# Generalized connections, spinors, and integrability of generalized structures on Courant algebroids 

Vicente Cortés and Liana David

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#### Abstract

We present a characterization, in terms of torsion-free generalized connections, for the integrability of various generalized structures (generalized almost complex structures, generalized almost hypercomplex structures, generalized almost Hermitian structures and generalized almost hyper-Hermitian structures) defined on Courant algebroids. We develop a new, self-contained, approach for the theory of Dirac generating operators for regular Courant algebroids. As an application we provide a criterion for the integrability of generalized almost Hermitian structures and generalized almost hyper-Hermitian structures defined on a regular Courant algebroid $E$, in terms of canonically defined differential operators on spinor bundles associated to $E$.


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## 1 Introduction

Generalized complex geometry is a well established field in present mathematics. It unifies complex and symplectic geometry and represents an important direction of current research in differential geometry and theoretical physics. In generalized geometry, the role of the tangent bundle $T M$ of a manifold $M$ is played by the generalized tangent bundle $\mathbb{T} M:=T M \oplus T^{*} M$, or, more generally, by a Courant algebroid. Many classical objects from differential geometry (including almost complex, almost Hermitian, almost hypercomplex and almost hyper-Hermitian structures) were defined and studied by several authors in this more general setting, see e.g. [14] (also [4, 5, 6]). (Generalized almost hypercomplex structures seem to have received less atention in the literature, as they were usually considered as part of a generalized almost hyper-Hermitian structure.)

While the integrability of such generalized structures is defined and understood in terms of the Courant bracket of the Courant algebroid (leading to the notions of generalized complex, generalized hypercomplex, generalized Kähler and generalized hyper-Kähler structures), it seems that characterizations in terms of torsion-free generalized connections (analogous to the standard characterizations of integrability from classical geometry) are missing.

In the first part of the present paper we fill this gap by answering this natural question. We shall do this by a careful analysis of the space of generalized connections which are 'adapted', i.e. preserve, a given generalized structure $Q$ on a Courant algebroid $E$. In analogy with the classical case, we introduce the notion of intrinsic torsion $t_{Q}$ of $Q$, see Definition 15. This will play an important role in our treatment, when $Q$ is a generalized almost complex structure or a generalized almost hypercomplex structure. We compute the intrinsic torsion for these structures and we prove that their integrability is equivalent to the existence of a torsion-free adapted generalized connection, see Theorem [23 and Theorem 33, In the same framework, we prove that a generalized almost Hermitian (respectively, almost hyper-Hermitian) structure on a Courant algebroid is generalized Kähler (respectively, hyper-

Kähler) if and only if it admits a torsion-free adapted generalized connection, see Theorem 37 and Theorem 41. Our treatment shows that in generalized geometry the torsion-free condition on a generalized connection $D$ adapted to a generalized structure $Q$ does in general no longer determine $D$ uniquely, even if the classical structure generalized by $Q$ has a unique torsion-free connection. This was noticed already in [11, 9] for generalized Riemannian metrics: for a given generalized Riemannian metric $G$, there is an entire family of generalized connections which are torsion-free and preserve $G$. We will show that the same holds for the structures considered in this paper, for instance, for generalized hypercomplex structures: the Obata connection [20, 3] (see also [12]) of a hypercomplex structure is replaced in generalized geometry by an entire family of generalized connections.

In more general terms, the theory developed here allows to decide whether a given generalized structure $Q$ on a Courant algebroid admits an adapted generalized connection with prescribed torsion and to describe the space of all such generalized connections, see Proposition 17, A torsion-free generalized connection adapted to $Q$, for instance, exists if and only if the intrinsic torsion of $Q$ vanishes and is unique if and only if the generalized first prolongation $\mathfrak{h}^{\langle 1\rangle}$ of the Lie algebra $\mathfrak{h}$ of the structure group is zero. As a further application of the generalized first prolongation we present an alternative proof for the uniqueness of the canonical connection of a Born structure defined in [8] (see Section (2.7).

In the second part of this paper we define a canonical (i.e. independent on the choice of generalized connection) Dirac generating operator on a Courant algebroid $E$. We restrict to the case when $E$ is regular, i.e. its anchor has constant rank. Regular Courant algebroids form an important class of Courant algebroids, which was studied systematically in [7]. A Dirac generating operator is a first order odd differential operator on a suitable irreducible graded spinor bundle which encodes the anchor and the Dorfman bracket of the Courant algebroid. We recover the crucial result of [1] (see also [13]) namely that any regular Courant algebroid $E$ admits a canonical Dirac generating operator $\not d$ and we express $\not d$ in terms of a dissection of $E$. Such an expression for the canonical Dirac generating operator does not seem to exist in the literature and allows a better understanding of the relation between $\not d$ and the structure of the regular Courant algebroid.

The third part of the paper is devoted to applications. Owing to the failure of uniqueness of generalized connections adapted to various generalized structures $Q$ on a regular Courant algebroid $E$ it is natural to search for characterizations of integrability of $Q$ which involve the canonical Dirac generating operator of $E$. This was done in [1] when $Q=\mathcal{J}$ is a generalized almost complex structure. Here we present a criterion for a generalized al-
most Hermitian structure $(G, \mathcal{J})$ on $E$ to be generalized Kähler in terms of two canonical Dirac operators and the pure spinors associated to $\left.\mathcal{J}\right|_{E_{ \pm}}$(where $E_{ \pm}$are the subbundles of $E$ determined by the generalized metric $G$ ), see Theorem 74. Our arguments use, besides the theory of Dirac generating operators, the results on generalized connections adapted to a generalized almost Hermitian structure, developed in the first part of the paper. A similar spinorial characterization for generalized hyper-Kähler structures on regular Courant algebroids is obtained in Corollary 76.

In the appendix, intended for completeness of our exposition, we briefly recall basic facts we need on the theory of $\mathbb{Z}_{2}$-graded algebras. We also recall, following [1], the integrability criterion for generalized almost complex structures on regular Courant algebroids mentioned above.

We end this introduction with various remarks on our assumptions. Along the paper we assume that the scalar product $\langle\cdot, \cdot\rangle$ of the Courant algebroid $E$ has neutral signature $(n, n)$. This simplifies the construction of a Dirac generating operator. In fact, the spinor bundles $S$ on which the Dirac generating operators act are then irreducible $\mathbb{Z}_{2}$-graded $\mathrm{Cl}(E)$-bundles, with the essential property that any $\mathrm{Cl}(E)$-bundle morphism $f: S \rightarrow S$ is a multiple of the identity. (More generally, considering signatures $(t, s)$ with $t-s \equiv 0$ or $2(\bmod 8)$, would ensure the same property, where $t$ stands for the index and the Clifford relation is $v^{2}=\langle v, v\rangle$.) Similar considerations in Part III of the paper constrain further the rank of $E$ to be a multiple of 16 , which is used in our characterization of the integrability of generalized almost Hermitian structures in terms of two other Dirac operators. It would be interesting to extend the theory developed in this paper to Courant algebroids of other ranks and signatures.

## Part I

## 2 Preliminary material

We start by reviewing the basic definitions we need on Courant algebroids, generalized connections, their torsion (see [9]) and the definition of the generalized structures we shall consider in this paper. We prove a basic property of the Nijenhuis tensor of a generalized almost complex structure (see Lemma 9), we introduce the notion of intrinsic torsion of a generalized structure $Q$ on a Courant algebroid and we describe the space of generalized connections
adapted to $Q$ as an affine space modeled on the space of sections of a vector bundle (see Proposition [17). The fibers of the latter vector bundle are isomorphic to the generalized first prolongation $\mathfrak{h}^{(1)}$ of the structure group $H$, a notion which will be defined for any Lie subgroup $H \subset \mathrm{O}(k, \ell)$. In our applications we will assume that the signature $(k, \ell)$ of the scalar product $\langle\cdot, \cdot\rangle$ of the Courant algebroid $E$ is of the form $(n, n)$, see Definition $\square$ below.

### 2.1 Courant algebroids

Definition 1. A Courant algebroid on a manifold $M$ is a vector bundle $E \rightarrow M$ equipped with a non-degenerate symmetric bilinear form of neutral signature $\langle\cdot, \cdot\rangle \in \Gamma\left(\operatorname{Sym}^{2}\left(E^{*}\right)\right)$ (called the scalar product), a bilinear operation $[\cdot, \cdot]$ (called the Dorfman bracket) on the space of smooth sections $\Gamma(E)$ of $E$ and a homomorphism of vector bundles $\pi: E \rightarrow T M$ (called the anchor), such that the following conditions are satisfied: for all sections $u, v, w \in \Gamma(E)$,

C1) $[u,[v, w]]=[[u, v], w]+[v,[u, w]]$;
C2) $\pi([u, v])=[\pi(u), \pi(v)]$;
C3) $[u, f v]=\pi(u)(f) v+f[u, v]$;
C4) $\pi(u)\langle v, w\rangle=\langle[u, v], w\rangle+\langle v,[u, w]\rangle$;
C5) $2[u, u]=\pi^{*} d\langle u, u\rangle$.
A Courant algebroid is called regular if the anchor $\pi$ has constant rank.
Here $\pi^{*}: T^{*} M \rightarrow E$ denotes the map obtained by dualizing $\pi: E \rightarrow T M$ and identifying $E^{*}$ with $E$ using the scalar product. Therefore C5) can be written in the equivalent way

$$
\begin{equation*}
2\langle[u, u], v\rangle=\pi(v)\langle u, u\rangle . \tag{1}
\end{equation*}
$$

Remark 2. As already pointed out in [22, 2], the axioms of a Courant algebroid can be reduced. One can show that axioms C2) and C3) from the definition of Courant algebroids follow from the other axioms. In fact, C2) can be checked by calculating $[\pi(u), \pi(v)]\langle w, e\rangle$, for any $w, e \in \Gamma(E)$, with the help of C4) and C1). Similarly, C3) can be checked by taking the scalar product with a section $w \in \Gamma(E)$ and evaluating the result with help of C 4 ).

Example 3. The fundamental example of a Courant algebroid is the generalized tangent bundle $\mathbb{T} M=T M \oplus T^{*} M$ of a smooth manifold $M$ with the canonical projection $\mathbb{T M} \rightarrow$ TM as anchor, the scalar product defined by $\langle X+\xi, Y+\eta\rangle=\frac{1}{2}(\xi(Y)+\eta(X))$, and Dorfman bracket by

$$
\begin{equation*}
[X+\xi, Y+\eta]=[X, Y]+\mathcal{L}_{X} \eta-\mathcal{L}_{Y} \xi+d(\xi(Y))+H(X, Y, \cdot) \tag{2}
\end{equation*}
$$

where $H \in \Omega^{3}(M)$ is a closed 3 -form and $X, Y \in \Gamma(T M), \xi, \eta \in \Gamma\left(T^{*} M\right)$. On the right-hand side, $[X, Y]$ stands for the usual Lie bracket of vector fields and $\mathcal{L}_{X} \eta$ for the Lie derivative of $\eta$ in the direction of $X$. It is known that every Courant algebroid for which the sequence $0 \rightarrow T^{*} M \xrightarrow{\pi^{*}} E \xrightarrow{\pi} T M \rightarrow 0$ is exact is isomorphic to a Courant algebroid of this form. Such Courant algebroids are called exact.

### 2.2 E-connections and generalized connections

Unless otherwise stated, $E$ will denote a Courant algebroid, with anchor $\pi$, scalar product $\langle\cdot, \cdot\rangle$ and Dorfman bracket $[\cdot, \cdot]$. Let $V \rightarrow M$ be a vector bundle.

Definition 4. i) An E-connection on $V$ is an $\mathbb{R}$-linear map

$$
D: \Gamma(V) \rightarrow \Gamma\left(E^{*} \otimes V\right), \quad v \mapsto D v
$$

which satisfies the following Leibniz rule

$$
D_{e}(f v)=\pi(e)(f) v+f D_{e} v
$$

for all $e \in \Gamma(E), v \in \Gamma(V)$, where $D_{e} v=(D v)(e)$.
ii) An $E$-connection on $E$ is called a generalized connection if is compatible with the scalar product:

$$
\begin{equation*}
\pi(u)\langle v, w\rangle=\left\langle D_{u} v, w\right\rangle+\left\langle v, D_{u} w\right\rangle \tag{3}
\end{equation*}
$$

for all $u, v, w \in \Gamma(E)$.
Example 5. A connection $\nabla$ on a vector bundle $V \rightarrow M$ induces an $E$ connection $D$ on $V$ by the formula

$$
D_{e} v:=\nabla_{\pi(e)} v,
$$

for all $e \in \Gamma(E), v \in \Gamma(V)$. If $\nabla$ is a connection on $E$, then the induced $E$-connection is a generalized connection if and only if $\langle\cdot, \cdot\rangle$ is $\nabla$-parallel along $\pi E$, that is $\nabla_{X}\langle\cdot, \cdot\rangle=0$ for all $X \in \pi(E)$. Conversely, if the Courant algebroid $E$ is regular, an $E$-connection $D$ on $V$ which satisfies $D_{e}=0$ for all $e \in \operatorname{ker} \pi$ is induced by a connection $\nabla$ on $V$ and a similar statement holds for generalized connections. In fact, an E-connection $D$ on $V$ such that $D_{e}=0$ for all $e \in \operatorname{ker} \pi$ induces a partial connection $\nabla^{F}: \Gamma(F) \times \Gamma(V) \rightarrow$

[^1]$\Gamma(V)$ along the distribution $F=\pi(E) \subset T M$ such that $D_{e}=\nabla_{\pi(e)}^{F}$ for all $e \in E$. Choosing a partial connection $\nabla^{F^{\prime}}: \Gamma\left(F^{\prime}\right) \times \Gamma(V) \rightarrow \Gamma(V)$ along a complementary distribution $F^{\prime} \subset T M$, we can define a connection $\nabla$ such that $D_{e}=\nabla_{\pi(e)}$ for all $e \in E$ by $\nabla_{X+Y}=\nabla_{X}^{F}+\nabla_{Y}^{F^{\prime}}$ for all $X \in F, Y \in F^{\prime}$. If $V=E$ is a regular Courant algebroid and $D$ happens to be a generalized connection on $E$ then the partial connection $\nabla^{F}$ is metric and we can choose $\nabla^{F^{\prime}}$ (and hence $\nabla$ ) to be metric as well.
Definition 6. The torsion $T^{D} \in \Gamma\left(\wedge^{2} E^{*} \otimes E\right)$ of a generalized connection $D$ is defined by
\[

$$
\begin{equation*}
T^{D}(u, v)=D_{u} v-D_{v} u-[u, v]+(D u)^{*} v, \forall u, v \in \Gamma(E), \tag{4}
\end{equation*}
$$

\]

where $(D u)^{*}$ is the metric adjoint of $D u \in \Gamma(\operatorname{End} E)$ with respect to $\langle\cdot, \cdot\rangle$.
Note that the tensoriality of $T^{D}(u, v)$ in $v$ is obvious since the operators $D_{u}$ and $[u, \cdot]$ satisfy the same Leibniz rule. The tensoriality in $u$ is a consequence of the skew-symmetry of $T^{D}$, which follows from

$$
\begin{equation*}
\langle[u, v]+[v, u], w\rangle=\pi(w)\langle u, v\rangle \tag{5}
\end{equation*}
$$

(polarization of (1)) and the compatibility of $D$ with the scalar product. Moreover, one can check that $T^{D}(u, v, w):=\left\langle T^{D}(u, v), w\right\rangle$ is totally skewsymmetric. From (4),

$$
\begin{align*}
& \langle[u, v], w\rangle=-T^{D}(u, v, w)+\left\langle D_{u} v-D_{v} u, w\right\rangle+\left\langle D_{w} u, v\right\rangle, \\
& \left\langle[u, v]_{C}, w\right\rangle=-T^{D}(u, v, w)+\left\langle D_{u} v-D_{v} u, w\right\rangle+\frac{1}{2}\left(\left\langle D_{w} u, v\right\rangle-\left\langle D_{w} v, u\right\rangle\right), \tag{6}
\end{align*}
$$

where $[u, v]_{C}:=\frac{1}{2}([u, v]-[v, u])$ is the Courant bracket.
We often identify $E$ with $E^{*}$ using the scalar product $\langle\cdot, \cdot\rangle$. By means of this identification, $\Lambda^{2} E \subset$ End $E$ is the subbundle of skew-symmetric endomorphisms of $E$, where

$$
\left(e_{1} \wedge e_{2}\right)\left(e_{3}\right):=\left\langle e_{1}, e_{3}\right\rangle e_{2}-\left\langle e_{2}, e_{3}\right\rangle e_{1}, e_{i} \in E .
$$

With this convention, any two generalized connections are related by $\tilde{D}=$ $D+\eta$, for $\eta \in \Gamma\left(E^{*} \otimes \Lambda^{2} E\right)$ and

$$
\begin{equation*}
T^{\tilde{D}}(u, v, w)=T^{D}(u, v, w)+\eta(u, v, w)+\eta(w, u, v)+\eta(v, w, u) . \tag{7}
\end{equation*}
$$

The space of torsion-free generalized connections is non-empty, see [9]. If $D$ and $\tilde{D}$ are torsion-free (or, more generally, have the same torsion), then $\eta$ is of the form

$$
\eta(u, v, w)=\sigma(u, v, w)-\sigma(u, w, v)
$$

for a section $\sigma \in \Gamma\left(S^{2} E \otimes E\right)$, see [9.

### 2.3 Generalized Riemannian metrics

We assume that $\operatorname{rank} E=2 n$.
Definition 7. A generalized Riemannian metric on $E$ is a rank $n$ vector subbundle $E_{+}$of $E$ on which $\langle\cdot, \cdot\rangle$ is positive definite.

The restriction of $\langle\cdot, \cdot\rangle$ to the orthogonal complement (with respect to $\langle\cdot, \cdot\rangle) E_{-}:=E_{+}^{\perp}$ is negative definite and $G:=\left.\langle\cdot, \cdot\rangle\right|_{E_{+} \times E_{+}}-\left.\langle\cdot, \cdot\rangle\right|_{E_{-} \times E_{-}}$is a positive definite metric on $E$.

Alternatively, a generalized Riemannian metric on a Courant algebroid $E$ can be defined as a positive definite metric $G$ on the vector bundle $E$ such that the endomorphism $G^{\text {end }}$, defined by $G(u, v)=\left\langle G^{\text {end }} u, v\right\rangle$, satisfies $\left(G^{\text {end }}\right)^{2}=\operatorname{Id}_{E}$. The bundles $E_{ \pm}$are the $\pm 1$-eigenbundles of $G^{\text {end }}$. Along the paper we denote by $e_{ \pm}:=\frac{1}{2}\left(\operatorname{Id} \pm G^{\text {end }}\right) e$ the $E_{ \pm}$-components of a vector $e \in E$ in the decomposition $E=E_{+} \oplus E_{-}$determined by a generalized metric $G$.

Given a generalized metric $G$ on $E$, there is always a torsion-free generalized connection which preserves $G$ (see [11, 9]). Such a connection is not unique if $n>1$ and is called a Levi-Civita connection of $G$. The nonuniqueness is due to the fact that although the first prolongation $\mathfrak{s o}(k, \ell)^{(1)}$ is always trivial (which is responsible for the uniqueness of the Levi-Civita connection of a pseudo-Riemannian manifold), the generalized first prolongation $\mathfrak{s o}(k, \ell)^{\langle 1\rangle}$ (see Definition 16 below) is non-trivial if $k+\ell>1$.

### 2.4 Generalized complex and hyper-complex structures

Definition 8. $A$ generalized almost complex structure on $E$ is an endomorphism $\mathcal{J} \in \Gamma(\operatorname{End} E)$ which satisfies $\mathcal{J}^{2}=-\operatorname{Id}_{E}$ and is orthogonal with respect to the scalar product $\langle\cdot, \cdot \cdot\rangle$ of $E$. We say that $\mathcal{J}$ is integrable (or is a generalized complex structure) if its Nijenhuis tensor

$$
\begin{equation*}
N_{\mathcal{J}}(u, v):=[\mathcal{J} u, \mathcal{J} v]_{C}-[u, v]_{C}-\mathcal{J}\left([\mathcal{J} u, v]_{C}+[u, \mathcal{J} v]_{C}\right) \tag{8}
\end{equation*}
$$

vanishes identically.
Let $L \subset E_{\mathbb{C}}$ be the (1,0)-bundle (i.e. the $i$-eigenbundle) of $\mathcal{J}$. Since $L$ is (totally) isotropic, the Dorfman bracket $[\cdot, \cdot]$ coincides with the Courant bracket $[\cdot, \cdot]_{C}$ when restricted to $L$ (from relation (5)). The integrability condition in the above definition is equivalent to the condition

$$
\begin{equation*}
[\Gamma(L), \Gamma(L)]=[\Gamma(L), \Gamma(L)]_{C} \subset \Gamma(L) . \tag{9}
\end{equation*}
$$

The following simple lemma will be useful for us.

Lemma 9. Let $\mathcal{J}$ be a generalized almost complex structure on $E$. Then $N_{\mathcal{J}}(u, v, w):=\left\langle N_{\mathcal{J}}(u, v), w\right\rangle$ is a 3-form on $E$.
Proof. From its definition, $N_{\mathcal{J}}(u, v, w)$ is skew-symmetric in the first two arguments. We now prove that it is skew-symmetric also in the last two arguments. From the definition of the Nijenhuis tensor,

$$
\begin{align*}
& 2\left(N_{\mathcal{J}}(u, v, w)+N_{\mathcal{J}}(u, w, v)\right) \\
& =\langle[\mathcal{J} u, \mathcal{J} v]-[\mathcal{J} v, \mathcal{J} u]-[u, v]+[v, u], w\rangle \\
& +\langle[\mathcal{J} u, v]-[v, \mathcal{J} u]+[u, \mathcal{J} v]-[\mathcal{J} v, u], \mathcal{J} w\rangle \\
& +\langle[\mathcal{J} u, \mathcal{J} w]-[\mathcal{J} w, \mathcal{J} u]-[u, w]+[w, u], v\rangle \\
& +\langle[\mathcal{J} u, w]-[w, \mathcal{J} u]+[u, \mathcal{J} w]-[\mathcal{J} w, u], \mathcal{J} v\rangle . \tag{10}
\end{align*}
$$

Using axiom C4) from the definition of Courant algebroids, we obtain

$$
\begin{aligned}
& \langle[\mathcal{J} u, \mathcal{J} v]-[u, v], w\rangle+\langle[\mathcal{J} u, v]+[u, \mathcal{J} v], \mathcal{J} w\rangle \\
& +\langle[\mathcal{J} u, \mathcal{J} w]-[u, w], v\rangle+\langle[\mathcal{J} u, w]+[u, \mathcal{J} w], \mathcal{J} v\rangle \\
& =\pi(\mathcal{J} u)\langle\mathcal{J} v, w\rangle+\pi(\mathcal{J} u)\langle v, \mathcal{J} w\rangle-\pi(u)\langle v, w\rangle+\pi(u)\langle\mathcal{J} v, \mathcal{J} w\rangle=0
\end{aligned}
$$

and relation (10) reduces to

$$
\begin{align*}
& 2\left(N_{\mathcal{J}}(u, v, w)+\left\langle N_{\mathcal{J}}(u, w, v)\right)\right. \\
& =-\langle[\mathcal{J} v, \mathcal{J} u]-[v, u], w\rangle-\langle[v, \mathcal{J} u]+[\mathcal{J} v, u], \mathcal{J} w\rangle \\
& -\langle[\mathcal{J} w, \mathcal{J} u]-[w, u], v\rangle-\langle[w, \mathcal{J} u]+[\mathcal{J} w, u], \mathcal{J} v\rangle . \tag{11}
\end{align*}
$$

From (5), we obtain

$$
\begin{equation*}
\langle[u, v], w\rangle=\pi(w)\langle u, v\rangle-\langle[v, u], w\rangle . \tag{12}
\end{equation*}
$$

Using (12), we rearrange the right hand side of (11) to contain all brackets with either $u$ or $\mathcal{J} u$ as the first argument. After cancelling terms,

$$
\begin{align*}
& 2\left(N_{\mathcal{J}}(u, v, w)+\left\langle N_{\mathcal{J}}(u, w, v)\right)\right. \\
& =\langle[\mathcal{J} u, \mathcal{J} v]-[u, v], w\rangle+\langle[\mathcal{J} u, v]+[u, \mathcal{J} v], \mathcal{J} w\rangle \\
& +\langle[\mathcal{J} u, \mathcal{J} w]-[u, w], v\rangle+\langle[\mathcal{J} u, w]+[u, \mathcal{J} w], \mathcal{J} v\rangle . \tag{13}
\end{align*}
$$

From C4) again,

$$
\begin{aligned}
2\left(N_{\mathcal{J}}(u, v, w)+\left\langle N_{\mathcal{J}}(u, w, v)\right)\right. & =\pi(\mathcal{J} u)\langle\mathcal{J} v, w\rangle-\pi(u)\langle v, w\rangle \\
& +\pi(\mathcal{J} u)\langle v, \mathcal{J} w\rangle+\pi(u)\langle\mathcal{J} v, \mathcal{J} w\rangle,
\end{aligned}
$$

which vanishes. We have proved that $N_{\mathcal{J}}=N_{\mathcal{J}}(u, v, w)$ is completely skewsymmetric. As it is $C^{\infty}(M)$-linear in $w$, it is $C^{\infty}(M)$-linear also in $u, v$.
Definition 10. A generalized almost hypercomplex structure on $E$ is a triple $\left(\mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ of anti-commuting generalized almost complex structures such that $\mathcal{J}_{3}=\mathcal{J}_{1} \mathcal{J}_{2}$. We say that $\left(\mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ is integrable (is a generalized hypercomplex structure) if all $\mathcal{J}_{i}$ are generalized (integrable) complex structures.

### 2.5 Generalized Kähler and hyper-Kähler structures

Definition 11. A generalized almost Hermitian structure on $E$ is a pair $(G, \mathcal{J})$ where $G$ is a generalized Riemannian metric and $\mathcal{J}$ a generalized almost complex structure on $E$, such that $G(\mathcal{J} u, \mathcal{J} v)=G(u, v)$ for all $u, v \in$ E.

The endomorphism $G^{\text {end }}$ of $E$ determined by $G$ (see Section 2.3) commutes with $\mathcal{J}_{1}:=\mathcal{J}$ and $\mathcal{J}_{2}:=G^{\text {end }} \mathcal{J}$ is a generalized almost complex structure. The generalized almost complex structures $\mathcal{J}_{1}$ and $\mathcal{J}_{2}$ preserve $E_{ \pm}$(the $\pm 1$-eigenbundles of $G^{\text {end }}$ ). Moreover, $\mathcal{J}_{2}= \pm \mathcal{J}_{1}$ on $E_{ \pm}$.

Definition 12. A generalized almost Hermitian structure $(G, \mathcal{J})$ is called a generalized Kähler structure if $\mathcal{J}_{1}=\mathcal{J}$ and $\mathcal{J}_{2}=G^{\text {end }} \mathcal{J}_{1}$ are integrable.

It can be shown that $(G, \mathcal{J})$ is a generalized Kähler structure if and only if $L \cap\left(E_{ \pm}\right)_{\mathbb{C}}$ are closed under the bracket $[\cdot, \cdot]$ of $E$ (see e.g. Proposition 2.17 of [15]), where $L$ denotes the ( 1,0 )-bundle of $\mathcal{J}$.

Definition 13. i) $A$ generalized almost hyper-Hermitian structure is a generalized almost hypercomplex structure $\left(\mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ together with a generalized Riemannian metric $G$, such that $\left(G, \mathcal{J}_{i}\right)$ is a generalized almost Hermitian structure, for all $i=1,2,3$.
ii) A generalized almost hyper-Hermitian structure $\left(G, \mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ is a generalized hyper-Kähler structure if $\left(G, \mathcal{J}_{i}\right)$ is a generalized Kähler structure, for all $i=1,2,3$.

### 2.6 Intrinsic torsion of a generalized structure

Given a linear Lie group $H \subset \mathrm{O}(n, n)$, a generalized $H$-structure on a Courant algebroid $E$ of rank $2 n$ is a reduction of the structure group of $E$ from $\mathrm{O}(n, n)$ to $H$. Along this section by a generalized $H$-structure we mean a generalized $H$-structure defined by a system of tensor fields. Examples of such structures considered in this paper are:

1. generalized almost complex structures $Q=\mathcal{J}, H=\mathrm{U}(m, m), n=2 m$,
2. generalized almost hypercomplex structures $Q=\left(\mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$, $H=$ $\mathrm{Sp}(m, m), n=4 m$,
3. generalized Riemannian metrics $Q=G, H=\mathrm{O}(n) \times \mathrm{O}(n)$,
4. generalized almost Hermitian structures $Q=(G, \mathcal{J}), H=\mathrm{U}(m) \times$ $\mathrm{U}(m), n=2 m$, and
5. generalized almost hyper-Hermitian structures $Q=\left(G, \mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$, $H=\operatorname{Sp}(m) \times \operatorname{Sp}(m), n=4 m$.

We will refer to these simply as generalized structures $Q$.
In analogy with the classical case (see e.g. [21]), given a generalized $H$ structure $Q$ on $E$ we may consider the space of generalized connections $D$ which preserve, or are adapted, to $Q$ (i.e. $D Q=0$ ). A generalized connection adapted to a generalized almost complex (respectively, hypercomplex) structure $Q=\mathcal{J}$ (respectively, $\left.Q=\left(\mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)\right)$ will be called complex (respectively, hypercomplex).

Lemma 14. Any generalized $H$-structure $Q$ on a Courant algebroid $E$ admits an adapted generalized connection.

Proof. There exists a connection $\nabla$ in the vector bundle $E$ for which the tensor fields $T_{i}, i \in I$ from $Q$ and the scalar product $\langle\cdot, \cdot\rangle$ of $E$ are parallel. To see this it suffices to observe that by assumption the $H$-structure consists precisely of all frames in which the tensor fields $T_{i}$ and $\langle\cdot, \cdot\rangle$ are represented by a certain system of tensors in $\mathbb{R}^{2 n}$ (independent of the base point $p \in M$ of the frame). The stabilizer of the latter system in GL( $2 n, \mathbb{R}$ ) is precisely the group $H$. By choosing a local frame in the $H$-structure and considering the connection for which this frame is parallel, we obtain a locally defined connection. Such connections can be glued by a partition of unity to a globally defined connection $\nabla$ with the claimed property. Extending $\nabla$ to a generalized connection as in Example 5 yields an adapted generalized connection.

The space of generalized connections adapted to a generalized $H$-structure $Q$ is an affine space, modelled on the vector space of sections of the bundle $E^{*} \otimes \operatorname{ad}_{Q}(E)$, where $\operatorname{ad}_{Q}(E)$ denotes the bundle of 2-forms $\alpha \in \Lambda^{2} E^{*}$, adapted to $Q$. By the latter condition we mean that $\mathrm{ad}_{\alpha}$ annihilates the tensor fields from $Q$, when $\alpha$ is considered as an endomorphism of $E$. When $Q$ belongs to the above list, we require that $[\alpha, \mathcal{J}]=0$ and $\left[\alpha, G^{\text {end }}\right]=0$ when $\mathcal{J}, G \in Q$. These conditions are equivalent to $\mathcal{J}^{*} \alpha=\alpha$ and $\left(G^{\text {end }}\right)^{*} \alpha=\alpha$. Note that the fiber $\operatorname{ad}_{Q}(E)_{x}, x \in M$, of the bundle $\operatorname{ad}_{Q}(E) \rightarrow M$ is a Lie algebra isomorphic to the Lie algebra $\mathfrak{h} \subset \mathfrak{s o}(n, n)$ of the structure group $H$. The map
$\partial_{Q}: E^{*} \otimes \operatorname{ad}_{Q}(E) \rightarrow \Lambda^{3} E^{*},\left(\partial_{Q} \eta\right)(u, v, w):=\eta(u, v, w)+\eta(w, u, v)+\eta(v, w, u)$
is called the algebraic torsion map. From (7), the image of $T^{D} \in \Gamma\left(\Lambda^{3} E^{*}\right)$ in the quotient bundle $\left(\Lambda^{3} E^{*}\right) / \mathrm{im} \partial_{Q}$ is independent of the choice of adapted generalized connection $D$.

Definition 15. Let $Q$ be a generalized $H$-structure on $E$ and $D$ an adapted generalized connection. The class of $T^{D}$ in $\left(\Lambda^{3} E^{*}\right) / \mathrm{im} \partial_{Q}$ is called the intrinsic torsion of $Q$.

We shall often consider the intrinsic torsion as a 3 -form on $E$, by choosing a suitable complement $C(E)$ of im $\partial_{Q}$ in $\Lambda^{3} E^{*}$ and identifying the quotient $\left(\Lambda^{3} E^{*}\right) / \mathrm{im} \partial_{Q}$ with $C(E)$.

Definition 16. The generalized first prolongation of a Lie algebra $\mathfrak{h} \subset \mathfrak{s o}(k, \ell)$ $\cong \Lambda^{2} V^{*}, V:=\mathbb{R}^{k+\ell}$, is the subspace

$$
\mathfrak{h}^{\langle 1\rangle}:=\left\{\eta \in V^{*} \otimes \mathfrak{h} \mid \partial \eta=0\right\} \subset V^{*} \otimes \Lambda^{2} V^{*}
$$

where $\partial: V^{*} \otimes \Lambda^{2} V^{*} \rightarrow \Lambda^{3} V^{*}$ is defined by:

$$
(\partial \eta)(u, v, w):=\eta(u, v, w)+\eta(w, u, v)+\eta(v, w, u), \quad u, v, w \in V .
$$

Proposition 17. Let $Q$ be a generalized $H$-structure on a Courant algebroid $E$ and $D_{0}$ an adapted generalized connection with torsion $T^{D_{0}}$. Given a section $T \in \Gamma\left(\Lambda^{3} E^{*}\right)$ there exists a generalized connection $D$ adapted to $Q$ with torsion $T^{D}=T$ if and only if $T-T^{D_{0}} \in \Gamma\left(\operatorname{im} \partial_{Q}\right)$. The generalized connection $D$ is unique up to addition of a section of $\operatorname{ker} \partial_{Q} \subset E^{*} \otimes \operatorname{ad}_{Q}(E)$. It is unique if and only if the generalized first prolongation $\mathfrak{h}^{\langle 1\rangle}$ of the Lie algebra $\mathfrak{h} \subset \mathfrak{s o}(n, n)$ of the structure group $H$ vanishes.

Proof. This is obtained by writing an arbitrary adapted generalized connection as $D=D_{0}+\eta$, where $\eta \in \Gamma\left(E^{*} \otimes \operatorname{ad}_{Q}(E)\right)$ and observing that $T^{D}=T^{D_{0}}+\partial_{Q} \eta$, see (7). The last statement follows from the fact that the fibres of the bundle ker $\partial_{Q}$ are isomorphic to $\mathfrak{h}^{\langle 1\rangle}$.

The next corollary follows easily from Proposition 17.
Corollary 18. In the setting of Proposition 17, $Q$ admits an adapted torsionfree generalized connection $D$ if and only if the intrinsic torsion of $Q$ vanishes. The generalized connection $D$ is unique if and only if $\mathfrak{h}^{\langle 1\rangle}=0$.

### 2.7 Application to Born geometry

The vanishing of the generalized first prolongation for certain Lie subalgebras of $\mathfrak{s o}(k, \ell)$ can be used to prove the uniqueness of certain connections (rather than generalized connections). As an example we mention the canonical connection in Born geometry defined in [8], which is related to string compactifications and more specifically to double field theory. Its uniqueness, proven
in [8], can be alternatively deduced from the vanishing of the generalized first prolongation of the diagonal $\mathfrak{s o}(n)$-subalgebra

$$
\Delta_{\mathfrak{s o}(n)}=\{(A, A) \in \mathfrak{s o}(n) \oplus \mathfrak{s o}(n) \mid A \in \mathfrak{s o}(n)\} \subset \mathfrak{s o}(n, n)
$$

The corresponding diagonally embedded $\mathrm{O}(n)$-subgroup $\Delta_{\mathrm{O}(n)} \subset \mathrm{O}(n, n)$ is precisely the automorphism group of the following data on $V=\mathbb{R}^{2 n}$ :

1. a scalar product $\eta=\langle\cdot, \cdot\rangle$ of neutral signature,
2. a positive definite scalar product $g$ and
3. a linear involution $K$,
which satisfy the following compatibility conditions:
4. $K$ is skew-symmetric with respect to $\eta$,
5. $J=\eta^{-1} g$ is an involution and
6. $J$ anti-commutes with $K$.

These properties imply that the triple ( $I:=K J, J, K$ ) is a para-hypercomplex structure on $V$, i.e. $I, J, K$ pairwise anti-commute, $K=I J$ and $-I^{2}=J^{2}=$ $K^{2}=\mathrm{Id}$. However, the structure is not para-hyper-Hermitian with respect to $\eta$ as only $K$ is skew-symmetric, whereas $I$ and $J$ are symmetric with respect to $\eta$. A quadruple ( $\eta, I, J, K$ ) with these properties is called a Born structure on $V$ if the symmetric bilinear form $g=\eta J=\eta(J \cdot, \cdot)$ is positive definite. So the data $(\eta, g, K)$ with the above compatibility relations $1 .-3$. is equivalent to a Born structure $(\eta, I, J, K)$ on $V$.

Given smooth tensor fields $(\eta, I, J, K)$ on a manifold $M$ such that $\left(\eta_{p}, I_{p}, J_{p}\right.$, $K_{p}$ ) is a Born structure on $T_{p} M$ for all $p \in M$, the data ( $\eta, I, J, K$ ) is called a Born structure on $M$. It is proven in [8] that every Born structure ( $\eta, I, J, K$ ) on a manifold $M$ admits a canonical compatible connection $\nabla$, called the Born connection, with vanishing generalized torsion $\mathcal{T}_{\nabla} \in \Omega^{3}(M)$, where

$$
\mathcal{T}_{\nabla}(X, Y, Z):=\eta\left(\nabla_{X} Y-\nabla_{Y} X-[X, Y]^{c}+(\nabla X)^{*} Y, Z\right)
$$

is defined in terms of the so-called canonical D-bracket

$$
[X, Y]^{c}=\nabla_{X}^{c} Y-\nabla_{Y}^{c} X+\left(\nabla^{c} X\right)^{*} Y
$$

Here $B^{*}$ denotes the $\eta$-adjoint of an endomorphism field $B$ and $\nabla^{c}:=$ $\nabla^{\eta}+\frac{1}{2} K \nabla^{\eta} K$ is the canonical connection compatible with the almost paraHermitian structure ( $\eta, K$ ), where $\nabla^{\eta}$ denotes the Levi-Civita connection of $\eta$. The Born connection is denoted $\nabla^{B}$ and is defined by

$$
\nabla_{X}^{B} Y:=\left[X_{-}, Y_{+}\right]_{+}^{c}+\left[X_{+}, Y_{-}\right]_{-}^{c}+\left(K\left[X_{+}, K Y_{+}\right]^{c}\right)_{+}+\left(K\left[X_{-}, K Y_{-}\right]^{c}\right)_{-},
$$

where $\pm$ stands for the projections onto the eigendistributions of $J$. It is clear that any two connections $\nabla^{\prime}$ and $\nabla$ compatible with the same Born structure and having the same generalized torsion differ by a section $A \in$ $\Gamma\left(T^{*} M \otimes \mathfrak{s o}(T M)\right)$ in the kernel of the total skew-symmetrization map $\partial$, such that $A_{p}$ belongs to the generalized first prolongation of the Lie algebra $\operatorname{aut}\left(T_{p} M, \eta_{p}, I_{p} . J_{p}, K_{p}\right) \cong \Delta_{\mathfrak{s o}(n)}$. This shows, in particular, that the uniqueness of $\nabla^{B}$ follows from the next lemma.

Lemma 19. $\Delta_{\text {so }(n)}^{\langle 1\rangle}=0$.
Proof. Let $\left(e_{1}, \ldots, e_{n}, e_{1}^{\prime}, \ldots, e_{n}^{\prime}\right)$ be a basis of $V$, orthonormal with respect to both $\langle\cdot, \cdot\rangle$ and $g$, where

$$
\left\langle e_{i}, e_{i}\right\rangle=g\left(e_{i}, e_{i}\right)=1,\left\langle e_{i}^{\prime}, e_{i}^{\prime}\right\rangle=-g\left(e_{i}^{\prime}, e_{i}^{\prime}\right)=-1,
$$

such that $K\left(e_{i}\right)=e_{i}^{\prime}, K\left(e_{i}^{\prime}\right)=e_{i}$. Then $A \in V^{*} \otimes \Delta_{\mathfrak{s o}(n)}$ is completely determined by two (1,2)-tensors $\left(A_{i j}^{k}\right)$ and $\left(A_{i j}^{\prime k}\right)$ on $\mathbb{R}^{n}$, where $A_{e_{i}} e_{j}=\sum_{k} A_{i j}^{k} e_{k}$ and $A_{e_{i}^{\prime}} e_{j}=\sum_{k} A_{i j}^{k} e_{k}$, since (from $A_{v} K=K A_{v}$ for any $v$ ) $A_{e_{i}} e_{j}^{\prime}=\sum_{k} A_{i j}^{k} e_{k}^{\prime}$ and $A_{e_{i}^{\prime}} e_{j}^{\prime}=\sum_{k} A_{i j}^{k} e_{k}^{\prime}$. From $\partial A=0$ we obtain

$$
0=\left\langle A_{e_{i}} e_{j}^{\prime}, e_{k}^{\prime}\right\rangle+\left\langle A_{e_{j}^{\prime}} e_{k}^{\prime}, e_{i}\right\rangle+\left\langle A_{e_{k}^{\prime}} e_{i}, e_{j}^{\prime}\right\rangle=-A_{i j}^{k}
$$

and similarly $A_{i j}^{k}=0$. This proves that $A=0$.

## 3 Generalized almost complex structures

### 3.1 Intrinsic torsion of a generalized almost complex structure

In this section we compute the intrinsic torsion $t_{\mathcal{J}}$ of a generalized almost complex structure $\mathcal{J}$ on a Courant algebroid $E$ and relate it to the Nijenhuis tensor $N_{\mathcal{J}}$. Before we need to recall basic facts on projectors. Recall that an endomorphism $P \in \Gamma(\operatorname{End} V)$ of a vector bundle $V$ is a projector onto a subbundle $V_{0} \subset V$ if $P^{2}=P$ and $\operatorname{im} P=V_{0}$. Then $V$ decomposes as $V=$ $V_{0} \oplus \operatorname{ker} P$ and $P$ (respectively Id $-P$ ) are the projections onto $V_{0}$ (respectively ker $P$ ) along this decomposition. In particular, there is a canonical choice of a complement of im $P$ in $V$, namely, ker $P$.

Consider now the algebraic torsion map $\partial_{\mathcal{J}}: E^{*} \otimes \Lambda_{\mathcal{J}}^{1,1} E^{*} \rightarrow \Lambda^{3} E^{*}$ of $\mathcal{J}$, defined by (14), where $\operatorname{ad}_{\mathcal{J}}(E)=\Lambda_{\mathcal{J}}^{1,1} E^{*}$ is the bundle of $\mathcal{J}$-invariant 2-forms on $E$.

Lemma 20. i) The map $\Pi_{\mathcal{J}} \in \Gamma\left(\operatorname{End}\left(\Lambda^{3} E^{*}\right)\right)$ defined by

$$
\begin{gather*}
\left(\Pi_{\mathcal{J}} \alpha\right)(u, v, w):= \\
\frac{1}{4}(\alpha(u, v, w)-\alpha(u, \mathcal{J} v, \mathcal{J} w)-\alpha(\mathcal{J} u, v, \mathcal{J} w)-\alpha(\mathcal{J} u, \mathcal{J} v, w)) \tag{15}
\end{gather*}
$$

is a projector onto the subbundle

$$
\Lambda_{\mathcal{J}}^{3} E^{*}:=\left\{\alpha \in \Lambda^{3} E^{*}, \alpha(\mathcal{J} u, v, w)=\alpha(u, \mathcal{J} v, w)=\alpha(u, v, \mathcal{J} w)\right\} .
$$

ii) The equality $\operatorname{im} \partial_{\mathcal{J}}=\operatorname{ker} \Pi_{\mathcal{J}}$ holds.

Proof. It is straightforward to check that $\operatorname{im} \Pi_{\mathcal{J}} \subset \Lambda_{\mathcal{J}}^{3} E^{*}$. Also, for all $\alpha \in \Lambda_{\mathcal{J}}^{3} E^{*}, \Pi_{\mathcal{J}}(\alpha)=\alpha$. We obtain that $\operatorname{im} \Pi_{\mathcal{J}}=\Lambda_{\mathcal{J}}^{3} E^{*}$ and $\Pi_{\mathcal{J}}^{2}=\Pi_{\mathcal{J}}$. Claim i) is proved. To prove claim ii), we notice that $\Pi_{\mathcal{J}} \circ \partial_{\mathcal{J}}=0$, i.e. $\operatorname{im} \partial_{\mathcal{J}} \subset \operatorname{ker} \Pi_{\mathcal{J}}$. Let

$$
\pi: \Lambda^{3} E^{*} \rightarrow E^{*} \otimes \Lambda_{\mathcal{J}}^{1,1} E^{*},(\pi \alpha)(u, v, w):=\alpha(u, v, w)+\alpha(u, \mathcal{J} v, \mathcal{J} w) .
$$

It is straightforward to check that $\Pi_{\mathcal{J}}=\operatorname{Id}_{\Lambda^{3} E^{*}}-\frac{1}{4} \partial_{\mathcal{J}} \circ \pi$, which implies $\operatorname{ker} \Pi_{\mathcal{J}} \subset \operatorname{im} \partial_{\mathcal{J}}$. We proved that $\operatorname{im} \partial_{\mathcal{J}}=\operatorname{ker} \Pi_{\mathcal{J}}$.

The next corollary follows from Lemma 20 and our comments before this lemma.

Corollary 21. With the notation from Lemma 20, $\mathrm{im}_{\mathcal{J}}$ is a complement of $\operatorname{im} \partial_{\mathcal{J}}$ in $\Lambda^{3} E^{*}$.

From Corollary 21, we can (and will) identify the quotient $\left(\Lambda^{3} E^{*}\right) /\left(\operatorname{im} \partial_{\mathcal{J}}\right)$ with $\operatorname{im} \Pi_{\mathcal{J}}=\Lambda_{\mathcal{J}}^{3} E^{*}$ and consider the intrinsic torsion $t_{\mathcal{J}}$ of $\mathcal{J}$ as a section of $\Lambda_{\mathcal{J}}^{3} E^{*}$. On the other hand, $N_{\mathcal{J}}(\mathcal{J} u, v)=-\mathcal{J} N_{\mathcal{J}}(u, v)$ (easy check), which implies that $N_{\mathcal{J}}$, considered as a 3 -form (see Lemma (9), is also a section of $\Lambda_{\mathcal{J}}^{3} E^{*}$. Up to a constant, $N_{\mathcal{J}}$ and $t_{\mathcal{J}}$ coincide. More precisely, we have the following result.

Corollary 22. The torsion $T^{D}$ of a generalized connection $D$ with $D \mathcal{J}=0$ satisfies
$T^{D}(u, v, w)-T^{D}(\mathcal{J} u, v, \mathcal{J} w)-T^{D}(u, \mathcal{J} v, \mathcal{J} w)-T^{D}(\mathcal{J} u, \mathcal{J} v, w)=N_{\mathcal{J}}(u, v, w)$,
for all $u, v, w \in \Gamma(E)$. In particular, $t_{\mathcal{J}}=\frac{1}{4} N_{\mathcal{J}}$ (viewed as 3 -forms on $E$ ).
Proof. Relation (16) follows from (8), together with

$$
\begin{equation*}
[u, v]_{C}=D_{u} v-D_{v} u+\frac{1}{2}\left((D u)^{*} v-(D v)^{*} u\right)-T^{D}(u, v) \tag{17}
\end{equation*}
$$

and $D \mathcal{J}=0$. Relation (16) can be written as $\Pi_{\mathcal{J}}\left(T^{D}\right)=\frac{1}{4} N_{\mathcal{J}}$, which implies the second statement.

### 3.2 Integrability using torsion-free generalized connections

In this section we prove the following theorem.
Theorem 23. A generalized almost complex structure $\mathcal{J}$ on $E$ is integrable if and only if there is a torsion-free generalized connection $D$ such that $D \mathcal{J}=0$.

Part of the statement of Theorem 23 follows from Corollary 22; ; if there is a torsion-free generalized connection $D$ such that $D \mathcal{J}=0$ then from relation (16) $\mathcal{J}$ is integrable. For the converse statement, let $\mathcal{J} \in \Gamma(\operatorname{End} E)$ be a generalized almost complex structure. We will construct a generalized complex connection whose torsion equals the intrinsic torsion of $\mathcal{J}$. Such a generalized connection will be torsion-free (and complex) when $\mathcal{J}$ is integrable. This will conclude the proof of Theorem 23. The next remark represents our motivation for the choice of generalized connection $\tilde{D}$ in Proposition 25.

Remark 24. i) Given an almost complex structure $J$ and a torsion-free connection $\nabla$ on a manifold $M$, the connection

$$
\begin{equation*}
\tilde{\nabla}_{X} Y=\nabla_{X} Y-\frac{1}{4}\{(\nabla J) X, J\} Y-\frac{1}{2} J\left(\nabla_{X} J\right) Y, \tag{18}
\end{equation*}
$$

where $\{A, B\}:=A B+B A$ denotes the anti-commutator of $A$ and $B$ and $(\nabla J) X \in \Gamma(\operatorname{End} T M)$ is defined by $Y \rightarrow\left(\nabla_{Y} J\right) X$, is complex $(\tilde{\nabla} J=0)$ and its torsion satisfies $T^{\tilde{\nabla}}(X, Y)=\frac{1}{8} N_{J}(X, Y)$ (see Theorem 3.4 of [18; remark the difference by a multiplicative factor between our definition for the Nijenhuis tensor and that of [18]). In particular, if $J$ is integrable, then $\tilde{\nabla}$ is torsion-free (and complex).

Now, for a generalized almost complex structure $\mathcal{J}$ and a generalized torsion-free connection $D$ on the Courant algebroid $E$, we may define the analogous expression

$$
\begin{equation*}
\tilde{D}_{u}^{\prime} v=D_{u} v-\frac{1}{4}\{(D \mathcal{J}) u, \mathcal{J}\} v-\frac{1}{2} \mathcal{J}\left(D_{u} \mathcal{J}\right) v . \tag{19}
\end{equation*}
$$

However, $\tilde{D}^{\prime}$ defined by (19) is not a generalized connection (while $\mathcal{J} D_{u} \mathcal{J}$ is skew-symmetric with respect to the scalar product $\langle\cdot, \cdot\rangle$ of $E$, the anticommutator $\{(D \mathcal{J}) u, \mathcal{J}\}$ is not and $\tilde{D}^{\prime}$ does not preserve $\langle\cdot, \cdot\rangle$, in general; we shall give more details on this argument in Lemma (26).
ii) On the other hand, the connections $\nabla$ and $\tilde{\nabla}$ from (18) extend (as usual connections) to the vector bundle $\mathbb{T} M=T M \oplus T^{*} M$ and define, according to Example 5, generalized connections $\nabla^{G}$ and $\tilde{\nabla}^{G}$ on the Courant algebroid $\mathbb{T} M$. More precisely, $\nabla_{u}^{G} v:=\nabla_{X} Y+\nabla_{X} \eta$, where $u:=X+\xi, v:=Y+\eta$,
$X, Y \in \mathfrak{X}(M), \xi, \eta \in \Omega^{1}(M)$, and similarly for $\tilde{\nabla}^{G}$. Now, if two arbitrary connections $\nabla^{2}$ and $\nabla^{1}$ on $T M$ are related by $\nabla^{2}=\nabla^{1}+A$, the associated generalized connections on $\mathbb{T} M$ are related by

$$
\begin{equation*}
\left(\nabla^{2}\right)_{u}^{G}=\left(\nabla^{1}\right)_{u}^{G}+A_{X}-A_{X}^{*}, \forall u=X+\xi \in \mathbb{T} M \tag{20}
\end{equation*}
$$

The connections $\nabla$ and $\tilde{\nabla}$ from (18) differ by $A_{X}:=B_{X}+C_{X}$, where

$$
B_{X}:=-\frac{1}{4}\{(\nabla J) X, J\}, C_{X}:=-\frac{1}{2} J\left(\nabla_{X} J\right) .
$$

The adjoint of $B_{X}$ is given by

$$
B_{X}^{*}=-\frac{1}{4}\left\{((\nabla J) X)^{*}, J^{*}\right\}=\frac{1}{4}\left\{((\nabla J) X)^{*},\left.\mathcal{J}_{J}\right|_{T^{*} M}\right\}
$$

and

$$
B_{X}-B_{X}^{*}=-\frac{1}{4}\left\{(\nabla J)(X)+((\nabla J) X)^{*}, \mathcal{J}_{J}\right\}
$$

where $\mathcal{J}_{J}=J-J^{*}$ is the generalized almost complex structure defined by $J$ (i.e. $\left.\mathcal{J}_{J}\right|_{T M}=J$ and $\left.\mathcal{J}_{J}\right|_{T^{*} M}=-J^{*}$, where in our convention $J^{*} \alpha=\alpha \circ J$ ). From (20) applied to $\nabla$ and $\tilde{\nabla}$ we obtain

$$
\begin{equation*}
\tilde{\nabla}_{u}^{G}=\nabla_{u}^{G}-\frac{1}{4}\left\{(\nabla J) X+((\nabla J) X)^{*}, \mathcal{J}_{J}\right\}+C_{X}-C_{X}^{*} . \tag{21}
\end{equation*}
$$

Remark that $(\nabla J) X+((\nabla J) X)^{*} \in \Gamma($ End $\mathbb{T} M)$ is symmetric with respect to $\langle\cdot, \cdot\rangle$.

Relations (18) and (21), together with the symmetry of $(\nabla J) X+((\nabla J) X)^{*}$ are our motivation for the definition of the generalized connection $\tilde{D}$ in the next proposition.

Proposition 25. Let $\mathcal{J}$ be a generalized almost complex structure and $D$ a torsion-free generalized connection on $E$. Define

$$
\begin{equation*}
\tilde{D}_{u} v=D_{u} v-\frac{1}{4}\left\{A_{u}^{\text {sym }}, \mathcal{J}\right\} v-\frac{1}{2} \mathcal{J}\left(D_{u} \mathcal{J}\right) v, \tag{22}
\end{equation*}
$$

where $A_{u}:=(D \mathcal{J}) u$ and $A_{u}^{\text {sym }}$ is its $\langle\cdot, \cdot\rangle$-symmetric part. Then $\tilde{D}$ is a generalized connection, which preserves $\mathcal{J}$. Its torsion is given by

$$
\begin{equation*}
T^{\tilde{D}}(u, v, w)=\frac{1}{4} N_{\mathcal{J}}(u, v, w) . \tag{23}
\end{equation*}
$$

In particular, if $\mathcal{J}$ is integrable, then $\tilde{D}$ is torsion-free (and complex).

We divide the proof of the above proposition into several lemmas.
Lemma 26. Equation (22) defines a generalized complex connection.
Proof. Note that $\mathcal{J} D_{u} \mathcal{J}$ is skew-symmetric with respect to the scalar product $\langle\cdot, \cdot\rangle$ of $E\left(D_{u} \mathcal{J}\right.$ is skew-symmetric and also $\mathcal{J} D_{u} \mathcal{J}$ is skew-symmetric, being the composition of two anti-commuting skew-symmetric endomorphisms). Similarly, $\left\{A_{u}^{\text {sym }}, \mathcal{J}\right\}$ is skew-symmetric, because $A_{u}^{\text {sym }}$ is symmetric and $\mathcal{J}$ is skew-symmetric. We obtain that $\tilde{D}_{u}$ and $D_{u}$ differ by a skew-symmetric endomorphism, i.e. $\tilde{D}$ is a generalized connection. The generalized connection

$$
\begin{equation*}
D_{u}^{(1)}:=D_{u}-\frac{1}{2} \mathcal{J} D_{u} \mathcal{J} \tag{24}
\end{equation*}
$$

preserves $\mathcal{J}$. As $\left\{A_{u}^{\text {sym }}, \mathcal{J}\right\}$ commutes with $\mathcal{J}$, we obtain that

$$
\begin{equation*}
\tilde{D}_{u}=D_{u}^{(1)}-\frac{1}{4}\left\{A_{u}^{\text {sym }}, \mathcal{J}\right\} \tag{25}
\end{equation*}
$$

preserves $\mathcal{J}$ as well.
It remains to prove relation (23). From (25),

$$
\begin{align*}
\left\langle\tilde{D}_{u} v, w\right\rangle=\left\langle D_{u}^{(1)} v, w\right\rangle & -\frac{1}{8}(\eta(\mathcal{J} v, u, w)+\eta(w, u, \mathcal{J} v)) \\
& +\frac{1}{8}(\eta(v, u, \mathcal{J} w)+\eta(\mathcal{J} w, u, v)) \tag{26}
\end{align*}
$$

where $\eta=\eta^{D, \mathcal{J}}$ is defined by

$$
\begin{equation*}
\eta(u, v, w):=\left\langle\left(D_{u} \mathcal{J}\right) v, w\right\rangle . \tag{27}
\end{equation*}
$$

Remark that $\eta$ has the symmetries

$$
\begin{equation*}
\eta(u, v, w)=-\eta(u, w, v), \eta(u, \mathcal{J} v, w)=\eta(u, v, \mathcal{J} w) . \tag{28}
\end{equation*}
$$

Lemma 27. The torsion of $D^{(1)}$ is given by

$$
\begin{equation*}
T^{D^{(1)}}(u, v, w)=\frac{1}{2} \sum_{(u, v, w) \text { cyclic }} \eta(u, v, \mathcal{J} w) \tag{29}
\end{equation*}
$$

where the sum is over cyclic permutations on $(u, v, w)$.
Proof. The claim follows from the torsion-free property of $D$ together with relations (7) and (24).

Lemma 28. The following relation holds:

$$
\begin{equation*}
N_{\mathcal{J}}(u, v, w)=\sum_{(u, v, w) \text { cyclic }}(\eta(u, v, \mathcal{J} w)+\eta(\mathcal{J} u, v, w)) . \tag{30}
\end{equation*}
$$

Proof. Using relations (8), (17) and $T^{D}=0$, we obtain

$$
\begin{aligned}
N_{\mathcal{J}}(u, v) & =\left(D_{\mathcal{J} u} \mathcal{J}\right) v-\left(D_{\mathcal{J} v} \mathcal{J}\right) u+\frac{1}{2}(D(\mathcal{J} u))^{*} \mathcal{J} v-\frac{1}{2}(D(\mathcal{J} v))^{*} \mathcal{J} u \\
& -\frac{1}{2}(D u)^{*} v+\frac{1}{2}(D v)^{*} u+\mathcal{J}\left(D_{v} \mathcal{J}\right) u-\frac{1}{2} \mathcal{J}(D(\mathcal{J} u))^{*} v+\frac{1}{2} \mathcal{J}(D v)^{*} \mathcal{J} u \\
& -\mathcal{J}\left(D_{u} \mathcal{J}\right) v-\frac{1}{2} \mathcal{J}(D u)^{*} \mathcal{J} v+\frac{1}{2} \mathcal{J}(D(\mathcal{J} v))^{*} u .
\end{aligned}
$$

Taking the inner product of the above equality with $w$ and using the symmetries (28) of $\eta$ we obtain

$$
\begin{align*}
N_{\mathcal{J}}(u, v, w)= & \left\langle N_{\mathcal{J}}(u, v), w\right\rangle=\eta(\mathcal{J} u, v, w)-\eta(\mathcal{J} v, u, w)+\eta(w, u, \mathcal{J} v) \\
& -\eta(v, u, \mathcal{J} w)+\eta(\mathcal{J} w, u, v)+\eta(u, v, \mathcal{J} w) \tag{31}
\end{align*}
$$

Taking in (31) cyclic permutations over $u, v, w$, using again the symmetries (28) of $\eta$ and that $N_{\mathcal{J}}(u, v, w)$ is completely skew we obtain (30).

The next lemma concludes the proof of Proposition 25 and Theorem [23,
Lemma 29. The torsion of $\tilde{D}$ satisfies relation (23).
Proof. From relations (26) and (7), we obtain

$$
\begin{aligned}
& T^{\tilde{D}}(u, v, w)-T^{D^{(1)}}(u, v, w)= \\
& -\frac{1}{8} \sum_{(u, v, w) \text { cyclic }}(\eta(\mathcal{J} v, u, w)+\eta(w, u, \mathcal{J} v)-\eta(v, u, \mathcal{J} w)-\eta(\mathcal{J} w, u, v)) \\
& =-\frac{1}{4} \sum_{(u, v, w) \text { cyclic }}(\eta(\mathcal{J} u, w, v)+\eta(u, v, \mathcal{J} w))
\end{aligned}
$$

where in the last equality we have used the symmetries (28) of $\eta$. From (29) we then obtain

$$
\begin{equation*}
T^{\tilde{D}}(u, v, w)=\frac{1}{4} \sum_{(u, v, w) \text { cyclic }}(\eta(u, v, \mathcal{J} w)+\eta(\mathcal{J} u, v, w)), \tag{32}
\end{equation*}
$$

which implies (23), from Lemma 28,

Remark 30. Given an almost complex structure $J$ and a connection $\nabla$ on $M$, we may consider the generalized connections $\nabla^{G}$ and $\widetilde{\left(\nabla^{G}\right)}$ on $T M \oplus T^{*} M$, where $\widetilde{\left(\nabla^{G}\right)}$ is defined by the right hand side of (22), with $D$ replaced by $\nabla^{G}$ and $\mathcal{J}$ replaced by $\mathcal{J}_{J}$, the generalized almost complex structure determined by $J$. On the other hand, we may also consider the (usual) connection $\tilde{\nabla}$ related to $\nabla$ by (18), and the associated generalized connection $(\tilde{\nabla})^{G}$ determined by $\tilde{\nabla}$. We remark that $(\tilde{\nabla})^{G} \neq \widetilde{\left(\nabla^{G}\right)}$, as $(\tilde{\nabla})_{\xi}^{G}=0$ but $\widetilde{\left(\nabla^{G}\right)_{\xi}}=$ $-\frac{1}{4}\left\{\left(\left(\nabla^{G} \mathcal{J}\right) \xi\right)^{\text {sym }}, \mathcal{J}\right\} \neq 0$ in general, for $\xi \in T^{*} M$. It is easy to write down the precise formula which connects $\widetilde{\left(\nabla^{G}\right)}$ to $(\tilde{\nabla})^{G}$.

## 4 Generalized almost hypercomplex structures

### 4.1 Intrinsic torsion of a generalized almost hypercomplex structure

Let $\left(\mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ be a generalized almost hypercomplex structure on a Courant algebroid $E$ and $\partial_{\mathbb{H}}: E^{*} \otimes \Lambda_{\mathbb{H}}^{1,1} E^{*} \rightarrow \Lambda^{3} E^{*}$ the algebraic torsion map defined by (14), where

$$
\operatorname{ad}_{\left(\mathcal{J}_{i}\right)}(E)=\Lambda_{\mathbb{H}}^{1,1} E^{*}:=\left\{\alpha \in \Lambda^{2} E^{*}, \alpha\left(\mathcal{J}_{i} u, \mathcal{J}_{i} v\right)=\alpha(u, v), u, v \in E, i=1,2,3\right\} .
$$

Lemma 31. The endomorphism $P \in \Gamma\left(\operatorname{End}\left(\Lambda^{3} E^{*}\right)\right)$ defined by

$$
\begin{equation*}
P:=\frac{2}{3} \sum_{i=1}^{3} \Pi_{\mathcal{J}_{i}} \tag{33}
\end{equation*}
$$

is a projector with $\operatorname{ker} P=\operatorname{im} \partial_{\mathbb{H}}$. In particular, im $P$ is a complement of $\operatorname{im} \partial_{\text {HI }}$ in $\Lambda^{3} E^{*}$.

Proof. For any generalized almost complex structure $\mathcal{J}$, define the endomorphism $\Pi_{\mathcal{J}}^{0,2}$ of $\Lambda^{2} E^{*} \otimes E$ by $\left\langle\left(\Pi_{\mathcal{J}}^{0,2} \alpha\right)(u, v), w\right\rangle=\left(\Pi_{\mathcal{J}} \alpha\right)(u, v, w)$, where on the right-hand side $\alpha$ is considered as an element of $\Lambda^{2} E^{*} \otimes E^{*}$ and $\Pi_{\mathcal{J}} \alpha \in \Lambda^{2} E^{*} \otimes E^{*}$ is defined by the formula (15). Then $\Pi_{\mathcal{J}}^{0,2}$ coincides with the operator given by relation (5) of [12]. We obtain that $(P \alpha)(u, v, w)=$ $\langle p(\alpha)(u, v), w\rangle$ where $p=\frac{2}{3} \sum_{i=1}^{3} \Pi_{\mathcal{J}_{i}}^{0,2} \in \Gamma\left(\operatorname{End}\left(\Lambda^{2} E^{*} \otimes E\right)\right)$ is the map from Lemma 1 of [12]. As $p$ is a projector, $P$ is also a projector. From Lemma 20 ii), im $\partial_{\mathbb{H}} \subset$ ker $P$. Let

$$
\begin{equation*}
\pi: \Lambda^{3} E^{*} \rightarrow E^{*} \otimes \Lambda_{\mathbb{H}}^{1,1} E^{*},(\pi \alpha)(u, v, w):=\alpha(u, v, w)+\sum_{i=1}^{3} \alpha\left(u, \mathcal{J}_{i} v, \mathcal{J}_{i} w\right) \tag{34}
\end{equation*}
$$

It is straightforward to check that

$$
\begin{equation*}
\partial_{\mathbb{H}} \circ \pi=6\left(\operatorname{Id}_{\Lambda^{3} E^{*}}-P\right), \tag{35}
\end{equation*}
$$

which implies $\operatorname{ker} P \subset \operatorname{im} \partial_{\mathbb{H}}$ and thus $\operatorname{im} \partial_{\mathbb{H}}=\operatorname{ker} P$. As $P$ is a projector, $\operatorname{im} \partial_{\mathbb{H}}=\operatorname{ker} P$ is a complement of $\operatorname{im} P$ in $\Lambda^{3} E^{*}$.

As in the previous section, we will identify $\left(\Lambda^{3} E^{*}\right) /\left(\operatorname{im} \partial_{\mathbb{H}}\right)$ with im $P$ and consider the intrinsic torsion $t_{\left(\mathcal{J}_{i}\right)}$ of $\left(\mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ as a section of im $P$.

Corollary 32. The intrinsic torsion $t_{\left(\mathcal{J}_{i}\right)}$ of $\left(\mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ is given by $\frac{1}{6} \sum_{i=1}^{3} N_{\mathcal{J}_{i}}$.
Proof. Let $D$ be a hypercomplex connection. By Corollary 22 we have that $\Pi_{\mathcal{J}_{i}}\left(T^{D}\right)=\frac{1}{4} N_{\mathcal{J}_{i}}$. So $t_{\left(\mathcal{J}_{i}\right)}=P\left(T^{D}\right)=\frac{1}{6} \sum_{i=1}^{3} N_{\mathcal{J}_{i}}$.

### 4.2 Integrability using torsion-free generalized connections

Our aim in this section is to prove the next theorem.
Theorem 33. A generalized almost hypercomplex structure $\left(\mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ on a Courant algebroid $E$ is integrable if and only if there is a torsion-free generalized connection $D$ such that $D \mathcal{J}_{i}=0$ for all $i=1,2,3$.

Part of the statement of Theorem 33 is obvious: if there is a torsion-free generalized connection which preserves all $\mathcal{J}_{i}$, then $\mathcal{J}_{i}$ are integrable from Theorem [23. The converse statement is proved in the next proposition.

Proposition 34. Let $\left(\mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ be a generalized almost hypercomplex structure and $D$ a generalized connection on $E$, such that $D \mathcal{J}_{1}=0$. Define

$$
\begin{equation*}
D^{(1)}:=D-\frac{1}{2} \mathcal{J}_{2} D \mathcal{J}_{2} \tag{36}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{D}:=D^{(1)}+\frac{1}{6} \pi\left(\left(P-\operatorname{Id}_{\Lambda^{3} E^{*}}\right) T^{D^{(1)}}\right), \tag{37}
\end{equation*}
$$

where $P$ is the map (33) and $\pi$ is the map (34). Then $D^{(1)}$ and $\tilde{D}$ are generalized hypercomplex connections. Moreover,

$$
T^{\tilde{D}}(u, v, w)=\frac{1}{6} \sum_{i=1}^{3} N_{\mathcal{J}_{i}}(u, v, w) .
$$

In particular, if $\left(\mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ is a generalized hypercomplex structure, then $\tilde{D}$ is torsion-free (and hypercomplex).

Proof. The existence of a generalized connection $D$ with $D \mathcal{J}_{1}=0$ follows from the proof of Lemma 26 (take any generalized connection, say $D^{(0)}$, and define $\left.D:=D^{(0)}-\frac{1}{2} \mathcal{J}_{1} D^{(0)} \mathcal{J}_{1}\right)$. The same argument shows that $D^{(1)}$ is a generalized connection which preserves $\mathcal{J}_{2}$. As $D \mathcal{J}_{1}=0$ and $\mathcal{J}_{2}$ anticommutes with $\mathcal{J}_{1}$, we obtain that $D_{u} \mathcal{J}_{2}$ anti-commutes with $\mathcal{J}_{1}$ as well, for any $u \in E$. Then

$$
D_{u}^{(1)} \mathcal{J}_{1}=D_{u} \mathcal{J}_{1}-\frac{1}{2}\left[\mathcal{J}_{2} D_{u} \mathcal{J}_{2}, \mathcal{J}_{1}\right]=-\frac{1}{2}\left[\mathcal{J}_{2} D_{u} \mathcal{J}_{2}, \mathcal{J}_{1}\right]=0 .
$$

As $D^{(1)} \mathcal{J}_{1}=D^{(1)} \mathcal{J}_{2}=0$ also $D^{(1)} \mathcal{J}_{3}=0$ and $D^{(1)}$ is hypercomplex. This proves the statement on $D^{(1)}$.

It remains to prove the statements on $\tilde{D}$. Let $\eta:=\frac{1}{6} \pi\left(\left(P-\operatorname{Id}_{\Lambda^{3} E^{*}}\right) T^{D^{(1)}}\right)$. With this notation, $\tilde{D}=D^{(1)}+\eta$. Since $\operatorname{im} \pi \subset E^{*} \otimes \Lambda_{\mathbb{H}}^{1,1} E^{*}$ and $D^{(1)} \mathcal{J}_{i}=0$, also $\tilde{D} \mathcal{J}_{i}=0$ for all $i=1,2,3$, i.e. $\tilde{D}$ is hypercomplex. The torsion of $\tilde{D}$ is given by $T^{\tilde{D}}=T^{D^{(1)}}+\partial_{\mathbb{H}} \eta$, where

$$
\begin{aligned}
\partial_{\mathbb{H}} \eta & =\frac{1}{6}\left(\partial_{\mathbb{H}} \circ \pi\right)\left(\left(P-\operatorname{Id}_{\Lambda^{3} E^{*}}\right) T^{D^{(1)}}\right)=-\left(P-\operatorname{Id}_{\Lambda^{3} E^{*}}\right)^{2}\left(T^{D^{(1)}}\right) \\
& =\left(P-\operatorname{Id}_{\Lambda^{3} E^{*}}\right)\left(T^{D^{(1)}}\right)
\end{aligned}
$$

and we used relation (35) and $P^{2}=P$. We obtain

$$
\begin{equation*}
T^{\tilde{D}}=T^{D^{(1)}}+(P-\operatorname{Id})\left(T^{D^{(1)}}\right)=P\left(T^{D^{(1)}}\right)=\frac{1}{6} \sum_{i=1}^{3} N_{\mathcal{J}_{i}}, \tag{38}
\end{equation*}
$$

as claimed. The last equality in (38) follows from $\Pi_{\mathcal{J}_{i}}\left(T^{D^{(1)}}\right)=\frac{1}{4} N_{\mathcal{J}_{i}}$ (see the proof of Corollary (22) and the fact that $D^{(1)}$ is hypercomplex.

## 5 Generalized almost Hermitian structures: integrability and torsion-free generalized connections

In this section we characterize the integrability of generalized almost Hermitian structures using Levi-Civita connections (see Theorem 37 below). We begin with the following simple lemma.

Lemma 35. Let $(G, \mathcal{J})$ be a generalized Kähler structure on $E$. Then

$$
\begin{equation*}
[u, \mathcal{J} v]_{+}-\mathcal{J}[u, v]_{+}=0, \forall u \in \Gamma\left(E_{-}\right), v \in \Gamma\left(E_{+}\right) . \tag{39}
\end{equation*}
$$

Above we denoted by $e_{ \pm}$the $E_{ \pm}$-components of a vector $e \in E$ in the decomposition $E=E_{+} \oplus E_{-}$determined by $G$.

Proof. From (12), if $\langle u, v\rangle=0$, then $[u, v]=-[v, u]$. From this remark and using that $E_{+}$and $E_{-}$are orthogonal with respect to $\langle\cdot, \cdot\rangle$, we can (and will) replace in (39) the Dorfman bracket $[\cdot, \cdot]$ by the Courant bracket $[\cdot, \cdot]_{C}$. Relation (39) is equivalent to

$$
\begin{equation*}
\left([u, v]_{C}\right)_{+} \in \Gamma\left(\left(E_{+}\right)_{\mathbb{C}} \cap L_{1}\right), \forall u \in \Gamma\left(\left(E_{-}\right)_{\mathbb{C}}\right), v \in \Gamma\left(\left(E_{+}\right)_{\mathbb{C}} \cap L_{1}\right), \tag{40}
\end{equation*}
$$

where we have denoted by $L_{1}$ the ( 1,0 )-bundle of $\mathcal{J}_{1}=\mathcal{J}$. Remark that $\left(E_{+}\right)_{\mathbb{C}} \cap L_{1}=\left(E_{+}\right)_{\mathbb{C}} \cap L_{2}$, where $L_{2}$ is the (1,0)-bundle of $\mathcal{J}_{2}=G^{\text {end }} \mathcal{J}$ (since $\mathcal{J}_{1}=\mathcal{J}_{2}$ on $\left.E_{+}\right)$. In (40) we distinguish two cases: a) $u \in \Gamma\left(L_{1} \cap\left(E_{-}\right)_{\mathbb{C}}\right)$; b) $u \in \Gamma\left(\bar{L}_{1} \cap\left(E_{-}\right)_{\mathbb{C}}\right)$. In case a), relation (40) follows from the integrability of $\mathcal{J}$. In case b), relation (40) follows from the integrability of $\mathcal{J}_{2}$.

Remark 36. When the Courant algebroid is exact the above lemma can be proved using the Bismut connection for generalized Kähler structures, constructed in [16]. More precisely, for a generalized Kähler structure $(G, \mathcal{J})$ on an exact Courant algebroid $E$, there is a unique generalized connection $D$ (called in [16] the Bismut connection), such that $D G=0, D \mathcal{J}=0$, and whose torsion is of type $(2,1)+(1,2)$ with respect to $\mathcal{J}$. The expression of $D$ is given in Theorem 3.1 of [16]. Its mixed components $D_{u} v$ and $D_{v} u$, for $u \in \Gamma\left(E_{+}\right)$and $v \in \Gamma\left(E_{-}\right)$, are $D_{u} v:=[u, v]_{-}$and $D_{v} u=[v, u]_{+}$. Relation (39) follows from $D \mathcal{J}=0$.

Theorem 37. A generalized almost Hermitian structure $(G, \mathcal{J})$ on $E$ is generalized Kähler if and only if there is a Levi-Civita connection of $G$ which preserves $\mathcal{J}$.

In one direction the statement is obvious: if there is a Levi-Civita connection $D$ of $G$ which preserves $\mathcal{J}$, then it preserves also $\mathcal{J}_{2}=G^{\text {end }} \mathcal{J}$ and we deduce that both $\mathcal{J}$ and $\mathcal{J}_{2}$ are integrable (from Theorem [23, because $D$ is torsion free). We obtain that $(G, \mathcal{J})$ is generalized Kähler.

We now prove the converse statement. Assume that $(G, \mathcal{J})$ is generalized Kähler. We are looking for a Levi-Civita connection of $G$ which preserves $\mathcal{J}$. Let $D$ be any Levi-Civita connection of $G$ and $\tilde{D}$ the generalized connection defined by (22). From Proposition [25, $\tilde{D}$ is torsion-free and complex ( $\tilde{D} \mathcal{J}=$ 0 ). We will modify $\tilde{D}$ in order to obtain a new generalized connection $\tilde{D}^{\prime}$, also torsion-free and complex, but which preserves the metric $G$ as well. Consider the generalized connection

$$
\tilde{D}_{u}^{\prime} v=\tilde{D}_{u} v+S_{u} v, \quad u, v \in \Gamma(E)
$$

where $S_{u} \in \Gamma(\operatorname{End} E)$ is defined by

$$
\left\langle S_{u} v, w\right\rangle=\sigma(u, v, w)-\sigma(u, w, v)
$$

for $\sigma \in \Gamma\left(S^{2}(E) \otimes E\right)$ to be determined. Then $\tilde{D}^{\prime}$ is torsion-free, because so is $\tilde{D}$ (see Section (2.2).

Lemma 38. The generalized connection $\tilde{D}^{\prime}$ preserves $G$ if and only if

$$
\begin{align*}
& \sigma\left(u, G^{\text {end }} v, w\right)-\sigma\left(u, w, G^{\text {end }} v\right)-\sigma\left(u, v, G^{\text {end }} w\right)+\sigma\left(u, G^{\text {end }} w, v\right) \\
& =\frac{1}{4}\left(\left\langle\left(A_{u}^{\text {sym }} \mathcal{J}+\mathcal{J} A_{u}^{\text {sym }}\right) G^{\text {end }} v, w\right\rangle-\left\langle\left(A_{u}^{\text {sym }} \mathcal{J}+\mathcal{J} A_{u}^{\text {sym }}\right) v, G^{\text {end }} w\right\rangle\right) \tag{41}
\end{align*}
$$

and is complex if and only if

$$
\begin{equation*}
\sigma(u, \mathcal{J} v, w)-\sigma(u, w, \mathcal{J} v)+\sigma(u, v, \mathcal{J} w)-\sigma(u, \mathcal{J} w, v)=0 \tag{42}
\end{equation*}
$$

for any $u, v, w \in E$.
Proof. For the first claim we use the definition of $\tilde{D}^{\prime}, D G=0$, and the skew-symmetry of $\mathcal{J} D_{u} \mathcal{J}$ with respect to $G$. We obtain that $\tilde{D}^{\prime} G=0$ if and only if $S_{u}-\frac{1}{4}\left\{A_{u}^{\text {sym }}, \mathcal{J}\right\}$ is skew-symmetric with respect to $G$, or commutes with $G^{\text {end }}$, and this is equivalent to (41). The second claim can be proved similarly: as $\tilde{D} \mathcal{J}=0$, we obtain that $\tilde{D}^{\prime} \mathcal{J}=0$ if and only if $S_{u} \mathcal{J}=\mathcal{J} S_{u}$, which is equivalent to (42).

Relation (41) is satisfied when $v, w \in E_{+}$and when $v, w \in E_{-}$. For $v \in E_{+}, w \in E_{-}$, it becomes

$$
\begin{equation*}
\sigma(u, v, w)-\sigma(u, w, v)=\frac{1}{4}\left\langle\left(A_{u}^{\text {sym }} \mathcal{J}+\mathcal{J} A_{u}^{\text {sym }}\right) v, w\right\rangle, v \in E_{+}, w \in E_{-} \tag{43}
\end{equation*}
$$

and for $v \in E_{-}, w \in E_{+}$it is equivalent to (43). It is also straightforward to check that if (43) is satisfied, then (42), with $v \in E_{+}, w \in E_{-}$(and any $u$ ), and with $v \in E_{-}, w \in E_{+}$(and any $u$ ), is also satisfied.

To summarize: we are looking for $\sigma \in \Gamma\left(S^{2} E \otimes E\right)$ such that relation (42) is satisfied when $v, w \in E_{+}$and when $v, w \in E_{-}$(and any $u$ ) and relation (43) is also satisfied.

Notation 39. We shall denote by $\sigma^{+++}, \sigma^{++-}$etc the restriction of $\sigma$ to $E_{+} \times E_{+} \times E_{+}, E_{+} \times E_{+} \times E_{-}$etc. Everytime we write e.g. $\sigma^{++-}(u, v, w)$ we assume that $u, v \in E_{+}$and $w \in E_{-}$and we use the same convention for the other components of $\sigma$.

Let

$$
E(u, v, w):=\frac{1}{4}\left\langle\left(A_{u}^{\text {sym }} \mathcal{J}+\mathcal{J} A_{u}^{\text {sym }}\right) v, w\right\rangle, u, v, w \in E
$$

which is skew-symmetric in $(v, w)$. Relation (43) with $u \in E_{+}$becomes

$$
\begin{equation*}
\sigma^{++-}(u, v, w)=\sigma^{+-+}(u, w, v)+E(u, v, w), u, v \in E_{+}, w \in E_{-} \tag{44}
\end{equation*}
$$

Letting $\sigma^{+-+}(u, w, v):=E(v, u, w)$ we obtain that $\sigma^{++-}$, defined by (44), is symmetric in the first two arguments. Similarly, relation (43) with $u \in E_{-}$ becomes

$$
\begin{equation*}
\sigma^{--+}(u, w, v)=\sigma^{-+-}(u, v, w)-E(u, v, w), u, w \in E_{-}, v \in E_{+} \tag{45}
\end{equation*}
$$

Letting $\sigma^{-+-}(u, v, w):=-E(w, v, u)$ we obtain that $\sigma^{--+}$, defined by (45), is symmetric in the first two arguments. Letting $\sigma^{+++}=0, \sigma^{---}=0$ $\sigma^{-++}(u, v, w):=\sigma^{+-+}(v, u, w)$ and $\sigma^{+--}(u, v, w):=\sigma^{-+-}(v, u, w)$ we obtain that $\sigma \in \Gamma\left(S^{2} E \otimes E\right)$ satisfies relation (43)).

Lemma 40. The section $\sigma$ defined above satisfies relation (42) for $v, w \in E_{+}$ and $v, w \in E_{-}$(and $\left.u \in E\right)$.

Proof. Relation (42) is satisfied when all arguments belong to $E_{+}$or to $E_{-}$, owing to $\sigma^{+++}=0$ and $\sigma^{---}=0$. Therefore, we need to prove that

$$
\begin{align*}
& \sigma^{-++}(u, \mathcal{J} v, w)-\sigma^{-++}(u, w, \mathcal{J} v)+\sigma^{-++}(u, v, \mathcal{J} w)-\sigma^{-++}(u, \mathcal{J} w, v)=0 \\
& \sigma^{+--}(u, \mathcal{J} v, w)-\sigma^{+--}(u, w, \mathcal{J} v)+\sigma^{+--}(u, v, \mathcal{J} w)-\sigma^{+--}(u, \mathcal{J} w, v)=0 . \tag{46}
\end{align*}
$$

We will check the first relation (46) (the second relation (46) can be checked similarly). For this, we remark that the first relation (46) is equivalent to

$$
\begin{equation*}
E(w, \mathcal{J} v, u)+E(\mathcal{J} w, v, u)=E(v, \mathcal{J} w, u)+E(\mathcal{J} v, w, u) \tag{47}
\end{equation*}
$$

for any $u \in E_{-}$and $v, w \in E_{+}$. We claim that both sides of (47) vanish. Indeed, from a straightforward computation,

$$
\begin{equation*}
8 E(u, v, w)=\eta(\mathcal{J} v, u, w)+\eta(w, u, \mathcal{J} v)-\eta(v, u, \mathcal{J} w)-\eta(\mathcal{J} w, u, v), \tag{48}
\end{equation*}
$$

where $\eta(u, v, w):=\left\langle\left(D_{u} \mathcal{J}\right) v, w\right\rangle$, and then

$$
\begin{equation*}
4(E(w, \mathcal{J} v, u)+E(\mathcal{J} w, v, u))=-\eta(u, w, v)-\eta(\mathcal{J} u, w, \mathcal{J} v) . \tag{49}
\end{equation*}
$$

Define $R$ to be the right hand side (modulo a minus sign) of the above equality, i.e.

$$
\begin{align*}
R(u, v, w):= & \eta(u, w, v)+\eta(\mathcal{J} u, w, \mathcal{J} v)=\left\langle\left(D_{u} \mathcal{J}\right) w, v\right\rangle+\left\langle\left(D_{\mathcal{J} u} \mathcal{J}\right) w, \mathcal{J} v\right\rangle \\
& =\left\langle D_{u}(\mathcal{J} w), v\right\rangle+\left\langle D_{u} w, \mathcal{J} v\right\rangle+\left\langle D_{\mathcal{J} u}(\mathcal{J} w), \mathcal{J} v\right\rangle-\left\langle D_{\mathcal{J} u} w, v\right\rangle, \tag{50}
\end{align*}
$$

where we have extended $u$ and $v, w$ to sections of $E_{-}$and $E_{+}$respectively, denoted by the same symbols. Using that $D$ is torsion-free and preserves $E_{ \pm}$, $u \in \Gamma\left(E_{-}\right)$and $v, w \in \Gamma\left(E_{+}\right)$, we obtain

$$
D_{u} w=D_{w} u+[u, w]-(D u)^{*} w,\left\langle D_{w} u, \mathcal{J} v\right\rangle=0,\left\langle(D u)^{*} w, \mathcal{J} v\right\rangle=0 .
$$

These relations imply $\left\langle D_{u} w, \mathcal{J} v\right\rangle=\langle[u, w], \mathcal{J} v\rangle$. In a similar way we compute $\left\langle D_{u}(\mathcal{J} w), v\right\rangle,\left\langle D_{\mathcal{J} u}(\mathcal{J} w), \mathcal{J} v\right\rangle$ and $\left\langle D_{\mathcal{J} u} w, v\right\rangle$ in terms of the Dorfman bracket and we obtain

$$
R(u, v, w)=\langle[u, \mathcal{J} w]-\mathcal{J}[u, w], v\rangle+\langle[\mathcal{J} u, \mathcal{J} w]-\mathcal{J}[\mathcal{J} u, w], \mathcal{J} v\rangle,
$$

which vanishes by Lemma 35 ,

## 6 Generalized almost hyper-Hermitian structures: integrability and torsion-free generalized connections

Theorem 41. A generalized almost hyper-Hermitian structure $\left(G, \mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ on a Courant algebroid $E$ is generalized hyper-Kähler if and only if there is a Levi-Civita connection $D$ of $G$ which satisfies $D \mathcal{J}_{i}=0$, for all $i=1,2,3$.

In one direction the statement is obvious: if there is a Levi-Civita connection $D$ of $G$ which is hypercomplex then $\left(G, \mathcal{J}_{i}\right)$ is generalized Kähler (see Theorem (37). For the converse statement, let $\left(G, \mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ be a generalized hyper-Kähler structure. We will construct a generalized hypercomplex connection, which is also a Levi-Civita connection of $G$. Let $D$ be a generalized Levi-Civita connection of $G$ with $D \mathcal{J}_{1}=0$ (which exists, from Theorem 37). Define the generalized connections

$$
\begin{equation*}
D_{u}^{(1)}:=D_{u}-\frac{1}{2} \mathcal{J}_{2} D_{u} \mathcal{J}_{2} \tag{51}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{D}_{u}:=D_{u}^{(1)}+\eta_{u}=D_{u}-\frac{1}{2} \mathcal{J}_{2} D_{u} \mathcal{J}_{2}+\eta_{u} \tag{52}
\end{equation*}
$$

where $\eta:=\frac{1}{6} \pi\left(\left(P-\operatorname{Id}_{\Lambda^{3} E^{*}}\right) T^{D^{(1)}}\right)$. From Proposition 34, $D^{(1)}$ and $\tilde{D}$ are hypercomplex and $\tilde{D}$ is torsion-free. Note also that $D^{(1)} G=0$, since $D G=0$ and $\mathcal{J}_{2} D_{u} \mathcal{J}_{2}$ is skew-symmetric with respect to $G$. We will modify $\tilde{D}$ in order to obtain a new generalized connection $\tilde{D}^{(1)}$ which is hypercomplex, torsionfree and preserves $G$ as well. Let

$$
\begin{equation*}
\tilde{D}_{u}^{(1)}:=\tilde{D}_{u}+\tilde{\eta}_{u}=D_{u}^{(1)}+\eta_{u}+\tilde{\eta}_{u}, \tag{53}
\end{equation*}
$$

where

$$
\tilde{\eta}(u, v, w)=\tilde{\sigma}(u, v, w)-\tilde{\sigma}(u, w, v),
$$

for a section $\tilde{\sigma} \in \Gamma\left(S^{2} E \otimes E\right)$ to be determined. Our argument will be similar to the one used in the proof of Theorem 37. As before, it will be divided into several lemmas. Obviously, $\tilde{D}^{(1)}$ is torsion-free (because so is $\tilde{D}$ ).

Lemma 42. The generalized connection $\tilde{D}^{(1)}$ preserves $G$ if and only if

$$
\begin{align*}
& \tilde{\sigma}\left(u, G^{\mathrm{end}} v, w\right)-\tilde{\sigma}\left(u, w, G^{\mathrm{end}} v\right)-\tilde{\sigma}\left(u, v, G^{\mathrm{end}} w\right)+\tilde{\sigma}\left(u, G^{\mathrm{end}} w, v\right) \\
& =-\left\langle\eta_{u} G^{\mathrm{end}} v, w\right\rangle+\left\langle\eta_{u} v, G^{\mathrm{end}} w\right\rangle \tag{54}
\end{align*}
$$

and is hypercomplex if and only if

$$
\begin{equation*}
\tilde{\sigma}\left(u, \mathcal{J}_{i} v, w\right)-\tilde{\sigma}\left(u, w, \mathcal{J}_{i} v\right)+\tilde{\sigma}\left(u, v, \mathcal{J}_{i} w\right)-\tilde{\sigma}\left(u, \mathcal{J}_{i} w, v\right)=0, \quad i=1,2, \tag{55}
\end{equation*}
$$

for any $u, v, w \in E$.
Proof. Relation (54) uses (53), $D^{(1)} G=0$, and that $G^{\text {end }}$ is symmetric with respect to $\langle\cdot, \cdot\rangle$. Relation (55)) uses again (53), that $\tilde{D}$ is hypercomplex and that $\mathcal{J}_{i}$ are skew-symmetric with respect to $\langle\cdot, \cdot\rangle$. (In fact, relation (55) expresses $\tilde{D}^{(1)} \mathcal{J}_{1}=\tilde{D}^{(1)} \mathcal{J}_{2}=0$; but then $\tilde{D}^{(1)} \mathcal{J}_{3}=0$ because $\left.\mathcal{J}_{3}=\mathcal{J}_{1} \mathcal{J}_{2}\right)$.

We remark that (54) is satisfied for $v, w \in E_{+}$and for $v, w \in E_{-}$, for any section $\tilde{\sigma}$ of $S^{2} E \otimes E$. Moreover, if it is satisfied for $v \in E_{+}, w \in E_{-}$, then it is satisfied for $v \in E_{-}, w \in E_{+}$as well (again, for any such $\tilde{\sigma}$ ). We conclude that (54) is equivalent to

$$
\begin{equation*}
\tilde{\sigma}(u, v, w)-\tilde{\sigma}(u, w, v)=-\eta(u, v, w), \tag{56}
\end{equation*}
$$

for $u \in E, v \in E_{+}$and $w \in E_{-}$(or to (56) with $E_{ \pm}$interchanged). Thus we are looking for $\tilde{\sigma} \in \Gamma\left(S^{2} E \otimes E\right)$ such that relations (55) and (56) hold.

Lemma 43. If relation (56) is satisfied, then relation (55) is satisfied for $v \in E_{+}, w \in E_{-}$and for $v \in E_{-}, w \in E_{+}$(and any $u \in E$ ).

Proof. From the definition of $\eta$ we know that $\eta_{u} \in \Gamma\left(\Lambda_{\mathbb{H}}^{1,1} E^{*}\right)$, that is $\left[\eta_{u}, \mathcal{J}_{i}\right]=$ 0 for all $i=1,2,3$. Evaluating $\left\langle\left[\eta_{u}, J_{i}\right] v, w\right\rangle=0$ on $v \in E_{ \pm}, w \in E_{\mp}$ and using (56) gives the desired result.

From the above arguments, we are looking for $\tilde{\sigma} \in \Gamma\left(S^{2} E \otimes E\right)$ such that (56) is satisfied for any $v \in E_{+}, w \in E_{-}$(and $u \in E$ ), and (55) is satisfied for any $v, w \in E_{+}$and any $v, w \in E_{-}$(and any $u \in E$ ). Relation (56) with $u \in E_{+}$gives

$$
\begin{equation*}
\tilde{\sigma}^{++-}(u, v, w)=\tilde{\sigma}^{+-+}(u, w, v)-\eta(u, v, w) . \tag{57}
\end{equation*}
$$

Setting $\tilde{\sigma}^{+-+}(u, w, v):=-\eta(v, u, w)$ we obtain that $\tilde{\sigma}^{++-}$, defined by (57), is symmetric in the first two arguments. Similarly, relation (56) with $u \in E_{-}$ gives

$$
\begin{equation*}
\tilde{\sigma}^{--+}(u, w, v)=\tilde{\sigma}^{-+-}(u, v, w)+\eta(u, v, w) . \tag{58}
\end{equation*}
$$

Setting $\tilde{\sigma}^{-+-}(u, v, w):=\eta(w, v, u)$ we obtain that $\tilde{\sigma}^{--+}$, defined by (58), is symmetric in the first two arguments. Finally, setting $\tilde{\sigma}^{-++}(u, v, w):=$ $\tilde{\sigma}^{+-+}(v, u, w), \tilde{\sigma}^{+--}(u, v, w):=\tilde{\sigma}^{-+-}(v, u, w), \tilde{\sigma}^{+++}:=0$ and $\tilde{\sigma}^{---}:=0$ we obtain a section $\tilde{\sigma} \in \Gamma\left(S^{2} E \otimes E\right)$ which satisfies relation (56). The next lemma concludes the proof of Theorem 41.

Lemma 44. The section $\tilde{\sigma}$ satisfies relation (55) with $v, w \in E_{+}$and $v, w \in$ $E_{-}$(and any $u \in E$ ).
Proof. We explain the argument when $v, w \in E_{+}$(when $v, w \in E_{-}$the argument is similar). Assume, therefore, that $v, w \in E_{+}$. Since $\tilde{\sigma}\left(E_{+}, E_{+}, E_{+}\right)=$ 0 , relation (55) is satisfied when $u \in E_{+}$. Let $u \in E_{-}$. Then relation (55) involves only the component $\tilde{\sigma}^{-++}$of $\tilde{\sigma}$. Using that

$$
\tilde{\sigma}^{-++}(u, v, w)=-\eta(w, v, u),
$$

we obtain that relation (55) with $u \in E_{-}, v, w \in E_{+}$is equivalent to

$$
\begin{equation*}
\eta\left(w, \mathcal{J}_{i} v, u\right)+\eta\left(\mathcal{J}_{i} w, v, u\right)=\eta\left(v, \mathcal{J}_{i} w, u\right)+\eta\left(\mathcal{J}_{i} v, w, u\right), i=1,2 . \tag{59}
\end{equation*}
$$

or to

$$
\begin{equation*}
\left\langle\left(\tilde{D}_{w}-D_{w}^{(1)}\right) \mathcal{J}_{i} v-\left(\tilde{D}_{\mathcal{J}_{i} v}-D_{\mathcal{J}_{i} v}^{(1)}\right) w, u\right\rangle+\left\langle\left(\tilde{D}_{\mathcal{J}_{i} w}-D_{\mathcal{J}_{i} w}^{(1)}\right) v-\left(\tilde{D}_{v}-D_{v}^{(1)}\right) \mathcal{J}_{i} w, u\right\rangle=0, \tag{60}
\end{equation*}
$$

where we used $\eta=\tilde{D}-D^{(1)}$. Remark that we can replace in (60) $D^{(1)}$ by $D$, since $D^{(1)}=D-\frac{1}{2} \mathcal{J}_{2} D \mathcal{J}_{2}$ and $\mathcal{J}_{2} D \mathcal{J}_{2}$ is a 1 -form with values in the bundle of skew-symmetric endomorphisms with respect to both $G$ and $\langle\cdot, \cdot\rangle$ and such endomorphisms preserve $E_{ \pm}$, which are orthogonal with respect to $\langle\cdot, \cdot\rangle$. Since $D$ and $\tilde{D}$ are torsion-free, for any $e_{i} \in E$,

$$
\left\langle\left(D_{e_{1}}-\tilde{D}_{e_{1}}\right) e_{2}-\left(D_{e_{2}}-\tilde{D}_{e_{2}}\right) e_{1}, e_{3}\right\rangle=\left\langle\left(\tilde{D}_{e_{3}}-D_{e_{3}}\right) e_{1}, e_{2}\right\rangle .
$$

In particular,

$$
\begin{aligned}
& \left\langle\left(\tilde{D}_{w}-D_{w}\right) \mathcal{J}_{i} v-\left(\tilde{D}_{\mathcal{J}_{i} v}-D_{\mathcal{J} v}\right) w, u\right\rangle=\left\langle\left(D_{u}-\tilde{D}_{u}\right) w, \mathcal{J}_{i} v\right\rangle \\
& \left\langle\left(\tilde{D}_{\mathcal{J}_{i} w}-D_{\mathcal{J}_{i} w}\right) v-\left(\tilde{D}_{v}-D_{v}\right) \mathcal{J}_{i} w, u\right\rangle=\left\langle\left(D_{u}-\tilde{D}_{u}\right) \mathcal{J}_{i} w, v\right\rangle .
\end{aligned}
$$

Therefore, we need to check that

$$
\begin{equation*}
E(u, v, w):=\left\langle\left(D_{u}-\tilde{D}_{u}\right) w, \mathcal{J}_{i} v\right\rangle+\left\langle\left(D_{u}-\tilde{D}_{u}\right) \mathcal{J}_{i} w, v\right\rangle=0 . \tag{61}
\end{equation*}
$$

For $i=1$ relation (61) is true, because $D \mathcal{J}_{1}=\tilde{D} \mathcal{J}_{1}=0$. For $i=2$, we use $\tilde{D} \mathcal{J}_{2}=0$ and that $D$ is torsion-free. With these remarks,

$$
\begin{aligned}
& E(u, v, w)=\left\langle D_{u} w, \mathcal{J}_{2} v\right\rangle+\left\langle D_{u}\left(\mathcal{J}_{2} w\right), v\right\rangle \\
& =\left\langle D_{w} u+[u, w]-(D u)^{*} w, \mathcal{J}_{2} v\right\rangle+\left\langle D_{\mathcal{J}_{2} w} u+\left[u, \mathcal{J}_{2} w\right]-(D u)^{*}\left(\mathcal{J}_{2} w\right), v\right\rangle,
\end{aligned}
$$

where we have extended $u$, respectively $v, w$ to sections of $E_{-}$, respectively $E_{+}$, denoted by the same symbols. Using that $D$ preserves $E_{ \pm}$and $u \in$ $\Gamma\left(E_{-}\right), v, w \in \Gamma\left(E_{+}\right)$, we obtain that

$$
E(u, v, w)=\left\langle\left[u, \mathcal{J}_{2} w\right]-\mathcal{J}_{2}[u, w], v\right\rangle=0,
$$

where the last equality follows from Lemma 35 .

## Part II

## 7 The space of local Dirac generating operators

Let $E$ be a Courant algebroid of rank $2 n$, with anchor $\pi: E \rightarrow T M$, scalar product $\langle\cdot, \cdot\rangle$ and Dorfman bracket $[\cdot, \cdot]$. We denote by $\mathrm{Cl}(E)$ the bundle of Clifford algebras over $(E,\langle\cdot, \cdot\rangle)$ with the Clifford relation $e^{2}=\langle e, e\rangle, e \in E$. Let $S \rightarrow M$ be a real vector bundle of irreducible $\mathrm{Cl}(E)$-modules. We will call $S$ a spinor bundle over $\mathrm{Cl}(E)$. The representation of $\mathrm{Cl}(E)$ on $S$, denoted by

$$
\begin{equation*}
\gamma: \operatorname{Cl}(E) \rightarrow \operatorname{End}(S), \quad a \mapsto \gamma(a):=\gamma_{a}, \tag{62}
\end{equation*}
$$

is an isomorphism of algebra bundles. To simplify notation, we shall sometimes write as for the Clifford action $\gamma_{a} s$ of $a \in \mathrm{Cl}(E)$ on $s \in S$.

Recall that the Clifford algebra bundle $\mathrm{Cl}(E)$ is $\mathbb{Z}_{2}$-graded. We denote the subbundle of degree $i \in \mathbb{Z}_{2}$ by $\mathrm{Cl}^{i}(E)$. Since $\langle\cdot, \cdot\rangle$ has neutral signature, the bundle $S$ has a compatible $\mathbb{Z}_{2}$-grading denoted by $S=S^{0} \oplus S^{1}$, where $S^{0}=\frac{1}{2} \gamma_{(1+\omega)} S$ and $S^{1}=\frac{1}{2} \gamma_{(1-\omega)} S$, with $\omega$ a volume element of $\mathrm{Cl}(E)$ (see e.g. Proposition 3.6 of [19]). Moreover, an argument analogous to the proof of Proposition 5.10 of [19] shows that the $\mathrm{Cl}^{0}(E)$-submodules $S^{0}$ and $S^{1}$ are pointwise inequivalent and irreducible. There is an induced $\mathbb{Z}_{2}$-grading on $\operatorname{End}(\Gamma(S))$ and, in particular, on the algebra of differential operators on $S$, which includes $\Gamma(\operatorname{End} S) \subset \operatorname{End}(\Gamma(S))$ as the subalgebra of operators of 0 -th order. We will denote by

$$
[A, B]=A B-(-1)^{\operatorname{deg} A \operatorname{deg} B} B A
$$

the super commutator of two homogeneous elements $A, B \in \operatorname{End}(\Gamma(S))$, where $\operatorname{deg} A$ stands for the degree of $A$.

Definition 45. A first order odd differential operator dd on a spinor bundle $S$ over $\operatorname{Cl}(E)$ is called a Dirac generating operator for $E$ if for all $f \in C^{\infty}(M)$ and $e, e_{1}, e_{2} \in \Gamma(E)$,
i) $\left.\left[[\not d, f], \gamma_{e}\right]\right]=\pi(e)(f)$,
ii) $\left[\left[d, \gamma_{e_{1}}\right], \gamma_{e_{2}}\right]=\gamma_{\left[e_{1}, e_{2}\right]}$ and
iii) $\not d^{2} \in C^{\infty}(M)$.

Note that given $(E,\langle\cdot, \cdot\rangle)$ and $\not d$ one can reconstruct the full Courant algebroid structure from i) and ii). This is why the operator $\not d d$ is called generating.
Proposition 46. Suppose that there is a Dirac generating operator d for $E$ on $S$. Then the set of Dirac generating operators for $E$ on $S$ has the structure of an affine space modelled on the vector space

$$
\begin{equation*}
V_{d d}:=\left\{e \in \Gamma(E) \mid\left[d, \gamma_{e}\right] \in C^{\infty}(M)\right\} . \tag{63}
\end{equation*}
$$

In particular, $V_{d}$ is independent of the choice of Dirac generating operator d.
Proof. We first check that $\not d^{\prime}:=\not d+\gamma_{e_{0}}$ is a Dirac generating operator for all $e_{0} \in V_{d}$. Since $\left[\gamma_{e_{0}}, f\right]=0$ and $\left[\gamma_{e_{0}}, \gamma_{e_{1}}\right]=2\left\langle e_{0}, e_{1}\right\rangle \in C^{\infty}(M)$ the properties i) and ii) in Definition 45 for $\not d$ immediately imply the same properties for $d^{\prime}$. Finally, the equation $\left(d^{\prime}\right)^{2}=d^{2}+\left[d, \gamma_{e_{0}}\right]+\left\langle e_{0}, e_{0}\right\rangle$ shows that property iii) holds for $d^{\prime}$ if it holds for $\not d d$ and $e_{0} \in V_{d}$.

Conversely, we show that given Dirac generating operators $\not d$ and $\not d^{\prime}$, there exists $e_{0} \in V_{d d}$ such that $L:=\not d^{\prime}-\not d=\gamma_{e_{0}}$. We first observe that $[L, f]$ is a 0 -th order operator of odd degree for all $f \in C^{\infty}(M)$. By property i) it satisfies $\left[[L, f], \gamma_{e}\right]=0$ for all $e \in \Gamma(E)$. This implies that $[L, f]$ commutes with $\mathrm{Cl}^{0}(E)$. Being of odd degree, it interchanges $S^{0}$ and $S^{1}$ and we deduce that $[L, f]=0$ since the irreducible $\mathrm{Cl}^{0}(\mathrm{E})$-modules $S^{0}$ and $S^{1}$ are inequivalent. This shows that the odd differential operator $L$ is of 0 -th order, that is $L=\gamma_{a}$ for some $a \in \Gamma\left(\mathrm{Cl}^{1}(E)\right)$.

Next we consider the even 0 -the order operator

$$
L^{\prime}:=\left[L, \gamma_{e}\right]=\gamma(a e+e a),
$$

$e \in \Gamma(E)$. It commutes with $\mathrm{Cl}^{1}(E)$, in virtue of property ii), and is hence a scalar operator (since $\langle\cdot, \cdot\rangle$ has neutral signature). We conclude that $a e+e a$ is a scalar in $\operatorname{Cl}(E)$, for any $e \in E$. This easily implies that $a=: e_{0} \in \Gamma(E)$, by a straightforward computation in which $e$ runs through the elements of an orthonormal frame. Now property iii) implies that $e_{0} \in V_{d}$.

The last claim is now obvious: if $\not d$ and $\not d^{\prime}$ are two Dirac generating operators then $\not d^{\prime}=\not d+\gamma_{e}$ for $e \in V_{d} \subset \Gamma(E)$ which implies $V_{d^{\prime}}=V_{d}$.

The next theorem is our main result in this section.
Theorem 47 (Alekseev-Xu). Let $E$ be a regular Courant algebroid. Every spinor bundle $S$ over $\mathrm{Cl}(E)$ admits locally a Dirac generating operator.

We divide the proof of Theorem 47 into several steps. Let $D$ be a generalized connection on $E$. The existence of $D$ is ensured by Example 5. The generalized connection $D$ induces an $E$-connection in $\mathrm{Cl}(E)$, which we denote again by $D$. Next we choose an $E$-connection $D^{S}$ on $S$ compatible with $D$ in the sense that

$$
D_{e}^{S}(a s)=\left(D_{e} a\right) s+a D_{e}^{S} s,
$$

for all $e \in \Gamma(E), a \in \Gamma(\mathrm{Cl}(E)), s \in \Gamma(S)$.
The existence of such a connection $D^{S}$ can be shown as follows. The bundle $(E,\langle\cdot, \cdot\rangle)$ admits locally a spin structure. To this structure we can associate a spinor bundle $\Sigma$ over some domain $U \subset M$. The $E$-connection $D$ on $E$ induces an $E$-connection $D^{\Sigma}$ on $\Sigma$. The connection form of $D^{\Sigma}$ with respect to a local trivialization of the spin structure is one half of the connection form of $D$ with respect to the corresponding local orthonormal frame of $E$. (Both forms can be considered as local sections of $E^{*} \otimes \mathfrak{s o}(E)$, after identifying $\mathfrak{s p i n}(E) \cong \mathfrak{s o}(E)$ via the adjoint representation ad : $\mathfrak{s p i n}(E) \rightarrow \mathfrak{s o}(E)$, $\operatorname{ad}_{u} v:=u v-v u$.) In more concrete terms, let $\left(e_{i}\right)$ be an orthonormal frame of $\left.E\right|_{U}$ and $\left(\sigma_{\alpha}\right)$ a frame of $\Sigma$ such that

$$
e_{i} \sigma_{\alpha}=\sum_{\beta} C_{i \alpha}^{\beta} \sigma_{\beta},
$$

where $C_{i \alpha}^{\beta}$ are constants. Let $\left(\omega_{i j}\right)$ be the (skew-symmetric) matrix of 1-forms defined by

$$
\begin{aligned}
D_{v}\left(e_{k}\right) & =-\epsilon_{k} \sum_{j} \omega_{j k}(v) e_{j}=2 \sum_{j<p} \omega_{p j}(v)\left(\epsilon_{p} e_{p}^{*} \otimes e_{j}-\epsilon_{j} e_{j}^{*} \otimes e_{p}\right)\left(e_{k}\right) \\
& =2 \sum_{j<p} \omega_{p j}(v)\left(e_{p} \wedge e_{j}\right)\left(e_{k}\right),
\end{aligned}
$$

where $\epsilon_{k}=\left\langle e_{k}, e_{k}\right\rangle \in\{ \pm 1\}$. Then $D_{e}^{\Sigma}\left(\sigma_{\alpha}\right):=\frac{1}{2} \sum_{i<j} \omega_{j i}(v) e_{i} e_{j} \sigma_{\alpha}$ is compatible with $\left.D\right|_{U} .\left(\right.$ Note that the element $\frac{1}{2} e_{i} e_{j} \in \mathfrak{s p i n}(E) \subset \mathrm{Cl}(\mathrm{E})(i \neq j)$ acts under the adjoint representation as $\epsilon_{j} e_{j}^{*} \otimes e_{i}-\epsilon_{i} e_{i}^{*} \otimes e_{j} \in \mathfrak{s o}(E)$ and the latter corresponds to the bivector $e_{j} \wedge e_{i} \in \Lambda^{2} E$.) Since $(E,\langle\cdot, \cdot\rangle)$ has neutral signature, $\left.S\right|_{U}$ and $\Sigma$ differ only by a real line bundle $L$ over $U:\left.S\right|_{U} \cong \Sigma \otimes L$. Choosing an $E$-connection in $L$, we obtain an $E$-connection $D^{S, U}$ in $S_{U}$ by taking the tensor product with the connection $D^{\Sigma}$. By considering an open
covering $\left(U_{i}\right)$ of $M$ and a corresponding partition of unity, we can glue the $E$-connections $D^{S, U_{i}}$ to an $E$-connection $D^{S}$.

The $E$-connection $D^{S}$ gives rise to a first order differential operator on $S$, which we call the Dirac operator:

$$
\begin{equation*}
\not D^{S}=\frac{1}{2} \sum_{i=1}^{2 n} \tilde{e}_{i} D_{e_{i}}^{S}, \tag{64}
\end{equation*}
$$

where $\left(e_{i}\right)$ is any local frame of $E$ and $\left(\tilde{e}_{i}\right)$ is the metrically dual frame, that is $\left\langle e_{i}, \tilde{e}_{j}\right\rangle=\delta_{i j}$.

Lemma 48. For any generalized connection $D$ and compatible $E$-connection $D^{S}$, the operator

$$
\begin{equation*}
d x=\not D^{S}+\frac{1}{4} \gamma_{T} \tag{65}
\end{equation*}
$$

satisfies conditions i) and ii) from Definition 45. Above $T \in \Gamma\left(\wedge^{3} E^{*}\right) \cong$ $\Gamma\left(\wedge^{3} E\right) \subset \Gamma(\mathrm{Cl}(E))$ denotes the torsion of $D$.

Proof. We compute $\left.\left[[\not d, f], \gamma_{v}\right]\right]=\left[\left[\not D^{S}, f\right], \gamma_{v}\right]$ for $f \in C^{\infty}(M)$ and $v \in \Gamma(E)$. We find

$$
\left[\not D^{S}, f\right]=\frac{1}{2} \sum_{i} \pi\left(e_{i}\right)(f) \gamma_{\tilde{e}_{i}}, \quad\left[\left[\not D^{S}, f\right], \gamma_{v}\right]=\sum_{i} \pi\left(e_{i}\right)(f)\left\langle\tilde{e}_{i}, v\right\rangle=\pi(v)(f)
$$

This shows that i) in Definition 45 is satisfied.
Next we compute

$$
\left[\not D^{S}, \gamma_{v}\right]=\frac{1}{2} \sum_{i} \gamma_{\tilde{e}_{i}} \gamma_{D_{e_{i}} v}+D_{v}^{S}, \quad\left[\left[\not D^{S}, \gamma_{v}\right], \gamma_{w}\right]=\frac{1}{2} \sum_{i}\left[\gamma_{\tilde{e}_{i}} \gamma_{D_{e_{i}} v}, \gamma_{w}\right]+\gamma_{D_{v} w},
$$

where $w \in \Gamma(E)$. A simple calculation in the Clifford algebra shows that for all $u, v, w \in E$ :

$$
u v w-w u v=-2\langle u, w\rangle v+2\langle v, w\rangle u .
$$

So
$\frac{1}{2} \sum_{i}\left[\gamma_{\tilde{e}_{i}} \gamma_{D_{e_{i}} v}, \gamma_{w}\right]=-\sum_{i}\left\langle\tilde{e}_{i}, w\right\rangle \gamma_{D_{e_{i}} v}+\sum_{i}\left\langle D_{e_{i}} v, w\right\rangle \gamma_{\tilde{e}_{i}}=-\gamma_{D_{w} v}+\sum_{i}\left\langle D_{e_{i}} v, w\right\rangle \gamma_{\tilde{e}_{i}}$ and using that $D$ has torsion $T$ we obtain

$$
\begin{aligned}
{\left[\left[\not D^{S}, \gamma_{v}\right], \gamma_{w}\right] } & =\gamma_{D_{v} w-D_{w} v}+\sum_{i}\left\langle D_{e_{i}} v, w\right\rangle \gamma_{\tilde{e}_{i}} \\
& =\gamma_{T(v, w)}+\gamma_{[v, w]}-\gamma_{(D v)^{*} w}+\sum_{i}\left\langle D_{e_{i}} v, w\right\rangle \gamma_{\tilde{e}_{i}} \\
& =\gamma_{T(v, w)}+\gamma_{[v, w]} .
\end{aligned}
$$

A simple calculation in the Clifford algebra shows that (for any 3-form $T$ )

$$
\left[\left[\gamma_{T}, \gamma_{v}\right], \gamma_{w}\right]=-4 \gamma_{T(v, w)} .
$$

Thus we can conclude that

$$
\left[\left[\not d, \gamma_{v}\right], \gamma_{w}\right]=\left[\left[\not D^{S}+\frac{1}{4} \gamma_{T}, \gamma_{v}\right], \gamma_{w}\right]=\gamma_{[v, w]} .
$$

So ii) in Definition 45 is also satisfied.
In order to conclude the proof of Theorem 47 we therefore need to find a locally defined generalized connection $D$ on $E$ and a compatible $E$-connection $D^{S}$ on $S$ such that condition iii) from Definition 45holds as well. To analyze condition iii) in Definition 45 we will use the following lemma.

Lemma 49. ([77) Let E be a regular Courant algebroid with scalar product $\langle\cdot, \cdot\rangle$ and anchor $\pi: E \rightarrow T M$.
i) The bundle $\operatorname{ker} \pi \subset E$ is a coisotropic subbundle of $E$, that is $(\operatorname{ker} \pi)^{\perp} \subset$ $\operatorname{ker} \pi$.
ii) The bundle $E$ decomposes as $E=\operatorname{ker} \pi \oplus \mathcal{F}$ where $\mathcal{F}$ is isotropic.
iii) The bundle $\operatorname{ker} \pi$ decomposes as $\operatorname{ker} \pi=(\operatorname{ker} \pi)^{\perp} \oplus \mathcal{G}$ where $\mathcal{G}$ is orthogonal to $\mathcal{F}$.
iv) The decomposition $E=\left((\operatorname{ker} \pi)^{\perp} \oplus \mathcal{F}\right) \oplus \mathcal{G}$ is orthogonal with respect to $\langle\cdot, \cdot\rangle$. The restrictions of $\langle\cdot, \cdot\rangle$ to the two factors $(\operatorname{ker} \pi)^{\perp} \oplus \mathcal{F}$ and $\mathcal{G}$ have neutral signature.

Proof. i) By the regularity assumption, we know that $\operatorname{ker} \pi \subset E$ is a subbundle. The image of $\pi^{*}: T^{*} M \rightarrow E$ (cf. Definition (1) is contained in $(\operatorname{ker} \pi)^{\perp}$ and therefore

$$
\begin{equation*}
\operatorname{im} \pi^{*}=(\operatorname{ker} \pi)^{\perp} \tag{66}
\end{equation*}
$$

by comparing ranks. We claim that $\pi \circ \pi^{*}=0$. Since (for $E \neq 0$ )

$$
T_{p}^{*} M=\operatorname{span}\left\{d f_{p} \mid f=\langle e, e\rangle, e \in \Gamma(E)\right\},
$$

the claim is a consequence of axiom C5) from Definition 1:

$$
\pi \circ \pi^{*} d\langle e, e\rangle=2 \pi[e, e]=2[\pi(e), \pi(e)]=0 .
$$

From (666) and $\pi \circ \pi^{*}=0$ we obtain $(\operatorname{ker} \pi)^{\perp} \subset \operatorname{ker} \pi$ as needed.
ii) Let $F:=\pi(E)$ and $s:=\operatorname{rank} F$. Let $\lambda_{0}: F \rightarrow E$ be a section of $\pi: E \rightarrow F$ and define $\rho: F \rightarrow F^{*}$ by $\rho(X)(Y):=\left\langle\lambda_{0}(X), \lambda_{0}(Y)\right\rangle$. Let

$$
\lambda: F \rightarrow E, \lambda:=\lambda_{0}-\frac{1}{2} \pi^{*} \circ \rho .
$$

As $\pi \circ \lambda_{0}=\operatorname{Id}_{F}$ and $\pi \circ \pi^{*}=0, \lambda$ is again a section and we deduce that $\mathcal{F}:=\lambda(F) \subset E$ is transverse to ker $\pi$ and hence is a complement of $\operatorname{ker} \pi$ in $E$ (as $\pi: E \rightarrow F$ is surjective). We now show that $\mathcal{F}$ is isotropic. For this, we compute

$$
\begin{aligned}
\langle\lambda(X), \lambda(Y)\rangle & =\left\langle\lambda_{0}(X), \lambda_{0}(Y)\right\rangle-\frac{1}{2}\left\langle\lambda_{0}(X), \pi^{*} \circ \rho(Y)\right\rangle \\
& -\frac{1}{2}\left\langle\lambda_{0}(Y), \pi^{*} \circ \rho(X)\right\rangle+\frac{1}{4}\left\langle\pi^{*} \circ \rho(X), \pi^{*} \circ \rho(Y)\right\rangle .
\end{aligned}
$$

Remark that

$$
\begin{equation*}
\left\langle\pi^{*} \circ \rho(X), \lambda_{0}(Y)\right\rangle=\rho(X)\left(\left(\pi \circ \lambda_{0}\right)(Y)\right)=\left\langle\lambda_{0}(X), \lambda_{0}(Y)\right\rangle, \tag{67}
\end{equation*}
$$

where we used the definition of $\rho$, and

$$
\left\langle\pi^{*} \circ \rho(X), \pi^{*} \circ \rho(Y)\right\rangle=\rho(X)\left(\pi \circ \pi^{*}(\rho(Y))\right)=0
$$

The above computations show that $\mathcal{F}$ is isotropic.
iii) and iv) We first notice that the scalar product $\langle\cdot, \cdot\rangle$ defines a nondegenerate pairing between $(\operatorname{ker} \pi)^{\perp}$ and $\mathcal{F}$. In fact, these two transversal subbundles are of the same rank and if $v \in(\operatorname{ker} \pi)^{\perp}$ is orthogonal to $\mathcal{F}$, then it is orthogonal to $\mathcal{F} \oplus \operatorname{ker} \pi=E$, which implies $v=0$. Since the two subbundles are isotropic, we obtain that the restriction of $\langle\cdot, \cdot\rangle$ to $(\operatorname{ker} \pi)^{\perp} \oplus \mathcal{F}$ is non-degenerate of neutral signature, as claimed. It follows that $\mathcal{G}:=$ $\left((\operatorname{ker} \pi)^{\perp} \oplus \mathcal{F}\right)^{\perp}=\operatorname{ker} \pi \cap \mathcal{F}^{\perp}$ is also non-degenerate of neutral signature.

The above lemma implies that for any $U \subset M$ sufficiently small, the bundle $\left.E\right|_{U}$ admits a frame $\left(p_{a}, q_{a}\right), a=1, \ldots, n$, such that $\left(p_{a}\right), a=1, \ldots, n$, span a maximally isotropic subbundle $\mathcal{P}$ of ker $\pi,\left(p_{a}\right), a=1, \ldots, s$, span $(\operatorname{ker} \pi)^{\perp},\left(q_{a}\right), a=1, \ldots, n$, span a maximally isotropic subbundle $\mathcal{Q}$ of $E$, $q_{a} \in \operatorname{ker} \pi$ for any $a \geq s+1,\left\langle p_{a}, q_{b}\right\rangle=\delta_{a b}$ and $\left[\pi\left(q_{a}\right), \pi\left(q_{b}\right)\right]=0$ for any $a$, $b$. For the latter condition we are using that the image of $\pi$ is an integrable distribution on $M$ (by the axiom C2) in Definition (1). More precisely, using Lemma 49 iv), this basis can be constructed in the following way: start with any basis $q_{a}, a=1, \cdots, s$, of $\mathcal{F}$ such that $\left[\pi\left(q_{a}\right), \pi\left(q_{b}\right)\right]=0$ for any $a$, $b$. Consider the basis $p_{a}, a=1, \cdots, s$, of $(\operatorname{ker} \pi)^{\perp}$ such that $\left\langle p_{a}, q_{b}\right\rangle=\delta_{a b}$ for any $a, b$. Finally, choose a basis $p_{a}, q_{a}, a=s+1, \cdots, n$, of $\mathcal{G}$ such that $\left\langle p_{a}, p_{b}\right\rangle=\left\langle q_{a}, q_{b}\right\rangle=0$ and $\left\langle p_{a}, q_{b}\right\rangle=\delta_{a b}$ for any $a, b=s+1, \cdots, n$. The following inclusions summarize the properties of the two complementary maximally isotropic subbundles $\mathcal{P}$ and $\mathcal{Q}$ of $E$ :

$$
(\operatorname{ker} \pi)^{\perp} \subset \mathcal{P} \subset \operatorname{ker} \pi, \quad \mathcal{F} \subset \mathcal{Q} \subset \mathcal{F}^{\perp}=\mathcal{F} \oplus \mathcal{G}
$$

The next corollary will be useful in the proof of Lemma 51 below.

Corollary 50. For any $\sigma \in \Gamma(\operatorname{ker} \pi), \sum_{a=1}^{n} \pi\left(q_{a}\right)\left\langle\sigma, p_{a}\right\rangle=0$.
Proof. Each term in the above sum vanishes: if $a \leq s$ then $p_{a} \in \Gamma\left((\operatorname{ker} \pi)^{\perp}\right)$ and $\left\langle\sigma, p_{a}\right\rangle=0$. If $a \geq s+1$, then $q_{a} \in \Gamma(\operatorname{ker} \pi)$ and $\pi\left(q_{a}\right)=0$.

Let $\nabla$ be the connection on $\left.E\right|_{U}$ such that the frame $\left(p_{a}, q_{a}\right)$ is parallel. Then $\nabla$ is flat, preserves the scalar product $\langle\cdot, \cdot\rangle$ of $E$ and $\left.S\right|_{U}$ admits a flat connection $\nabla^{S}$ compatible with $\nabla$. Then $\nabla$ induces a generalized connection $D$ on $\left.E\right|_{U}$ and $\nabla^{S}$ induces an $E$-connection $D^{S}$ on $\left.S\right|_{U}$ which is compatible with $D$.

The next lemma concludes the proof of Theorem 47.
Lemma 51. The operator (65) constructed using $D$ and $D^{S}$ satisfies $\phi^{2} \in$ $C^{\infty}(U)$ and is a Dirac generating operator.
Proof. The Dirac operator $\not D^{S}$ has the expression

$$
\not D^{S}=\frac{1}{2} \sum_{a}\left(p_{a} D_{q_{a}}^{S}+q_{a} D_{p_{a}}^{S}\right)=\frac{1}{2} \sum_{a} p_{a} D_{q_{a}}^{S},
$$

since $\pi\left(p_{a}\right)=0$. To see this it is sufficient to remark that the frame $\left(\tilde{q}_{a}, \tilde{p}_{b}\right)$ dual to the frame $\left(q_{a}, p_{b}\right)$ is precisely $\left(p_{a}, q_{b}\right)$. Its square is given by

$$
\begin{equation*}
\left(\not D^{S}\right)^{2}=\frac{1}{4} \sum_{a} p_{a} p_{b} D_{q_{a}}^{S} D_{q_{b}}^{S}=\frac{1}{4} \sum_{a}\left\langle p_{a}, p_{b}\right\rangle \nabla_{\pi\left(q_{a}\right)}^{S} \nabla_{\pi\left(q_{b}\right)}^{S}=0, \tag{68}
\end{equation*}
$$

where we used $\nabla^{S} p_{a}=0$, the flatness of $\nabla^{S}$ and $\left[\pi\left(q_{a}\right), \pi\left(q_{b}\right)\right]=0$ for any $a$, b.

Next, we compute $\not D^{S} \gamma_{T}+\gamma_{T} \not D^{S}=\left[\not D^{S}, \gamma_{T}\right]$. We write the torsion $T$ of $D$ as $T=\frac{1}{6} \sum T^{i j k} e_{i j k} \in \mathrm{Cl}(E)$, where $\left(e_{i}\right)$ is a $D$-parallel orthonormal frame and $e_{i j k}:=e_{i} e_{j} e_{k}$. The coefficients $T^{i j k}$ are given by

$$
\begin{equation*}
T^{i j k}=T\left(\tilde{e}_{i}, \tilde{e}_{j}, \tilde{e}_{k}\right)=\epsilon_{i} \epsilon_{j} \epsilon_{k} T\left(e_{i}, e_{j}, e_{k}\right)=-\epsilon_{i} \epsilon_{j} \epsilon_{k}\left\langle\left[e_{i}, e_{j}\right], e_{k}\right\rangle \tag{69}
\end{equation*}
$$

where $\left(\tilde{e}_{i}\right)$ is the frame of $\left.E\right|_{U}$ metrically dual to $\left(e_{i}\right)$, i.e. $\tilde{e}_{i}=\epsilon_{i} e_{i}$ with $\epsilon_{i}=\left\langle e_{i}, e_{i}\right\rangle$. Using the abbreviation $\gamma_{e_{i} e_{j} e_{k}}=\gamma_{i j k}$, we write

$$
\begin{aligned}
& 12\left[\not D^{S}, \gamma_{T}\right]=\sum_{i, j, k, \ell}\left[\gamma_{\tilde{e}_{\ell}} D_{e_{\ell}}^{S}, T^{i j k} \gamma_{i j k}\right] \\
& =\sum_{i, j, k, \ell} \pi\left(e_{\ell}\right)\left(T^{i j k}\right) \gamma_{\tilde{e}_{\ell}} \gamma_{i j k}+\sum_{i, j, k, \ell} T^{i j k} \gamma_{\tilde{e}_{\ell}} \gamma(\underbrace{D_{e_{\ell}} e_{i j k}}_{=0})+\sum_{i, j, k, \ell} T^{i j k}\left[\gamma_{\tilde{e}_{\ell}}, \gamma_{i j k}\right] D_{e_{\ell}}^{S} .
\end{aligned}
$$

Note that, for any fixed $\ell$,

$$
\sum_{i, j, k} T^{i j k}\left[\gamma_{\tilde{e} \ell}, \gamma_{i j k}\right]=6 \sum_{j, k} T^{\ell j k} \gamma_{j k} .
$$

Hence

$$
\sum_{i, j, k, \ell} T^{i j k}\left[\gamma_{\tilde{e} \ell}, \gamma_{i j k}\right] D_{e_{\ell}}^{S}=-6 \sum_{j, k} \epsilon_{j} \epsilon_{k} \gamma_{j k} D_{\left[e_{j}, e_{k}\right]}^{S} .
$$

To compute the last term we choose the orthonormal frame $\left(e_{i}\right)$ to be

$$
\left(e_{i}\right)_{i=1, \ldots, 2 n}=\left(\frac{1}{\sqrt{2}}\left(p_{a}+q_{a}\right), \frac{1}{\sqrt{2}}\left(p_{a}-q_{a}\right)\right)_{a=1, \ldots, n}
$$

where $\left(p_{a}, q_{a}\right)$ is the frame constructed above. Then $\pi\left[e_{j}, e_{k}\right]=\left[\pi e_{j}, \pi e_{k}\right]=0$, because $\left[\pi\left(q_{a}\right), \pi\left(q_{b}\right)\right]=0$. This implies that $D_{\left[e_{j}, e_{k}\right]}^{S}=0$, showing that

$$
\begin{equation*}
\left[\not D^{S}, \gamma_{T}\right]=\frac{1}{12} \sum_{i, j, k, \ell} \pi\left(e_{\ell}\right)\left(T^{i j k}\right) \gamma_{\tilde{e}_{\ell}} \gamma_{i j k} . \tag{70}
\end{equation*}
$$

From (68) and (70), we obtain

$$
\begin{equation*}
\not d^{2}=\frac{1}{4}\left[\not D^{S}, \gamma_{T}\right]+\frac{1}{16} \gamma_{T}^{2}=\frac{1}{16}\left(\frac{1}{3} \sum_{i, j, k, \ell} \pi\left(e_{\ell}\right)\left(T^{i j k}\right) \gamma_{\tilde{e_{\ell}}} \gamma_{i j k}+\gamma_{T}^{2}\right) . \tag{71}
\end{equation*}
$$

We compute

$$
\gamma_{T}^{2}=\frac{1}{4} \sum_{i, j, \ell, m, n} \epsilon_{\ell} T^{\ell i j} T^{\ell m n} \gamma_{i j m n}=\frac{1}{4} \sum_{i, j, \ell, m, n}^{\prime} \epsilon_{\ell} T^{\ell i j} T^{\ell m n} \gamma_{i j m n}-\sum_{i, j, r} \epsilon_{i} \epsilon_{j} \epsilon_{r}\left(T^{i j r}\right)^{2},
$$

where the primed sum is only over pairwise distinct indices. Similarly,

$$
\sum_{i, j, k, \ell} \pi\left(e_{\ell}\right)\left(T^{i j k}\right) \gamma_{\tilde{e}_{\ell}} \gamma_{i j k}=\sum_{i, j, k, \ell}^{\prime} \pi\left(e_{\ell}\right)\left(T^{i j k}\right) \gamma_{\tilde{e}_{\ell}} \gamma_{i j k}+3 \sum_{j, k, \ell} \pi\left(e_{\ell}\right)\left(T^{\ell j k}\right) \gamma_{j k} .
$$

On the other hand, for any $j$ and $k$ fixed,

$$
\begin{aligned}
\sum_{\ell} \pi\left(e_{\ell}\right)\left(T^{\ell j k}\right) & =-\epsilon_{j} \epsilon_{k} \sum_{\ell} \pi\left(e_{\ell}\right)\left\langle\left[e_{j}, e_{k}\right], e_{\ell}\right\rangle \epsilon_{\ell} \\
& =-\epsilon_{j} \epsilon_{k} \sum_{a} \pi\left(q_{a}\right)\left\langle\left[e_{j}, e_{k}\right], p_{a}\right\rangle=0
\end{aligned}
$$

where we used (69) and Corollary 50 (with $\sigma=\left[e_{j}, e_{k}\right]$, which is a section of $\operatorname{ker} \pi$ ). Combining the above relations we obtain
$\not d^{2}=\frac{1}{16}\left(\frac{1}{3} \sum_{i, j, k, \ell}^{\prime} \pi\left(e_{\ell}\right)\left(T^{i j k}\right) \gamma_{\tilde{e} \ell} \gamma_{i j k}+\frac{1}{4} \sum_{i, j, \ell, m, n}^{\prime} \epsilon_{\ell} T^{\ell i j} T^{\ell m n} \gamma_{i j m n}-\sum_{i, j, r} \epsilon_{i} \epsilon_{j} \epsilon_{r}\left(T^{i j r}\right)^{2}\right)$.

We aim to prove that

$$
\begin{equation*}
\not d^{2}=-\frac{1}{16} \sum_{i, j, k} \epsilon_{i} \epsilon_{j} \epsilon_{k}\left(T^{i j k}\right)^{2} . \tag{72}
\end{equation*}
$$

For this, we need to show that

$$
\begin{equation*}
\frac{1}{3} \sum_{i, j, k, \ell}^{\prime} \epsilon_{\ell} \pi\left(e_{\ell}\right)\left(T^{i j k}\right) \gamma_{\ell i j k}+\frac{1}{4} \sum_{i, j, \ell, m, n}^{\prime} \epsilon_{\ell} T^{\ell i j} T^{\ell m n} \gamma_{i j m n}=0 . \tag{73}
\end{equation*}
$$

To prove (73) we use axiom C1) of Definition (1) where indices of tensor components are metrically raised and lowered according with standard conventions: for any $i, j, k$ fixed,

$$
\begin{aligned}
0= & {\left[e_{i},\left[e_{j}, e_{k}\right]\right]-\left[\left[e_{i}, e_{j}\right], e_{k}\right]-\left[e_{j},\left[e_{i}, e_{k}\right]\right] } \\
= & \sum_{\ell}\left(-\left[e_{i}, T_{j k}{ }^{\ell} e_{\ell}\right]+\left[T_{i j}{ }^{\ell} e_{\ell}, e_{k}\right]+\left[e_{j}, T_{i k}{ }^{\ell} e_{\ell}\right]\right) \\
= & \sum_{\ell}\left(-\pi\left(e_{i}\right)\left(T_{j k}{ }^{\ell}\right)+\pi\left(e_{j}\right)\left(T_{i k}{ }^{\ell}\right)\right) e_{\ell}+\sum_{\ell, m}\left(T_{j k}^{\ell} T_{i \ell}{ }^{m}-T_{i k}{ }^{\ell} T_{j \ell}{ }^{m}\right) e_{m} \\
& +\underbrace{\left.\sum_{\ell}\left(-\left[e_{k}, T_{i j}{ }^{\ell} e_{\ell}\right]\right)+\pi^{*} d\left\langle T_{i j}{ }^{\ell} e_{\ell}, e_{k}\right\rangle\right)}_{\pi^{*} d T_{i j k}-\sum_{\ell} \pi\left(e_{k}\right)\left(T_{i j}{ }^{\ell}\right) e_{\ell}+\sum_{\ell, m} T_{i j}{ }^{\ell} T_{k \ell}{ }^{m} e_{m}} \\
= & \sum_{\ell} \pi\left(e_{\ell}\right)\left(T_{i j k}\right) \tilde{e}_{\ell}-\sum_{(i, j, k) \text { cyclic }} \sum_{\ell}\left(\pi\left(e_{i}\right)\left(T_{j k}{ }^{\ell}\right) e_{\ell}-\sum_{m} T_{i j}{ }^{\ell} T_{k \ell}{ }^{m} e_{m}\right),
\end{aligned}
$$

where we have used that $\pi^{*} d f=\sum_{\ell} \pi\left(e_{\ell}\right)(f) \tilde{e}_{\ell}$ for all $f \in C^{\infty}(M)$. Therefore, for any $i, j, k, \ell$ fixed,

$$
\pi\left(e_{\ell}\right)\left(T_{i j k}\right) \tilde{e}_{\ell}-\sum_{(i, j, k) \text { cyclic }}\left(\pi\left(e_{i}\right)\left(T_{j k}^{\ell}\right) e_{\ell}-\sum_{s} T_{i j}{ }^{s} T_{k s}{ }^{\ell} e_{\ell}\right)=0 .
$$

Taking now $i, j, k, \ell$ pairwise distinct, multiplying the above equality with $\gamma^{i j k}$ and summing over (pairwise distinct) $i, j, k, \ell$, we obtain

$$
4 \sum_{i, j, k, \ell}^{\prime} \pi\left(e_{\ell}\right)\left(T_{i j k}\right) \gamma^{\ell i j k}+3 \sum_{i, j, k, \ell, m}^{\prime} T_{i j}^{\ell} T_{k \ell m} \gamma^{m i j k}=0
$$

which is precisely (73) after re-organising the indices. We proved relation (72) which implies in particular that $d^{2} \in C^{\infty}(U)$. From Lemma 48, $d$ is a Dirac generating operator for $E$.

Combining Proposition 46 with Theorem 47 we obtain:

Corollary 52. Let $E$ be a regular Courant algebroid on a manifold $M$ and $S$ a spinor bundle over $\mathrm{Cl}(E)$. For any sufficiently small open subset $U \subset M$, the set of Dirac generating operators for $\left.E\right|_{U}$ on $\left.S\right|_{U}$ has the structure of an affine space modelled on the vector space

$$
\left.V_{d}\right|_{U}:=\left\{e \in \Gamma\left(\left.E\right|_{U}\right),\left[d, \gamma_{e}\right] \in C^{\infty}(U)\right\}
$$

where d is an arbitrarily chosen Dirac generating operator on $\left.S\right|_{U}$.

## 8 The canonical Dirac generating operator

Let $E$ be a regular Courant algebroid, $S$ a spinor bundle over $\operatorname{Cl}(E), D$ a generalized connection on $E$ and $D^{S}$ a compatible $E$-connection on $S$. We begin by analyzing the dependence of the Dirac operator $D^{S}$ defined by (64) on the data $\left(D, D^{S}\right)$.

Let $D^{\prime}=D+A$ be another generalized connection on $E$, where $A \in$ $\Gamma\left(E^{*} \otimes \mathfrak{s o}(E)\right)$.

Proposition 53. The following holds.
(i) The torsions $T^{\prime}$ and $T$ of $D^{\prime}$ and $D$ are related by:

$$
\begin{equation*}
T^{\prime}=T+\alpha, \tag{74}
\end{equation*}
$$

where $\alpha \in \Gamma\left(\wedge^{3} E^{*}\right)$ is given by

$$
\begin{equation*}
\alpha(u, v, w)=\sum_{(u, v, w) \text { cyclic }}\left\langle A_{u} v, w\right\rangle . \tag{75}
\end{equation*}
$$

(ii) The E-connection

$$
\left(D^{S}\right)^{\prime}:=D^{S}-\frac{1}{2} A
$$

is compatible with the generalized connection $D^{\prime}$. Here $A$ is considered as a map $E \rightarrow \wedge^{2} E^{*} \cong \wedge^{2} E \subset \mathrm{Cl}(E)$, so that $A_{e}$ acts by Clifford multiplication on $S$ for all $e \in E$.
(iii) The Dirac operators $D^{S}$ and $\left(D^{S}\right)^{\prime}$ associated with $\left(D, D^{S}\right)$ and $\left(D^{\prime}\right.$, $\left.\left(D^{S}\right)^{\prime}\right)$ are related by

$$
\begin{equation*}
\left(\not D^{S}\right)^{\prime}=\not D^{S}-\frac{1}{4} \gamma_{\alpha}-\frac{1}{4} \gamma_{v_{A}}, \tag{76}
\end{equation*}
$$

where $v_{A}=\operatorname{tr}_{\langle\cdot,\rangle\rangle} A=\sum A_{e_{i}} \tilde{e}_{i} \in \Gamma(E)$.

Proof. (i) is relation (7).
(ii) To check the compatibility let $e, v \in \Gamma(E)$ and $s \in \Gamma(S)$ :

$$
\begin{aligned}
\left(D^{S}\right)_{e}^{\prime}(v s) & =D_{e}^{S}(v s)-\frac{1}{2} A_{e}(v s)=D_{e}(v) s+v D_{e}^{S} s-\frac{1}{2}\left(A_{e} v\right) s \\
& =D_{e}(v) s+v D_{e}^{S} s-\frac{1}{2}\left[A_{e}, v\right] s-\frac{1}{2} v A_{e} s \\
& =D_{e}^{\prime}(v) s+v\left(D^{S}\right)_{e}^{\prime} s .
\end{aligned}
$$

In the fourth equality we used that the commutator $\left[A_{e}, v\right] \in \Gamma(E)$ in the Clifford algebra is related to the evaluation $A_{e}(v)$ of $A_{e} \in \Gamma(\mathfrak{s o}(E))$ on $v$ by the formula

$$
\begin{equation*}
A_{e}(v)=-\frac{1}{2}\left[A_{e}, v\right] . \tag{77}
\end{equation*}
$$

(iii) With respect to an orthonormal frame $\left(e_{i}\right)$ we write

$$
A=\frac{1}{2} \sum_{i, j, k} A^{i j k} e_{i} \otimes\left(e_{j} \wedge e_{k}\right), \alpha=\frac{1}{6} \sum_{i, j, k} \alpha^{i j k} e_{i} \wedge e_{j} \wedge e_{k}
$$

where $A^{i j k}:=A\left(\tilde{e}_{i}, \tilde{e}_{j}, \tilde{e}_{k}\right)$ and $\alpha^{i j k}:=\alpha\left(\tilde{e}_{i}, \tilde{e}_{j}, \tilde{e}_{k}\right)$, where $E$ and $E^{*}$ are identified with the help of the scalar product. In particular, $A_{e_{i}}=\frac{1}{2} \epsilon_{i} \sum_{j, k} A^{i j k} e_{j} \wedge$ $e_{k}$ is identified with $\frac{1}{2} \epsilon_{i} \sum A^{i j k} e_{j} e_{k}$ in $\mathrm{Cl}(E)$. Similarly, $\alpha$ is identified with $\frac{1}{6} \sum \alpha^{i j k} e_{i} e_{j} e_{k}$ in $\mathrm{Cl}(E)$. With this notation,

$$
\begin{aligned}
\left(\not D^{S}\right)^{\prime}-\not D^{S} & =\frac{1}{2} \sum_{i} \tilde{e}_{i}\left(-\frac{1}{2} A_{e_{i}}\right)=-\frac{1}{4} \sum_{i} \tilde{e}_{i} A_{e_{i}}=-\frac{1}{8} \sum_{j \neq k} A^{i j k} \gamma_{i j k} \\
& =-\frac{1}{8} \sum_{i, j, k}^{\prime} A^{i j k} \gamma_{i j k}-\frac{1}{4} \sum_{j, k} A^{j j k} \epsilon_{j} \gamma_{k} .
\end{aligned}
$$

Remark that

$$
\sum_{i, j, k}^{\prime} A^{i j k} \gamma_{i j k}=\frac{1}{3} \sum_{i, j, k}^{\prime}\left(\sum_{(i, j, k) \text { cyclic }} A^{i j k} \gamma_{i j k}\right)=\frac{1}{3} \sum_{i, j, k}^{\prime} \alpha^{i j k} \gamma_{i j k}=2 \gamma_{\alpha}
$$

(where in the second equality we used (75)) and similarly

$$
\sum_{j, k} A^{j j k} \epsilon_{j} \gamma_{k}=\sum_{j, k}\left\langle A\left(e_{j}, \tilde{e}_{j}\right), \tilde{e}_{k}\right\rangle \gamma_{k}=\gamma_{v_{A}} .
$$

Combining the above relations we obtain (761).

Theorem 54. Let $D$ be any generalized connection on $E$ with torsion $T, D^{S}$ any compatible E-connection on $S$ and $\not D^{S}$ the corresponding Dirac operator. For any sufficiently small open subset $U \subset M$, there is a section $e_{0} \in \Gamma\left(\left.E\right|_{U}\right)$ such that

$$
\not d=\not D^{S}+\frac{1}{4} \gamma_{T}+\gamma_{e_{0}}
$$

is a Dirac generating operator on $\left.S\right|_{U}$. The section $e_{0}$ is unique up to addition of a section of $\left.E\right|_{U}$ which belongs to $\left.V_{d}\right|_{U}=\left.V_{\not D^{S}+\frac{1}{4} \gamma_{T}}\right|_{U}$.

Proof. By Proposition 533, the operator $\not D^{S}+\frac{1}{4} \gamma_{T}$ changes only by a term of the form $\gamma_{v}$ for some $v \in \Gamma(E)$ if we pass from the data $\left(D, D^{S}\right)$ to $\left(D^{\prime},\left(D^{S}\right)^{\prime}\right)$, where $D^{\prime}$ is another generalized connection and $\left(D^{S}\right)^{\prime}$ is the $E$-connection compatible with $D^{\prime}$, constructed in that proposition. Since $\langle\cdot, \cdot\rangle$ has neutral signature, any $\mathrm{Cl}(E)$-bundle endomorphism of $S$ is a multiple of the identity. This implies that any other $E$-connection on $S$ compatible with $D^{\prime}$ is of the form

$$
\left(D^{S}\right)^{\prime}+\varphi \otimes \operatorname{Id}_{S}
$$

for some section $\varphi \in \Gamma\left(E^{*}\right)$. This changes $\left(\not D^{S}\right)^{\prime}$ to $\left(\not D^{S}\right)^{\prime}+\frac{1}{2} \gamma_{\varphi}$. We obtain that the operator $\not D^{S}+\frac{1}{4} \gamma_{T}$ depends only on $D$ modulo operators of the form $\gamma_{e}, e \in \Gamma(E)$. On the other hand, in the proof of Theorem47 we constructed a Dirac generating operator on $\left.S\right|_{U}$, for any sufficiently small $U \subset M$, which was of the form $D_{U}^{S}+\frac{1}{4} \gamma_{T_{U}}$, where $D_{U}^{S}$ was an $E$-connection on $\left.S\right|_{U}$ compatible with a generalized connection $D_{U}$ on $\left.E\right|_{U}$ with the torsion $T_{U}$. This implies that $D^{S}+\frac{1}{4} \gamma_{T}$ and $D_{U}^{S}+\frac{1}{4} \gamma_{T_{U}}$ differ only by an operator of the form $\gamma_{e_{0}}$, where $e_{0} \in \Gamma\left(\left.E\right|_{U}\right)$. It follows that $\not D^{S}+\frac{1}{4} \gamma_{T}+\gamma_{e_{0}}$ coincides with $\not D_{U}^{S}+\frac{1}{4} \gamma_{T_{U}}$ and hence is a Dirac generating operator on $\left.S\right|_{U}$. The uniqueness of $e_{0}$ up to a section of $\left.V_{d}\right|_{U}$ was established in Proposition 46 .

Given a generalized connection $D$ on $E$ we would now like to define a Dirac generating operator canonically associated with $D$. Let $S$ be a spinor bundle over $\mathrm{Cl}(E)$ and

$$
\begin{equation*}
\mathcal{S}:=S \otimes\left(\Lambda^{r_{S}} S^{*}\right)^{\frac{1}{r_{S}}} \tag{78}
\end{equation*}
$$

where $r_{S}:=\operatorname{rank} S$ is even. (For a line bundle $L$ with transition functions $g_{i j}$, we define $L^{\frac{1}{r}}$ to be the line bundle with transition functions $\left(g_{i j}\right)^{\frac{1}{r}}$. Hence $L^{\frac{1}{r}}$ exists when $L$ admits a trivialisation atlas with positive transition functions, or, equivalently, when $L$ is trivial). Therefore, $\left(\Lambda^{r_{S}} S\right)^{\frac{1}{r_{S}}}$ (or its dual) exists when $S$ is orientable, an assumption we shall make on $S$. We call the bundle $\mathcal{S}$ the canonical spinor bundle.

Remark 55. The terminology 'canonical spinor bundle' for $\mathcal{S}$ is justified by the fact that if $S$ and $S^{\prime}$ are two isomorphic (orientable) spinor bundles then $\mathcal{S}$ and $\mathcal{S}^{\prime}$ are canonically isomorphic. To explain this statement consider two (not necessarily isomorphic) orientable spinor bundles $S$ and $S^{\prime}$. Since $\langle\cdot, \cdot\rangle$ has neutral signature, we can write $S^{\prime}=S \otimes L$, where $L$ is a line bundle and $r=r_{S}$ denotes the common rank of $S$ and $S^{\prime}$. Since $r$ is even, $L^{r}$ is trivial. Then $\mathcal{S}^{\prime}=\mathcal{S} \otimes\left(L^{-r}\right)^{\frac{1}{r}} \otimes L$. If $L$ is trivial then $\left(L^{-r}\right)^{\frac{1}{r}}$ is canonically isomorphic to $L^{*}$ and $\left(L^{-r}\right)^{\frac{1}{r}} \otimes L=\operatorname{Hom}(L, L)$ is canonically trivial. We obtain that $\mathcal{S}^{\prime}$ and $\mathcal{S}$ are canonically isomorphic in this case. (If $L$ is not trivial, nonetheless $L^{r}$ is and we can still define $\left(L^{-r}\right)^{\frac{1}{r}}$, which is trivial and is not isomorphic to $L^{*}$.)

Proposition 56. Let $D$ be a generalized connection on $E$ with torsion $T$ and $D^{S}$ an $E$-connection on $S$ compatible with $D$. Then $D^{S}$ induces a connection $D^{\mathcal{S}}$ on $\mathcal{S}$, which is compatible with $D$ and depends only on $D$. In particular, the corresponding Dirac operator $\not D^{\mathcal{S}}$ and $\not D^{\mathcal{S}}+\frac{1}{4} \gamma_{T}$ depend only on $D$.

Proof. It is clear that the $E$-connection $D^{\mathcal{S}}$ on $\mathcal{S}$ induced by $D^{S}$ is again compatible with $D$. Any other $E$-connection on $S$ compatible with $D$ is of the form $D^{S}+\varphi \otimes \operatorname{Id}_{S}$ for some section $\varphi \in \Gamma\left(E^{*}\right)$. The latter induces the connection $D^{\mathcal{S}}+\varphi-\frac{1}{r_{S}} r_{S} \varphi=D^{\mathcal{S}}$ on $\mathcal{S}$. Similarly, tensoring $\left(S, D^{S}\right)$ with a line bundle with connection changes neither $\mathcal{S}$ nor $D^{\mathcal{S}}$.

In order to define a Dirac generating operator independent of $D$ we consider as in 9 the following canonical weighted spinor bundle

$$
\begin{equation*}
\mathbb{S}:=\mathcal{S} \otimes L, \quad L:=\left|\Lambda^{\mathrm{top}} T^{*} M\right|^{\frac{1}{2}}, \tag{79}
\end{equation*}
$$

where $\left|\Lambda^{\text {top }} T^{*} M\right|$ denotes the (trivial) line bundle of densities (the transition functions of which are $\left|g_{i j}\right|$, where $g_{i j}$ are the transition functions of $\Lambda^{\mathrm{top}} T^{*} M$ ). The line bundle $L$ carries an induced $E$-connection $D^{L}$ defined by

$$
\begin{equation*}
D_{v}^{L} \mu:=\mathcal{L}_{\pi(v)} \mu-\frac{1}{2} \operatorname{div}_{D}(v) \mu \tag{80}
\end{equation*}
$$

where $v \in \Gamma(E), \mu \in \Gamma(L)$, and $\operatorname{div}_{D}(v):=\operatorname{tr} D v$. The next lemma will be used in the proof of Corollary 60 below.

Lemma 57. If $D$ is the generalized connection from Lemma 51, then the $E$-connection $D^{L}$ defined by (80) is induced by a usual connection on $L$.

Proof. We need to show that $\operatorname{div}_{D}(v)=0$ for any $v \in \Gamma$ (ker $\pi$ ) (see Example 5). Consider the frame $\left(p_{a}, q_{a}\right)$ constructed after Lemma 49 and recall that
it is parallel with respect to the connection $\nabla_{\pi(e)}=D_{e}$ on $E$. Its dual frame is $\left(\tilde{p}_{a}, \tilde{q}_{b}\right)=\left(q_{a}, p_{b}\right)$ and
$\operatorname{div}_{D}(v)=\sum_{a}\left(\left\langle D_{p_{a}}(v), q_{a}\right\rangle+\left\langle D_{q_{a}}(v), p_{a}\right\rangle\right)=\sum_{a}\left\langle\nabla_{\pi\left(q_{a}\right)}(v), p_{a}\right\rangle=\sum_{a} \pi\left(q_{a}\right)\left\langle v, p_{a}\right\rangle$
where we used $\pi\left(p_{a}\right)=0$ and $\nabla p_{a}=0$. From Corollary 50 we obtain $\operatorname{div}_{D}(v)=0$ as needed.
Theorem 58. Let $D$ be a generalized connection on $E$ with torsion $T, D^{\mathbb{S}}=$ $D^{\mathcal{S}} \otimes D^{L}$ the induced compatible E-connection on the canonical weighted spinor bundle $\mathbb{S}$, and $\perp D$ the corresponding Dirac operator on $\mathbb{S}$. Then the operator

$$
\begin{equation*}
\not D+\frac{1}{4} \gamma_{T}: \Gamma(\mathbb{S}) \rightarrow \Gamma(\mathbb{S}) \tag{81}
\end{equation*}
$$

is independent of $D$.
Proof. Replacing $D$ by another generalized connection $D^{\prime}=D+A$ changes $\operatorname{div}_{D}$ to $\operatorname{div}_{D^{\prime}}=\operatorname{div}_{D}-\left\langle v_{A}, \cdot\right\rangle$ and hence $D^{L}$ to $\left(D^{\prime}\right)^{L}=D^{L}+\frac{1}{2}\left\langle v_{A}, \cdot\right\rangle$. (See Proposition 53 for the definition of $v_{A}$.) On the other hand, from Proposition 53, on any spinor bundle, in particular on the canonical spinor bundle $\mathcal{S}=$ $S \otimes\left(\operatorname{det} S^{*}\right)^{\frac{1}{r_{S}}}, \not D^{\mathcal{S}}+\frac{1}{4} \gamma_{T}$ changes to $\left(\not D^{\mathcal{S}}\right)^{\prime}+\frac{1}{4} \gamma_{T^{\prime}}=\not D^{\mathcal{S}}+\frac{1}{4} \gamma_{T}-\frac{1}{4} \gamma_{v_{A}}$. Here we used relation (74) between the torsions of $D$ and $D^{\prime}$ and that $\left(D^{\mathcal{S}}\right)^{\prime}=$ $D^{\mathcal{S}}-\frac{1}{2} A$, which follows from $\left(D^{S}\right)^{\prime}=D^{S}-\frac{1}{2} A$ (see the next remark). On $\mathbb{S}=\mathcal{S} \otimes L$,

$$
\not D(s \otimes l)=\left(\not \mathcal{D}^{\mathcal{S}} s\right) \otimes l+\frac{1}{2} \tilde{e}_{i} s \otimes D_{e_{i}}^{L} l, s \in \Gamma(\mathcal{S}), l \in \Gamma(L)
$$

and a similar expression holds for the Dirac operator $\not D^{\prime}$ on $\mathbb{S}$ computed with the generalized connection $D^{\prime}$. We deduce that

$$
\begin{aligned}
& \left(\not D+\frac{1}{4} \gamma_{T}\right)(s \otimes l)=\left(\not D^{\mathcal{S}} s+\frac{1}{4} T s\right) \otimes l+\frac{1}{2} \tilde{e}_{i} s \otimes D_{e_{i}}^{L} l \\
& \left(\not D^{\prime}+\frac{1}{4} \gamma_{T^{\prime}}\right)(s \otimes l)=\left(\not D^{\mathcal{S}} s+\frac{1}{4} T s-\frac{1}{4} v_{A} s\right) \otimes l+\frac{1}{2} \tilde{e}_{i} s \otimes\left(D^{\prime}\right)_{e_{i}}^{L} l .
\end{aligned}
$$

From $\left(D^{\prime}\right)^{L}=D^{L}+\frac{1}{2}\left\langle v_{A}, \cdot\right\rangle$ we obtain that (81) is invariant on $\mathbb{S}$.
Remark 59. The statement is a consequence of the following algebraic fact: let $V$ be an $n$-dimensional vector space, $V \oplus V^{*}$ with its standard metric $\langle v+\xi, w+\eta\rangle=\frac{1}{2}(\eta(v)+\xi(w))$ and $A \in \Lambda^{2}\left(V \oplus V^{*}\right)=\mathfrak{s p i n}\left(V \oplus V^{*}\right) \subset$ $\mathrm{Cl}\left(V \oplus V^{*}\right)$. Recall that $S:=\Lambda V^{*}$ is the unique irreducible $\mathrm{Cl}\left(V \oplus V^{*}\right)$ module, with spinor representation

$$
\begin{equation*}
(v+\xi) \cdot \alpha:=\iota_{v} \alpha+\xi \wedge \alpha \tag{82}
\end{equation*}
$$

where $\iota_{v} \alpha:=\alpha(v, \cdot)$ denotes the interior product. The Clifford action of $A$ on $S$ extents to an action of $A$ on $\Lambda^{t o p} S$, defined by

$$
\begin{equation*}
A \cdot\left(\alpha_{1} \wedge \cdots \wedge \alpha_{N}\right):=\left(A \cdot \alpha_{1}\right) \wedge \alpha_{2} \cdots \wedge \alpha_{N}+\cdots+\alpha_{1} \wedge \cdots \alpha_{N-1} \wedge\left(A \cdot \alpha_{N}\right) \tag{83}
\end{equation*}
$$

where $\left\{\alpha_{i}\right\}$ is a basis of $S$ and we claim that this action is trivial. Indeed, take a basis $\left\{e_{i}\right\}$ of $V$ and dual basis $\left\{e_{i}^{*}\right\}$ of $V^{*}$. In terms of such bases, $A$ is a linear combination of $\left(e_{i} \pm e_{i}^{*}\right) \wedge\left(e_{j} \pm e_{j}^{*}\right)$ (for any $\left.i \neq j\right)$ and $\left(e_{i}+e_{i}^{*}\right) \wedge\left(e_{j}-e_{j}^{*}\right)$ (for any $i, j$ ). Let $\left\{\alpha_{i}\right\}$ be the basis of $S$ induced by $\left\{e_{i}^{*}\right\}$. The spinor representation (82) easily implies that $A \cdot\left(\alpha_{1} \wedge \cdots \wedge \alpha_{N}\right)=0$ by computing the trace of $A$ as an operator on $S$. This does also follow from the fact that $\mathfrak{s p i n}(n, n) \cong \mathfrak{s o}(n, n)$ is semi-simple (for $n>1$ ) and that a semi-simple Lie algebra has no non-trivial one-dimensional representations.

Corollary 60. ([1]) The operator $\not d=\not D+\frac{1}{4} \gamma_{T}$ on $\mathbb{S}$ constructed in Proposition 58 is a Dirac generating operator for $E$, independent of $D$.

Proof. From Theorem [58, it remains to show that $\not d$ is a Dirac generating operator. As this is a local property, we need to show that for any sufficiently small open subset $U \subset M$, the restriction $\left.\not \subset\right|_{U}: \Gamma\left(\left.\mathbb{S}\right|_{U}\right) \rightarrow \Gamma\left(\left.\mathbb{S}\right|_{U}\right)$ is a Dirac generating operator. This follows by noticing that $\left.\not d\right|_{U}$, being independent of the choice of generalized connection, coincides with the Dirac generating operator constructed in Lemma [51, with spinor bundle $\mathbb{S}$. (From Lemma 57, if $D$ is the generalized connection from Lemma 51 induced by the (flat, metric) connection $\nabla$ on $E$ then $D^{L}$ is induced by a usual connection on $L$ and we obtain that the $E$-connection $D^{\mathbb{S}}=D^{\mathcal{S}} \otimes D^{L}$ compatible with $D$ is also induced by a usual connection, which is compatible with $\nabla$.)

Definition 61. The operator $\not d: \Gamma(\mathbb{S}) \rightarrow \Gamma(\mathbb{S})$ is called the canonical Dirac generating operator associated to $E$.

Remark 62. Our canonical Dirac generating operator coincides with the Dirac generating operator constructed in Theorem 4.1 of [1]. This follows from formula (53) of [1], by noticing a difference of a minus sign between our definition for the torsion of a generalized connection and that from [1] (see Section 3.2 of this reference). Our Dirac generating operator also coincides (up to a multiplicative constant factor) with the Dirac generating operator from Proposition 5.12 of 13 (in formula (5.14) of 13 the term $\frac{1}{2} \gamma\left(C_{\nabla}\right)$ should have a minus sign).

## 9 Standard form of the canonical Dirac generating operator

In this section (see Theorem 67) we provide an alternative expression for the canonical Dirac generating operator $\not d$ of a regular Courant algebroid $E$, which uses the structure of regular Courant algebroids, as described in [7]. From Lemma 49, there is a vector bundle isomorphism

$$
\begin{equation*}
I: E \rightarrow F^{*} \oplus \mathcal{G} \oplus F \tag{84}
\end{equation*}
$$

where we recall that $F=\pi(E) \subset T M$ is an integrable distribution and $\mathcal{G} \subset$ ker $\pi \subset E$ is a subbundle. The isomorphism $I$ maps the anchor $\pi: E \rightarrow T M$ of $E$ to the map $\rho(\xi+r+X)=X$ and the scalar product of $E$ to a scalar product

$$
\begin{equation*}
\langle\xi+r+X, \eta+s+y\rangle=\frac{1}{2}(\xi(Y)+\eta(X))+(r, s)^{\mathcal{G}} \tag{85}
\end{equation*}
$$

where $\xi+r+X, \eta+s+Y \in F^{*} \oplus \mathcal{G} \oplus F$, and the scalar product $(\cdot, \cdot)^{\mathcal{G}}$ on $\mathcal{G}$ is of neutral signature. The bundle $\mathcal{G}$, together with $(\cdot, \cdot)^{\mathcal{G}}$, is canonically associated to $E$. More precisely, $\left(\mathcal{G},(\cdot, \cdot)^{\mathcal{G}}\right)$ is isomorphic to $(\operatorname{ker} \pi) /(\operatorname{ker} \pi)^{\perp}$ with scalar product induced by the scalar product of $E$. Moreover, $\mathcal{G}$ is a bundle of Lie algebras, with Lie bracket $[\cdot, \cdot]^{G}$ induced from the Dorfman bracket of $E$. The scalar product $(\cdot, \cdot)^{\mathcal{G}}$ is invariant with respect to $[\cdot, \cdot]^{\mathcal{G}}$, that is the adjoint representation of the Lie algebra is by skew-symmetric endomorphisms. In fact, these properties follow immediately from $\mathcal{G} \subset \operatorname{ker} \pi$. An isomorphism (84) as above is called a dissection of $E$ [7].

The Dorfman bracket $[\cdot, \cdot]$ of $F^{*} \oplus \mathcal{G} \oplus F$ induced from $E$ via a dissection satisfies

$$
\begin{equation*}
\operatorname{Pr}_{\mathcal{G}}\left[r_{1}, r_{2}\right]=\left[r_{1}, r_{2}\right]^{\mathcal{G}}, r_{i} \in \Gamma(\mathcal{G}), \tag{86}
\end{equation*}
$$

where $\operatorname{Pr}_{\mathcal{G}}$ is the natural projection from $F^{*} \oplus \mathcal{G} \oplus F$ on $\mathcal{G}$ (we shall use a similar notation $\operatorname{Pr}_{F^{*}}$ for the natural projection on $F^{*}$ ).

Therefore, we may (and will) assume that the given regular Courant algebroid is of the form $E=F^{*} \oplus \mathcal{G} \oplus F$, with anchor $\rho(\xi+r+X)=X$, metric given by (85) and Dorfman bracket $[\cdot, \cdot]$ satisfying (86). As proved in Lemma 2.1 of [7], the Dorfman bracket of $E$ is determined by its components

$$
\begin{aligned}
& \nabla: \Gamma(F) \times \Gamma(\mathcal{G}) \rightarrow \Gamma(\mathcal{G}), \nabla_{X}(r):=\operatorname{Pr}_{\mathcal{G}}[X, r] \\
& R: \Gamma(F) \times \Gamma(F) \rightarrow \Gamma(\mathcal{G}), R(X, Y):=\operatorname{Pr}_{\mathcal{G}}[X, Y] \\
& \mathcal{H}: \Gamma(F) \times \Gamma(F) \times \Gamma(F) \rightarrow C^{\infty}(M), \mathcal{H}(X, Y, Z):=\left(\operatorname{Pr}_{F^{*}}[X, Y]\right)(Z) .
\end{aligned}
$$

Note that here $[X, Y]$ stands for the Dorfman bracket

$$
[X, Y]=\mathcal{H}(X, Y, \cdot)+R(X, Y)+\mathcal{L}_{X} Y
$$

of $X, Y$ as sections of $F \subset E$, whereas, for the rest of this section, the commutator of vector fields will be always denoted by $\mathcal{L}_{X} Y$. The map $\nabla$ is an $F$-connection on $\mathcal{G}$, i.e. it satisfies

$$
\nabla_{X}(f r)=X(f) r+f \nabla_{X} r, \quad \nabla_{f X} r=f \nabla_{X} r,
$$

for all $X \in \Gamma(F), r \in \Gamma(\mathcal{G}), f \in C^{\infty}(M)$. The map $R$ is a 2 -form on $F$ with values in $\mathcal{G}$ and $\mathcal{H}$ is a 3 -form on $F$. The properties of the triple $(\nabla, R, \mathcal{H})$ are described in Theorem 2.3 of [7].

The next lemma was proved in [7] and can be checked directly (we remark a difference of sign between our definition for the torsion of a generalized connection and that from [7). By a torsion-free connection on $F$ we mean a map $\nabla^{F}: \Gamma(F) \times \Gamma(F) \rightarrow \Gamma(F)$ which satisfies the usual properties of a connection and $\nabla_{X} Y-\nabla_{Y} X=\mathcal{L}_{X} Y$ for any $X, Y \in \Gamma(F)$.

Lemma 63. (77) Let $\nabla^{F}$ be a torsion-free connection on $F$. Then

$$
\begin{equation*}
\nabla_{\xi+r+X}^{E}(\eta+s+Y):=\left(\nabla_{X}^{F} \eta-\frac{1}{3} \mathcal{H}(X, Y, \cdot), \nabla_{X} s+\frac{2}{3}[r, s]^{\mathcal{G}}, \nabla_{X}^{F} Y\right) \tag{87}
\end{equation*}
$$

is a generalized connection on $E$ with torsion given by

$$
\begin{align*}
T^{\nabla^{E}}(u, v, w)= & -\mathcal{H}(X, Y, Z)-(R(X, Y), t)^{\mathcal{G}}-(R(Y, Z), r)^{\mathcal{G}}-(R(Z, X), s)^{\mathcal{G}} \\
& +\left([r, s]^{\mathcal{G}}, t\right)^{\mathcal{G}} \tag{88}
\end{align*}
$$

for any $u=\xi+r+X, v=\eta+s+Y, w=\zeta+t+Z$ (where $\xi, \eta, \zeta \in F^{*}$, $r, s, t \in \mathcal{G}, X, Y, Z \in F)$.

Remark 64. For any regular Courant algebroid $E$ with anchor $\pi: E \rightarrow$ $T M$, the quotient $\mathcal{A}:=E /(\operatorname{ker} \pi)^{\perp}$ inherits a Lie algebroid structure from the Dorfman bracket of $E$. This Lie algebroid is called in [7] the ample Lie algebroid associated to $E$. A dissection induces a bundle isomorphism $\mathcal{A} \cong \mathcal{G} \oplus F$ and a Lie algebroid structure on $\mathcal{G} \oplus F$ (inherited from the Lie algebroid structure of $\mathcal{A}$ ). The restriction of the 3 -form $\Omega:=-T^{\nabla^{E}}$ to $\mathcal{G} \oplus F$ is closed with respect to the Lie algebroid differential of $\mathcal{G} \oplus F$ and its cohomology class is independent on the chosen dissection. It is called the Severa class of $E$, as it coincides with the well-known Severa class when $E$ is exact [7].

Let $\Lambda F^{*}$ be the bundle of forms on $F$. It is a spinor bundle over $\mathrm{Cl}\left(F^{*} \oplus\right.$ $F)$, where $F^{*} \oplus F$ has scalar product $\langle\xi+X, \eta+Y\rangle=\frac{1}{2}(\eta(X)+\xi(Y)$ ), with spinor representation (82). We assume that $F$ is orientable. Then $\Lambda F^{*}$ is orientable as well. Let $S_{\mathcal{G}}$ be an orientable spinor bundle over $\mathrm{Cl}(\mathcal{G})$, where
$\mathcal{G}$ is considered with the scalar product $(\cdot, \cdot)^{\mathcal{G}}$. The $\mathbb{Z}_{2}$-graded tensor product $S:=\Lambda F^{*} \hat{\otimes} S_{\mathcal{G}}$ is an orientable spinor bundle over $\mathrm{Cl}(E)=\mathrm{Cl}\left(F^{*} \oplus F\right) \hat{\otimes} \mathrm{Cl}(\mathcal{G})$. (Basic facts concerning the $\mathbb{Z}_{2}$-graded tensor product are reviewed in more detail in appendix; in particular, see relation (108) for the Clifford action of $E$ on $S$ ). Assuming that $M$ is orientable, $L=\left|\Lambda^{\text {top }} T^{*} M\right|^{\frac{1}{2}}$ is canonically isomorphic to $\left(\Lambda^{\operatorname{top}} T^{*} M\right)^{\frac{1}{2}}$ and the canonical spinor bundle over $\mathrm{Cl}(E)$ is given by

$$
\begin{equation*}
\mathbb{S}=\Lambda F^{*} \otimes\left(\Lambda^{\mathrm{top}} F \otimes \Lambda^{\mathrm{top}} T^{*} M\right)^{1 / 2} \otimes \mathcal{S}_{\mathcal{G}}=\Lambda F^{*} \otimes\left(\Lambda^{\mathrm{top}} \mathrm{Ann} F\right)^{1 / 2} \otimes \mathcal{S}_{\mathcal{G}} \tag{89}
\end{equation*}
$$

where $\mathcal{S}_{\mathcal{G}}=S_{\mathcal{G}} \otimes\left(\Lambda^{\text {top }} S_{\mathcal{G}}^{*}\right)^{1 / g}$ is the weighted spinor bundle over $\mathrm{Cl}(\mathcal{G})$, $g:=\operatorname{rank}\left(S_{\mathcal{G}}\right)$, and we dropped the hats in the tensor products to simplify notation. In (89) we used the definition (79) of the canonical spinor bundle $\mathbb{S}$ and the vector bundle isomorphisms

$$
\begin{equation*}
\Lambda^{n_{1} n_{2}}\left(V_{1} \otimes V_{2}\right)=\left(\Lambda^{n_{1}} V_{1}\right)^{n_{2}} \otimes\left(\Lambda^{n_{2}} V_{2}\right)^{n_{1}}, \Lambda^{N}(\Lambda V)=\left(\Lambda^{n} V\right)^{N / 2} \tag{90}
\end{equation*}
$$

which hold for any $V, V_{1}, V_{2}$, where $n_{i}:=\operatorname{rank} V_{i}, n:=\operatorname{rank} V, N:=$ $\operatorname{rank}(\Lambda V)$. We also used the fact that $\left(\left(\Lambda^{\text {top }} F\right)^{f / 2}\right)^{1 / f}$ is canonically isomorphic to $\left(\Lambda^{\operatorname{top}} F\right)^{1 / 2}$ when $F$ is orientable and $f:=\operatorname{rank} F$. In the second equality (89) we identified $\left(\Lambda^{\text {top }} T^{*} M\right) \otimes \Lambda^{\text {top }} F$ to $\Lambda^{\text {top }}$ Ann $F$, where Ann $F \subset T^{*} M$ is the annihilator of $F$. The gradation of $\mathbb{S}$ is induced from the gradations of $\Lambda F^{*}$ and $S_{\mathcal{G}}$ (the line bundles $\left(\Lambda^{\text {top }} \text { Ann } F\right)^{1 / 2}$ and $\left(\Lambda^{\text {top }} S_{\mathcal{G}}^{*}\right)^{1 / g}$ have zero degree). Recall the definition of the $E$-connection $D^{\mathbb{S}}$ from Theorem 58 computed from a generalized connection $D$.
Lemma 65. The $E$-connection $\nabla^{\mathbb{S}}$ on $\mathbb{S}$ computed from the generalized connection $\nabla^{E}$ defined in Lemma 63 has the following expression: for any $\omega \in \Gamma\left(\Lambda F^{*}\right), \tau \in \Gamma\left(\left(\Lambda^{\text {top }} \mathrm{Ann} F\right)^{1 / 2}\right)$ and $s \in \Gamma\left(\mathcal{S}_{\mathcal{G}}\right)$ we have

$$
\begin{align*}
& \nabla_{u}^{\mathbb{S}}(\omega \otimes \tau \otimes s)=\left(\nabla_{X}^{F} \omega\right) \otimes \tau \otimes s+\omega \otimes\left(\mathcal{L}_{X} \tau\right) \otimes s \\
& +\frac{1}{3}\left(\iota_{X} \mathcal{H}\right) \wedge \omega \otimes \tau \otimes s+\omega \otimes \tau \otimes\left(\nabla_{X}^{0, \mathcal{S}_{\mathcal{G}}} s-\frac{1}{3}\left(\operatorname{ad}_{r}\right)(s)\right) . \tag{91}
\end{align*}
$$

Above $u=\xi+r+X, \mathcal{L}_{X} \tau$ denotes the Lie derivative of $\tau$ in the direction of $X \in \Gamma(F), \nabla^{0, \mathcal{S}_{\mathcal{G}}}$ is an $F$-connection on $\mathcal{S}_{\mathcal{G}}$, induced by an $F$-connection $\nabla^{0, S_{\mathcal{G}}}$ on $S_{\mathcal{G}}$ compatible with the $F$-connection $\nabla$ of $\mathcal{G}$ and $\operatorname{ad}_{r} \in \mathfrak{s o}(\mathcal{G})$ is considered as a 2 -form on $\mathcal{G}$, which acts by Clifford multiplication on $\mathcal{S}_{\mathcal{G}}$.

Proof. We remark that $\nabla^{E}=\nabla^{F^{*}+F}+\nabla^{\mathcal{G}}$, where $\nabla^{F^{*}+F}$ and $\nabla^{\mathcal{G}}$ are $E$ connections on $F^{*} \oplus F$ and $\mathcal{G}$ respectively, defined by

$$
\begin{aligned}
& \nabla_{u}^{F^{*}+F}(\eta+Y)=\nabla_{X}^{F} \eta-\frac{1}{3} \mathcal{H}(X, Y, \cdot)+\nabla_{X}^{F} Y, \\
& \nabla_{u}^{\mathcal{G}}(s):=\nabla_{X} s+\frac{2}{3}[r, s]^{\mathcal{G}}
\end{aligned}
$$

The $F$-connection $\nabla^{F}$ induces an $F$-connection (also denoted by $\nabla^{F}$ ) on $\Lambda F^{*}$ (and on $\Lambda F^{*} \otimes\left(\Lambda^{\text {top }} F\right)^{1 / 2}$, see below). A straightforward computation shows that the $E$-connection

$$
\begin{equation*}
\nabla_{u}^{F, \text { spin }}:=\nabla_{X}^{F}+\frac{1}{3}\left(\iota_{X} \mathcal{H}\right) \wedge \tag{92}
\end{equation*}
$$

on $\Lambda F^{*}$ is compatible with $\nabla^{F^{*}+F}$. Note that for all $A \in \mathfrak{s o}\left(\mathcal{G}_{p}\right)$ and $s \in$ $\mathcal{G}_{p}, p \in M$ we have the relation $A(s)=-\frac{1}{2}[A, s]$ where $[A, s]$ denotes the commutator in the Clifford algebra and $A(s)$ is the evaluation of $A$ on $s$ (see relation (77)). Applying this to $A=\mathrm{ad}_{r}$ we see, like in the proof of Proposition 53 ii), that the $E$-connection

$$
\begin{equation*}
\nabla_{u}^{\mathcal{G}, \text { spin }}:=\nabla_{X}^{0, S_{\mathcal{G}}}-\frac{1}{3} \operatorname{ad}_{r} \tag{93}
\end{equation*}
$$

on $S_{\mathcal{G}}$ is compatible with $\nabla^{\mathcal{G}}$. We obtain that $\nabla^{F, \text { spin }} \otimes \nabla^{\mathcal{G}}$, spin is an $E$ connection on $\Lambda F^{*} \otimes S_{\mathcal{G}}$ compatible with the generalized connection $\nabla^{E}=$ $\nabla^{F^{*}+F}+\nabla^{\mathcal{G}}$. Since $\operatorname{trace}\left(\nabla^{E} u\right)=\operatorname{trace}\left(\nabla^{F} X\right)$, from (80) we obtain

$$
\left(\nabla^{E}\right)_{u}^{L}=\mathcal{L}_{X}-\frac{1}{2} \operatorname{trace}\left(\nabla^{F} X\right)
$$

on $L=\left|\Lambda^{\text {top }} T^{*} M\right|^{1 / 2}$, and, from the definition of $\nabla^{\mathbb{S}}$,

$$
\begin{equation*}
\nabla^{\mathbb{S}}:=\nabla^{F, \mathrm{spin}} \otimes\left(\nabla^{E}\right)^{L} \otimes \nabla^{\mathcal{G}, \mathrm{spin}} \tag{94}
\end{equation*}
$$

where we used the decomposition (89) of $\mathbb{S}$ (the first equality), and we preserved the same symbols $\nabla^{F \text {,spin }}$ and $\nabla^{\mathcal{G}}$,spin for the $E$-connections induced by $\nabla^{F, \text { spin }}$ and $\nabla^{\mathcal{G}}$, spin on $\Lambda F^{*} \otimes\left(\Lambda^{\operatorname{top}} F\right)^{1 / 2}$ and $\mathcal{S}_{\mathcal{G}}$ respectively. In order to compute $\nabla^{\mathbb{S}}$ we shall compute $\nabla^{F, \text { spin }} \otimes\left(\nabla^{E}\right)^{L}$ and $\nabla^{\mathcal{G}}$, spin separately.

We begin with $\nabla^{F \text {,spin }} \otimes\left(\nabla^{E}\right)^{L}$, which is an $E$-connection on $\Lambda F^{*} \otimes$ $\left(\Lambda^{\text {top }} F\right)^{1 / 2} \otimes L$. We remark that relation (92) holds also on $\Lambda F^{*} \otimes\left(\Lambda^{\text {top }} F\right)^{1 / 2}$ (the endomorphism $\omega \rightarrow \frac{1}{3}\left(\iota_{X} \mathcal{H}\right) \wedge \omega$ of $\Lambda F^{*}$ being trace-free). Let $\omega \in$ $\Gamma\left(\Lambda F^{*}\right), l \in \Gamma\left(\left(\Lambda^{\operatorname{top}} F\right)^{1 / 2}\right)$ and $\mu \in \Gamma(L)$. Then

$$
\begin{aligned}
& \left(\nabla^{F, \text { spin }} \otimes\left(\nabla^{E}\right)^{L}\right)_{u}(\omega \otimes l \otimes \mu)=\nabla_{X}^{F, \text { spin }}(\omega \otimes l) \otimes \mu+\omega \otimes l \otimes\left(\nabla^{E}\right)_{X}^{L} \mu \\
& =\left(\nabla_{X}^{F}(\omega \otimes l)+\frac{1}{3}\left(i_{X} \mathcal{H} \wedge \omega\right) \otimes l\right) \otimes \mu+\omega \otimes l \otimes\left(\mathcal{L}_{X} \mu-\frac{1}{2} \operatorname{trace}\left(\nabla^{F} X\right) \mu\right),
\end{aligned}
$$

and this relation can be simplified as follows: using $\nabla_{X}^{F}(\omega \otimes l)=\left(\nabla_{X}^{F} \omega\right) \otimes$ $l+\omega \otimes \nabla_{X}^{F} l$ and

$$
\nabla_{X}^{F} l=\mathcal{L}_{X} l+\frac{1}{2} \operatorname{trace}\left(\nabla^{F} X\right) l
$$

we obtain

$$
\begin{equation*}
\left(\nabla^{F, \mathrm{spin}} \otimes\left(\nabla^{E}\right)^{L}\right)_{u}(\omega \otimes l \otimes \mu)=\left(\nabla_{X}^{F} \omega\right) \otimes l \otimes \mu+\omega \otimes \mathcal{L}_{X}(l \otimes \mu)+\frac{1}{3}\left(\iota_{X} \mathcal{H} \wedge \omega\right) \otimes l \otimes \mu \tag{95}
\end{equation*}
$$

Next, we compute $\nabla^{\mathcal{G}}$, spin . We claim that relation (93) holds also on $\mathcal{S}_{\mathcal{G}}$ (with $\nabla^{0, S_{\mathcal{G}}}$ replaced by $\nabla^{0, \mathcal{S}_{\mathcal{G}}}$, the $E$-connection on $\mathcal{S}_{\mathcal{G}}$ induced by $\nabla^{0, S_{\mathcal{G}}}$ ). This follows from the fact that the Clifford multiplication by $\operatorname{ad}_{r} \in \Gamma\left(\Lambda^{2} \mathcal{G}\right)$ on $S_{\mathcal{G}}$ is trace-free. The latter is a consequence of Remark 59, So we have proven:

$$
\begin{equation*}
\nabla_{u}^{\mathcal{G}, \text { spin }}=\nabla_{X}^{0, \mathcal{S}_{\mathcal{G}}}-\frac{1}{3} \operatorname{ad}_{r} \quad \text { on } \quad \mathcal{S}_{\mathcal{G}} . \tag{96}
\end{equation*}
$$

Combining (94), (95) and (96), we obtain (91).
Notation 66. For any $\tau \in \Gamma\left(\left(\Lambda^{\text {top }} \operatorname{Ann} F\right)^{1 / 2}\right), \mathcal{L}_{X} \tau$ is $C^{\infty}(M)$-linear in $X$, when $X \in \Gamma(F)$ (as $\mathcal{L}_{f X} \omega=f \mathcal{L}_{X} \omega$, for any $\omega \in \Gamma($ Ann $F)$ ). The map $\Gamma(F) \ni X \rightarrow \mathcal{L}_{X} \tau$ is a 1 -form on $F$ with values in $\left(\Lambda^{\text {top }} \text { Ann } F\right)^{1 / 2}$, which will be denoted by $\mathcal{L}(\tau)$.

We arrive now at what we call the standard form for the canonical Dirac generating operator in terms of the data encoding the regular Courant algebroid.

Theorem 67. Let $E$ be a regular Courant algebroid with anchor $\pi: E \rightarrow T M$ such that $F:=\pi(E)$ and $T M$ are orientable. In terms of a dissection of $E$, the canonical Dirac generating operator is given by

$$
\begin{align*}
\not d(\omega \otimes \tau \otimes s) & =\left(d^{F} \omega\right) \otimes \tau \otimes s+\mathcal{L}(\tau) \wedge \omega \otimes s+\nabla^{0, S_{\mathcal{G}}}(s) \wedge \omega \otimes \tau \\
& -(\mathcal{H} \wedge \omega) \otimes \tau \otimes s+\frac{1}{4}(-1)^{|\omega|+1} \omega \otimes \tau \otimes C s \\
& +(-1)^{|\omega|+1} \bar{R}(\omega \otimes \tau \otimes s), \tag{97}
\end{align*}
$$

where $\omega \in \Gamma\left(\Lambda F^{*}\right)$, $\tau \in \Gamma\left(\left(\Lambda^{\operatorname{top}} \operatorname{Ann} F\right)^{1 / 2}\right)$ and $s \in \Gamma\left(\mathcal{S}_{\mathcal{G}}\right)$. Above $d^{F}$ : $\Gamma\left(\Lambda^{\bullet} F^{*}\right) \rightarrow \Gamma\left(\Lambda^{\bullet+1} F^{*}\right)$ is the exterior derivative along the integrable distribution $F, C \in \Gamma\left(\Lambda^{3} \mathcal{G}^{*}\right)$ is the Cartan form $C(u, v, w)=\left([u, v]^{\mathcal{G}}, w\right)^{\mathcal{G}}$ of $\mathcal{G}$ viewed as a section of $\mathrm{Cl}(\mathcal{G})$, Cs denotes its Clifford action on $s$ and

$$
\begin{equation*}
\bar{R}(\omega \otimes \tau \otimes s)=\frac{1}{2} \sum_{i, j, k}\left(R\left(X_{i}, X_{j}\right), r_{k}\right)^{\mathcal{G}}\left(\alpha_{i} \wedge \alpha_{j} \wedge \omega\right) \otimes \tau \otimes \tilde{r}_{k} s, \tag{98}
\end{equation*}
$$

where $\left(X_{i}\right)$ is a basis of $F,\left(\alpha_{i}\right)$ is the dual basis, i.e. $\alpha_{i}\left(X_{j}\right)=\delta_{i j},\left(r_{i}\right)$ is a basis of $\mathcal{G}$ and $\left(\tilde{r}_{i}\right)$ is the dual basis with respect to $(\cdot, \cdot)^{\mathcal{G}}$.

Proof. The bases of $E$

$$
\begin{align*}
& \left(e_{i}\right):=\left(X_{1}, \cdots, X_{n}, r_{1}, \cdots, r_{m}, \alpha_{1}, \cdots, \alpha_{n}\right) \\
& \left(\tilde{e}_{i}\right):=\left(2 \alpha_{1}, \cdots, 2 \alpha_{n}, \tilde{r}_{1}, \cdots, \tilde{r}_{m}, 2 X_{1}, \cdots, 2 X_{n}\right) \tag{99}
\end{align*}
$$

are dual with respect to the scalar product (85) of $E$. Using Lemma 65 and the definition of the Clifford action, we obtain

$$
\begin{aligned}
\sum_{i} \tilde{e}_{i} \nabla_{e_{i}}^{\mathbb{S}}(\omega \otimes \tau \otimes s) & =\sum_{i}\left(2\left(\alpha_{i} \wedge \nabla_{X_{i}}^{F} \omega\right) \otimes \tau \otimes s+2\left(\alpha_{i} \wedge \omega\right) \otimes \mathcal{L}_{X_{i}} \tau \otimes s\right. \\
& +\frac{2}{3}\left(\alpha_{i} \wedge \iota_{X_{i}} \mathcal{H} \wedge \omega\right) \otimes \tau \otimes s+2\left(\alpha_{i} \wedge \omega\right) \otimes \tau \otimes \nabla_{X_{i}}^{0, \mathcal{S}_{\mathcal{G}}} s \\
& \left.+(-1)^{|\omega|+1} \frac{1}{3} \omega \otimes \tau \otimes\left(\tilde{r}_{i} \operatorname{ad}_{r_{i}}\right)(s)\right)
\end{aligned}
$$

But

$$
\begin{align*}
& \sum_{i} \alpha_{i} \wedge \nabla_{X_{i}}^{F} \omega=d^{F} \omega, \sum_{i} \alpha_{i} \wedge \iota_{X_{i}} \mathcal{H}=3 \mathcal{H}, \sum_{i} \alpha_{i} \otimes \mathcal{L}_{X_{i}} \tau=\mathcal{L}(\tau) \\
& \sum_{i} \tilde{r}_{i} \operatorname{ad}_{r_{i}}=\frac{1}{2} \sum_{i, j, k}\left(\operatorname{ad}_{r_{i}}\left(r_{j}\right), r_{k}\right)^{\mathcal{G}} \tilde{r}_{i} \tilde{r}_{j} \tilde{r}_{k}=3 C \tag{100}
\end{align*}
$$

where in the first equality (100) we used that $\nabla^{F}$ is torsion-free and the last equality holds in the $\operatorname{Clifford}$ algebra $\mathrm{Cl}(\mathcal{G})$. We obtain that

$$
\begin{align*}
& \frac{1}{2} \sum_{i} \tilde{e}_{i} \nabla_{e_{i}}^{\mathbb{S}}(\omega \otimes \tau \otimes s)=\left(d^{F} \omega\right) \otimes \tau \otimes s+\mathcal{L}(\tau) \wedge \omega \otimes s \\
& +\left(\nabla^{0, \mathcal{S}_{\mathcal{G}}} s\right) \wedge \omega \otimes \tau+(\mathcal{H} \wedge \omega) \otimes \tau \otimes s+\frac{1}{2}(-1)^{|\omega|+1} \omega \otimes \tau \otimes C s \tag{101}
\end{align*}
$$

Combining (101) with the definition of $d d$ we obtain

$$
\begin{align*}
& \not d(\omega \otimes \tau \otimes s)=\left(d^{F} \omega\right) \otimes \tau \otimes s+\mathcal{L}(\tau) \wedge \omega \otimes s+\left(\nabla^{0, \mathcal{S}_{\mathcal{G}}} s\right) \wedge \omega \otimes \tau \\
& +(\mathcal{H} \wedge \omega) \otimes \tau \otimes s+\frac{1}{2}(-1)^{|\omega|+1} \omega \otimes \tau \otimes C s+\frac{1}{4} T^{\nabla^{E}}(\omega \otimes \tau \otimes s) \tag{102}
\end{align*}
$$

In order to conclude our proof we need to express $T^{\nabla^{E}}$ as a section of $\Lambda^{3} E$ and to compute $T^{\nabla^{E}}(\omega \otimes \tau \otimes s)$, where $T^{\nabla^{E}} \in \Gamma\left(\Lambda^{3} E\right) \subset \Gamma(\mathrm{Cl}(E))$ acts by Clifford multiplication on $\omega \otimes \tau \otimes s$. Using the bases (99), we write

$$
\begin{aligned}
T^{\nabla^{E}}= & \frac{1}{6} \sum_{i, j, k} T^{\nabla^{E}}\left(e_{i}, e_{j}, e_{k}\right) \tilde{e}_{i} \wedge \tilde{e}_{j} \wedge \tilde{e}_{k} \\
= & \frac{4}{3} \sum_{i, j, k} T^{\nabla^{E}}\left(X_{i}, X_{j}, X_{k}\right) \alpha_{i} \wedge \alpha_{j} \wedge \alpha_{k} \\
& +2 \sum_{i, j, k} T^{\nabla^{E}}\left(X_{i}, X_{j}, r_{k}\right) \alpha_{i} \wedge \alpha_{j} \wedge \tilde{r}_{k}+C
\end{aligned}
$$

where we used that $T^{\nabla^{E}}\left(\alpha_{i}, \cdot, \cdot\right)=0$ and $T^{\nabla^{E}}\left(r_{i}, r_{j}, X_{k}\right)=0$ from relation (88). Again from relation (88),

$$
\sum_{i, j, k} T^{\nabla^{E}}\left(X_{i}, X_{j}, X_{k}\right) \alpha_{i} \wedge \alpha_{j} \wedge \alpha_{k}=-\sum_{i, j, k} \mathcal{H}\left(X_{i}, X_{j}, X_{k}\right) \alpha_{i} \wedge \alpha_{j} \wedge \alpha_{k}=-6 \mathcal{H}
$$

and

$$
T^{\nabla^{E}}\left(X_{i}, X_{j}, r_{k}\right) \alpha_{i} \wedge \alpha_{j} \wedge \tilde{r}_{k}=-\left(R\left(X_{i}, X_{j}\right), r_{k}\right)^{\mathcal{G}} \alpha_{i} \wedge \alpha_{j} \wedge \tilde{r}_{k}
$$

We have proven that $T^{\nabla^{E}}$, as a section of $\Lambda^{3} E$, is given by

$$
T^{\nabla^{E}}=-8 \mathcal{H}-2 \sum_{i, j, k}\left(R\left(X_{i}, X_{j}\right), r_{k}\right)^{\mathcal{G}} \alpha_{i} \wedge \alpha_{j} \wedge \tilde{r}_{k}+C .
$$

This implies

$$
\begin{align*}
& T^{\nabla^{E}}(\omega \otimes \tau \otimes s)=-8(\mathcal{H} \wedge \omega) \otimes \tau \otimes s+(-1)^{|\omega|} \omega \otimes \tau \otimes C s \\
& +2(-1)^{|\omega|+1} \sum_{i, j, k}\left(R\left(X_{i}, X_{j}\right), r_{k}\right)^{\mathcal{G}}\left(\alpha_{i} \wedge \alpha_{j} \wedge \omega\right) \otimes \tau \otimes \tilde{r}_{k} s . \tag{103}
\end{align*}
$$

We conclude by combining (102) with (103).
Example 68. Consider a regular Courant algebroid $E$ for which $\mathcal{G}$ is trivial and the anchor $\pi$ is surjective (i.e. $F=T M$ ). A dissection defines an isomorphism between $E$ and the Courant algebroid $T M \oplus T M^{*}$ from Example 3 (see Lemma 2.1 of [1]). From Theorem [67, the canonical Dirac generating operator is defined on $\Gamma\left(\Lambda T^{*} M\right)$ by $\phi(\omega)=d \omega-\mathcal{H} \wedge \omega$ (since Ann $T M$ and $\mathcal{S}_{\mathcal{G}}$ are trivial). We recover the well-known expression of the Dirac generating operator for exact Courant algebroids, see e.g. [14].

## Part III

## 10 Generalized almost Hermitian structures: integrability and spinors

In [1] an integrability criterion for a generalized almost complex structure $\mathcal{J}$ on a regular Courant algebroid $E$ using the canonical Dirac generating operator $\not d: \Gamma(\mathbb{S}) \rightarrow \Gamma(\mathbb{S})$ of $E$ was developed. More precisely, Theorem 6.4 of [1] (which is stated in the more general setting of almost Dirac structures)
implies that $\mathcal{J}$ is integrable if and only if one (equivalenty, any) pure spinor $\eta \in \Gamma\left(\mathbb{S}_{\mathbb{C}}\right)$ associated to $\mathcal{J}$ is projectively closed, i.e. there is $v \in \Gamma\left(E_{\mathbb{C}}\right)$ such that $\not \mathscr{d}(\eta)=\gamma_{v} \eta$. (In order to simplify notation, we use the same symbols $\not d$ and $\gamma$ for their complex linear extensions). Recall that $\eta$ (which is uniquely determined up to a non-vanishing multiplicative function) is defined by the condition $\gamma_{v} \eta=0$, for any $v \in L$, where $L \subset E_{\mathbb{C}}$ is the ( 1,0 )-bundle of $\mathcal{J}$. (For a detailed treatment of pure spinors associated to orthogonal complex structures and their relation to twistors see e.g. [19], Chapter 4, Section 9). For completeness of our exposition we recall in appendix the proof of Theorem 6.4. of [1] together with integrability criterion for generalized almost complex structures mentioned above (see Corollary 78).

As an application of the theory from the previous sections, we now characterize the integrability of a generalized almost Hermitian structure $(G, \mathcal{J})$ on $E$ in terms of suitably chosen Dirac operators. Let $E=E_{+} \oplus E_{-}$be the decomposition of $E$ determined by $G$. As $\mathcal{J}$ preserves $E_{ \pm}, n=\operatorname{rank} E_{+}=$ rank $E_{-}$is even, where $2 n$ is the rank of $E$. Recall that the restriction $\left.\langle\cdot, \cdot\rangle\right|_{E_{ \pm}}$ of the scalar product $\langle\cdot, \cdot\rangle$ of $E$ to $E_{ \pm}$is respectively positive definite (in case of $E_{+}$) and negative definite (in case of $E_{-}$). We consider $E_{ \pm}$endowed with the scalar products $\left.\langle\cdot, \cdot\rangle\right|_{E_{ \pm}}$and we denote by $\mathrm{Cl}\left(E_{ \pm}\right)$the bundle of Clifford algebras over $\left(E_{ \pm},\left.\langle\cdot, \cdot\rangle\right|_{E_{ \pm}}\right)$.

We assume that there are given irreducible $\mathrm{Cl}\left(E_{ \pm}\right)$-bundles $S_{ \pm}$with Schur algebra $\mathbb{R}$. The latter condition means that any vector bundle morphism $f_{ \pm}: S_{ \pm} \rightarrow S_{ \pm}$which commutes with the $\mathrm{Cl}\left(E_{ \pm}\right)$-action is a multiple of the identity.

Lemma 69. Under the above assumption, $n=8 p$ for $p \in \mathbb{N}_{\geq 0}$. Moreover, $S_{ \pm}$ has a $\mathbb{Z}_{2}$-gradation $S_{ \pm}=S_{ \pm}^{0} \oplus S_{ \pm}^{1}$ which makes it a $\mathbb{Z}_{2}$-graded $\mathrm{Cl}\left(E_{ \pm}\right)$-bundle.

Proof. As $n$ is even and the metrics $\pm\left.\langle\cdot, \cdot\rangle\right|_{E_{ \pm}}$are positive definite, a quick inspection of Table 1 from [19] (see page 29), implies that $n=8 p$, for $p \in \mathbb{N}_{\geq 0}$ : only for such even $n$ both Clifford algebras $\mathrm{Cl}_{n, 0}$ and $\mathrm{Cl}_{0, n}$ are matrix (or direct sum of two matrix) algebras over the field $\mathbb{K}=\mathbb{R}$ and hence $S_{ \pm}$ have Schur algebra $\mathbb{R}$. (It is worth to remark a difference of sign between our conventions on Clifford algebras and those from [19] ; more precisely, the Clifford algebra of $(V, q)$ is defined in [19] using the Clifford relation $v^{2}=-q(v) 1$, rather than $\left.v^{2}=q(v) 1\right)$. The gradation of $S_{ \pm}$is defined by $S_{ \pm}^{0}:=\frac{1}{2} \gamma_{\left(1+\omega_{ \pm}\right)} S_{ \pm}$and $S_{ \pm}^{1}:=\frac{1}{2} \gamma_{\left(1-\omega_{ \pm}\right)} S_{ \pm}$, where $\omega_{ \pm}$is a volume element of $\mathrm{Cl}\left(E_{ \pm}\right)$. For more details, see e.g. Proposition 3.6 in Chapter I of [19].

Since $S_{ \pm}$are irreducible $\mathbb{Z}_{2}$-graded $\mathrm{Cl}\left(E_{ \pm}\right)$-bundles, $S_{+} \hat{\otimes} S_{-}$is an irreducible $\mathbb{Z}_{2}$-graded $\mathrm{Cl}(E)$-bundle, with Clifford action given by

$$
\begin{equation*}
\gamma_{v}\left(s_{+} \otimes s_{-}\right)=\gamma_{v_{+}}\left(s_{+}\right) \otimes s_{-}+(-1)^{\left|s_{+}\right|} s_{+} \otimes \gamma_{v_{-}}\left(s_{-}\right), \tag{104}
\end{equation*}
$$

for any $v=v_{+}+v_{-} \in E$ (see appendix for more details).
Define now the $\mathbb{Z}_{2}$-graded $\mathrm{Cl}\left(E_{ \pm}\right)$-bundles $\mathcal{S}_{ \pm}:=S_{ \pm} \otimes\left(\operatorname{det} S_{ \pm}^{*}\right)^{\frac{1}{{ }^{S_{ \pm}}}}$(where $r_{S_{+}}=r_{S_{-}}$is the rank of $S_{+}$and $S_{-}$). From the first isomorphism (90) we obtain that the graded tensor product $\mathcal{S}_{+} \hat{\otimes} \mathcal{S}_{-}$is isomorphic, as a $\mathbb{Z}_{2}$-graded $\mathrm{Cl}(E)$-bundle, to the canonical spinor bundle $\mathcal{S}$ over $\mathrm{Cl}(E)$ defined by (78).
Lemma 70. If $\eta_{ \pm} \in \Gamma\left(\left(\mathcal{S}_{ \pm}\right)_{\mathbb{C}}\right)$ are pure spinors associated to $\left.\mathcal{J}\right|_{E_{ \pm}}$, then $\eta_{+} \otimes \eta_{-} \in \Gamma\left(\mathcal{S}_{\mathbb{C}}\right)$ is a pure spinor associated to $\mathcal{J}$.

Proof. Let $L$ be the ( 1,0 )-bundle of $\mathcal{J}$ and

$$
\left(E_{ \pm}\right)_{\mathbb{C}} \cap L=L_{\eta_{ \pm}}:=\left\{v \in\left(E_{ \pm}\right)_{\mathbb{C}} \mid \gamma_{v} \eta_{ \pm}=0\right\} .
$$

Both $L_{\eta_{+}} \oplus L_{\eta_{-}}$and $L_{\eta_{+} \otimes \eta_{-}}$are subbundles of $\left(E_{+}\right)_{\mathbb{C}} \oplus\left(E_{-}\right)_{\mathbb{C}}=E_{\mathbb{C}}$. From (104), $L_{\eta_{+}} \oplus L_{\eta_{-}} \subset L_{\eta_{+} \otimes \eta_{-}}$. Since $L_{\eta_{+} \otimes \eta_{-}}$is an isotropic subbundle of $E_{\mathbb{C}}$, its rank is at most $n$. By comparing ranks we obtain $L_{\eta_{+}} \oplus L_{\eta_{-}}=L_{\eta_{+} \otimes \eta_{-}}$. As $L=L_{\eta_{+}} \oplus L_{\eta_{-}}$we obtain $L=L_{\eta_{+} \otimes \eta_{-}}$as needed.

Definition 71. The pure spinors $\eta_{ \pm}$from the above lemma are called pure spinors associated to $(G, \mathcal{J})$.

Let $D$ be a generalized Levi-Civita connection of $G$. Since $D$ preserves $G$, it also preserves $E_{ \pm}$. Let $D^{ \pm}$be the $E$-connections on $E_{ \pm}$induced by $D$. Choose $E$-connections $D^{S_{ \pm}}$on $S_{ \pm}$compatible with $D^{ \pm}$. The existence of $D^{S_{ \pm}}$can be proved as in Section 7. Namely, with respect to a local orthonormal frame of $E_{ \pm}$and the corresponding frame of $S_{ \pm}$, we define $D^{S_{ \pm}}$ to have connection matrix equal to one half of the connection matrix of $D$. In particular, $D^{S_{ \pm}}$preserves the grading of $S_{ \pm}$, i.e. $D_{e}^{S_{ \pm}} \Gamma\left(S_{ \pm}^{0}\right) \subset \Gamma\left(S_{ \pm}^{0}\right)$ and $D_{e}^{S_{ \pm}} \Gamma\left(S_{ \pm}^{1}\right) \subset \Gamma\left(S_{ \pm}^{1}\right)$, for any $e \in \Gamma(E)$. If $\tilde{D}^{S_{ \pm}}$is another $E$-connection compatible with $D^{ \pm}$, then $\tilde{D}^{S_{ \pm}}=D^{S_{ \pm}}+\lambda^{ \pm} \otimes \operatorname{Id}_{S_{ \pm}}$, for $\lambda^{ \pm} \in \Gamma\left(E^{*}\right)$ (from our assumption stated before Lemma 69).

We denote by $D^{\mathcal{S}_{ \pm}}$the $E$-connections induced by $D^{S_{ \pm}}$on $\mathcal{S}_{ \pm}$and by $D^{\mathcal{S}}=D^{\mathcal{S}_{+}} \otimes D^{\mathcal{S}_{-}}$their tensor product, which is an $E$-connection on the tensor product bundle $\mathcal{S}=\mathcal{S}_{+} \otimes \mathcal{S}_{-}$, defined in the standard way (independent on gradations). Recall that we use the convention $v_{ \pm} s_{ \pm}$for the Clifford action $\gamma_{v_{ \pm}} s_{ \pm}$. Similarly, to simplify notation, we write $v\left(s_{+} \otimes s_{-}\right)$instead of $\gamma_{v}\left(s_{+} \otimes s_{-}\right)$, for any $v \in E$.
Lemma 72. The tensor product E-connection $D^{\mathcal{S}}:=D^{\mathcal{S}_{+}} \otimes D^{\mathcal{S}_{-}}$on $\mathcal{S}=$ $\mathcal{S}_{+} \hat{\otimes} \mathcal{S}_{-}$is compatible with the Clifford multiplication of $\mathrm{Cl}(E)$ on $\mathcal{S}$.
Proof. Using (104) and that $D^{\mathcal{S}_{+}}$preserves the grading of $\mathcal{S}_{+}$, we obtain

$$
D_{e}^{\mathcal{S}}\left(v\left(s_{+} \otimes s_{-}\right)\right)=D_{e}(v)\left(s_{+} \otimes s_{-}\right)+v D_{e}^{\mathcal{S}}\left(s_{+} \otimes s_{-}\right)
$$

for any $v \in \Gamma(E)$ and $s_{ \pm} \in \Gamma\left(\mathcal{S}_{ \pm}\right)$, as needed.

In the above setting, there are three Dirac operators which need to be considered: two Dirac operators $\mathscr{D}^{\mathcal{S}_{ \pm}}: \Gamma\left(\mathcal{S}_{ \pm}\right) \rightarrow \Gamma\left(\mathcal{S}_{ \pm}\right)$computed using the $E$-connections $D^{\mathcal{S}_{ \pm}}$of the $\mathrm{Cl}\left(E_{ \pm}\right)$-bundles $\mathcal{S}_{ \pm}$, defined by

$$
\not D^{\mathcal{S}_{ \pm}}\left(s_{ \pm}\right):=\frac{1}{2} \sum_{i} \tilde{e}_{i}^{ \pm} D_{e_{i}^{ \pm}}^{\mathcal{S}_{ \pm}} s_{ \pm}
$$

where $\left\{e_{1}^{ \pm}, \cdots, e_{n}^{ \pm}\right\}$is a basis of $E_{ \pm}$and $\left\{\tilde{e}_{1}^{ \pm}, \cdots, \tilde{e}_{n}^{ \pm}\right\}$is the metric dual basis, i.e. $\tilde{e}_{i}^{ \pm}$belongs to $E_{ \pm}$and $\left\langle e_{i}^{ \pm}, \tilde{e}_{j}^{ \pm}\right\rangle=\delta_{i j}$. The third Dirac operator $\not D^{\mathcal{S}}$ is the one from Section 8, computed using the $E$-connection $D^{\mathcal{S}}$ on the $\mathrm{Cl}(E)$-bundle $\mathcal{S}$. Since $\left\{e_{1}^{+}, \cdots, e_{n}^{+}, e_{1}^{-}, \cdots, e_{n}^{-}\right\}$and $\left\{\tilde{e}_{1}^{+}, \cdots, \tilde{e}_{n}^{+}, \tilde{e}_{1}^{-}, \cdots, \tilde{e}_{n}^{-}\right\}$ are bases of $E$ dual with respect to $\langle\cdot, \cdot\rangle, \not D^{\mathcal{S}}$ is given by

$$
\not \mathcal{S}^{\mathcal{S}}\left(s_{+} \otimes s_{-}\right)=\frac{1}{2} \sum_{i} \tilde{e}_{i}^{+} D_{e_{i}^{+}}^{\mathcal{S}}\left(s_{+} \otimes s_{-}\right)+\frac{1}{2} \sum_{i} \tilde{e}_{i}^{-} D_{e_{i}^{-}}^{\mathcal{S}}\left(s_{+} \otimes s_{-}\right) .
$$

The following lemma can be checked directly from definitions.
Lemma 73. The operators $D^{\mathcal{S}}, \not D^{\mathcal{S}_{+}}$and $D^{\mathcal{S}_{-}}$are related by

$$
\begin{align*}
& \not D^{\mathcal{S}}\left(s_{+} \otimes s_{-}\right)=\left(\not D^{\mathcal{S}_{+}} s_{+}\right) \otimes s_{-}+(-1)^{\left|s_{+}\right|} s_{+} \otimes\left(\not D^{\mathcal{S}_{-}} s_{-}\right) \\
& +\frac{1}{2} \sum_{i}\left(\tilde{e}_{i}^{+} s_{+} \otimes\left(D_{e_{i}^{+}}^{\mathcal{S}_{-}} s_{-}\right)+(-1)^{\left|s_{+}\right|}\left(D_{e_{i}^{-}}^{\mathcal{S}_{+}} s_{+}\right) \otimes \tilde{e}_{i}^{-} s_{-}\right), \tag{105}
\end{align*}
$$

where $s_{ \pm} \in \Gamma\left(\mathcal{S}_{ \pm}\right)$.
In the next theorem we use the notation $D_{e_{\mp}}^{\mathcal{S}_{ \pm}}\left(\eta_{ \pm}\right) \equiv \eta_{ \pm}$(for $e_{\mp} \in E_{\mp}$ ) if $D_{e_{\mp}}^{\mathcal{S}_{ \pm}}\left(\eta_{ \pm}\right)=f \eta_{ \pm}$for a function $f=f\left(e_{\mp}\right) \in C^{\infty}(M, \mathbb{C})$ which depends on $e_{\mp}$. By $D^{\mathcal{S}_{ \pm}} \eta_{ \pm} \equiv \eta_{ \pm}$we mean $D^{\mathcal{S}_{ \pm}} \eta_{ \pm} \in \gamma_{\left(E_{ \pm}\right)_{\mathrm{C}}}\left(\eta_{ \pm}\right)$.
Theorem 74. In the above setting, the generalized almost Hermitian structure $(G, \mathcal{J})$ on the regular Courant algebroid $E$ is generalized Kähler if and only if there is a Levi-Civita connection $D$ of $G$ such that

$$
\begin{equation*}
\not D^{\mathcal{S}_{ \pm}} \eta_{ \pm} \equiv \eta_{ \pm}, \quad D_{e_{\mp}}^{\mathcal{S}_{ \pm}} \eta_{ \pm} \equiv \eta_{ \pm} \tag{106}
\end{equation*}
$$

for any $e_{\mp} \in \Gamma\left(E_{\mp}\right)$. Here $E=E_{+} \oplus E_{-}$is the decomposition determined by $G$ and $\eta_{ \pm} \in \Gamma\left(\mathcal{S}_{ \pm}\right)$are pure spinors associated to $(G, \mathcal{J})$.
Proof. Let $D$ be a Levi-Civita connection of $G$. From (81), $d d=D^{\mathbb{S}}+\frac{1}{4} \gamma_{T}=$ $\not D^{\mathbb{S}}$ since $D$ is torsion-free and, using $D^{\mathbb{S}}=D^{\mathcal{S}} \otimes D^{L}$,

$$
\not D^{\mathbb{S}}(s \otimes l)=\not D^{\mathcal{S}}(s) \otimes l+\frac{1}{2} \sum_{i} \tilde{e}_{i} s \otimes D_{e_{i}}^{L} l, \quad \forall s \in \Gamma(\mathcal{S}), \forall l \in \Gamma(L),
$$

where $\left(e_{i}\right)$ is a basis of $E$ and $\left(\tilde{e}_{i}\right)$ the dual basis with respect to $\langle\cdot, \cdot\rangle$. We obtain that a pure spinor $\eta \otimes l$ from $\mathbb{S}=\mathcal{S} \otimes L$ is projectively closed if and only if $\not D^{\mathcal{S}} \eta \in \gamma_{E_{\mathbb{C}}} \eta$.

Assume now that relations (106) hold, with $D^{\mathcal{S}_{ \pm}}$and $D^{\mathcal{S}_{ \pm}}$computed starting with $D$. From (105), we deduce that the pure spinor $\eta=\eta_{+} \otimes \eta_{-}$ associated to $\mathcal{J}$ satisfies $\not D^{\mathcal{S}} \eta \in \gamma_{E_{\mathbb{C}}} \eta$, i.e. $\mathcal{J}$ is integrable (see Corollary 78). In a similar way, we show that $\mathcal{J}_{2}=G^{\text {end }} \mathcal{J}$ is integrable. For this, we use the fact $\eta_{+} \otimes \bar{\eta}_{-}$is a pure spinor associated to $\mathcal{J}_{2}$ and $D^{\mathcal{S}_{-}} \bar{\eta}_{-} \in \gamma_{\left(E_{-}\right)} \bar{\eta}_{-}$(because $\not D^{\mathcal{S}_{-}}: \Gamma\left(\left(\mathcal{S}_{-}\right)_{\mathbb{C}}\right) \rightarrow \Gamma\left(\left(\mathcal{S}_{-}\right)_{\mathbb{C}}\right)$ is the complex linear extension of its restriction to $\mathcal{S}_{-}$and hence commutes with the natural conjugation of $\left.\left(\mathcal{S}_{-}\right)_{\mathbb{C}}\right)$. We obtain that $(G, \mathcal{J})$ is generalized Kähler.

Conversely, assume now that $\mathcal{J}$ is integrable and let $D$ be a Levi-Civita connection of $G$, which preserves $\mathcal{J}$ (the existence of $D$ is ensured by Theorem 37). The relation

$$
\begin{equation*}
D_{e}^{\mathcal{S}_{+}}\left(v \eta_{+}\right)=\left(D_{e}^{+} v\right) \eta_{+}+v D_{e}^{\mathcal{S}_{+}} \eta_{+}, e \in \Gamma(E), \tag{107}
\end{equation*}
$$

together with the fact that $D$ preserves $L_{\eta_{+}}=L \cap\left(E_{+}\right)_{\mathbb{C}}$ imply that $v D_{e}^{\mathcal{S}_{+}} \eta_{+}=$ 0 , for any $v \in \Gamma\left(L_{\eta_{+}}\right)$, i.e. $D_{e}^{\mathcal{S}_{+}} \eta_{+}=\lambda(e) \eta_{+}$. For $e:=e_{-} \in \Gamma\left(E_{-}\right)$, this relation means $D_{e_{+}}^{\mathcal{S}_{+}} \eta_{+} \equiv \eta_{+}$. For $e:=e_{i}^{+}$it implies $\not D^{\mathcal{S}_{+}} \eta_{+}=\lambda_{+} \eta_{+}$where $\lambda_{+}:=$ $\left.\lambda\right|_{E_{+}}$is a section of $E_{+}^{*} \cong E_{+}$(identified using $\left.\langle\cdot, \cdot\rangle\right|_{E_{+}}$) and $\lambda_{+} \eta_{+}=\gamma_{\lambda_{+}} \eta_{+}$ is the Clifford action of $\lambda_{+} \in E_{+}$on $\eta_{+}$. We proved $\not D^{\mathcal{S}_{+}} \eta_{+} \equiv \eta_{+}$as needed. The same argument with $\mathcal{S}_{+}$and $\mathcal{S}_{-}$interchanged shows that all relations (106) are satisfied.

Lemma 75. Relations (106) are independent of the choice of Levi-Civita connection.

Proof. Let $D$ be a Levi-Civita connection. Since $D$ is torsion-free and preserves $E_{ \pm}$, for any $e_{-} \in \Gamma\left(E_{-}\right)$and $v_{+}, w_{+} \in \Gamma\left(E_{+}\right)$,

$$
\begin{aligned}
& 0=T^{D}\left(e_{-}, v_{+}, w_{+}\right)=\left\langle D_{e_{-}} v_{+}-D_{v_{+}} e_{-}-\left[e_{-}, v_{+}\right], w_{+}\right\rangle+\left\langle v_{+}, D_{w_{+}} e_{-}\right\rangle \\
& =\left\langle D_{e_{-}} v_{+}-\left[e_{-}, v_{+}\right], w_{+}\right\rangle,
\end{aligned}
$$

which implies $D_{e_{-}} v_{+}=\left[e_{-}, v_{+}\right]_{+}$. In particular, $D_{e_{-}}^{+}=\left.D_{e_{-}}\right|_{\Gamma\left(E_{+}\right)}$is independent of the choice of Levi-Civita connection $D$, for any $e_{-} \in \Gamma\left(E_{-}\right)$. We obtain that any two $E$-connections $D^{\mathcal{S}_{+}}$and $\tilde{D}^{\mathcal{S}_{+}}$on $\mathcal{S}_{+}$, compatible with any two Levi-Civita connections of $G$, satisfy $\tilde{D}_{e_{-}}^{\mathcal{S}_{+}}=D_{e_{-}}^{\mathcal{S}_{+}}+\lambda_{-}\left(e_{-}\right) \operatorname{Id}_{\mathcal{S}_{+}}$, for $\lambda_{-} \in E_{-}^{*}$. This implies that the condition $D_{e_{-}}^{\mathcal{S}_{+}} \eta_{+} \equiv \eta_{+}$is independent of the choice of $D$. In a similar way we prove the statement for $D_{e_{+}}^{\mathcal{S}_{-}} \eta_{-} \equiv \eta_{-}$.

Next, consider two Levi-Civita connections $D$ and $\tilde{D}=D+A$ of $G$. The arguments from Propositions 53 and 56 show that $D^{\mathcal{S}_{+}}$(hence, also $\not D^{\mathcal{S}_{+}}$) depends only on $D^{+}$and

$$
\tilde{D D}^{\mathcal{S}_{+}}=\not D^{\mathcal{S}_{+}}-\frac{1}{4} \gamma_{\alpha^{+}}-\frac{1}{4} \gamma_{v_{A^{+}}}
$$

where $\alpha \in \Lambda^{3} E_{+}^{*}$ is given by $\alpha^{+}(u, v, w):=\sum_{(u, v, w) \text { cyclic }}\left\langle A_{u} v, w\right\rangle, u, v, w \in$ $E_{+}$and $v_{A^{+}}:=\sum_{i=1}^{n} A_{e_{i}^{+}} \tilde{e}_{i}^{+} \in \Gamma\left(E_{+}\right)$, where $\left\{e_{1}^{+}, \cdots, e_{n}^{+}\right\}$and $\left\{\tilde{e}_{1}^{+}, \cdots, \tilde{e}_{n}^{+}\right\}$ are $\langle\cdot, \cdot\rangle$-dual bases of $E_{+}$. As $D$ and $\tilde{D}$ are torsion-free, $\alpha^{+}=0$ and we obtain $\tilde{D D}^{\mathcal{S}_{+}} \eta_{+}=\not D^{\mathcal{S}_{+}} \eta_{+}-\frac{1}{4} v^{A^{+}} \eta_{+}$. This implies that the condition $\not D^{\mathcal{S}_{+}} \eta_{+} \equiv \eta_{+}$is independent of the choice of $D$. In a similar way we prove the statement for $\not D^{\mathcal{S}_{-}} \eta_{-} \equiv \eta_{-}$.

From Theorem 74 combined with Lemma 75 we obtain the following characterization for the integrability of generalized almost hyper-Hermitan structures.

Corollary 76. A generalized almost hyper-Hermitian structure $\left(G, \mathcal{J}_{1}, \mathcal{J}_{2}, \mathcal{J}_{3}\right)$ is generalized hyper-Kähler if and only if conditions (106) from Theorem 74 hold for each of the pure spinors $\eta_{ \pm}^{i}$ associated to $\left(G, \mathcal{J}_{i}\right), i=1,2,3$. The conditions are independent of the choice of Levi-Civita connection of $G$.

## 11 Appendix

## 11.1 $\mathbb{Z}_{2}$-graded algebras and Clifford algebras

Recall that if $A=A^{0} \oplus A^{1}$ and $B=B^{0} \oplus B^{1}$ are $\mathbb{Z}_{2}$-graded vector spaces, then the tensor product $A \otimes B$ inherits a $\mathbb{Z}_{2}$-gradation

$$
(A \otimes B)^{0}:=A^{0} \otimes B^{0}+A^{1} \otimes B^{1},(A \otimes B)^{1}:=A^{0} \otimes B^{1}+A^{1} \otimes B^{0} .
$$

We denote by $A \hat{\otimes} B$ the vector space $A \otimes B$ together with this gradation. If, moreover, $A$ and $B$ are $\mathbb{Z}_{2}$-graded algebras, then $A \hat{\otimes} B$ inherits the structure of a $\mathbb{Z}_{2}$-graded algebra with multiplication on homogeneous elements defined by

$$
(a \otimes b)(\tilde{a} \otimes \tilde{b}):=(-1)^{|b| a \tilde{a} \mid} a \tilde{a} \otimes b \tilde{b},
$$

where $|a|,|\tilde{b}| \in\{0,1\}$ are the degrees of $a$ and $\tilde{b}$.
We say that a $\mathbb{Z}_{2}$-graded vector space $S=S^{0} \oplus S^{1}$ is a $\mathbb{Z}_{2}$-graded $A$ module if it is a representation space for $A$ and the action of $A$ on $S$ is compatible with gradations, i.e. $A^{i} \cdot S^{j} \subset S^{i+j}$ for any $i, j \in \mathbb{Z}_{2}$.

Finally, if $S$ and $S^{\prime}$ are $\mathbb{Z}_{2}$-graded $A$ - and $B$-modules respectively, then their graded tensor product $S \hat{\otimes} S^{\prime}$ is a $\mathbb{Z}_{2}$-graded $A \hat{\otimes} B$-module with action given by

$$
\gamma_{a \otimes b}\left(s \otimes s^{\prime}\right):=(-1)^{|b||s|} a(s) \otimes b\left(s^{\prime}\right),
$$

where $s \in S, s^{\prime} \in S^{\prime}, a \in A, b \in B,|b|:=\operatorname{deg}(b),|s|:=\operatorname{deg}(s)$ are the degrees of the homogeneous elements $b$ and $s$.

We apply these facts to Clifford algebras and their representations. Assume that $\left(V_{+}, q_{+}\right)$and $\left(V_{-}, q_{-}\right)$are two vector spaces with scalar products and let $\left(V:=V_{+} \oplus V_{-}, q:=q_{+}+q_{-}\right)$be their direct sum. As $\mathrm{Cl}\left(V_{ \pm}\right)$are $\mathbb{Z}_{2}$-graded algebras we can consider $\mathrm{Cl}\left(V_{+}\right) \hat{\otimes} \mathrm{Cl}\left(V_{-}\right)$which is also a $\mathbb{Z}_{2}$-graded algebra, and as such is isomorphic to $\mathrm{Cl}(V)$ (for the latter statement see e.g. Chapter I of [19]). The isomorphism between $\mathrm{Cl}(V)$ and $\mathrm{Cl}\left(V_{+}\right) \hat{\otimes} \mathrm{Cl}\left(V_{-}\right)$is obtained by extending the morphism $V \rightarrow \mathrm{Cl}\left(V_{+}\right) \hat{\otimes} \mathrm{Cl}\left(V_{-}\right)$which maps any $v=v_{+}+v_{-} \in V_{+} \oplus V_{-}$to $v_{+} \otimes 1+1 \otimes v_{2}$.

Let $S_{ \pm}$be $\mathbb{Z}_{2}$-graded $\mathrm{Cl}\left(V_{ \pm}\right)$-modules. From above, the graded tensor product $S_{+} \hat{\otimes} S_{-}$is a $\mathbb{Z}_{2^{-}}$graded $\mathrm{Cl}\left(V_{+}\right) \hat{\otimes} \mathrm{Cl}\left(V_{-}\right)$-module, hence also a $\mathbb{Z}_{2^{-}}$ graded $\mathrm{Cl}(V)$-bundle. Any $v=v_{+}+v_{-} \in V_{+} \oplus V_{-} \subset \mathrm{Cl}(V)$ acts on $S_{+} \hat{\otimes} S_{-}$ as

$$
\begin{equation*}
\gamma_{v_{+}+v_{-}}\left(s_{+} \otimes s_{-}\right)=\gamma_{v_{+}}\left(s_{+}\right) \otimes s_{-}+(-1)^{\left|s_{+}\right|} s_{+} \otimes \gamma_{v_{-}}\left(s_{-}\right) . \tag{108}
\end{equation*}
$$

### 11.2 Integrability of generalized almost complex structures and spinors

Let $E$ be a regular Courant algebroid of rank $2 n$ with anchor $\pi: E \rightarrow T M$ and Dirac generating operator $d: \Gamma(\mathbb{S}) \rightarrow \Gamma(\mathbb{S})$. An almost Dirac structure of $E_{\mathbb{C}}$ is an isotropic (complex) subbundle of $E_{\mathbb{C}}$ of rank $n$. It is integrable (or a Dirac structure) if is is closed under the (complex linear extension of) the Dorfman bracket of $E$. It is well-known that there is a one to one correspondence between almost Dirac structures on $E_{\mathbb{C}}$ and classes of projectively equivalent pure spinors of $\mathbb{S}_{\mathbb{C}}$, as follows. For a non-vanishing section $\eta \in \Gamma\left(\mathbb{S}_{\mathbb{C}}\right)$ we define

$$
L_{\eta}:=\left\{v \in E_{\mathbb{C}} \mid \gamma_{v} \eta=0\right\} .
$$

The spinor $\eta$ is called pure if $L_{\eta}$ is a vector bundle of rank $n$. A simple computation shows that $L_{\eta}$ is isotropic. Being of rank $n, L_{\eta}$ is an almost Dirac structure. It is called the null bundle of $\eta$. The assignment $L_{\eta} \rightarrow[\eta]$ is a one-to-one correspondence between almost Dirac structures of $E_{\mathbb{C}}$ and classes of projectively equivalent pure spinors of $\mathbb{S}$ (two pure spinors $\eta_{1}$ and $\eta_{2}$ defined on an open set $U \subset M$ are projectively equivalent if $\eta_{2}=f \eta_{1}$ for a non-vanishing function $f$ on $U$ ). Note that any pure spinor which is projectively equivalent to a projectively closed spinor is also projectively
closed. This follows from $\not d(f \eta)=\gamma_{\pi^{*}(d f)}(\eta)+f d(\eta)$, for any $\eta \in \Gamma\left(S_{\mathbb{C}}\right)$ and $f \in C^{\infty}(M, \mathbb{C})$.

Theorem 77. ([1]) An almost Dirac structure $L$ of $E_{\mathbb{C}}$ is a Dirac structure if and only if, locally, any pure spinor $\eta$ associated to $L$ is projectively closed.

Proof. Assume that $\eta$ is projectively closed and let $e \in \Gamma\left(E_{\mathbb{C}}\right)$ such that $\not d(\eta)=\gamma_{e} \eta$. Let $v, w \in \Gamma(L)$. Using condition ii) from Definition 45 and $\gamma_{v} \eta=\gamma_{w} \eta=0$, we obtain

$$
\begin{equation*}
\left.\gamma_{[v, w]} \eta=\left[\left[d, \gamma_{v}\right], \gamma_{w}\right]\right] \eta=-\gamma_{w} \gamma_{v} d(\eta)=-\gamma_{w} \gamma_{v} \gamma_{e} \eta . \tag{109}
\end{equation*}
$$

On the other hand,

$$
\begin{equation*}
\gamma_{w} \gamma_{v} \gamma_{e}=-\gamma_{w} \gamma_{e} \gamma_{v}+2\langle v, e\rangle \gamma_{w} . \tag{110}
\end{equation*}
$$

Combining (109) with (110) and using $\gamma_{v} \eta=\gamma_{w} \eta=0$, we obtain $\gamma_{[v, w]} \eta=0$. This proves that $L$ is a Dirac structure.

Conversely, assume that $L$ is a Dirac structure. Then, for any $v, w \in$ $\Gamma(L),[v, w] \in \Gamma(L)$ and $\gamma_{[v, w]} \eta=0$. This implies, using condition ii) from Definition 45, [[d, $\left.\left.\gamma_{v}\right], \gamma_{w}\right] \eta=0$, or $\gamma_{w}\left[d, \gamma_{v}\right] \eta=0, \forall w \in \Gamma(L)$. We obtain that $\left[d, \gamma_{v}\right] \eta$, which is equal to $\gamma_{v} d(\eta)$, is a multiple of $\eta$, i.e. $\gamma_{v} d(\eta)=\lambda(v) \eta$ for $\lambda(v) \in C^{\infty}(M, \mathbb{C})$. Remark that $\lambda \in \Gamma\left(L^{*}\right)$. Extend $\lambda$ to a (complex linear) 1-form on $E_{\mathbb{C}}$ and let $v_{0} \in \Gamma\left(E_{\mathbb{C}}\right)$, such that $2 v_{0}$ is dual to this 1-form with respect to the complex linear extension of $\langle\cdot, \cdot\rangle$. Then

$$
\lambda(v) \eta=2\left\langle v, v_{0}\right\rangle \eta=\gamma_{v} \gamma_{v_{0}} \eta+\gamma_{v_{0}} \gamma_{v} \eta=\gamma_{v} \gamma_{v_{0}} \eta .
$$

The above computations show that $\gamma_{v}\left(\mathbb{d}(\eta)-\gamma_{v_{0}} \eta\right)=0$, for any $v \in \Gamma(L)$, which implies $\not d(\eta)-\gamma_{v_{0}} \eta=g \eta$ for $g \in C^{\infty}(M, \mathbb{C})$. As $\not d$ and $\gamma_{v_{0}}$ are odd operators and pure spinors are chiral, i.e. either even or odd, we conclude $g=0$. This shows that $\not d(\eta)=\gamma_{v_{0}} \eta$, i.e. $\eta$ is projectively closed.

Let $\mathcal{J}$ be a generalized almost complex structure on $E$. The ( 1,0 )-bundle $L \subset E_{\mathbb{C}}$ of $\mathcal{J}$ is isotropic with respect to $\langle\cdot, \cdot\rangle$ and satisfies $L \oplus \bar{L}=E_{\mathbb{C}}$. In particular, $\operatorname{rank} L=n$ and $L$ is an almost Dirac structure. A pure spinor $\eta \in \Gamma(\mathbb{S})$ is called associated to $\mathcal{J}$ if $L=L_{\eta}$. From Theorem [77 we obtain:

Corollary 78. A generalized almost complex structure $\mathcal{J}$ on a regular Courant algebroid is integrable if and only if, locally, one (equivalently, any) pure spinor associated to $\mathcal{J}$ is projectively closed.

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V. Cortés: vicente.cortes@math.uni-hamburg.de

Department of Mathematics and Center for Mathematical Physics, University of Hamburg, Bundesstrasse 55, D-20146, Hamburg, Germany.
L. David: liana.david@imar.ro

Institute of Mathematics 'Simion Stoilow' of the Romanian Academy, Calea Grivitei no. 21, Sector 1, 010702, Bucharest, Romania.


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[^1]:    ${ }^{1}$ Recall that the notion of a partial connection is defined by the same properties as that of a connection, namely tensoriality in the first argument and Leibniz rule in the second.

