \mathcal{C}_g(z.X, X)$$

defines a non-unital E_2^G-algebra in differential graded vector spaces, i.e. a non-unital differential graded little bundles algebra. We refer to $(\mathcal{V}_g^{\mathcal{C},z})_{g \in G}$ as the G-equivariant derived Verlinde algebra of $\mathcal{C}$ and the central element $z \in Z(G)$.

Remark 5.11. The homology $H_*(\mathcal{V}^{\mathcal{C},z}) = \bigoplus_{g \in G} H_*(\mathcal{V}_g^{\mathcal{C},z})$ of the G-equivariant Verlinde algebra is a differential graded algebra, but in contrast to the non-equivariant case it is not graded commutative. Instead, for $x \in H_p(\mathcal{V}_g^{\mathcal{C},z})$ and $y \in H_q(\mathcal{V}_h^{\mathcal{C},z})$

$$xy = (-1)^{pq}(g.y)x \ .$$

Proof of Theorem 5.10. In analogy to (5.1), we define $\mathcal{L}^z \,\mathsf{Proj}\, \mathcal{C}_g$ as the simplicial vector space given by

$$\mathcal{L}_n^z \,\mathsf{Proj}\, \mathcal{C}_g = \bigoplus_{X_0, \ldots, X_n \in \mathsf{Proj}\, \mathcal{C}_g} \mathcal{C}_g(X_1, X_0) \otimes \cdots \otimes \mathcal{C}_g(X_n, X_{n-1}) \otimes \mathcal{C}_g(z.X_0, X_n) \ .$$

in degree n.

As in the proof of Theorem 5.5, it is now sufficient to prove that $G \ni g \longmapsto \mathcal{L}_n^z \operatorname{\mathsf{Proj}} \mathcal{C}_g$ is a $k[B\Pi\bar{E}_2^G]$-algebra in simplicial vector spaces, where $\bar{E}_2^G$ is the non-unital little bundles operad.

The action needed to establish this consists of maps

$$k\left[B\Pi\bar{E}_2^G\right]\binom{h}{\underline{g}} \otimes \mathcal{L}^z \operatorname{\mathsf{Proj}} \mathcal{C}_{\underline{g}} \longrightarrow \mathcal{L}^z \operatorname{\mathsf{Proj}} \mathcal{C}_h \ ,$$
$$\text{where} \quad \mathcal{L}^z \operatorname{\mathsf{Proj}} \mathcal{C}_{\underline{g}} := \mathcal{L}^z \operatorname{\mathsf{Proj}} \mathcal{C}_{g_1} \otimes \cdots \otimes \mathcal{L}^z \operatorname{\mathsf{Proj}} \mathcal{C}_{g_n} \ , \quad \underline{g} = (g_1, \dots, g_n)$$

of simplicial vector spaces for $n \geq 0$ which are formed by linear maps

$$k\left[B_p\Pi\bar{E}_2^G\right]\binom{h}{\underline{g}} \otimes \mathcal{L}_p^z \operatorname{\mathsf{Proj}} \mathcal{C}_{\underline{g}} \longrightarrow \mathcal{L}_p^z \operatorname{\mathsf{Proj}} \mathcal{C}_h \tag{5.15}$$

for any simplicial degree $p \geq 0$.

When combining [MW19, Theorem 4.13] with duality and the exactness of the tensor product, we see that $\operatorname{\mathsf{Proj}} \mathcal{C}$ is a $k[\Pi\bar{E}_2^G]$-algebra (compare to Lemma 5.4). This fact will be used to construct the maps (5.15): For an element

$$g := \left(\varphi_0 \xrightarrow{c_0} \varphi_1 \xrightarrow{c_1} \dots \xrightarrow{c_{p-1}} \varphi_p\right) \in B_p\Pi\bar{E}_2^G\binom{h}{\underline{g}}$$

the object φ_j will act as a linear functor $\varphi_j : \mathcal{C}_{\underline{g}} \longrightarrow \mathcal{C}_h$ (denoted by the same symbol), where $\mathcal{C}_{\underline{g}} := \mathcal{C}_{g_1} \otimes \cdots \otimes \mathcal{C}_{g_n}$ and $0 \leq j \leq p$, and the c_j will yield natural isomorphisms $\varphi_j \overset{c_j}{\cong} \varphi_{j+1}$ for $0 \leq j \leq p-1$. We write a pure tensor $\mathcal{X}$ of $\mathcal{L}_p^z \operatorname{\mathsf{Proj}} \mathcal{C}_{\underline{g}}$ as a family

$$\mathcal{X} := \left\{ \begin{array}{ccccccc|ccc} X_{1,p} & \xrightarrow{f_{1,p}} & X_{1,p-1} & \xrightarrow{f_{1,p-1}} & \dots & \xrightarrow{f_{1,1}} & X_{0,0} & z.X_{1,0} & \xrightarrow{f_{1,0}} & X_{1,p} \\ \vdots & & & & \ddots & & \vdots & \vdots & & \vdots \\ X_{n,p} & \xrightarrow{f_{n,p}} & X_{n,p-1} & \xrightarrow{f_{n,p-1}} & \dots & \xrightarrow{f_{n,1}} & X_{n,0} & z.X_{n,0} & \xrightarrow{f_{n,0}} & X_{n,p} \end{array} \right\}$$

where the morphisms in the ℓ-th row live in $\mathcal{C}_{g_\ell}$, $1 \leq \ell \leq n$. Note that unlike in the non-equivariant case, these are not loops (not even strings), and we have indicated the 'interruption' by a vertical line. It is now a crucial observation that z commutes with every little bundles operation $o : \mathcal{C}_{\underline{g}} \longrightarrow \mathcal{C}_h$ in the sense that there is a canonical natural isomorphism $z.o(-) \cong o.(z.-)$. This follows from the fact that G acts by monoidal functors on $\mathcal{C}$ and that z lies in the center of G. Using again the shorthands $\underline{X}_j := (X_{1,j}, \dots, X_{n,j}) \in \mathcal{D}^{\otimes n}$, $\underline{f}_j := f_{1,j} \otimes \cdots \otimes f_{n,j}$ for $0 \leq j \leq p$ and $c_p := c_{p-1} \dots c_0$, we define the image $g.\mathcal{X}$ of $g \otimes \mathcal{X}$ under (5.15) by

$$g.\mathcal{X} := \left\{ \varphi_p(\underline{X}_p) \xrightarrow{c_{p-1}^{-1}\varphi_p(\underline{f}_p)} \varphi_{p-1}(\underline{X}_{p-1}) \longrightarrow \dots \longrightarrow \varphi_0(\underline{X}_0) \,\middle|\, z.\varphi_0(\underline{X}_0) \cong \varphi_0(z.\underline{X}_0) \xrightarrow{c_p\varphi_0(\underline{f}_0)} \varphi_p(\underline{X}_p) \right\} .$$

Direct, but tedious computations similar to those in the proof of Theorem 5.1 show that these maps are simplicial and form an operad action. □

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