On the growth rate of dichromatic numbers of finite subdigraphs

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Abstract

Chris Lambie-Hanson proved recently that for every function $f : \mathbb{N} \to \mathbb{N}$ there is an $\aleph_1$-chromatic graph G of size $2^{\aleph_1}$ such that every $(n+3)$-chromatic subgraph of G has at least $f(n)$ vertices. Previously, this fact was just known to be consistently true due to P. Komjáth and S. Shelah. We investigate the analogue of this question for directed graphs. In the first part of the paper we give a simple method to construct for an arbitrary $f : \mathbb{N} \to \mathbb{N}$ an uncountably dichromatic digraph D of size $2^{\aleph_0}$ such that every $(n+2)$-dichromatic subgraph of D has at least $f(n)$ vertices. In the second part we show that it is consistent with arbitrary large continuum that in the previous theorem "uncountably dichromatic" and "of size $2^{\aleph_0}$" can be replaced by "κ-dichromatic" and "of size κ" respectively where κ is universally quantified with bounds $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$.

1 Introduction

The investigation of the finite subgraphs of uncountably chromatic graphs was initiated by Erdős and Hajnal in the 1970s. First they were trying to construct uncountably chromatic graphs avoiding short cycles. After finding out that it is impossible they showed in [1] that every uncountably chromatic graph must contain every finite bipartite graph as a subgraph and exactly those are the "obligatory" finite subgraphs. An old conjecture of Erdős and Hajnal has been recently justified by Chris Lambie-Hanson:

Theorem 1.1 (C. Lambie-Hanson [2])**.** *For every function $f : \mathbb{N} \to \mathbb{N}$ there is an $\aleph_1$-chromatic graph G of size $2^{\aleph_1}$ such that every $(n+3)$-chromatic subgraph of G has at least $f(n)$ vertices.*

P. Komjáth and S. Shelah showed earlier the consistency of the statement (guaranteeing an $\aleph_1$-sized G) in [3]. They also proved that consistently, for every graph G of chromatic number at least $\aleph_2$ one can find graphs of arbitrary large chromatic number whose finite subgraphs are already induced subgraphs of G. If we replace $\aleph_2$ by $\aleph_1$ the resulting statement (known as Taylor conjecture) becomes consistently false.

As a directed analogue of the chromatic number V. Neumann-Lara defined the dichromatic number of a digraph D in [7] as the smallest cardinal κ such that $V(D)$ can be partitioned into κ many sets each of spanning an acyclic subdigraph of D. He and Erdős conjectured that having chromatic number greater than $f(k)$ implies to have orientation with dichromatic number greater than k for a suitable $f : \mathbb{N} \to \mathbb{N}$. This old conjecture is still wide open, even the existence of

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$f(3)$ is unknown. Several results about dichromatic number are similar with the corresponding theorems about chromatic numbers. For example it was shown first by Bokal et al. in [9] using probabilistic methods and later by Severino in [8] in a constructive way that there are digraphs with arbitrary large finite dichromatic number avoiding directed cycles up to a given length. Considering the infinite analogue of the question, D. T. Soukup showed that in contrast to the the behaviour of uncountably chromatic graphs, it is consistent that a digraph is uncountably dichromatic but avoids directed cycles up to a prescribed length (see Theorem 3.5 of [4]). Later it was shown that this is already true in ZFC (see [5]), more precisely, for every $n \in \mathbb{N}$ and infinite cardinal κ there is a κ-dichromatic digraph avoiding directed cycles of length up to n. Even so, Soukup's forcing construction has additional strong properties and it has the flexibility to handle stronger statements. We will develop it further to prove Theorem 1.3. Another result in [4] says that the statement "every graph of size and chromatic number $\aleph_1$ has an $\aleph_1$-dichromatic orientation" is independent of ZFC. It suggests that maybe the infinite version of the Erdős-Neumann Lara conjecture, where we consider arbitrary cardinals instead of natural numbers, is more approachable than the original.

Observe that avoiding short directed cycles can be formulated as a lower bound on the size of 2-dichromatic subgraphs. It seems natural to have such a bound for the n-chromatic subgraphs for each $n \in \mathbb{N}$ simultaneously. The first result of the paper is the following directed analogue of Theorem 1.1:

Theorem 1.2. *For every $f : \mathbb{N} \to \mathbb{N}$, there is an uncountably dichromatic digraph D of size $2^{\aleph_0}$ such that every $(n+2)$-dichromatic subdigraph of D has at least $f(n)$ vertices.*

Under the continuum hypothesis the digraphs in Theorem 1.2 have "optimal" size (i.e. equal to their dichromatic number) and settle the problem for $\mathfrak{c} := 2^{\aleph_0}$. Our second result tells that it is consistent with arbitrarily large continuum that the analogue statement holds for every infinite $\kappa \leq \mathfrak{c}$ with optimal sized digraphs. More precisely:

Theorem 1.3. *There is a ccc forcing $\mathbb{P}$ of size $\mathfrak{c}$ such that $V^{\mathbb{P}} \models$ for every infinite cardinal $\kappa \leq \mathfrak{c}$ and for every $f : \mathbb{N} \to \mathbb{N}$ there is a κ-dichromatic digraph D on κ such that every $(n+2)$-dichromatic subdigraph of D has at least $f(n)$ vertices.*

Note that being ccc and having size $\mathfrak{c}$ ensures the preservation of all cardinals and it keeps the continuum the same thus the forcing really accomplishes what we promised.

The paper is organized as follows. In the next section we introduce the necessary notation. The third section is subdivided into two parts in which we prove Theorems 1.2 and 1.3 respectively.

2 Notation

For an ordered pair $\langle u, v \rangle$ we write simply $\boldsymbol{uv}$. The range of a function f is denoted by $\textsf{ran}(\boldsymbol{f})$. The concatenation of sequences s and z is $\boldsymbol{s}^\frown\boldsymbol{z}$ where sequences of length one are not distinguished in notation from their only elements. The Cartesian product of the sets X_i is $\boldsymbol{\bigtimes_i X_i}$. The variable $\boldsymbol{\kappa}$ is used for infinite cardinals, $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}$ for ordinals and $\boldsymbol{\omega}$ stands for the set of natural numbers. The set subsets of X of size κ is denoted by $[\boldsymbol{X}]^{\boldsymbol{\kappa}}$ while $[\boldsymbol{X}]^{<\boldsymbol{\kappa}}$ stands for the subsets smaller than κ. About forcing we use the standard terminology and notation except that the ground model and the generic extension by generic filter $\boldsymbol{G}$ are denoted by $\boldsymbol{M}$ and $\boldsymbol{M}[\boldsymbol{G}]$ respectively instead of the more common V (which we preserve for vertex sets).

A digraph $\boldsymbol{D}$ is a set of ordered pairs without loops (i.e., without elements of the form vv). A directed cycle C of size n ($2 \leq n < \omega$) is a digraph of the form $\{v_0v_1, v_1v_2, \ldots, v_{n-1}v_0\}$

where the v_i are pairwise distinct and $n \geq 2$. The **digirth** of a D is the size of its smallest directed cycle if there are any, otherwise ∞. A colouring of the vertex set $\boldsymbol{V(D)}$ of D is **chromatic** (with respect to D) if there is no monochromatic directed cycle. The **dichromatic number** $\boldsymbol{\chi(D)}$ of D is the smallest cardinal κ such that D admits a chromatic colouring with κ many colours. For $U \subseteq V(D)$, $\boldsymbol{D[U]}$ denotes the subdigraph induced by U in D. Let $\boldsymbol{f_D}(n) := \min\{|U| : U \subseteq V(D) \wedge \chi(D[U]) = n\}$ where $\min \varnothing$ is considered ∞. Note that if $\chi(D) \geq \aleph_0$ then by standard compactness arguments D has a finite n-dichromatic subdigraph for every $n < \omega$ and hence f_D has only finite values.

3 Main results

3.1 The growing rate of $\mathrm{f_D}$ for uncountably dichromatic D

Theorem 3.1. *For every $f : \omega \to \omega$, there is an uncountably dichromatic digraph D of size $2^{\aleph_0}$ such that every $(n+2)$-dichromatic subdigraph of D has at least $f(n)$ vertices.*

Proof. To prove Theorem 3.1, it is enough to construct for every non-decreasing $g : \omega \to \omega$ an uncountably dichromatic digraph D of size $2^{\aleph_0}$ such that whenever $H \subseteq D$ with $|V(H)| < g(n)$ for some $n < \omega$, we have $\chi(H) \leq 2^n$. Clearly, it is enough to consider only induced subdigraphs.

Let $\boldsymbol{V} := \bigtimes_{n<\omega}[0, g(n) - 1]$ and for $u \neq v \in V$, let $uv \in \boldsymbol{D}$ if for the smallest m for which $u(m) \neq v(m)$ we have $v(m) = u(m) + 1 \mod g(m)$. For $s \in \bigtimes_{k<n}[0, g(k) - 1]$, let $\boldsymbol{V_s} := \{v \in V : v \upharpoonright n = s\}$.

Observation 3.2. For every $s \in \bigtimes_{k<n}[0, g(k) - 1]$, $D[V_s]$ has digirth $g(n)$.

Lemma 3.3. $\chi(D) > \aleph_0$

Proof. Suppose for a contradiction that $c : V(D) \to \omega$ is a chromatic colouring of D. Colour 0 cannot appear in all the sets V_i $(i < g(0))$ because otherwise by picking one 0-coloured vertex from each of those sets we would obtain a monochromatic directed cycle. We choose an i_0 such that colour 0 is not used in V_{i_0}. Colour 1 cannot appear in all the sets $V_{i_0^\frown i}$ $(i < g(1))$ because of similar reasons hence there is some $i_1 < g(1)$ such that colours 0 and 1 are not used in $V_{i_0^\frown i_1}$. By recursion we get a sequence $v := (i_n)_{n<\omega} \in V(D)$ such that none of the colours are used by c to colour vertex v which is a contradiction. □

Lemma 3.4. $f_D \geq f$

Proof. Let $n < \omega$ be fixed and take a $U \subseteq V$ of size less than $g(n)$. We need to show $\chi(D[U]) \leq 2^n$. In the case $n = 0$, Observation 3.10 says that D has digirth $g(0)$, thus $D[U]$ is acyclic. For the case $n > 0$, we define a chromatic colouring $c : U \to \{0,1\}^n$ by setting $\boldsymbol{c(u)} := (\mathsf{sgn}(u(0)), \mathsf{sgn}(u(1)), \ldots, \mathsf{sgn}(u(n-1)))$ for $u \in U$ (here $\mathsf{sgn}(0) = 0$ and $\mathsf{sgn}(n) = 1$ for $n > 0$). To prove that c is a chromatic colouring, suppose that C is a directed cycle in $D[U]$ and let s be the longest common initial segment of the vertices in $V(C)$. Since $|V(C)| \leq |U| < g(n)$, Observation 3.2 guarantees that $n > |s| =: \boldsymbol{m}$. From the structure of D is clear that we must have $V(C) \cap V_{s^\frown i} \neq \varnothing$ for every $i < g(m)$. Then for a $u_0 \in V(C) \cap V_{s^\frown 0}$ and $u_1 \in V(C) \cap V_{s^\frown 1}$, $c(u_0) \neq c(u_1)$ (because $c(u_0)(m) = 0 \neq 1 = c(u_1)(m)$). Thus c is a chromatic colouring and hence $\chi(D[U]) \leq 2^n$. □

□

Remark 3.5. For every $f : \omega \to \omega$ there is an $\aleph_0$-dichromatic digraph D on ω such that $f_D \geq f$. Indeed, it follows from Theorem 3.1 via compactness arguments that for a fixed f for every $n < \omega$ there is a finite $(n+2)$-dichromatic digraph D_n for which $f_{D_n} \geq f$. By taking disjoint copies of the digraphs D_n we obtain a desired D.

3.2 A consistency result about $\leq 2^{\aleph_0}$-dichromatic digraphs

We restate the result here in a slightly stronger form. Let us remind that in light of Remark 3.5 we do not have to bother with the case $\kappa = \aleph_0$.

Theorem 3.6. *There is a ccc forcing $\mathbb{P}$ of size $\mathfrak{c}$ such that $\Vdash_{\mathbb{P}}$ for every uncountable cardinal $\kappa \leq \mathfrak{c}$ and for every $f : \omega \to \omega$, there is a digraph D on κ with $f_D \geq f$ such that for every uncountable $U \subseteq V(D)$ we have $\chi(D[U]) = |U|$, in particular $\chi(D) = \kappa$.*

Proof. Without loss of generality we can assume that f is non-decreasing. We start with some basic estimations that we need later. If D, H are digraphs then a function $\varphi : V(D) \to V(H)$ is a **semihomomorphism** from D to H if for each $uv \in D$ either $\varphi(u)\varphi(v) \in H$ or $\varphi(u) = \varphi(v)$. A semihomomorphism is **acyclic** if for every $v \in \mathsf{ran}(\varphi)$, $\varphi^{-1}(v)$ spans an acyclic subdigraph of D (shortly $\chi(D[\varphi^{-1}(v)]) = 1$).

Proposition 3.7. *If D, H are digraphs and φ is an acyclic semihomomorphism from D to H, then $\chi(D) \leq \chi(H)$.*

Proof. If c is a chromatic colouring of H then so is $c \circ \varphi$ for D. □

Proposition 3.8. *Let D, H be digraphs and assume that there is a semihomomorphism φ from D to H where $\chi(D[\varphi^{-1}(v)]) = k_v + 1$ for $v \in \mathsf{ran}(\varphi)$. Then $\chi(D) \leq \chi(H) + \sum_{v \in \mathsf{ran}(\varphi)} k_v$.*

Proof. For $v \in \mathsf{ran}(\varphi)$, fix a chromatic colouring c_v of $D[\varphi^{-1}(v)]$ with the colours $\{0, \ldots, k_v\}$. For every $u \in V(D)$ with $c_{\varphi(u)}(u) \neq 0$, colour u with the ordered pair $\langle \varphi(u), c_{\varphi(u)}(u) \rangle$. Delete all the u we already coloured. Since from each $D[\varphi^{-1}(v)]$ all but the vertices u with $c_v(u) = 0$ have been deleted, the restriction of φ to the remaining digraph is an acyclic semihomomorphism. By Proposition 3.8, it has a chromatic colouring with the colours, say $\{0, \ldots, \chi(H) - 1\}$. We defined a chromatic colouring of D with $\chi(H) + \sum_{v \in \mathsf{ran}(\varphi)} k_v$ colours witnessing the desired inequality. □

To continue the proof of Theorem 3.6, let κ be an uncountable cardinal and let $f : \omega \to \omega$. We define $\boldsymbol{\mathbb{P}_{\kappa,f}}$ to be the poset where $p \in \mathbb{P}_{\kappa,f}$ if p is a digraph with $V(p) \in [\kappa]^{<\aleph_0}$ satisfying $f_p \geq f$ and $q \leq p$ if $p \subseteq q$.

Lemma 3.9. *$\mathbb{P}_{\kappa,f}$ satisfies ccc.*

Proof. Let $\{p_\alpha : \alpha < \omega_1\} \subseteq \mathbb{P}_{\kappa,f}$. By the Δ-system lemma, there is an uncountable $\boldsymbol{U} \subseteq \omega_1$ such that $\{V(p_\alpha) : \alpha \in U\}$ forms a Δ-system with root $\boldsymbol{R}$. We consider for $\alpha \in U$ the following first order structure $\boldsymbol{\mathcal{A}_\alpha}$ on ground set $V(q_\alpha)$: we have a binary relation defined by the digraph p_α, a linear order $\in$ given by the fact that the elements of $V(q_\alpha)$ are ordinals and constants for each element of R. Up to isomorphism there are just finitely many such a first order structures on a finite ground set therefore there is an uncountable $\boldsymbol{U'} \subseteq U$ such that for $\alpha \in U'$, the $\mathcal{A}_\alpha$ are pairwise isomorphic. Note that the isomorphism between two $\mathcal{A}_\alpha$ is uniquely determined by the linear order, and its restriction to R is the identity.

We show that for $\beta \neq \delta \in U'$, $p_\beta \cup p_\delta \in \mathbb{P}_{\kappa,f}$. Let $V(p_\beta) = \{\beta_0, \ldots, \beta_{n-1}\}$ and $V(p_\delta) = \{\delta_0, \ldots, \delta_{n-1}\}$ where the enumerations are in increasing order. Consider $\boldsymbol{\varphi} : p_\beta \cup p_\delta \to p_\delta$ where

$\varphi(\beta_i) := \delta_i$ and $\varphi(\delta_i) := \delta_i$. Since $\mathcal{A}_\beta$ and $\mathcal{A}_\delta$ are isomorphic, φ is a semihomomorphism. The inverse image of a vertex δ_i is $\{\beta_i, \delta_i\}$ which is a singleton if $\delta_i \in R$ and a vertex pair without any edge between them otherwise. Hence φ is an acyclic semihomomorphism. Let $\boldsymbol{W} \subseteq V(p_\beta \cup p_\delta)$ be arbitrary and $\boldsymbol{W^*} := \varphi[W]$. We write $\boldsymbol{k}$ and $\boldsymbol{k^*}$ for $\chi((p_\beta \cup p_\delta)[W])$ and $\chi((p_\beta \cup p_\delta)[W^*])$ respectively. Then $|W^*| \geq f(k^*)$ because $p_\delta \in \mathbb{P}_{\kappa,f}$. Proposition 3.7 guarantees $k^* \geq k$ from which $f(k^*) \geq f(k)$ follows since f is assumed to be non-decreasing. By combining these facts, we obtain

$$|W| \geq |W^*| \geq f(k^*) \geq f(k).$$

Since W was arbitrary, we may conclude that $f_{p_\beta \cup p_\delta} \geq f$ and hence $p_\beta \cup p_\delta \in \mathbb{P}_{\kappa,f}$. □

Suppose that G is a $\mathbb{P}_{\kappa,f}$-generic filter and let us define $\boldsymbol{D} := \bigcup G$.

Observation 3.10. D is a digraph on κ satisfying $f_D \geq f$.

Lemma 3.11. $\chi(D[U]) = |U|$ *holds for every uncountable* $U \subseteq V(D)$.

Proof. It is enough to show that for every $U \in [\kappa]^{\aleph_1}$ the digraph $D[U]$ contains some directed cycle. Let $U \subseteq V(D)$ be uncountable forced by the condition $\boldsymbol{p} \in G$. It is enough to show that $\boldsymbol{S} := \{q \in \mathbb{P}_{\kappa,f} : q \Vdash \chi(\dot{D}[\dot{U}]) \geq 2\}$ is dense below p (note that $\chi(D[U]) \geq 2$ means that U spans some directed cycle in D). Let $\boldsymbol{r} \leq p$ be given. For every $\alpha \in U$, we pick an $\boldsymbol{q_\alpha} \leq r$ such that $q_\alpha \Vdash \check{\alpha} \in \dot{U}$ and $\alpha \in V(q_\alpha)$. We proceed similarly as in the proof of Lemma 3.9. By applying the Δ-system lemma, we trim U to an uncountable $\boldsymbol{U'} \subseteq U$ where $\{V(p_\alpha) : \alpha \in U\}$ form a Δ-system with root $\boldsymbol{R}$. For $\alpha \in U'$, let $\boldsymbol{\mathcal{B}_\alpha}$ be the first order structure on ground set $V(q_\alpha)$ where we have a constant that stands for α in $\mathcal{B}_\alpha$, a binary relation defined by the digraph q_α, a linear order $\in$ given by the fact that the elements of $V(q_\alpha)$ are ordinals and one constant for each element of R. Up to isomorphism there are just finitely many such a first order structures on a finite ground set therefore there is an uncountable $\boldsymbol{U''} \subseteq U'$ such that for $\alpha \in U''$, the $\mathcal{B}_\alpha$ are pairwise isomorphic. Let $\boldsymbol{n}$ be the common size of the ground sets of the structures $\{\mathcal{B}_\alpha : \alpha \in U''\}$. Note that $U'' \cap R = \varnothing$ otherwise we would have $U'' \subseteq R$ contradicting the fact that R is finite. We pick $\boldsymbol{m} := n + \max_{k \leq n} f(k+1) - f(k)$ many elements, say $\boldsymbol{\alpha_0}, \dots, \boldsymbol{\alpha_{m-1}}$, of U'' and define the directed cycle $\boldsymbol{C} := \{\alpha_0\alpha_1, \alpha_1\alpha_2, \dots, \alpha_{m-1}\alpha_0\}$. To simplify the notation, from now on we write simply $\boldsymbol{q_i}$ instead of q_{α_i} and let us define $\boldsymbol{q} := C \cup \bigcup_{i<m} q_i$.

Claim 3.12. $q \in \mathbb{P}_{\kappa,f}$.

Proof. The only nontrivial part of the claim is that $f_q \geq f$ holds. Let $V(q_i) = \{\beta_{i,j} : j < n\}$ where the enumeration is in increasing order. There is a $j_0 < n$ such that $\alpha_i = \beta_{i,j_0}$ for every $i < m$ because the $\mathcal{B}_{\alpha_i}$ are pairwise isomorphic. Consider the function $\varphi : V(q) \to V(q_0)$ where $\varphi(\beta_{i,j}) = \beta_{0,j}$. The inverse image of a $v \in V(q_0)$ with respect to φ is: $\{v\}$ if $v \in R$, the directed cycle C if $v = \alpha_0$ and an independent set of size m otherwise. Combining this with the fact that $\mathcal{B}_{\alpha_i}$ are pairwise isomorphic, we may conclude that φ is a semihomomorphism from q to q_0. Let $\boldsymbol{W} \subseteq V(q)$ be arbitrary and $\boldsymbol{W^*} := \varphi[W]$. We write $\boldsymbol{k}$ and $\boldsymbol{k^*}$ for $\chi(q[W])$ and $\chi(q[W^*])$ respectively. If $V(C) \nsubseteq W$, then $\varphi \upharpoonright W$ is an acyclic semihomomorphism from $q[W]$ to $q_0[W^*]$ and hence $k \leq k^*$. By combining this with the facts that $f_{q_0} \geq f$ and f is non-decreasing, we obtain

$$|W| \geq |W^*| \geq f(k^*) \geq f(k).$$

If $V(C) \subseteq W$ then $|W| \geq |V(C)| = m$ and hence $|W| - |W^*| \geq m - n$. In this case we apply Proposition 3.8 with $\varphi \upharpoonright W$. Using the terminology of the Proposition, $k_v = 1$ if $v = \alpha_0$ and $k_v = 0$ otherwise thus $k \leq k^* + 1$. By the choice of m we obtain

$$|W| \geq |W^*| + (m-n) \geq f(k^*) + (m-n) \geq f(k^* + 1) \geq f(k).$$

□

Clearly $q \leq q_i$ for $i < m$ and therefore $q \Vdash \bigwedge_{i<m} \check{\alpha}_i \in \dot{U}$. Because of $C \subseteq q$ we also have $q \Vdash \chi(\dot{D}[\dot{U}]) \geq 2$. Since $r \leq p$ was arbitrary and $q \leq q_i \leq r$, we may conclude that S is dense below p. □

We build the $\mathbb{P}$ in Theorem 3.6 as the $\mathbb{P}_{\mathfrak{c}}$ of a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta)_{\alpha \leq \mathfrak{c}, \beta < \mathfrak{c}}$ of ccc posets of size at most $\mathfrak{c}$ which ensures that $\mathbb{P}$ is ccc and $|\mathbb{P}| = \mathfrak{c}$. We let every (non-trivial) factor $\dot{\mathbb{Q}}_\beta$ to be $\dot{\mathbb{P}}_{\check{\kappa}, \dot{f}}$ for some $\aleph_0 < \kappa \leq 2^{\aleph_0}$ and for a nice $\mathbb{P}_\delta$-name $\dot{f}$ of a function $f : \omega \to \omega$ where $\delta < \beta$. Lemma 3.9 ensures that the factors are really ccc. By standard bookkeeping techniques, the iteration can be organized in the way that for every $\mathbb{P}$-generic filter G, uncountable $\kappa \leq \mathfrak{c}$ and $f : \omega \to \omega$ living in $M[G]$, there is a factor $\dot{\mathbb{Q}}_\beta = \dot{\mathbb{P}}_{\check{\kappa}, \dot{f}}$ in the iteration. Then by Observation 3.10, the digraph D given by the intermediate forcing $\mathbb{Q}_\beta = \mathbb{P}_{\kappa, f}$ has size κ and satisfies $f_D \geq f$. To justify $M[G] \models \forall U \in [\kappa]^{\aleph_1} : \chi(D[U]) \geq 2$, consider the forcing $\mathbb{P}_{\geq \beta}$ over the intermediate extension $M[G_{<\beta}]$). From this point the proof goes the same way as the proof of Lemma 3.11 by working formally with the whole $\mathbb{P}_{\geq \beta}$ instead of just $\mathbb{Q}_\beta = \mathbb{P}_{\kappa, f}$. More precisely, whenever we deal with a condition p in the original proof, we consider now just its initial coordinate $p(\beta)$.

□

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