ZMP-HH/19-13
Hamburger Beiträge zur Mathematik 796

On invertible 2-dimensional framed and r-spin topological field theories

Lóránt Szegedy*

Fachbereich Mathematik, Universität Hamburg
Bundesstraße 55, 20146 Hamburg, Germany

July 2019

Abstract

We classify invertible 2-dimensional framed and r-spin topological field theories by computing the homotopy groups and the k-invariant of the corresponding bordism categories. By a recent result of Kreck, Stolz and Teichner the first homotopy groups are given by the so called SKK groups. We compute them explicitly using the combinatorial model of framed and r-spin surfaces of Novak, Runkel and the author.

1 Introduction

Invertible topological field theories (TFTs) have recently gained attention as they are predicted to describe short range entangled topological phases of matter [Kap, FH]. The latter are defined as deformation classes of gapped Hamiltonians while the former are symmetric monoidal functors from the category of d-dimensional bordisms with G-tangential structure $\mathcal{B}ord_d^G$ to the category of super vector spaces, (or to any other symmetric monoidal category), which land in the category of super lines (or respectively in the Picard subgroupoid of the target category).

A G-tangential structure on an oriented d-dimensional manifold is given by a group homomorphism $G \to SO_d$ and a map from the manifold into BG which factors the classifying map of the tangent bundle. A G-structure is called stable if it extends in an appropriate way to a G'-tangential structure in one dimensional higher after stabilising the tangent

*Email: lorant.szegedy@uni-hamburg.de

bundle (Definition 2.7). In [FH] fully extended TFTs with stable tangential structures and with values in the category of super lines have been identified by maps of spectra and these have been classified using the computational power of stable homotopy theory.

Non-extended invertible TFTs factor through the fundamental groupoid $||\mathcal{B}ord_d^G||$ of the classifying space of the bordism category. Hence the classification of non-extended invertible TFTs can be formulated into understanding Picard groupoids and functors between them. By a result of [Sín, JS, JO], Picard groupoids are classified by their zeroth and first homotopy groups and their k-invariant, i.e. two abelian groups and a group homomorphism between them. Functors of Picard groupoids are classified by group homomorphisms between the homotopy groups which commute with the k-invariants and extensions of the zeroth homotopy group of the source category by the first homotopy group of the target category.

In [KST] the zeroth and first homotopy groups of $||\mathcal{B}ord_d^G||$ in arbitrary dimensions and with arbitrary tangential structure have been identified with the bordism group Ω_{d-1}^G one dimension lower and with the so called SKK group SKK_d^G [KKNO] respectively. The latter is defined as the group completion of closed d-dimensional manifolds with G-structures with disjoint union as addition modulo the four-term SKK relation.

In case the tangential structure is stable there exists a surjective group homomorphism $SKK_d^G \twoheadrightarrow \Omega_d^{G'}$ to the bordism group in the same dimension [KST], which allows for a computation of the SKK group. If the tangential structure in question is not stable, e.g. framings in dimension not equal to 1 or 3, to our knowledge there is no general method of computing the SKK groups.

In the present work we consider 2-dimensional invertible TFTs with framings and r-spin structures. The latter are tangential structures corresponding to the r-fold cover of SO_2 which are not stable unless $r = 1$ (which correspond to orientations) or $r = 2$. Our main result in Theorem 3.8 lists the corresponding bordism groups and SKK groups explicitly and it is proven using the SKK relations and the combinatorial model of framed and r-spin surfaces of [Nov, RS].

This provides a full classification of invertible 2-dimensional framed and r-spin TFTs with arbitrary target (Theorem 3.9). If we consider super lines as the target category this result can be interpreted as follows.

Theorem 1.1.

1. *The group of isomorphism classes of invertible 2-dimensional framed TFTs with target super lines is $\mathbb{Z}/2$ generated by the TFT computing the Arf invariant.*
2. *The group of isomorphism classes of invertible 2-dimensional r-spin TFTs with target super lines is*
   - $\mathbb{C}^\times$ *generated by the TFT computing the Euler characteristic, if r is odd,*
   - $\mathbb{Z}/2 \times \mathbb{C}^\times$ *generated by the Euler and the Arf TFT, if r is even.*

The the rest of the paper is organised as follows. In Section 2 we review the notion of invertible TFTs with tangential structures, the classification of Picard groupoids and functors of them, and finally we introduce the SKK groups. In Section 3 we turn to dimension 2 and after a brief recollection of notions on framed and r-spin surfaces we compute the corresponding bordism and SKK groups. In Appendix A we give the proof of Lemma 3.5 which is central for our computation of the SKK groups.

Acknowledgments

The author would like to thank Manuel Araújo, Bertram Arnold, Mark Penney and Peter Teichner for helpful discussions and comments. The author gratefully acknowledges the Max Planck Institute for Mathematics for hospitality and financial support. The author is partially funded by the DFG Research Training Group 1670 "Mathematics Inspired by String Theory and Quantum Field Theory".

2 Invertible topological field theories

In this section we review the notion of invertible topological field theories with tangential structure. We then turn to the classification of them which by a result of [KST] boils down to computing bordism groups and the so called SKK groups [KKNO].

Let $d \geq 1$ be an integer, G a topological group and $\xi : G \to SO_d$ a continuous group homomorphism. A *G-tangential structure (or G-structure)* on a d-dimensional oriented manifold Σ is a homotopy class of maps $\varphi : \Sigma \to BG$ such that the diagram

$$\begin{array}{ccc} & & BG \\ & \overset{\varphi}{\nearrow} & \big\downarrow{\scriptstyle B\xi} \\ \Sigma & \xrightarrow[T\Sigma]{} & BSO_d \end{array} \tag{2.1}$$

commutes up to homotopy, where the map $T\Sigma$ is the one corresponding to the tangent bundle of Σ. Similarly, we define a G-structure on a $(d-1)$-dimensional manifold S as a G-structure on $S \times \mathbb{R}$.

Consider two $(d-1)$-dimensional closed oriented manifolds S_0 and S_1. A *d-dimensional bordism* $\Sigma : S_0 \to S_1$ is a compact oriented d-dimensional manifold together with orientation preserving embeddings $\iota_i : S_i \to \Sigma$ which identify the disjoint union of S_0 and S_1 with reversed orientation with the boundary of Σ. A *d-dimensional G-bordism* $(\Sigma, \varphi) : (S_0, \varphi_0) \to (S_1, \varphi_1)$ is a bordism together with a G-structure φ on Σ and G-structures φ_i on S_i, so that $\varphi_i|_{S_i} = \varphi \circ \iota_i$. The *category of d-dimensional G-bordisms* $\mathcal{B}ord_d^G$ has objects closed $(d-1)$-dimensional oriented G-manifolds and morphisms diffeomorphism classes of d-dimensional G-bordisms. For more details on this definition see e.g. [Tur, ST, KST]. The category $\mathcal{B}ord_d^G$ is symmetric monoidal with the disjoint union as tensor product.

Let $\mathcal{C} = (\mathcal{C}, \otimes, 1_\mathcal{C}, \sigma)$ be a symmetric monoidal category with tensor product $\otimes$, tensor unit $1_\mathcal{C}$ and symmetry σ. A d-dimensional *topological field theory with G-structure (TFT)* is a symmetric monoidal functor $Z : \mathcal{B}ord_d^G \to \mathcal{C}$ [Ati, Seg1, Seg2]. A *Picard groupoid* is a symmetric monoidal groupoid in which every object has an inverse with respect to the tensor product. A TFT is *invertible* if its image lies in the Picard subgroupoid $\mathcal{C}^\times$ of $\mathcal{C}$. We write $\mathrm{Fun}_\otimes^{\mathrm{inv}}(\mathcal{B}ord_d^G, \mathcal{C})$ for the category of invertible TFTs.

Denote with $||\mathcal{C}||$ the groupoid completion of $\mathcal{C}$, i.e. the fundamental groupoid of the classifying space of $\mathcal{C}$. We have an essentially surjective functor $\mathcal{C} \to ||\mathcal{C}||$. If $\mathcal{C}$ has duals then $||\mathcal{C}||$ is in fact a Picard groupoid.

Proposition 2.1. *For an invertible TFT $Z : \mathcal{B}ord_d^G \to \mathcal{C}$ there is a unique symmetric monoidal functor $\tilde{Z} : ||\mathcal{B}ord_d^G|| \to \mathcal{C}^\times$ so that*

$$\begin{array}{ccc} \mathcal{B}ord_d^G & \xrightarrow{Z} & \mathcal{C} \\ \downarrow & \searrow^{Z} & \uparrow \\ ||\mathcal{B}ord_d^G|| & \xrightarrow[\tilde{Z}]{} & \mathcal{C}^\times \end{array} \tag{2.2}$$

commutes.

Recall that for an object X in a Picard groupoid $\mathcal{C}$ its inverse X^{-1} is its dual $X^\vee$. We write $\mathrm{ev}_X : X \otimes X^{-1} \to 1_\mathcal{C}$ and $\mathrm{coev}_X : 1_\mathcal{C} \to X^{-1} \otimes X$ for the evaluation and the coevaluation.

Let $\mathcal{C}$ be a Picard groupoid. The *zeroth homotopy group of* $\mathcal{C}$ is the abelian group $\pi_0(\mathcal{C})$ of isomorphism classes of objects. The *first homotopy group of* $\mathcal{C}$ is the abelian group $\pi_1(\mathcal{C})$ of automorphisms of the tensor unit of $\mathcal{C}$. The *k-invariant of* $\mathcal{C}$ is the group homomorphism $k_\mathcal{C} : \pi_0(\mathcal{C}) \otimes \mathbb{Z}/2 \to \pi_1(\mathcal{C})$ given by $k_\mathcal{C}(X) := \mathrm{ev}_X \circ \sigma_{X^{-1},X} \circ \mathrm{coev}_X$.

Theorem 2.2 ([Sín, JS, JO])**.**

1. *Picard groupoids are classified by the zeroth and first homotopy groups π_0 and π_1 and the k-invariant $\kappa : \pi_0 \otimes \mathbb{Z}/2 \to \pi_1$.*

2. *The set of isomorphism classes of functors $\mathcal{C} \to \mathcal{D}$ of Picard groupoids is in bijection with the set of triples (f_0, f_1, α), where $f_i \in \mathrm{Hom}(\pi_i(\mathcal{C}), \pi_i(\mathcal{D}))$ $i = 0, 1$ are group homomorphisms, which make the diagram*

$$\begin{array}{ccc} \pi_0(\mathcal{C}) \otimes \mathbb{Z}/2 & \xrightarrow{f_0} & \pi_0(\mathcal{D}) \otimes \mathbb{Z}/2 \\ {\scriptstyle k_\mathcal{C}}\downarrow & & \downarrow{\scriptstyle k_\mathcal{D}} \\ \pi_1(\mathcal{C}) & \xrightarrow{f_1} & \pi_1(\mathcal{D}) \end{array} \tag{2.3}$$

   commute and $\alpha \in \mathrm{Ext}(\pi_0(\mathcal{C}), \pi_1(\mathcal{D}))$.

We note that the triple $(f_0, f_1, \alpha = 0)$ determines a strict symmetric monoidal functor. The different choices of $\alpha \in \mathrm{Ext}(\pi_0(\mathcal{C}), \pi_1(\mathcal{D}))$ parametrise different monoidal structures for the same underlying functor.

Example 2.3. There are 2 symmetric braidings on the monoidal category of $\mathbb{Z}/2$-graded vector spaces over $\mathbb{C}$, one with the usual flip map and one which is -1 times the flip on purely odd components. The corresponding Picard groupoids are

- $\mathcal{L}ine^{\times}_{\mathbb{Z}/2}$: $\pi_0 = \mathbb{Z}/2$, $\pi_1 = \mathbb{C}^\times$, $k([\mathbb{C}^{0|1}]) = +1$,
- $\mathcal{SL}ine^{\times}_{\mathbb{Z}/2}$: $\pi_0 = \mathbb{Z}/2$, $\pi_1 = \mathbb{C}^\times$, $k([\mathbb{C}^{0|1}]) = -1$,

where π_0 is generated by the 1-dimensional odd vector space $\mathbb{C}^{0|1}$.

Definition 2.4 ([KKNO, KST]). Let $S \in \mathcal{B}ord^G_d$, $M_i : S \to \emptyset$ and $N_i : \emptyset \to S$ for $i = 1, 2$ be morphisms in $\mathcal{B}ord^G_d$. The *SKK group* SKK^G_d is the group completion of the monoid of closed d-dimensional manifolds with disjoint union as product modulo diffeomorphisms and the SKK relations which are of the form

$$M_1 \circ N_1 \sqcup M_2 \circ N_2 \sim M_1 \circ N_2 \sqcup M_2 \circ N_1 \ . \tag{2.4}$$

We write $[\Sigma] \in SKK^G_d$ for the class of the closed d-dimensional G-manifold Σ.

Theorem 2.5 ([KST]). *For $||\mathcal{B}ord^G_d||$ the zeroth and first homotopy groups are given by the bordism group and the SKK group:*

$$\pi_0\left(||\mathcal{B}ord^G_d||\right) = \Omega^G_{d-1} \quad \textit{and} \quad \pi_1(||\mathcal{B}ord^G_d||) = SKK^G_d \ . \tag{2.5}$$

As a corollary we get that isomorphism classes of invertible TFTs with target $\mathcal{SL}ine$ are in bijection with SKK-invariants, i.e. with $\mathrm{Hom}(SKK^G_d, \mathbb{C}^\times)$. This can be seen by noticing that since the k-invariant of $\mathcal{SL}ine$ is injective, f_1 completely determines f_0 and that by the divisibility of $\mathbb{C}^\times$ the group $\mathrm{Ext}(\Omega^G_{d-1}, \mathbb{C}^\times)$ vanishes.

An important example of an SKK invariant is the Euler characteristic:

Lemma 2.6 ([KST]). *The Euler characteristic is a group homomorphism:*

$$\chi : SKK^G_d \to \mathbb{Z} \ . \tag{2.6}$$

Stable tangential structures

In the rest of this section we will consider stable tangential structures which we define now.

Definition 2.7. We call tangential structures corresponding to $\xi : G \to SO_d$ *stable* if there is a topological group G' containing G as a subgroup and a continuous group homomorphism $\xi' : G' \to SO_{d+1}$ such that the diagram

$$\begin{array}{ccc} G & \hookrightarrow & G' \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi'} \\ SO_d & \hookrightarrow & SO_{d+1} \end{array} \tag{2.7}$$

commutes and the action of G' on $S^d = SO_{d+1}/SO_d$ induces a homeomorphism $G'/G \to S^d$.

For a stable tangential structure there is a bijection between G-structures on a d-dimensional oriented manifold Σ and G'-structures on $\Sigma \times \mathbb{R}$. In this case we can also define the bordism group of d-dimensional G-manifolds $\Omega_d^{G'}$, where the equivalence relation is being G'-bordant.

Remark 2.8. Note that if G is the trivial group then the corresponding tangential structure is a framing which is only stable in dimension 1 or 3.

Theorem 2.9 ([KST])**.** *If a tangential structure is stable then there is a surjective group homomorphism*

$$SKK_d^G \twoheadrightarrow \Omega_d^{G'} \ . \tag{2.8}$$

So in the stable case one can go on and study the map (2.8) to compute SKK_d^G as in [KST]. In the next section we will consider tangential structures which are not stable and where we need to do something different.

3 Two-dimensional framed and r-spin TFTs

In this section we introduce the notion of framed and r-spin surfaces and recall some properties of the respective bordism categories. Then we compute the corresponding SKK groups explicitly and give the classification of 2-dimensional framed and r-spin TFTs.

We start by sketching some definitions from [RS, Sec. 2]. Let $r \in \mathbb{Z}_{\geq 0}$. The *$r$-spin group* $Spin_2^r$ is the r-fold cover for $r > 0$ and the universal cover for $r = 0$ of SO_2. We write $\xi : Spin_2^r \to SO_2$ for the covering map. An *r-spin structure* on a surface Σ is the tangential structure on Σ with respect to $\xi : Spin_2^r \to SO_2$.

We will work with a skeletal version of the r-spin bordism category, which we also denote with $\mathcal{B}ord_2^{Spin_2^r}$. It has objects r-spin circles, i.e. disjoint unions of pairs $(S^1, x) = S_x^1$, where $x \in \mathbb{Z}/r$. The morphisms are diffeomorphism classes of bordisms with r-spin structure. Every r-spin bordism Σ comes with two maps $\kappa_{\text{in/out}} : \pi_0(\partial_{\text{in/out}}\Sigma) \to \mathbb{Z}/r$ giving the types of the in- and outgoing boundary components. (In [RS] the map $\lambda : \pi_0(\partial_{\text{in}}\Sigma) \to \mathbb{Z}/r$ is related to $\kappa_{\text{in}} : \pi_0(\partial_{\text{in}}\Sigma) \to \mathbb{Z}/r$ via $\lambda = 1 - \kappa_{\text{in}}$ and similarly the map $\mu : \pi_0(\partial_{\text{out}}\Sigma) \to \mathbb{Z}/r$ is related to κ_{out} via $\mu = 1 - \kappa_{\text{out}}$.)

Remark 3.1. By [RS, Prop. 2.2] 0-spin structures correspond to framings, i.e. $\mathcal{B}ord_2^{Spin_2^0}$ is equivalent to the framed 2-dimensional bordism category. Recall from Remark 2.8 that framings are not stable in dimension 2.

Proposition 3.2 ([RS, Prop. 2.19])**.** *Let Σ be a connected bordism of genus g with b_{in} ingoing and b_{out} outgoing boundary components and $\kappa^{\text{in/out}} : \pi_0(\partial\Sigma)^{\text{in/out}} \to \mathbb{Z}/r$. There exist r-spin structures on Σ if and only if*

$$\chi(\Sigma) = 2 - 2g - b_{\text{in}} - b_{\text{out}} \equiv \sum_{j=1}^{b_{\text{out}}} \kappa^{\text{out}}(j) - \sum_{l=1}^{b_{\text{in}}} \kappa^{\text{in}}(l) \ \ (mod\ r) \ . \tag{3.1}$$

Let us write $\Sigma_{g,b}^{\mathrm{in}}$ (resp. $\Sigma_{g,b}^{\mathrm{out}}$) for a connected bordism of genus g with b ingoing (resp. outgoing) boundary components only and Σ_g for $b = 0$.

Theorem 3.3 ([RS, Thm. 1.1])**.** *If r is even then there is an invertible r-spin TFT $\mathcal{Z}_{C\ell}$ which computes the Arf invariant. For Σ_g with g satisfying (3.1) and an r-spin structure φ on Σ_g we have*

$$\mathcal{Z}_{C\ell}(\Sigma_g, \varphi) = 2^{\chi(\Sigma_g)/2}(-1)^{\mathrm{Arf}(\varphi)} \ . \tag{3.2}$$

Example 3.4.

1. There exist r-spin structures on the sphere if and only if $r = 1$ or $r = 2$.

2. There exist r-spin structures on the torus for every value of r. The isomorphism classes of r-spin structures on a fixed torus are in bijection with $\mathbb{Z}/r^2$ and we write $T(s,t)$ for an r-spin torus corresponding to $(s,t) \in \mathbb{Z}/r^2$. The mapping class group of the torus $SL(2,\mathbb{Z})$ acts on $\mathbb{Z}/r^2$ via the standard action and the orbits, i.e. diffeomorphism classes of r-spin tori are in bijection with the divisors of r.

3. Let $\tilde{r} := r/\gcd(r,2)$. There exist r-spin structures on Σ_g if and only if

$$g \equiv 1 \pmod{\tilde{r}} \ . \tag{3.3}$$

   We will write $U_l = \Sigma_{1+l\tilde{r}}$. If $1 + l\tilde{r} \geq 2$, there is one mapping class group orbit of r-spin surfaces with underlying surface U_l for r odd and two for r even. The latter two are distinguished by the Arf invariant and we denote these r-spin surfaces by $U_l^{(+)}$ (Arf invariant $+1$) and $U_l^{(-)}$ (Arf invariant -1).

4. The disc $\Sigma_{0,1}$ has a unique r-spin structure up to isomorphism. In case the boundary is outgoing then it is of type $+1$, if it is ingoing then it is of type -1.

After this recollection of notions we turn to the computation of the group $SKK_2^{Spin_2^r}$. For this we will look at some SKK-relations. The following lemma will be proved in Appendix A using the combinatorial model of r-spin surfaces of [RS].

Lemma 3.5. *In $SKK_2^{Spin_2^r}$ we have the relations*

$$[T(\kappa,u_1)] + [T(\kappa,u_2)] + [T(\kappa,u_3)] + [T(\kappa,u_4)] = [T(\kappa,u_1+u_2)] + [T(\kappa,u_3+u_4)] \ , \tag{3.4}$$

$$[T(1,u)] = 0 \ , \tag{3.5}$$

for every $\kappa, u, u_i \in \mathbb{Z}/r$ $(i = 1,\ldots,4)$. The k-invariant of S^1_κ is $T(\kappa,0)$.

Lemma 3.6. *The subgroup $\mathcal{T}^{(r)} \subset SKK_2^{Spin_2^r}$ generated by r-spin tori is $\mathbb{Z}/2$ for r even and trivial for r odd.*

Proof. By (3.4) in Lemma 3.5 for $u_1 = u_3 = 0$ we get

$$2[T(\kappa,0)] = 0 \tag{3.6}$$

for every $\kappa \in \mathbb{Z}/r$. Since for arbitrary $s,t \in \mathbb{Z}/r$ we have $[T(s,t)]$=$[T(\gcd(s,t),0)]$, every r-spin torus has 2-torsion in $SKK_2^{Spin_2^r}$.

Again by (3.4), now for $u_1 = 0$, $u_3 = 1$ and using (3.5) we get

$$[T(\kappa,0)] + [T(\kappa,u)] = [T(\kappa,u+1)] \ . \tag{3.7}$$

Combining the latter with (3.6) we get

$$[T(\kappa,u)] = [T(\kappa,u+2)] \ . \tag{3.8}$$

Therefore every r-spin torus is equal to $[T(0,0)]$ or $[T(1,0)]$, the latter being zero by (3.5) of Lemma 3.5.

If r is odd then $[T(0,0)] = [T(1,0)]$ and hence $\mathcal{T}^{(r)} = \{0\}$.

For r even Theorem 3.3 gives an SKK-invariant $SKK_2^{Spin_2^r} \to /Cb^\times$ computing the Euler characteristic and the Arf invariant. It has value $-1 \in \mathbb{C}^\times$ on $[T(0,0)]$, which shows that in $SKK_2^{Spin_2^r}$ the element $[T(0,0)]$ is non-trivial and hence $\mathcal{T}^{(r)} \simeq \mathbb{Z}/2$.

□

The following lemma can be proven using the SKK-relations, Lemma 3.5 and Theorem 3.3, i.e. the fact that the Arf invariant is compatible with glueing.

Lemma 3.7. *Recall the r-spin surfaces U_l (r odd) and $U_l^{(\pm)}$ (r even) from Example 3.4. We have*

$$[U_l^{(+)}] + [T(0,0)] = [U_l^{(-)}] \ , \tag{3.9}$$

$$[U_l] + [U_j] = [U_{l+j}] \quad \text{and} \quad [U_l^{(+)}] + [U_j^{(+)}] = [U_{l+j}^{(+)}] \ , \tag{3.10}$$

for every $l,j \in \mathbb{Z}_{\geq 0}$. If $r \leq 2$ we furthermore have

$$[\Sigma_{g+1}] + [S^2] = [\Sigma_g] \quad \text{and} \quad [\Sigma_{g+1}^{(+)}] + [S^2] = [\Sigma_g^{(+)}] \ , \tag{3.11}$$

for every $g \in \mathbb{Z}_{\geq 0}$.

Theorem 3.8. *The zeroth and first homotopy groups and the k-invariant of $||\mathcal{B}ord_2^{Spin_2^r}||$ are the following:*

r	π_0	π_1	$k : \pi_0 \to \pi_1$
0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	id
> 0, *even*	$\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}/2$	$x \mapsto (0,x)$
> 0, *odd*	$\{0\}$	$\mathbb{Z}$	0

Proof. We start with π_1, which by Theorem 2.5 is $SKK_2^{Spin_2^r}$. Note that the cases $r=1$ and $r=2$ could be treated using Theorem 2.9. Here we present a computation which applies for arbitrary values of r.

By Lemma 2.6 the Euler characteristic is a group homomorphism $\chi : SKK_2^{Spin_2^r} \to \mathbb{Z}$. We claim that the kernel of χ is $\mathcal{T}^{(r)}$ the subgroup generated by r-spin tori. This can be seen by observing that any element in $SKK_2^{Spin_2^r}$ can be brought to the form $[U_l^{(\varepsilon)}] - [U_{l'}^{(\varepsilon)}]$ (or $[U_l] - [U_{l'}]$) for $l, l' \in \mathbb{Z}_{\geq 1}$ if $r > 2$ or to a multiple of $[S^2]$ if $r \leq 2$ up to r-spin tori using Lemma 3.7.

If $r = 0$ then $\chi = 0$ (cf. Example 3.4) and by Lemma 3.6 we have $SKK_2^{Spin_2^0} = \mathbb{Z}/2$.

If $r > 0$ then by Proposition 3.2 the values of χ are divisible by $2\tilde{r}$, where $\tilde{r} = r/\gcd(2, r)$ and $\chi/(2\tilde{r})$ is surjective so we have a short exact sequence

$$\mathcal{T}^{(r)} \hookrightarrow SKK_2^{Spin_2^r} \xrightarrow{\chi/(2\tilde{r})} \mathbb{Z} \; . \tag{3.12}$$

We can define a section of $\chi/(2\tilde{r})$ as follows. Let $j = \varepsilon|j| \in \mathbb{Z}$ and let us define

$$\begin{aligned} \varphi : \mathbb{Z} &\to SKK_2^{Spin_2^r} \\ j &\mapsto \begin{cases} j[S^2] & ; \text{if } r = 1, 2, \\ \varepsilon[U_{|j|}] & ; \text{if } r > 2 \text{ is odd}, \\ \varepsilon[U_{|j|}^{(+)}] & ; \text{if } r > 2 \text{ is even}. \end{cases} \end{aligned} \tag{3.13}$$

By Lemma 3.7 φ is a group homomorphism and it is clearly a section of $\chi/(2\tilde{r})$, so (3.12) splits which together with Lemma 3.6 proves the first part of the theorem.

We continue with computing $\pi_0 = \Omega_1^{Spin_2^r}$. Let $\kappa_i \in \mathbb{Z}/r$ be fixed for $i = 1, \ldots, n$. By (3.1) it is possible to choose g such that

$$2 - 2g - (n+1) + \sum_{i=1}^{n} \kappa_i \equiv \begin{cases} 1 \pmod{r} & ; \text{if } r \text{ is odd}, \\ 0 \text{ or } 1 \pmod{r} & ; \text{if } r \text{ is even}, \end{cases} \tag{3.14}$$

so that there exists an r-spin bordism

$$\bigsqcup_{i=1}^{n} S^1_{\kappa_i} \to S^1_\kappa \tag{3.15}$$

with underlying surface $\Sigma_{g,n+1}$ with n ingoing and one outgoing boundary component and where $\kappa = 1$ for r odd or $\kappa \in \{0, 1\}$ for r even. That is, in $\Omega_1^{Spin_2^r}$ every element is equal to $[S^1_0]$ or to $[S^1_1]$.

Recall that the disc gives an r-spin bordism $S^1_1 \to \emptyset$, so if r is odd then $\Omega_1^{Spin_2^r} = \{0\}$ and if r is even then $\Omega_1^{Spin_2^r}$ is generated by $[S^1_0]$.

It is easy to see from the previous discussion that $[S^1_0]$ has 2-torsion. For the rest of the proof let us assume that r is even. By Lemma 3.5 the k-invariant of $[S^1_0]$ is $[T(0, 0)] \in \pi_1$, which is non-zero showing that $[S^1_0]$ is non-zero. Altogether we get that $\Omega_1^{Spin_2^r} = \mathbb{Z}/2$ for r even.

□

Theorem 3.9. *The group of isomorphism classes of invertible r-spin TFTs with values in $\mathcal{C}$ is given by*

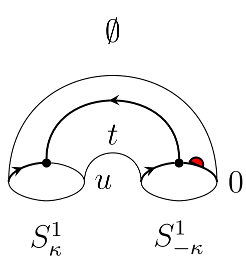
r	$\pi_0(\mathrm{Fun}_\otimes^{\mathrm{inv}}(\mathcal{B}ord_2^{Spin_2^r}, \mathcal{C}))$
0	$\mathrm{Hom}(\mathbb{Z}/2, \pi_0(\mathcal{C}^\times)) \times \mathrm{Ext}(\mathbb{Z}/2, \pi_1(\mathcal{C}^\times))$
> 0, *even*	$\mathrm{Hom}(\mathbb{Z}/2, \pi_0(\mathcal{C}^\times)) \times \mathrm{Ext}(\mathbb{Z}/2, \pi_1(\mathcal{C}^\times)) \times \pi_1(\mathcal{C}^\times)$
> 0, *odd*	$\pi_1(\mathcal{C}^\times)$

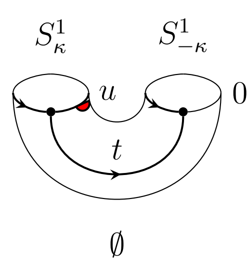
Proof. We use Theorem 2.2 to compute $\pi_0(\mathrm{Fun}_\otimes^{\mathrm{inv}}(\mathcal{B}ord_2^{Spin_2^r}, \mathcal{C}))$. If $r = 0$ then $f_0 \in \mathrm{Hom}(\mathbb{Z}/2, \pi_0(\mathcal{C}^\times))$ determines f_1. If $r > 0$ is even then by (2.3) we can write $f_1 : \mathbb{Z} \times \mathbb{Z}/2 \to \pi_1(\mathcal{C}^\times)$ as $f_1(x, y) = k_{\mathcal{C}^\times} \circ f_0(y) + f_1(x, 0)$. So f_1 is completely determined by $f_0 \in \mathrm{Hom}(\mathbb{Z}/2, \pi_0(\mathcal{C}^\times))$ and by $f_1(1, 0) \in \pi_1(\mathcal{C}^\times)$. If $r > 0$ is odd then $f_0 = 0$ and $f_1 \in Hom(\mathbb{Z}, \pi_1(\mathcal{C}^\times)) \simeq \pi_1(\mathcal{C}^\times)$. In this case $\mathrm{Ext}(\{0\}, \pi_1(\mathcal{C}^\times)) = \{0\}$. □

A Proof of Lemma 3.5

In this section we will use the combinatorial model of r-spin surfaces of [Nov, RS] to prove Lemma 3.5. We do not wish to present all details of the combinatorial model, we just note that it consists of a cell decomposition of the surface in question together with a marking, and refer the reader to [RS, Sec. 2.3]. A marking consists of an edge orientation and an element in $\mathbb{Z}/r$ for each edge and for each face a choice of an edge before glueing the faces along the edges. There are certain moves between different marked cell decompositions which describe isomorphic r-spin surfaces, these can be found in [RS, Fig. 4 and 5].

We continue by introducing some notation. Consider the r-spin cylinder $C^{\mathrm{in}}(\kappa, u)$ (resp. $C^{\mathrm{out}}(\kappa, u)$) with two ingoing (resp. outgoing) boundary components with boundary type κ and $-\kappa$ given by the following marked cell decomposition:

$C^{\mathrm{in}}(\kappa, u) =$ 

$C^{\mathrm{out}}(\kappa, u) =$ 

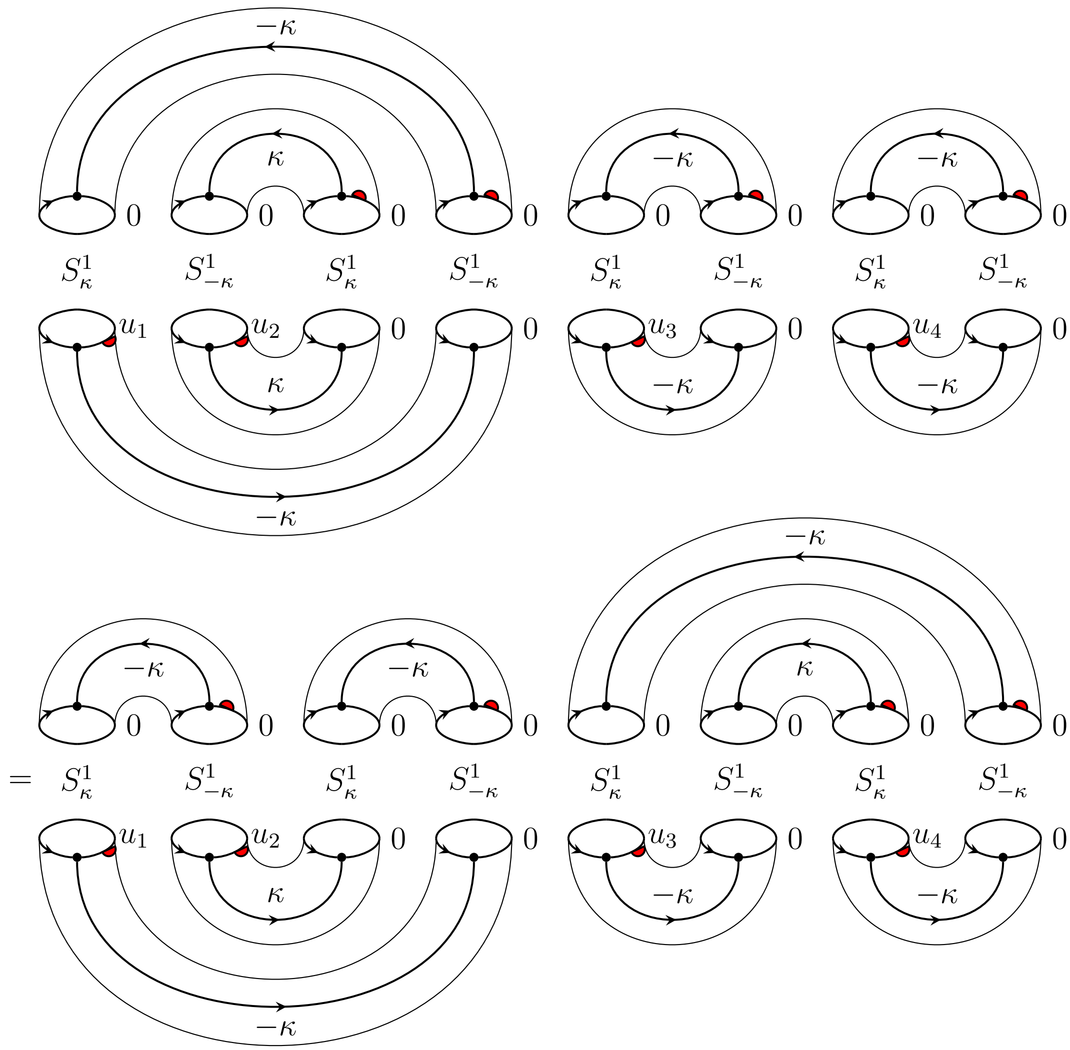
. (A.1)

Let

$$T(\kappa, u) := C^{\mathrm{in}}(\kappa, 0) \circ C^{\mathrm{out}}(\kappa, u) \tag{A.2}$$

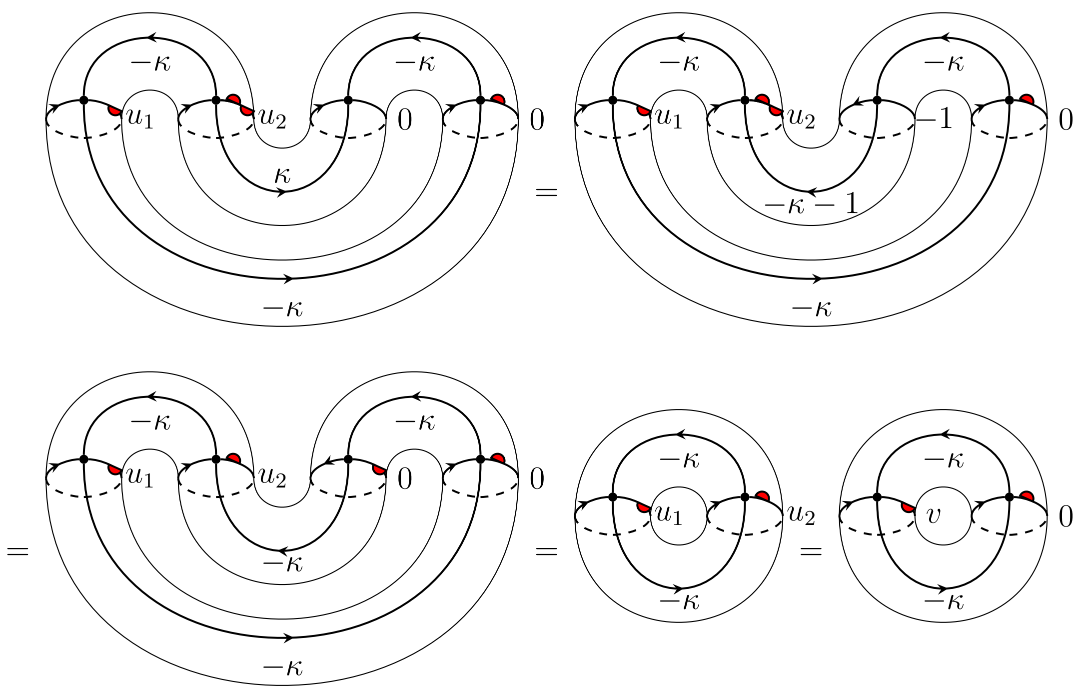
be the r-spin torus obtained from the composition of the cylinders in (A.1).

Proof of Lemma 3.5. We start by proving (3.4). Let us consider the equation in $SKK_2^{Spin_2^r}$:



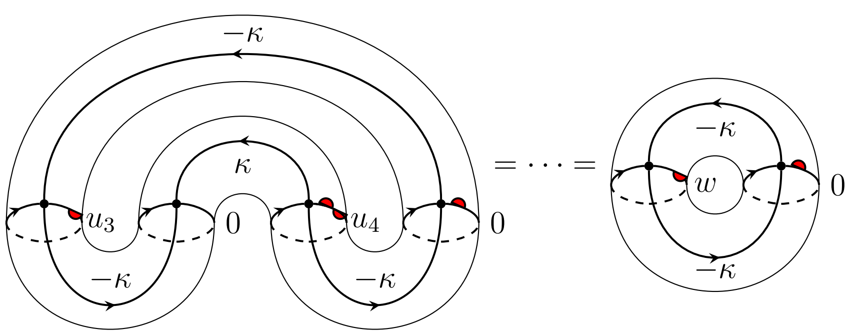
(A.3)

The first term on the right hand side of (A.3) is



(A.4)

with $v = u_1 - u_2$, where we used the moves of [RS, Fig. 4 and 5]. The second term on the right hand side of (A.3) is



(A.5)

with $w = u_3 + u_4$. Finally, using the $SL(2,\mathbb{Z})$ action on the set of isomorphism classes of r-spin structures on the torus, we have $[T(-\kappa, u_2)] = [T(\kappa, -u_2)]$. Putting all the above together we obtain (3.4).

We continue with proving (3.5). Pick arbitrary r-spin structures on $\Sigma_{1,1}^{\mathrm{in}}$ and on $\Sigma_{1,1}^{\mathrm{out}}$.

Now consider the equation in $SKK_2^{Spin_2^r}$:

$$\begin{aligned} &\left(\Sigma_{0,1}^{\mathrm{in}} \sqcup \Sigma_{1,1}^{\mathrm{in}} \sqcup C^{\mathrm{in}}(1,0)\right) \circ \left(\Sigma_{1,1}^{\mathrm{out}} \sqcup \Sigma_{0,1}^{\mathrm{out}} \sqcup C^{\mathrm{out}}(1,u)\right) \\ =& \left(C^{\mathrm{in}}(1,0) \sqcup \Sigma_{0,1}^{\mathrm{in}} \sqcup \Sigma_{1,1}^{\mathrm{in}}\right) \circ \left(C^{\mathrm{out}}(1,u) \sqcup \Sigma_{1,1}^{\mathrm{out}} \sqcup \Sigma_{0,1}^{\mathrm{out}}\right) \ , \end{aligned} \tag{A.6}$$

where we cut along $S^1_{-1} \sqcup S^1_1 \sqcup S^1_{-1} \sqcup S^1_1$. Notice, since up to isomorphism there is a unique r-spin structure on the disc the two leftmost r-spin tori of the left hand side are equal to the two r-spin tori on the right hand side, giving (3.5) directly.

Finally we compute the k-invariant of the r-spin circle $[S^1_\kappa]$ for $\kappa \in \mathbb{Z}/r$. The duality morphisms for S^1_κ are $C^{\mathrm{in}}(\kappa,0)$ and $C^{\mathrm{out}}(\kappa,0)$ from (A.1). The composition

$$k([S^1_\kappa]) = [C^{\mathrm{in}}(\kappa,0) \circ \sigma_{S^1_{-\kappa},S^1_\kappa} \circ C^{\mathrm{out}}(\kappa,0)] \tag{A.7}$$

is an r-spin torus $[T(\kappa,0)]$ after applying a Dehn-twist.

□

References

[Ati] M.F. Atiyah, *Topological quantum field theory.* Publications Mathématiques de l'IHÉS **68** (1988) 175–186.

[FH] D.S. Freed and M.J. Hopkins, *Reflection positivity and invertible topological phases.* 1604.06527 [hep-th].

[JO] N. Johnson and A.M. Osorno, *Modeling stable one-types.* Theory and Applications of Categories **26** (2012) 520–537, [1201.2686 [math.AT]].

[JS] A. Joyal and R. Street, *Braided tensor categories.* Adv. Math. **20** (1993) 102.

[Kap] A. Kapustin, *Symmetry Protected Topological Phases, Anomalies, and Cobordisms: Beyond Group Cohomology.* 1403.1467 [cond-mat.str-el].

[KKNO] U. Karras, M. Kreck, W.D. Neumann, and E. Ossa. *Cutting and pasting of manifolds: SK-groups.* Publish or Perish, inc., (1973).

[KST] M. Kreck, S. Stolz, and P. Teichner. *Invertible topological field theories are SKK invariants.* (To appear).

[Nov] S. Novak. *Lattice topological field theories in two dimensions.* PhD thesis, Universität Hamburg, 2015, http://ediss.sub.uni-hamburg.de/volltexte/2015/7527.

[RS] I. Runkel and L. Szegedy, *Topological field theory on r-spin surfaces and the Arf invariant.* 1802.09978 [math.QA].

[Seg1] G.B. Segal. *The Definition Of Conformal Field Theory*. In *Differential Geometrical Methods in Theoretical Physics*, pages 165–171. Kluwer, 1988.

[Seg2] G.B. Segal. *Two-Dimensional Conformal Field Theories And Modular Functors*. In *Mathematical physics*, pages 22–37. Swansea, 1988.

[Sín] H.X. Sính. *Gr-catégories*. PhD thesis, Université Paris VII, 1975.

[ST] S. Stolz and P. Teichner. *Supersymmetric field theories and generalized cohomology*. In H. Sati and U. Schreiber, editors, *Mathematical Foundations of Quantum Field Theory and Perturbative String Theory*, volume 83, pages 279–340. AMS, 2011, [1108.0189 [math.AT]].

[Tur] V. Turaev. *Homotopy Quantum Field Theory*. European Mathematical Society, (2010). With Appendices by M. Müger and A. Virelizier.