The Complete Lattice of Erdős-Menger Separations

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Abstract

F. Escalante and T. Gallai studied in the seventies the structure of different kind of separations and cuts between a vertex pair in a (possibly infinite) graph. One of their results is that if there is a finite separation, then the optimal (i.e. minimal sized) separations form a finite distributive lattice with respect to a natural partial order. Furthermore, any finite distributive lattice can be represented this way.

If there is no finite separation then cardinality is a too rough measure to capture being "optimal". Menger's theorem provides a structural characterization of optimality if there is a finite separation. We use this characterization to define Erdős-Menger separations even if there is no finite separation. The generalization of Menger's theorem to infinite graphs (which was not available until 2009) ensures that Erdős-Menger separations always exist. We show that they form a complete lattice with respect to the partial order given by Escalante and every complete lattice can be represented this way.

1 Introduction

The investigation of the structure of several type of separations (i.e. vertex cuts) and cuts in graphs has been started in the seventies by F. Escalante and T. Gallai. For their original papers see [1] and [2] and for an English survey about these and further results in this area we recommend the chapter "Lattices Related to Separation in Graphs" of [3] by R. Halin.

Among other results, it was discovered by Escalante that if there is a finite separation between two vertex sets in a given graph, then the optimal (minimal sized) separations form a finite distributive lattice with respect to a natural partial order. Furthermore, any finite distributive lattice can be represented this way. Without having a finite separation it was unclear which separations we should consider "optimal". By Menger's theorem, a finite separation S between two vertex sets is optimal if and only if there is a system of disjoint paths joining them such that S consists of choosing exactly one vertex from each of these paths. Based on this characterisation, the concept of optimal separation can be interpreted without having a finite separation. The generalization of Menger's theorem to infinite graphs (see [4]) ensures that this definition makes sense, this kind of separation always exists. Since the infinite version of Menger's theorem was conjectured by P. Erdős, we call them Erdős-Menger separations. Our main result is that the Erdős-Menger separations always form a complete lattice and every complete lattice can be represented as an Erdős-Menger separation lattice. We are working with digraphs but our results remain true in undirected graphs as well with obvious modification of the proofs. The paper is structured as follows. We introduce few notation in the next section. The main result

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is discussed in the third section. Finally at the Appendix we show by an example that to the contrary of the finite case the Erdős-Menger separation lattice is not necessarily a sublattice of the minimal separation lattice.

2 Notation

Let $D = (V, E)$ be a possibly infinite digraph and $A, B \subseteq V$. Later we will omit D from our notation whenever it is fixed or clear from the context. A finite directed path P is an $\boldsymbol{A \to B}$ **path** if exactly its first vertex is in A and exactly its last is in B. Let $\boldsymbol{\mathfrak{D}_D(A, B)}$ consist of the systems $\mathcal{P}$ of (pairwise) disjoint $A \to B$ paths. We write $\boldsymbol{V_{\text{first}}(\mathcal{P})}$ for the set of the first vertices of the paths in $\mathcal{P}$ and we define $\boldsymbol{V_{\text{last}}(\mathcal{P})}$ analogously. Let us write $\boldsymbol{\mathfrak{M}_D(A, B)}$ for the set of the minimal AB-separations in D, i.e., those $S \subseteq V$ for which every $A \to B$ path in D meets S and S is $\subseteq$-minimal with respect to this property. We consider the following relation $\trianglelefteq$ on $\mathfrak{M}_D(A, B)$. Let $S \trianglelefteq T$ if S separates T from A (i.e. S meets every $A \to T$ path of D). It is known that $\mathfrak{M}_D(A, B)$ is a complete lattice (see Proposition 3.3), we use **inf** and **sup** always with respect to this lattice. A vertex set S is orthogonal to a system $\mathcal{P}$ of disjoint paths (we write $\boldsymbol{S \perp \mathcal{P}}$) if S consists of choosing exactly one vertex from each path of $\mathcal{P}$. The formal definition of the **Erdős-Menger separations** is the following.

$$\boldsymbol{\mathfrak{S}_D(A, B)} := \{S \in \mathfrak{M}_D(A, B) : \exists \mathcal{P} \in \mathfrak{D}_D(A, B) \text{ with } S \perp \mathcal{P}\}.$$

The non-emptiness of $\mathfrak{S}_D(A, B)$ in the general case is guaranteed by the Aharoni-Berger theorem (see [4]). Finally let

$$\boldsymbol{\mathfrak{S}_D^-(A, B)} := \{S \in \mathfrak{M}_D(A, B) : \exists \mathcal{P} \in \mathfrak{D}_D(A, S) \text{ with } V_{\text{last}}(\mathcal{P}) = S\}$$
$$\boldsymbol{\mathfrak{S}_D^+(A, B)} := \{S \in \mathfrak{M}_D(A, B) : \exists \mathcal{P} \in \mathfrak{D}_D(S, B) \text{ with } V_{\text{first}}(\mathcal{P}) = S\}.$$

3 Main result

3.1 Preliminaries

We will need some of the basic facts discovered by Escalante. He formulated originally these results for undirected graphs and for separations between vertex pairs in his paper [1] (which is in German). We will give here all the necessary details to make the paper self-contained . From now on let a digraph $D = (V, E)$ and $A, B \subseteq V$ be fixed. If a statement is "symmetric", then we prove just one half of it without mentioning this every time explicitly.

The role of A and B are seemingly not symmetric in the definition of $\trianglelefteq$ (the definition based on A and does not mention B). The following Proposition restore the symmetry.

Proposition 3.1. *Let $S, T \in \mathfrak{M}(A, B)$. Then T separates S from A if and only if S separates B from T.*

Proof. Assume that T separates S from A. Let P be a $T \to B$ path starting at u. Pick an $A \to T$ path Q terminating at u (it exists by the minimality of T). The path Q cannot meet S before u because T separates S from A. Let R be the path that we obtain by uniting Q and P. It is an $A \to B$ path therefore it meets S. Thus P meets S. □

Proposition 3.2. *$\trianglelefteq$ is a partial order.*

Proof. The reflexivity and transitivity are obvious. Let $S, T \in \mathfrak{M}(A,B)$ and assume that $S \trianglelefteq T$ and $T \trianglelefteq S$. Let $u \in T$ be arbitrary and pick an $A \to B$ path P which meets T only at u. Then S cannot have a vertex on P before u because $T \trianglelefteq S$. On the other hand, S cannot have a vertex on P after u since T separates B from S (use Proposition 3.1 and $S \trianglelefteq T$). It follows that $u \in S$ thus $S \supseteq T$ and by minimality $S = T$. □

Proposition 3.3. *$(\mathfrak{M}(A,B), \trianglelefteq)$ is a complete lattice, where for a nonempty $\mathcal{S} \subseteq \mathfrak{M}(A,B)$, $\inf \mathcal{S}$ consists of those $s \in \bigcup \mathcal{S}$ which are reachable from A without touching any other element of $\bigcup \mathcal{S}$.*

Proof. The set we claimed to be $\inf \mathcal{S}$, say S, separates every element of $\mathcal{S}$ from A. Furthermore, if a $T \in \mathfrak{M}(A,B)$ separates all the separations in $\mathcal{S}$ from A, then it separates S from A as well.

It remains to check the $\subseteq$-minimality of S. Let $s \in S$ be arbitrary. We need to find an $A \to B$ path that meets S only at s. By the definition of S, there is an $A \to s$ path P which avoids $\bigcup \mathcal{S} \setminus \{s\}$. Pick a $T \in \mathcal{S}$ for which $s \in T$. Since T is a minimal separation, there is a $S \to B$ path Q starting at s. The vertices $V(Q) \setminus \{s\}$ are not reachable from A without touching T thus they are not in S. Hence by uniting P and Q we obtain a desired $A \to B$ path. □

Theorem 3.4 (Escalante)**.** *If $\mathfrak{M}(A,B)$ has a finite element, then $\mathfrak{S}(A,B)$ is a finite distributive sublattice of $\mathfrak{M}(A,B)$.*

Proof. Let a nonempty $\mathcal{S} \subseteq \mathfrak{S}(A,B)$ be given. We fix a maximal-sized element $\mathcal{P}$ of $\mathfrak{D}(A,B)$. Note that an $S \in \mathfrak{M}(A,B)$ is in $\mathfrak{S}(A,B)$ iff $S \perp \mathcal{P}$. Every vertex in $\inf \mathcal{S}$ is coming from an optimal separation and hence used by $\mathcal{P}$. Let $P \in \mathcal{P}$ be arbitrary and let s be the first vertex of P which is in $\inf \mathcal{S}$. There is a $S \in \mathcal{S}$ such that $s \in S$. Since $S \perp \mathcal{P}$, all the vertices of P after s are separated from A by S and hence cannot be in $\inf \mathcal{S}$. Therefore $\inf \mathcal{S} \perp \mathcal{P}$ which means $S \in \mathfrak{S}(A,B)$. Finally let H be the digraph consists of A, B and the paths in $\mathcal{P}$. Then $\mathfrak{S}(A,B)$ is a sublattice of the finite, distributive lattice $\mathfrak{M}_H(A,B)$, thus it is distributive. □

3.2 The complete lattice of the Erdős-Menger separations

Theorem 3.5. *For every digraph $D = (V,E)$ and $A, B \subseteq V$, $\mathfrak{S}_D(A,B)$ is a nonempty complete lattice (with respect to the restriction of $\trianglelefteq$).*

Proof. The non-emptiness of the subposet $\mathfrak{S}(A,B)$ of $\mathfrak{M}(A,B)$ is exactly the following theorem.

Theorem 3.6 (R. Aharoni and E. Berger, [4])**.** *For any (possibly infinite) digraph $D = (V,E)$ and $A, B \subseteq V$, $\mathfrak{S}_D(A,B) \neq \varnothing$.*

Proposition 3.7. *If $S \in \mathfrak{S}^+(A,B)$, then $\mathfrak{S}(A,S) = \{T \in \mathfrak{S}(A,B) : T \trianglelefteq S\}$.*

Proof: Let $T \in \mathfrak{S}(A,S)$. Take a $\mathcal{P} \in \mathfrak{D}(A,S)$ with $T \perp \mathcal{P}$. Since $S \in \mathfrak{S}^+(A,B)$, we can continue forward the paths $\mathcal{P}$ to obtain an element of $\mathfrak{D}(A,B)$ which shows $T \in \mathfrak{S}(A,B)$. Assume now that $T \in \mathfrak{S}(A,B)$ with $T \trianglelefteq S$. Take a $\mathcal{Q} \in \mathfrak{D}(A,B)$ with $T \perp \mathcal{Q}$. The initial segments of the paths $\mathcal{Q}$ up to S show $T \in \mathfrak{S}(A,S)$. ■

Lemma 3.8. *$\mathfrak{S}^+(A,B)$ is closed under the* inf *operation of $\mathfrak{M}(A,B)$ and $\mathfrak{S}^-(A,B)$ is closed under* sup*.*

Proof: Let $\{S_\xi\}_{\xi<\kappa} \subseteq \mathfrak{S}^+(A,B)$ be nonempty and let $S_{<\alpha} := \inf\{S_\xi : \xi < \alpha\}$. For every $0 < \alpha \leq \kappa$ and every $s \in S_{<\alpha}$ we define a path P_s that goes from s to B such that for each α the paths $\{P_s : s \in S_{<\alpha}\}$ are disjoint and hence witness $S_{<\alpha} \in \mathfrak{S}^+(A,B)$.

For $\alpha = 1$ we pick an arbitrary path-system that witnesses $S_0 \in \mathfrak{S}^+(A,B)$. If α is a limit ordinal and P_s is defined whenever $s \in S_{<\xi}$ for some $\xi < \alpha$, then from the characterisation of inf (see Proposition 3.3) it follows that P_s is defined for every $s \in S_{<\alpha}$. If $s \neq s' \in S_{<\alpha}$, then for every large enough ξ we have $s, s' \in S_{<\xi}$, thus by the induction hypothesis P_s and $P_{s'}$ are disjoint. Suppose now that $\alpha = \beta + 1$. Every $s \in S_{<\beta+1} \setminus S_{<\beta}$ is in S_β hence we may fix a $\{Q_s : s \in S_{<\beta+1} \setminus S_{<\beta}\} \in \mathfrak{D}(S_\beta, B)$ where Q_s goes from s to B. Since $S_{<\beta}$ separates B from $S_{<\beta+1}$ (see Proposition 3.1), each Q_s meets $S_{<\beta}$. Assume that the first common vertex of Q_s with $S_{<\beta}$ is s'. Note that $s' \notin S_{<\beta+1}$ because S_β separates $s' \notin S_\beta$ from A. Unite the initial segment of Q_s up to s' and $P_{s'}$ to obtain P_s. ■

Claim 3.9. *$\mathfrak{S}(A,B)$ has a smallest and a largest element, namely* $\inf \mathfrak{S}^+(A,B)$ *and* $\sup \mathfrak{S}^-(A,B)$.

Proof: Let $S := \inf \mathfrak{S}^+(A,B)$. By Lemma 3.8, $S \in \mathfrak{S}^+(A,B)$. By Proposition 3.7, $\mathfrak{S}(A,S) = \{T \in \mathfrak{S}(A,B) : T \trianglelefteq S\}$. Since $\mathfrak{S}(A,B) \subseteq \mathfrak{S}^+(A,B)$, the set $\{T \in \mathfrak{S}(A,B) : T \trianglelefteq S\}$ cannot have an element strictly smaller than S. By Theorem 3.6, $\mathfrak{S}(A,S) \neq \varnothing$, thus its only element must be S. ■

Let $\mathcal{S} \subseteq \mathfrak{S}(A,B)$ be nonempty. Since $\mathfrak{S}(A,B) \subseteq \mathfrak{S}^+(A,B)$ and by Lemma 3.8 $\mathfrak{S}^+(A,B)$ is closed under the inf operation of $\mathfrak{M}(A,B)$, $\inf \mathcal{S} =: S \in \mathfrak{S}^+(A,B)$. Being smaller or equal to all the elements of $\mathcal{S}$ means being smaller or equal to S. By Proposition 3.7, the lower bounds of S in $\mathfrak{S}(A,B)$ are exactly the elements of $\mathfrak{S}(A,S)$ which has a largest element by Claim 3.9. It is the desired largest lower bound of $\mathcal{S}$ with respect to the poset $\mathfrak{S}(A,B)$.

Remark 3.10. Theorem 3.5 remains true if the graph is undirected or if we consider cuts instead of separations. The proof is essentially the same.

□

3.3 Representation as Erdős-Menger separation lattice

Theorem 3.11. *Every complete lattice is representable as an Erdős-Menger separation lattice.*

Proof. We reduce our theorem to the following theorem of Escalante.

Theorem 3.12 (Escalante, [1])**.** *For every complete lattice L, there is a digraph $D = (V,E)$ and $A, B \subseteq V$ such that $\mathfrak{M}_D(A,B)$ is isomorphic to L.*

Remark 3.13. For an English source, see Theorem 6 on page 157 of [3]. It has been formulated originally for undirected graphs.

Let L be a given complete lattice. First we choose $D = (V,E), A, B$ according to Theorem 3.12. The only thing we need to do is to blow up the vertices of this system. Indeed, consider $V' = V \times \kappa$ where $\kappa := |V| + \aleph_0$ and draw an edge from (u,α) to (v,β) iff $uv \in E$ to obtain $D' = (V', E')$. We define A' to be $A \times \kappa$ and B' to be $B \times \kappa$.

Note that if $(v,\alpha) \in S \in \mathfrak{M}_{D'}(A',B')$ then necessarily $\{v\} \times \kappa \subseteq S$ otherwise (v,α) would be omittable in S contradicting its $\subseteq$-minimality. It implies that $T' \in \mathfrak{M}_{D'}(A',B')$ iff there is a $T \in \mathfrak{M}_D(A,B)$ such that $T' = T \times \kappa$. Therefore $\mathfrak{M}_{D'}(A',B') \cong \mathfrak{M}_D(A,B)$. It is enough to show that $\mathfrak{M}_{D'}(A',B') = \mathfrak{S}_{D'}(A',B')$. To prove the non-trivial inclusion, take an arbitrary $T' \in \mathfrak{M}_{D'}(A',B')$. Then $T' = T \times \kappa$ for some $T \in \mathfrak{M}_D(A,B)$. For $t \in T$, we can pick an $A \to B$ path $P_t = v_0, \ldots, v_i, t, v_{i+1}, \ldots, v_{n_t}$ in D where $v_j \notin T$. Take an injection $f : T \times \kappa \to \kappa$. Let $P_{(t,\alpha)}$ that we obtain from P_t by replacing t with (t,α) and v_j by $(v_j, f(t,\alpha))$. It is easy to check that the path-system $\{P_{(t,\alpha)} : (t,\alpha) \in T'\}$ exemplifies $T' \in \mathfrak{S}_{D'}(A',B')$. □

3.4 Appendix

We show that $\mathfrak{S}_D(A,B)$ is not necessarily a sublattice of $\mathfrak{M}_D(A,B)$. Consider the digraph D and vertex sets A, B at Figure 1. We have $S := \{\dots, b_{-2}, b_{-1}, u, a_1, a_2, \dots\} \in \mathfrak{S}(A,B)$ witnessed by

$$\{\dots a_{-2}b_{-2}, a_{-1}b_{-1}, a_0ub_1, a_1b_2, a_2b_3, \dots\} \in \mathfrak{D}(A,B).$$

We also have $T := \{\dots, a_{-2}, a_{-1}, v, b_1, b_2, \dots\} \in \mathfrak{S}(A,B)$ witnessed by

$$\{\dots a_{-2}b_{-3}, a_{-1}b_{-2}, a_0vb_{-1}, a_1b_1, a_2b_2, \dots\} \in \mathfrak{D}(A,B).$$

Here $\inf\{S,T\} = (A \setminus \{a_0\}) \cup \{u,v\}$.

But an $A \to B$ path through u must start at a_0 as well as an $A \to B$ path through v, thus $(A \setminus \{a_0\}) \cup \{u,v\} \notin \mathfrak{S}(A,B)$.

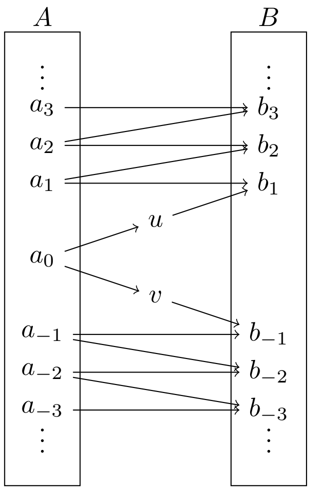

Figure 1: A system with $S,T \in \mathfrak{S}(A,B)$ where $\inf\{T,S\} \notin \mathfrak{S}(A,B)$

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