arXiv:1907.09338v1 [math.CO] 22 Jul 2019

A CANTOR-BERNSTEIN-TYPE THEOREM FOR SPANNING TREES IN INFINITE GRAPHS

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Abstract. We show that if a graph admits a packing and a covering both consisting of λ many spanning trees, where λ is some infinite cardinal, then the graph also admits a decomposition into λ many spanning trees. For finite λ the analogous question remains open, however, a slightly weaker statement is proved.

1. Introduction

The graphs in this paper may have parallel edges but not loops. A *spanning tree* of a graph G is a connected, acyclic subgraph $T \subseteq G$ containing all vertices of G. Given a cardinal λ, a λ*-packing (of* G*)* is a collection of λ many edge-disjoint spanning trees in G, a λ*-covering (of* G*)* is a collection of λ many spanning trees whose union covers the edge set of G, and a λ*-decomposition (of* G*)* is a collection of λ many spanning trees whose edge sets partition the edge set of G.

The purpose of this note is to establish the following Cantor-Bernstein-type theorem for decomposing infinite graphs into spanning trees:

Theorem 1.1. *Let λ be an infinite cardinal. Then a graph admits a λ-decomposition if and only if it admits both a λ-packing and a λ-covering.*

Perhaps interestingly, the λ in Theorem 1.1 does not need to be unique: For example, it is not hard to show directly that $K_{\aleph_1}$, the complete graph on $\aleph_1$ vertices, admits decompositions both into $\aleph_0$ or $\aleph_1$ many spanning trees. This effect can get arbitrarily pronounced, see Proposition 3.2 below.

Our proof of Theorem 1.1 relies on two well-known characterisations of when G admits a λ-packing or λ-covering for an infinite cardinal λ. Firstly, for λ-packings, we have the following characterisation in terms of the edge-connectivity of G.

Theorem 1.2 (Laviolette, [6, Corollary 14])**.** *Let λ be an infinite cardinal. Then a graph admits a λ-packing if and only if it has edge-connectivity at least λ.*

2010 *Mathematics Subject Classification.* 05C63, 05C40.

Key words and phrases. spanning trees, colouring number, packing, covering.

The third author acknowledges support by the Alexander von Humboldt Foundation and partially by OTKA 129211.

The analogous statement for finite λ fails dramatically: there are infinite graphs of arbitrarily large finite edge-connectivity which do not even contain two edge-disjoint spanning trees, see [1].

Theorem 1.2 was originally obtained by Laviolette as corollary to his theory on "bond-faithful decompositions" which required the generalised continuum hypothesis (GCH). The use of GCH to obtain these bond-faithful decompositions was subsequently removed by Soukup [8, Theorem 6.3] using the technique of elementary submodels. In Section 2, we will give a short direct proof of Theorem 1.2, not relying on the "bond-faithful decomposition" result.

The characterisation of the existence of λ-coverings relies on the following notion introduced by Erdős and Hajnal [5], which we adapt here slightly to take parallel edges into account: The *colouring number* $\mathrm{col}(G)$ of a graph $G = (V, E)$ is the smallest cardinal μ such that there exists a well-ordering $<^*$ of V such that for every $v \in V$ the cardinality of the set of edges between v and $\{w \in V : w <^* v\}$ is strictly less than μ. We call any well-ordering $<^*$ that witnesses the colouring number of a graph *good*. The relation of the colouring number to λ-coverings is the following:

Theorem 1.3 (Erdős and Hajnal, [5, Theorem 9]). *Let λ be an infinite cardinal. Then a graph admits a λ-covering if and only if it is connected and has colouring number at most λ^+.*

The original proof of Theorem 1.3, stated only for simple graphs, is quite oblique; it is reduced to a claim in an earlier paper by the same authors [4], the proof of which in turn is omitted, stating only that it follows from similar methods as a proof of Fodor, which itself is not entirely elementary.

For this reason, we will also provide a short proof of Theorem 1.3 in Section 2. Our proof has the additional feature that as a byproduct it yields that every graph has a good well-order of the shortest possible order type, $|V(G)|$. Previously this had to be deduced from Theorem 1.3 together with a result of Erdős and Hajnal in [4, Theorem 8.6], or by employing the main theorem in [2] which characterises the colouring number of a simple graph in terms of forbidden subgraphs.

The structure of the paper is then as follows. In Section 2 we provide short proofs of Theorems 1.2 and 1.3. In Section 3 we prove Theorem 1.1, and finally in Section 4 we discuss an open problem, namely whether Theorem 1.1 also holds for finite λ.

2. ELEMENTARY PROOFS OF LAVIOLETTE AND ERDŐS-HAJNAL

In this section, we provide elementary proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. The forward implication is trivial. For the converse, consider a graph G of infinite edge-connectivity λ. Let $V(G) = \{v_j : j < \kappa\}$. We will construct a family $\mathcal{T} = (T_i : i < \lambda)$ of edge-disjoint spanning subgraphs (which will then contain the desired trees) in κ many steps as follows: for each $t < \kappa$, we find families $\mathcal{T}_t = (T_i(t)\colon i < \lambda)$ of edge-disjoint connected subgraphs of G, all on the same vertex set $V_t \subset V$ which satisfies $\{v_j\colon j < t\} \subseteq V_t$. Moreover, we make sure that for every $i < \lambda$ we have $T_i(t) \subseteq T_i(t')$ whenever $t < t'$. Taking $T_i = \bigcup_{t<\kappa} T_i(t)$ yields the desired family $\mathcal{T}$.

It remains to describe the construction. Initially we let $V_0 = \varnothing$. In a limit step we may simply take unions. At a successor step, suppose that in some step $t < \kappa$ the family $\mathcal{T}_t$ is already defined. If $v_t \in V_t$, let $\mathcal{T}_{t+1} = \mathcal{T}_t$. Otherwise, consider the graph G_t where we contract V_t to a single vertex x_t and delete all resulting loops. Since G has edge-connectivity λ, so does G_t. Hence, by greedily adding new paths, we can find a sequence $(S_k\colon k < \lambda)$ of edge-disjoint, connected subgraphs of G_t, all of size strictly less than λ, such that $x_t, v_t \in S_0$ and $V(S_k) \subseteq V(S_{k'})$ whenever $k < k'$. Let $V'_t := \bigcup_{k<\lambda} V(S_k)$. Next, partition λ into λ many subsets $(O_i\colon i < \lambda)$ each of cardinality λ, and define $H_i = \bigcup_{k\in O_i} S_k$, a connected subgraph of G_t with vertex set V'_t. If for each $i < \lambda$ we let $T_i(t+1)$ be the subgraph of G with vertex set $V_{t+1} := V_t \cup (V'_t \setminus \{x_t\})$ and edge set $E(T_i(t)) \cup E(H_i)$, then $\mathcal{T}_{t+1}$ is as desired. □

Proof of Theorem 1.3. If the colouring number of G is less than λ^+, then, following Erdős and Hajnal, we can decompose G into forests in the following manner: Let $(v_i\colon i < \kappa)$ be a good well-order of $V(G)$, i.e. one where for each i the set E_i of 'backwards edges' from v_i (edges between v_i and some v_j where $j < i$) has cardinality at most λ. For each $i < \kappa$ let us pick an arbitrary injection $f_i : E_i \to \lambda$ and for each $k < \lambda$ let $T_k = \bigcup_{i<\lambda} f_i^{-1}(k)$. In words, for each i we pick an arbitrary rainbow colouring of E_i with (at most) λ many colours, and then consider the monochromatic edge sets. Since $\bigcup_{i<\kappa} E_i = E(G)$, the family $(T_k\colon k < \lambda)$ covers all edges of G. To see that each T_k is a forest, note that every cycle C in G has a vertex $v_i \in V(C)$ of maximal index i. This, however, implies $|C \cap E_i| = 2$, and so $C \nsubseteq T_k$ for any k. Finally, since G is connected, each forest can be extended to a spanning tree, and hence G admits a λ-covering.

For the converse implication, suppose there exists a family of λ many spanning trees $(T_i\colon i < \lambda)$ which covers $E(G)$. First we note that there are at most λ many parallel edges between any two vertices of G, since at most one such edge is in each T_i. If $|E(G)| \leq \lambda$ then any well-ordering of $V(G)$ witnesses that $\operatorname{col}(G) \leq \lambda^+$. Hence we may assume that $|E(G)| > \lambda$ which, by the previous comment, implies $|V(G)| > \lambda$. Let us root each T_i arbitrarily and let $\leq_i$ be the corresponding tree order on $V(G)$, cf. [3, §1.5]. For a vertex x, recall that $\lceil x \rceil_i = \{v\colon v \leq_i x\}$ denotes the vertex set of the path from the root to x in T_i. Consider the following closure operation of a given vertex set $X \subseteq V(G)$: Let $X_0 = X$ and for each $n \in \mathbb{N}$ put $X_{n+1} := \bigcup \{\lceil x \rceil_i\colon x \in X_n,\ i < \lambda\}$.

Let $\mathrm{cl}(X) = \bigcup_{n\in\mathbb{N}} X_n$ be the *closure* of X. We say a set $Y \subseteq V(G)$ is *closed* if $\mathrm{cl}(Y) = Y$, and it is clear that $\mathrm{cl}(X)$ is closed for every $X \subseteq V$. Since there are only λ many trees T_i, and $\lceil x \rceil_i$ is finite for each i, it follows that whenever X is closed and $Y \supseteq X$ is such that $|Y \setminus X| \leq \lambda$ then there is a closed set $Z \supseteq Y$ with $|Z \setminus X| \leq \lambda$.

Now let $(v_i \colon i < \kappa)$ be a well-ordering of $V(G)$ of length $\kappa = |V(G)|$ and define an increasing sequence of closed sets $(V_i \colon i < \kappa)$ by $V_0 = \varnothing$, $V_{i+1} = \mathrm{cl}\{V_i \cup \{\min_j \{v_j \colon v_j \notin V_i\}\}\}$ for each $i < \kappa$, and $V_i = \bigcup_{j<i} V_j$ for $i < \kappa$ a limit. In particular, we have $\bigcup_{i<\kappa} V_i = V(G)$ and $|V_{i+1} \setminus V_i| \leq \lambda$ for each $i < \kappa$. Let us well-order each set $V_{i+1} \setminus V_i$ arbitrarily, and concatenate these orderings to form a well-order $<^*$ of V. We claim that this well-ordering of order type $|V(G)|$ witnesses $\mathrm{col}(G) \leq \lambda^+$. Indeed, let $v \in V$ be arbitrary. There is a unique i such that $v \in V_{i+1} \setminus V_i$, and hence every 'backwards edge' (with respect to $<^*$) from v has both endpoints in V_{i+1}. We will show that there at at most λ many such edges.

Firstly, since $|V_{i+1} \setminus V_i| \leq \lambda$, there are at most $\lambda \cdot \lambda = \lambda$ many edges between $V_{i+1} \setminus V_i$ and v. Furthermore, suppose $e = (x, v)$ is an edge between V_i and v. There is some j such that $e \in E(T_j)$ and, since V_i is closed under the tree-order generated by any T_j and $v \notin V_i$, it follows that $x \leq_j v$. However, there is a unique edge $(x, v) \in E(T_j)$ such that $x \leq_j v$. It follows that there are at most λ many edges between V_i and v □

We remark that only the backwards implication used that λ is infinite.

Corollary 2.1. *Every graph has a good well-ordering of order-type* $|V(G)|$. □

3. A Cantor-Bernstein theorem for spanning trees in infinite graphs

Theorem 3.1. *Let λ be a cardinal (finite or infinite) and let G be a graph with col(G) $\leq \lambda^+$ which admits λ-packing. Then G admits a λ-decomposition.*

Proof. Let $(v_i \colon i < \kappa)$ be a good well-ordering of $V(G)$. For each $i < \kappa$ let E_i be the set $\{(v_j, v_i) \in E(G) \colon j < i\}$ of 'backwards edges' in this ordering at v_i. Then $(E_i \colon i < \kappa)$ is a partition of $E(G)$ and $|E_i| \leq \lambda$ for each $i < \kappa$. Let us well-order each of the sets E_i arbitrarily in order type $|E_i|$ and concatenate these orderings to form a well-order $\prec$ of E.

By assumption, there exists a family $(T_i \colon i < \lambda)$ of λ many edge-disjoint spanning trees of G. If $\bigcup_{i<\lambda} T_i = E(G)$, then $(T_i \colon i < \lambda)$ is a λ-decomposition. Our aim will be to exchange a yet uncovered edge $f \in E(G) \setminus \bigcup_{i<\lambda} T_i$ with some later edge $e \succ f$ from some T_i such that at each stage in our process we maintain the property that $(T_i \colon i < \lambda)$ is a λ-packing. By an appropriate book-keeping procedure, we guarantee that each edge is eventually covered.

Let us initialise by setting $T_i(0) = T_i$ for each $i < \lambda$. Suppose that we have already constructed a λ-packing $\mathcal{T}_t = (T_i(t) \colon i < \lambda)$ where $t < \kappa$. In step t we consider e_t. If $e_t \in \bigcup_{i<\lambda} E(T_i(t))$, then we set $T_i(t+1) = T_i(t)$ for each $i < \lambda$. Otherwise, $e_t \notin \bigcup_{i<\lambda} T_i(t)$. Then $e_t \in E_i$ for some i and by construction there are fewer than λ many edges $e \in E_i$ such that $e \prec e_t$, and hence there is some $k < \lambda$ such that $T_k(t)$ contains no edges $e \in E_i$ with $e \prec e_t$. Since $T_k(t)$ is a spanning tree, there is a unique cycle $C \subseteq T_k(t) + e_t$. Since C is finite, it contains a $\prec$-maximal edge f. Moreover, since $T_k(t)$ contains no edges $e \in E_i$ with $e \prec e_t$ it follows that $f \neq e_t$: if j is maximal such that $C \cap E_j \neq \varnothing$ then $|C \cap E_j| = 2$, since C is a cycle. Then, if $j = i$ it follows that $e_t \prec f$ by our choice of $T_k(t)$ and if $j > i$ then clearly $e_t \prec f$ since all of E_i precedes E_j.

Now let $T_k(t+1) = T_k(t) - f + e_t$, which is again a spanning tree, and $T_i(t+1) := T_i(t)$ for all $k \neq i < \lambda$. Finally for each limit ordinal $\tau < \kappa$ we let

$$T_i(\tau) = \{e\colon \text{ there exists } t_0 < \tau \text{ such that } e \in T_i(t) \text{ for all } t_0 < t < \tau\}$$

We claim that for every $t \leq \kappa$ the family $\mathcal{T}_t$ is indeed a λ-packing. Since this property is clearly preserved at successor steps, it remains to check that it holds at limit steps.

As it is clear that if each $\mathcal{T}_t$ is a family of edge-disjoint subgraphs for $t < \tau$, then $\mathcal{T}_\tau$ is a family of edge-disjoint subgraphs, it is sufficient to show that each $T_i(\tau)$ is in fact a spanning tree. That each $T_i(\tau)$ is acyclic is clear, as any finite cycle would have to appear at some successor step. To see that $T_i(\tau)$ is connected and spanning, it suffices to show that it contains an edge from each bond of G.

Given a bond $F \subset E(G)$ let us consider the set of edges $F_i(t) := E(T_i(t)) \cap F$. We claim that the sequence $f_i(t) := \min_\prec F_i(t)$ is $\prec$-non-increasing in t. Indeed, suppose we delete the $\prec$-minimal edge f of $F_i(t)$ from $T_i(t)$ at step t. Note that by the construction there is a cycle C with $\prec$-maximal edge f such that $C - f \subset T_i(t+1)$. Then $C \cap F$ is non-empty because it contains f and therefore, since $|C \cap F|$ must be even, there is some $e \neq f$ in $C \cap F$. It follows from the $\prec$-maximality of f in C that $e \prec f$. Furthermore, $e \in F_i(t+1)$ since $C - f \subset T_i(t+1)$, from which $f_i(t+1) \prec f_i(t)$ follows. Hence for each bond F and each limit ordinal τ, the sequence $(f_i(t)\colon t < \tau)$ is constant after some $t_0 < \tau$, and therefore $f_i(t_0) \in F \cap T_i(\tau)$.

It remains to verify that $\mathcal{T}_\kappa$ is a λ-decomposition. Since it is a λ-packing by the above, it suffices to show that $\bigcup_{i<\lambda} E(T_i(\kappa)) = E(G)$. However for each $t < \lambda$ we have $e_t \in E(T_k(t+1))$ for some k by construction. Furthermore, at any later stage s we only ever remove an edge f with $e_t \prec e_s \prec f$. It follows that $e_t \in E(T_k(s))$ for all $s > t$ and hence $e_t \in E(T_k(\kappa))$. □

Theorem 1.1 then follows from Theorems 3.1 and 1.3. We conclude this section by observing that the effect of a graph having λ-decompositions for different λ's can get arbitrarily pronounced:

Proposition 3.2. *For every infinite cardinal κ there is a graph that admits a λ-decomposition for any choice of λ with $2 \leq \lambda \leq \kappa$.*

Construction. We construct the desired graph G as an increasing union of graphs $G_n = (V_n, E_n)$ by recursion on $n \in \mathbb{N}$ as follows.

Let $G_0 = K_2$ be the complete graph on two vertices. We form G_{n+1} by adding κ many new $u - v$ paths of length two to G_n for every $u \neq v \in V_n$, internally disjoint from each other and from V_n. Finally, we set $G := \bigcup_{n\in\mathbb{N}} G_n$ which by construction has (edge-)connectivity κ. If we well-order each $V_{n+1} \setminus V_n$ arbitrarily and concatenate these orders, we obtain a well-ordering witnessing $\operatorname{col}(G) = 3$, as by construction, every newly added vertex in step n has degree two. Since κ was infinite, it follows from Theorem 1.2 that G has a κ-packing, and hence a λ-packing for all $\lambda \leq \kappa$. Therefore, the assertion of the proposition follows from Theorem 3.1. □

4. An open problem

It remains an interesting question whether the assertion of our main theorem also holds for finite λ. For finite graphs, a simple counting argument (every spanning tree has precisely $|G| - 1$ edges) shows that Theorem 1.1 holds when both the graph and λ are finite. Hence, the question remains what happens for infinite graphs and finite λ. We note that our main technical result, Theorem 3.1, did not require that λ is infinite. However, in order to deduce Theorem 1.1 from it we needed to apply Theorem 1.3, which only holds for infinite λ. When λ is finite, only the following, slightly weaker version of Theorem 1.3 holds, which is best possible as can be seen in the case of complete graphs.

Theorem 4.1 (Erdős and Hajnal, [5, Theorem 11])**.** *If G is a graph (finite or infinite) with a k-covering for some $k \in \mathbb{N}$, then col$(G) \leq 2k$.*

The following is then a consequence of Theorems 3.1 and 4.1.

Corollary 4.2. *For every $k \geq 1$, every graph with a k-covering and a $(2k-1)$-packing has a $(2k-1)$-decomposition.*

Hence, if one were to seek a proof for Theorem 1.1 for finite $k = \lambda$, one would need to use the assumption of the existence of a k-covering more efficiently than simply relying on the rather weak consequence that $\operatorname{col}(G) \leq 2k$. One such possibility might be offered by the following characterisation due to Nash-Williams (where the assertion for infinite graphs follows from the finite version by a straightforward compactness argument):

Theorem 4.3 (Nash-Williams [7])**.** *For every $k \in \mathbb{N}$, a graph G admits a k-covering if and only if for every non-empty finite $U \subseteq V(G)$ the number of edges in $G[U]$ is at most $k(|U|-1)$.*

However, we did not succeed in proving a theorem in the vein of Theorem 3.1 using Nash-Williams's condition.

Finally, we remark that in order to prove the assertion of Theorem 1.1 for finite $\lambda = k$, it suffices to consider countable graphs: Indeed, to see that the general case follows from the countable case, consider some uncountable graph G with a k-packing $\{T_1, \dots, T_k\}$ and a k-covering $\{T_{k+1}, \dots, T_{2k}\}$. Starting with $W_0 = \varnothing$, by greedily adding finite paths from the different trees in turn for ω many substeps, we find an increasing, continuous collection $(W_i \colon i < |G|)$ of subsets of V with $\bigcup_i W_i = V$ such that $W_{i+1} \setminus W_i$ is countable, and each $T_j[W_i]$

is an induced subtree of T_j for all $j \in [2k]$ and $i < |G|$. Then each minor $G_i = G[W_{i+1}]/G[W_i]$ has a k-packing and k-covering given by the trees $T_j[W_{i+1}]/T_j[W_i]$. Applying the countable assertion to each G_i yields a k-decomposition $\{S_1(i), \ldots, S_k(i)\}$ of G_i. Clearly, the subtrees S_j of G for $j \in [k]$ given by $E(S_j) = \bigcup_i E(S_j(i))$ are as desired.

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